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ORIGINAL PAPER

Chebyshev spectral collocation method for system of nonlinear Volterra integral equations

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Abstract
We investigate Chebyshev spectral collocation method for system of nonlinear Volterra integral equations. We choose Chebyshev Gauss points as collocation points, and approximate integral terms by Legendre Gauss quadrature formula. The provided convergence analysis shows that numerical errors decay exponentially, which is one of the most prominent features of spectral methods. Numerical experiments are carried out to confirm theoretical results. We have never seen any paper investigating the spectral method for the system of nonlinear Volterra integral equations (VIEs). The present method and corresponding convergence analysis would be useful in studying numerical methods for the system of integral equations and partial differential equations.

Keywords Spectral collocation method · System of nonlinear VIEs · Convergence analysis · Numerical experiments

1 Introduction

Volterra integral equations (VIEs) arise in many practice applications, such as population dynamics and spread of epidemics [2], wave problems [7], and semi-conductor devices [17]. There are many numerical methods for the system of nonlinear VIEs, such as a single-term Walsh series method [15], product integration methods [12], restarted Adomian method [8], block by block method [13], fast Runge-Kutta methods [5], Chebyshev wavelet methods [1], and piecewise polynomial collocation method [3]. Spectral methods are classic high-precision methods. One of their

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most prominent features is the exponential convergence of the resulting approximations. These methods are widely applied to solve partial differential equations [4] and Volterra integral equations [6, 10, 11, 16]. Based on these works of our predecessors, we have investigated piecewise spectral collocation methods for the system of linear VIEs in [9]. Continuing these researches, we study Chebyshev spectral collocation methods for the system of nonlinear VIEs in the present paper.

Consider the following system of nonlinear VIEs:

$$\mathbf{Y}(t) = \mathbf{g}(t) + \int_0^t \mathbf{K}(t, s)\mathbf{f}(s, \mathbf{Y}(s))ds, t \in [0, 1], \tag{1}$$

where,

$$\mathbf{g}(t) := [g_1(t), g_2(t), \cdots, g_Q(t)]',$$

$$\mathbf{Y}(t) := [Y_1(t), Y_2(t), \cdots, Y_Q(t)]',$$

$$\mathbf{K}(t, s) := \begin{bmatrix} K_{pq}(t, s) \\ {\scriptstyle 1\le p,q\le Q} \end{bmatrix}, (t, s) \in \Delta := \{(t, s) : 0 \le s \le t \le 1\},$$

$$\mathbf{f}(s, \mathbf{Y}(s)) := [f_1(s, \mathbf{Y}(s)), f_2(s, \mathbf{Y}(s)), \cdots, f_Q(s, \mathbf{Y}(s))]',$$

$$f_p(s, \mathbf{Y}(s)) := f_p(s, Y_1(s), Y_2(s), \cdots, Y_Q(s)), p = 1, 2, \cdots, Q,$$

$$(s, \mathbf{Y}(s)) \in \Omega := [0, 1] \times R^Q.$$

VIEs (1) can be written as the following form:

$$\mathbf{Y}(t) = \mathbf{g}(t) + \int_0^t \mathbf{K}(t, s)\mathbf{y}(s)ds, \tag{2}$$

$$\mathbf{y}(t) = \mathbf{f}(t, \mathbf{g}(t) + \int_0^t \mathbf{K}(t, s)\mathbf{y}(s)ds), \tag{3}$$

where,

$$\mathbf{y}(t) := [y_1(t), y_2(t), \cdots, y_Q(t)]'.$$

In general cases, if given functions in VIEs (1) are smooth, then corresponding solutions are smooth [3].

We choose Chebyshev Gauss points of order N in interval $[0, 1]$ as collocation points, and approximate integral terms by Legendre Gauss quadrature formula of order N. Finally, we obtain the numerical solution $\mathbf{y}^N(t)$ to (3). Replacing $\mathbf{y}(t)$ in (2) by $\mathbf{y}^N(t)$, we get the solution $\mathbf{Y}^N(t)$ to (1). Provided convergence analysis shows that the convergence order of the present method is $N^{(1/2)-m}$, where m is the index of regularity of given functions. This result implies that, given functions possess more regularity, numerical decays faster. We carry out numerical experiments to confirm theoretical results. It is worth noting that N and m are independent of each other. This is the key difference between the present method and piecewise polynomial collocation methods [3].

This paper is organized as follows. Numerical method is demonstrated in Section 2. Useful lemmas are prepared in Section 3. Convergence is analyzed in Section 4. Numerical experiments are carried out in Section 5. Finally, we end with conclusion and future work in Section 6.

2 Numerical method

Let,

$$t_i := (x_i + 1)/2,$$

where $x_i, i = 0, 1, \cdots, N$ are Chebyshev Gauss points of order N in interval $[-1, 1]$. VIEs (3) hold at t_i:

$$\mathbf{y}(t_i) = \mathbf{f}(t_i, \mathbf{g}(t_i) + \int_0^{t_i} \mathbf{K}(t_i, s)\mathbf{y}(s)ds), i = 0, 1, \cdots, N. \tag{4}$$

Let y_{pi} be the approximation of $y_p(t_i)$. Then, $y_p(t)$ can by approximated by:

$$y_p^N(t) := \sum_{j=0}^{N} y_{pi} L_j(t),$$

where, $L_j(t)$ is the jth Lagrange interpolation basic function associated with collocation points $t_i, i = 0, 1, \cdots, N$. Thus, $\mathbf{y}(t)$ can by approximated by:

$$\mathbf{y}^N(t) := [y_1^N(t), y_2^N(t), \cdots, y_Q^N(t)]'.$$

Therefore, (4) can by approximated by:

$$\mathbf{y}_i \approx \mathbf{f}(t_i, \mathbf{g}(t_i) + \int_0^{t_i} \mathbf{K}(t_i, s)\mathbf{y}^N(s)ds), i = 0, 1, \cdots, N, \tag{5}$$

where,

$$\mathbf{y}_i := [y_{1i}, y_{2i}, \cdots, y_{Qi}]'.$$

Approximate integral terms in the right-hand side of (5) by Legendre Gauss quadrature formula of order N:

$$\mathbf{y}_i = \mathbf{f}(t_i, \mathbf{g}(t_i) + \frac{t_i}{2} \sum_{k=0}^{N} \mathbf{K}(t_i, s(t_i, v_k))\mathbf{y}^N(s(t_i, v(k)))w_k), i = 0, 1, \cdots, N, \tag{6}$$

where, v_k and w_k are Legendre Gauss and weights of order N in interval $[-1, 1]$, and variable transformation $s(t, v)$ is defined by:

$$s(t, v) := \frac{t}{2}(v + 1), v \in [-1, 1].$$

Solving system (6), we obtain $y_{pi}, i = 0, 1, \cdots, N, p = 1, 2, \cdots, Q$, thus obtaining $\mathbf{y}^N(t)$. Let $\mathbf{y}^N(t)$. Replace $\mathbf{y}(t)$ in (2). We obtain an approximation to (2):

$$\mathbf{Y}(t) \approx \mathbf{g}(t) + \int_0^{t} \mathbf{K}(t, s)\mathbf{y}^N(s)ds. \tag{7}$$

Approximate integral term in the right-hand side of (7) by Legendre Gauss quadrature formula of order N:

$$\mathbf{Y}(t) \approx \mathbf{g}(t) + \frac{t}{2}\sum_{k=0}^{N}\mathbf{K}(t, s(t, v_k))\mathbf{y}^N(s(t, v(k)))w_k. \tag{8}$$

Define:

$$\mathbf{Y}_i := \mathbf{g}(t_i) + \frac{t_i}{2}\sum_{k=0}^{N}\mathbf{K}(t_i, s(t_i, v_k))\mathbf{y}^N(s(t_i, v(k)))w_k, i = 0, 1, \cdots, N, \tag{9}$$

where,

$$\mathbf{Y}_i = [Y_{1i}, Y_{2i}, \cdots, Y_{Qi}]' \approx \mathbf{Y}(t_i).$$

Finally, we obtain the numerical solution to (1):

$$\mathbf{Y}^N(t) := [Y_1^N(t), Y_2^N(t), \cdots, Y_Q^N(t)], \tag{10}$$

where,

$$Y_p^N(t) := \sum_{j=0}^{N} Y_{pj}L_j(t).$$

3 Useful lemmas

In order to state lemmas easily, we define some function spaces and symbols.

Denote the space of continuous functions on domain $\widetilde{\Omega}$ by $C(\widetilde{\Omega})$, equipped with norm $\|\cdot\|_{L^\infty(\widetilde{\Omega})}$. Denote by $C^m(\widetilde{\Omega})$ the space of functions possessing continuous derivatives of order m. The derivatives of order m of $u(t)$ with respect to variable t are denoted by ∂_t^m. Polynomial of degree not exceeding N on domain $\widetilde{\Omega}$ is denoted by $P_N(\widetilde{\Omega})$. The interpolation operator associated with $t_i, i = 0, 1, \cdots, N$ is denoted by I_N, namely:

$$I_N(u)(t) := \sum_{j=0}^{N} u(t_j)L_j(t), u(t) \in C(0, 1). \tag{11}$$

Identity operator is denoted by I, namely $I(u)(t) \equiv u(t)$.

Assume that matrix of P rows and Q columns is

$$\mathbf{A}(x) := \begin{bmatrix} a_{pq}(x) \\ {\scriptstyle 1\le p\le P,\ 1\le q\le Q} \end{bmatrix},$$

whose elements $a_{pq}(x),\ 1 \le p \le P, 1 \le q \le Q$ are functions on domain $\widetilde{\Omega}$. Let

$$\mathbf{A}(x) \in C(\widetilde{\Omega}) \text{ if } a_{pq}(x) \in C(\widetilde{\Omega}), 1 \le p \le P, 1 \le q \le Q,$$

$$\mathbf{A}(x) \in C^m(\widetilde{\Omega}) \text{ if } a_{pq}(x) \in C^m(\widetilde{\Omega}), 1 \le p \le P, 1 \le q \le Q,$$

$$\mathbf{A}(x) \in P_N(\widetilde{\Omega}) \text{ if } a_{pq}(x) \in P_N(\widetilde{\Omega}), 1 \le p \le P, 1 \le q \le Q.$$

Integrating matrix $A(x)$ on domain D means integrating each elements of $A(x)$ on domain D, namely:

$$\int_D \mathbf{A}(x)dx := \left[\int_D a_{pq}(x)dx \right]_{1\le p\le P, 1\le q\le Q}, D \subseteq \widetilde{\Omega}.$$

Interpolating matrix $A(x)$ means interpolating each elements of $A(x)$:

$$I_N(\mathbf{A})(x) := \left[I_N(a_{pq})(x) \right]_{1\le p\le P, 1\le q\le Q}, \mathbf{A}(x) \in C(\widetilde{\Omega}).$$

Taking derivatives of matrix $A(x)$ means taking derivatives of each element of $A(x)$:

$$\partial_x^m \mathbf{A}(x) := \left[\partial_x^m a_{pq}(x) \right]_{1\le p\le P, 1\le q\le Q}.$$

Define,

$$|\mathbf{A}(x)| := \max_{\substack{1\le p\le P \\ 1\le q\le Q}} |a_{pq}(x)|.$$

This means that $|\mathbf{A}(x)|$ is a nonnegative real function. Define:

$$\|\mathbf{A}\|_{L^\infty(\widetilde{\Omega})} := \max_{x\in\widetilde{\Omega}} |\mathbf{A}(x)|.$$

In order to easily state the proof in convergence analysis in Section 4, we introduce some new symbols for the given function $\mathbf{f}$ in VIEs (1), which is exactly denoted by $\mathbf{f}(z_1, \mathbf{z}_2)$, $\mathbf{z}_2 = (\theta_1, \theta_2, \cdots, \theta_Q)$. The partial derivative of $\mathbf{f}$ with respect to z_1 is denoted by:

$$\partial_{z_1}^1 \mathbf{f}(z_1, \mathbf{z}_2) := \left[\partial_{z_1}^1 f_p(z_1, \mathbf{z}_2) \right]_{1\le p\le Q},$$

while the partial derivative with respect to $\mathbf{z}_2$ means taking partial derivatives to each θ_q, $q = 1, 2, \cdots, Q$, namely:

$$\partial_{\mathbf{z}_2}^1 \mathbf{f}(z_1, \mathbf{z}_2) := \left[\partial_{\theta_q}^1 f_p(z_1, \mathbf{z}_2) \right]_{1\le p,q\le Q}. \tag{12}$$

Similarly,

$$\partial_{z_1\mathbf{z}_2}^2 \mathbf{f}(z_1, \mathbf{z}_2) := \left[\partial_{z_1\theta_q}^2 f_p(z_1, \mathbf{z}_2) \right]_{1\le p,q\le Q}, \tag{13}$$

and

$$\partial_{\mathbf{z}_2}^2 \mathbf{f}(z_1, \mathbf{z}_2) := \left[\partial_{\theta_q\theta_r}^2 f_p(z_1, \mathbf{z}_2) \right]_{1\le p,q,r\le Q}. \tag{14}$$

Additionally, we define:

$$|\partial^2_{\mathbf{z}_2}\mathbf{f}(z_1,\mathbf{z}_2)| := \max_{1\le p,q,r\le Q} |\partial^2_{\theta_q\theta_r} f_p(z_1,\mathbf{z}_2)| \tag{15}$$

and

$$\|\partial^2_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)} := \max_{(z_1,\mathbf{z}_2)\in\Omega} |\partial^2_{\mathbf{z}_2}\mathbf{f}(z_1,\mathbf{z}_2)|. \tag{16}$$

Lemma 1 *Assume that $\boldsymbol{A}(t) \in C(0,1)$, $\partial_t^1\overline{\boldsymbol{A}}(t,s) \in C(\Delta)$ are matrixes of $P \times Q$, $\boldsymbol{B}(t) \in C(0,1)$ is matrix of $Q \times 1$. Then, for $t \in [0,1]$:*

$$|\boldsymbol{A}(t)\boldsymbol{B}(t)| \le Q|\boldsymbol{A}(t)||\boldsymbol{B}(t)|, \tag{17}$$

$$|\int_0^t \boldsymbol{A}(s)\boldsymbol{B}(s)ds| \le Q\int_0^t |\boldsymbol{A}(s)||\boldsymbol{B}(s)|ds, \tag{18}$$

$$|\partial_t^1 \int_0^t \overline{\boldsymbol{A}}(t,s)\boldsymbol{B}(s)ds| \le Q(|\overline{\boldsymbol{A}}(t,t)||\boldsymbol{B}(t)| + \int_0^t |\partial_t^1\overline{\boldsymbol{A}}(t,s)||\boldsymbol{B}(s)|ds). \tag{19}$$

Proof We first prove (17). Assume that:

$$\mathbf{A}(t) = \begin{bmatrix} a_{pq}(t) \\ {\scriptstyle 1\le p\le P,1\le q\le Q} \end{bmatrix}, \mathbf{B}(t) = \begin{bmatrix} b_p(t) \\ {\scriptstyle 1\le p\le P} \end{bmatrix}.$$

Note that:

$$\mathbf{A}(t)\mathbf{B}(t) = \begin{bmatrix} \sum_{q=1}^{Q} a_{pq}(t)b_q(t) \\ {\scriptstyle 1\le p\le P} \end{bmatrix}.$$

By Cauchy inequality,

$$\begin{aligned} |\mathbf{A}(t)\mathbf{B}(t)| &= \max_{1\le p\le P}\left|\sum_{q=1}^{Q} a_{pq}(t)b_q(t)\right| \\ &\le \max_{1\le p\le P}\left(\sum_{q=1}^{Q}|a_{pq}(t)|^2\right)^{1/2}\left(\sum_{q=1}^{Q}|b_q(t)|^2\right)^{1/2} \\ &\le \max_{1\le p\le P}\left(\sum_{q=1}^{Q}|\mathbf{A}(t)|^2\right)^{1/2}\left(\sum_{q=1}^{Q}|\mathbf{B}(t)|^2\right)^{1/2} \\ &\le Q|\mathbf{A}(t)||\mathbf{B}(t)|, \end{aligned} \tag{20}$$

which leads to (17).

Now, we begin to prove (18). It is clear that

$$\left|\int_0^t \mathbf{A}(s)\mathbf{B}(s)ds\right| = \max_{1\le p\le P}\left|\int_0^t \sum_{q=1}^{Q} a_{pq}(s)b_q(s)ds\right|$$
$$\le \max_{1\le p\le P}\int_0^t \sum_{q=1}^{Q} |a_{pq}(s)||b_q(s)|ds. \tag{21}$$

By Cauchy inequality,

$$\begin{aligned}
&\max_{1\le p\le P}\int_0^t \sum_{q=1}^{Q} |a_{pq}(s)||b_q(s)|ds\\
&\le \max_{1\le p\le P}\int_0^t \left(\sum_{q=1}^{Q} |a_{pq}^2(s)|\right)^{1/2}\left(\sum_{q=1}^{Q} |b_q(s)|^2\right)^{1/2} ds\\
&\le Q\max_{1\le p\le P}\int_0^t |\mathbf{A}(s)||\mathbf{B}(s)|ds\\
&= Q\int_0^t |\mathbf{A}(s)||\mathbf{B}(s)|ds,
\end{aligned} \tag{22}$$

which leads to (18).

Now, we begin to prove (19). Assume that

$$\overline{\mathbf{A}}(t,s) = \left[\underset{1\le p\le P, 1\le q\le Q}{\overline{a}_{pq}(t,s)}\right].$$

Then,

$$\partial_t^1\int_0^t \overline{\mathbf{A}}(t,s)\mathbf{B}(s)ds = \left[\sum_{q=1}^{Q}\partial_t^1\int_0^t \overline{a}_{pq}(t,s)b_q(s)ds\right]_{1\le p\le P}. \tag{23}$$

Note that:

$$\begin{aligned}
&\left|\partial_t^1\int_0^t \overline{\mathbf{A}}(t,s)\mathbf{B}(s)ds\right|\\
&= \max_{1\le p\le P}\left|\sum_{q=1}^{Q}\partial_t^1\int_0^t \overline{a}_{pq}(t,s)b_q(s)ds\right|\\
&= \max_{1\le p\le P}\left|\sum_{q=1}^{Q}\overline{a}_{pq}(t,t)b_q(t) + \int_0^t \partial_t^1\overline{a}_{pq}(t,s)b_q(s)ds\right|\\
&\le \max_{1\le p\le P}\left(\sum_{q=1}^{Q}|\overline{a}_{pq}(t,t)||b_q(t)| + \int_0^t \sum_{q=1}^{Q}|\partial_t^1\overline{a}_{pq}(t,s)||b_q(s)|ds\right).
\end{aligned} \tag{24}$$

By Cauchy inequality,

$$\begin{aligned}
&\sum_{q=1}^{Q} |\overline{a}_{pq}(t,t)||b_q(t)| + \int_0^t \sum_{q=1}^{Q} |\partial_t^1 \overline{a}_{pq}(t,s)||b_q(s)|ds \\
&\leq \left(\sum_{q=1}^{Q} |\overline{a}_{pq}(t,t)|^2\right)^{1/2} \left(\sum_{q=1}^{Q} |b_q(t)|^2\right)^{1/2} \\
&\quad + \int_0^t \left(\sum_{q=1}^{Q} (\partial_t^1 \overline{a}_{pq}(t,s))^2\right)^{1/2} \left(\sum_{q=1}^{Q} |b_q(s)2\right)^{1/2} ds \\
&\leq Q(|\overline{\mathbf{A}}(t,t)||\mathbf{B}(t)| + \int_0^t |\partial_t^1 \overline{\mathbf{A}}(t,s)||\mathbf{B}(s)|ds),
\end{aligned} \tag{25}$$

which together with (24) yields (19). □

Lemma 2 *Assume that* $\boldsymbol{U}(t), \overline{\boldsymbol{U}}(t) \in C(0,1)$. *Then,*

$$|\boldsymbol{f}(t,\boldsymbol{U}(t)) - \boldsymbol{f}(t,\overline{\boldsymbol{U}}(t))| \leq Q\|\partial_{z_2}^1 \boldsymbol{f}\|_{L^\infty(\Omega)} |\boldsymbol{U}(t) - \overline{\boldsymbol{U}}(t)|, t \in [0,1], \tag{26}$$

$$|\partial_{z_1}^1 \boldsymbol{f}(t,\boldsymbol{U}(t)) - \partial_{z_1}^1 \boldsymbol{f}(t,\overline{\boldsymbol{U}}(t))| \leq Q\|\partial_{z_1 z_2}^2 \boldsymbol{f}\|_{L^\infty(\Omega)} |\boldsymbol{U}(t) - \overline{\boldsymbol{U}}(t)|, t \in [0,1], \tag{27}$$

and

$$|(\partial_{z_2}^1 \boldsymbol{f}(t,\boldsymbol{U}(t)) - \partial_{z_2}^1 \boldsymbol{f}(t,\overline{\boldsymbol{U}}(t)))| \leq Q\|\partial_{z_2}^2 \boldsymbol{f}\|_{L^\infty(\Omega)} |\boldsymbol{U}(t) - \overline{\boldsymbol{U}}(t))|, t \in [0,1]. \tag{28}$$

Proof We only prove (26) and (28) because (27) can be proved by the analysis similarly to (26).

Now, we begin to prove (26) . According to differential mean value theorems:

$$\begin{aligned}
|\mathbf{f}(t,\mathbf{U}(t)) - \mathbf{f}(t,\overline{\mathbf{U}}(t))| &= \left|\left[f_p(t,\mathbf{U}(t)) - f_p(t,\overline{\mathbf{U}}(t)) \right]_{1\leq p\leq Q}\right| \\
&= \left|\left[\partial_{\theta_q}^1 f_p(t,\xi(\mathbf{U}(t) - \overline{\mathbf{U}}(t)) \right]_{1\leq p,q\leq Q} \right. \\
&\qquad \left. \times (\mathbf{U}(t) - \overline{\mathbf{U}}(t))\right|, \xi \in (0,1).
\end{aligned} \tag{29}$$

By Lemma 1,

$$\begin{aligned}
&\left|\left[\underset{1\le p,q\le Q}{\partial^1_{\theta_q} f_p(t,\xi(\mathbf{U}(t)-\overline{\mathbf{U}}(t))}\right](\mathbf{U}(t)-\overline{\mathbf{U}}(t))\right| \\
&= \left|\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\xi(\mathbf{U}(t)-\overline{\mathbf{U}}(t))(\mathbf{U}(t)-\overline{\mathbf{U}}(t))\right| \\
&\le Q\left|\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\xi(\mathbf{U}(t)-\overline{\mathbf{U}}(t))\right||\mathbf{U}(t)-\overline{\mathbf{U}}(t)| \\
&\le Q\|\partial^1_{\mathbf{z}_2}\mathbf{f}(\cdot,\xi(\mathbf{U}(\cdot)-\overline{\mathbf{U}}(\cdot))\|_{L^\infty(0,1)}|\mathbf{U}(t)-\overline{\mathbf{U}}(t)| \\
&\le Q\|\partial^1_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}|\mathbf{U}(t)-\overline{\mathbf{U}}(t)|. \qquad (30)
\end{aligned}$$

Combining (29) with (30) leads to (27).

Now, we begin to prove (28). Use the symbol (12):

$$\begin{aligned}
&|\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\mathbf{U}(t))-\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\overline{\mathbf{U}}(t))| \\
&= \left|\left[\underset{1\le p,q\le Q}{\partial^1_{\theta_q} f_p(t,\mathbf{U}(t))-\partial^1_{\theta_q} f_p(t,\overline{\mathbf{U}}(t))}\right]\right| \\
&= \max_{1\le p,q\le Q}|\partial^1_{\theta_q} f_p(t,\mathbf{U}(t))-\partial^1_{\theta_q} f_p(t,\overline{\mathbf{U}}(t))|. \qquad (31)
\end{aligned}$$

By differential mean value theorem:

$$\begin{aligned}
&|\partial^1_{\theta_q} f_p(t,\mathbf{U}(t))-\partial^1_{\theta_q} f_p(t,\overline{\mathbf{U}}(t))| \\
&= \left|\left[\underset{1\le r\le Q}{\partial^2_{\theta_q\theta_r} f_p(t,\xi(\mathbf{U}(t)-\overline{\mathbf{U}}(t)))}\right](\mathbf{U}(t)-\overline{\mathbf{U}}(t))\right|, \\
&\xi\in(0,1),\ 1\le p,q\le Q. \qquad (32)
\end{aligned}$$

By Lemma 1:

$$\begin{aligned}
&\left|\left[\underset{1\le r\le Q}{\partial^2_{\theta_q\theta_r} f_p(t,\xi(\mathbf{U}(t)-\overline{\mathbf{U}}(t)))}\right](\mathbf{U}(t)-\overline{\mathbf{U}}(t))\right| \\
&\le Q\max_{1\le r\le Q}|\partial^2_{\theta_q\theta_r} f_p(t,\xi(\mathbf{U}(t)-\overline{\mathbf{U}}(t)))||\mathbf{U}(t)-\overline{\mathbf{U}}(t)|. \qquad (33)
\end{aligned}$$

By the symbol (16):

$$\begin{aligned}
&\max_{1\le r\le Q}|\partial^2_{\theta_q\theta_r} f_p(t,\xi(\mathbf{U}(t)-\overline{\mathbf{U}}(t)))| \\
&\le \max_{1\le p,q,r\le Q}|\partial^2_{\theta_q\theta_r} f_p(t,\xi(\mathbf{U}(t)-\overline{\mathbf{U}}(t)))| \\
&= \|\partial^2_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)},\ 1\le p,q\le Q. \qquad (34)
\end{aligned}$$

Combining (31) with (32), (33), and (34) leads to (28). □

Lemma 3 [4, 14] *For* $u(t) \in C^m(0,1)$, $v(t) \in C(0,1)$, *we have*

$$\|(I - I_N)(u)\|_{L^\infty(0,1)} \le CN^{(1/2)-m}\|\partial_t^m u\|_{L^\infty(0,1)}, \tag{35}$$

and

$$\|I_N(v)\|_{L^\infty(0,1)} \le C\log(N)\|v\|_{L^\infty(0,1)} \tag{36}$$

where I_N *is defined by* (11).

Lemma 4 *If* $\boldsymbol{A}(t) \in C^m(0,1)$, *then:*

$$\|(I - I_N)(\boldsymbol{A})\|_{L^\infty(0,1)} \le CN^{(1/2)-m}\|\partial_t^m \boldsymbol{A}\|_{L^\infty(0,1)}. \tag{37}$$

If $\boldsymbol{A}(t) \in C(0,1)$, *then:*

$$\|I_N(\boldsymbol{A})\|_{L^\infty(0,1)} \le C\log(N)\|\boldsymbol{A}\|_{L^\infty(0,1)}. \tag{38}$$

Proof We first prove (37). Note that:

$$(I - I_N)(\mathbf{A})(t) = \left[\underset{1\le p\le P, 1\le q\le Q}{(I - I_N)(a_{pq})(t)}\right]. \tag{39}$$

By Lemma 3:

$$\begin{aligned}
\|(I - I_N)(\mathbf{A})\|_{L^\infty(0,1)} &= \max_{t\in[0,1]} |(I - I_N)(\mathbf{A})(t)| \\
&= \max_{t\in[0,1]} \max_{\substack{1\le p\le P\\1\le q\le Q}} |(I - I_N)(a_{pq})(t)| \\
&\le \max_{t\in[0,1]} \max_{\substack{1\le p\le P\\1\le q\le Q}} \|(I - I_N)(a_{pq})\|_{L^\infty(0,1)} \\
&\le CN^{(1/2)-m} \max_{t\in[0,1]} \max_{\substack{1\le p\le P\\1\le q\le Q}} \|\partial_t^m a_{pq}\|_{L^\infty(0,1)} \\
&= CN^{(1/2)-m}\|\partial_t^m \mathbf{A}\|_{L^\infty(0,1)},
\end{aligned} \tag{40}$$

which leads to (37).

Now, we begin to prove (38). Note that:

$$|I_N(\mathbf{A})(t)| = \max_{\substack{1\le p\le P\\1\le q\le Q}} |I_N(a_{pq})(t)|. \tag{41}$$

By Lemma 3:

$$\begin{aligned}
\|I_N(\mathbf{A})\|_{L^\infty(0,1)} &= \max_{t\in[0,1]} |I_N(\mathbf{A})(t)| = \max_{t\in[0,1]} \max_{\substack{1\le p\le P\\1\le q\le Q}} |I_N(a_{pq})(t)| \\
&\le \max_{t\in[0,1]} \max_{\substack{1\le p\le P\\1\le q\le Q}} \|I_N(a_{pq})\|_{L^\infty(0,1)} \le C\log(N) \max_{t\in[0,1]} \max_{\substack{1\le p\le P\\1\le q\le Q}} \|a_{pq}\|_{L^\infty(0,1)} \\
&\le C\log(N)\|\mathbf{A}\|_{L^\infty(0,1)},
\end{aligned} \tag{42}$$

which leads to (38). □

Lemma 5 [4]*Assume that $u(x) \in C^m(-1,1)$, and $P(x)$ is polynomial of degree not exceeding N. Then:*

$$\left|\int_{-1}^{1} u(s)P(s)ds - \sum_{k=0}^{N} u(v_k)P(v_k)w_k\right| \le CN^{-m}\|\partial_t^m u\|_{L^\infty(-1,1)}\|P\|_{L^\infty(-1,1)}, \tag{43}$$

where $v_k, w_k, k = 0, 1, \cdots, N$ are Legendre Gauss points of order N on interval $[-1, 1]$.

Lemma 6 *Assume that $\boldsymbol{A}(t) \in C^m(-1,1)$ is matrix of $P \times Q$, and $\boldsymbol{B}(t) \in P_N(-1,1)$ is matrix of $Q \times 1$. Then:*

$$\left|\int_{-1}^{1} \boldsymbol{A}(v)\boldsymbol{B}(v)dv - \sum_{k=0}^{N} \boldsymbol{A}(v_k)\boldsymbol{B}(v_k)w_k\right| \\ \le CQN^{-m}\|\partial_t^m \boldsymbol{A}\|_{L^\infty(-1,1)}\|\boldsymbol{B}\|_{L^\infty(-1,1)}, \tag{44}$$

where $v_k, w_k, k = 0, 1, \cdots, N$ are Legendre Gauss points of order N on interval $[-1, 1]$*, and:*

$$\sum_{k=0}^{N} \boldsymbol{A}(v_k)\boldsymbol{B}(v_k)w_k := \left[\sum_{q=1}^{Q}\sum_{k=0}^{N} a_{pq}(v_k)b_q(v_k)w_k\right]_{1\le p\le P}.$$

Proof Let

$$d_p := \sum_{q=1}^{Q}\left(\int_{-1}^{1} a_{pq}(v)b_q(v)dv - \sum_{k=0}^{N} a_{pq}(v_k)b_q(v_k)w_k\right),$$

$$D := [d_1, d_2, \cdots, d_P]'.$$

Then,

$$D = \int_{-1}^{1} \mathbf{A}(v)\mathbf{B}(v)dv - \sum_{k=0}^{N} \mathbf{A}(v_k)\mathbf{B}(v_k)w_k.$$

By Lemma 5:

$$\begin{aligned} |d_p| &\le \sum_{q=1}^{Q}\left|\int_{-1}^{1} a_{pq}(v)b_q(v)dv - \sum_{k=0}^{N} a_{pq}(v_k)b_q(v_k)w_k\right| \\ &\le \sum_{q=1}^{Q} CN^{-m}\|\partial_t^m a_{pq}\|_{L^\infty(-1,1)}\|b_q\|_{L^\infty(-1,1)} \\ &\le CN^{-m}\sum_{q=1}^{Q}\|\partial_t^m a_{pq}\|_{L^\infty(-1,1)}\|b_q\|_{L^\infty(-1,1)} \\ &\le CQN^{-m}\|\partial_t^m \mathbf{A}\|_{L^\infty(-1,1)}\|\mathbf{B}\|_{L^\infty(-1,1)}. \end{aligned} \tag{45}$$

Thus,

$$|D| = \max_{1\le p\le P} |d_p| \le CQN^{-m} \|\partial_t^m \mathbf{A}\|_{L^\infty(-1,1)} \|\mathbf{B}\|_{L^\infty(-1,1)}, \tag{46}$$

which leads to (44). □

Lemma 7 [3]*Assume that $u(x)$, $v(t) \in C(0,1)$, $M > 0$, and*

$$|u(t)| \le |v(t)| + M\int_0^t |u(s)|ds. \tag{47}$$

Then, there exists constant C such that:

$$\|u\|_{L^\infty(0,1)} \le C\|v\|_{L^\infty(0,1)}.$$

4 Convergence analysis

Theorem 1 *Assume that $\boldsymbol{Y}(t)$ is the solution to (1) with given functions satisfying $\boldsymbol{g}(t) \in C^m(0,1)$, $\boldsymbol{K}(t,s) \in C^m(\Delta)$, and $\boldsymbol{f}(z_1,z_2) \in C^m(\Omega)$. $\boldsymbol{Y}^N(t)$ defined by (10) is the numerical solution to (1). Then,*

$$\|\boldsymbol{Y} - \boldsymbol{Y}^N\|_{L^\infty(0,1)} \le CN^{(1/2)-m}, \tag{48}$$

where $N+1$ is the number of collocation points used in the numerical method in Section 2, *and constant C is related to $\|\partial_t^m \boldsymbol{Y}\|_{L^\infty(0,1)}$, $\|\partial_s^m \boldsymbol{K}\|_{L^\infty(\Delta)}$, $\|\partial_t^1 \boldsymbol{K}\|_{L^\infty(\Delta)}$, $\|\boldsymbol{f}\|_{L^\infty(\Omega)}$, $\|\partial_t^m \boldsymbol{f}(\cdot, \boldsymbol{Y}(\cdot))\|_{L^\infty(0,1)}$.*

This theoretical result shows that convergence order of the present method is $N^{(1/2)-m}$. It is exponential convergence which is one of the most prominent features of spectral methods. It is worth noting that N and m are independent of each other. Employing more collocation points (increasing N) or given functions possessing more regularities (increasing m) will enhance the accuracy of the numerical solution.

Proof Note that VIEs (2) hold at t_i,

$$\mathbf{Y}(t_i) = \mathbf{g}(t_i) + \int_0^{t_i} \mathbf{K}(t_i, s)\mathbf{y}(s)ds. \tag{49}$$

Subtract (9) from (49):

$$\mathbf{Y}(t_i) - \mathbf{Y}_i = \mathbf{E}_1(t_i) + \mathbf{E}_2(t_i), \tag{50}$$

where,

$$\mathbf{e}_0(t) := \mathbf{y}(t) - \mathbf{y}^N(t),$$

$$\mathbf{E}_1(t) := \int_0^t \mathbf{K}(t,s)\mathbf{e}_0(s)ds,$$

$$\mathbf{E}_2(t) := \int_0^t \mathbf{K}(t_i, s)\mathbf{y}^N(s))ds - \frac{t_i}{2}\sum_{k=0}^{N} \mathbf{K}(t_i, s(t_i, v_k))\mathbf{y}^N(s(t_i, v_k))w_k.$$

Multiply $L_i(t)$ to both sides of (50) and sum up from $i = 0$ to N:

$$I_N(\mathbf{Y})(t) - \mathbf{Y}^N(t) = I_N(\mathbf{E}_1)(t) + I_N(\mathbf{E}_2)(t), \tag{51}$$

which can be written as:

$$\mathbf{Y}(t) - \mathbf{Y}^N(t) = (I - I_N)(\mathbf{Y})(t) + (I_N - I)(\mathbf{E}_1)(t) + I_N(\mathbf{E}_2)(t) + \mathbf{E}_1(t). \tag{52}$$

Next, we estimate each term in the right-hand side of (52).

We first estimate $(I - I_N)(\mathbf{Y})(t)$ in (52). By Lemma 4:

$$\|(I - I_N)(\mathbf{Y})\|_{L^\infty(0,1)} \leq N^{(1/2)-m}\|\partial_t^m \mathbf{Y}\|_{L^\infty(0,1)}. \tag{53}$$

Now, we begin to estimate $(I_N - I)(\mathbf{E}_1)(t)$ in (52). By Lemma 4 with $m = 1$:

$$\|(I - I_N)(\mathbf{E}_1)\|_{L^\infty(0,1)} \leq CN^{-1/2}\|\partial_t^1 \mathbf{E}_1\|_{L^\infty(0,1)}. \tag{54}$$

By Lemma 1:

$$\|\partial_t^1 \mathbf{E}_1\|_{L^\infty(0,1)} \leq C\|\mathbf{e}_0\|_{L^\infty(0,1)}(\|\mathbf{K}\|_{L^\infty(\Delta)} + \|\partial_t^1 \mathbf{K}\|_{L^\infty(\Delta)}). \tag{55}$$

Combine (54) with (55):

$$\|(I - I_N)(\mathbf{E}_1)\|_{L^\infty(0,1)} \leq CN^{-1/2}\|\mathbf{e}_0\|_{L^\infty(0,1)}(\|\mathbf{K}\|_{L^\infty(\Delta)} + \|\partial_t^1 \mathbf{K}\|_{L^\infty(\Delta)}). \tag{56}$$

Now, we begin to estimate $I_N(\mathbf{E}_2)(t)$ in (52). By Lemma 4:

$$\|I_N(\mathbf{E}_2\|_{L^\infty(0,1)} \leq C\log(N)\|\mathbf{E}_2\|_{L^\infty(0,1)}. \tag{57}$$

By Lemma 6:

$$\begin{aligned}
|\mathbf{E}_2(t)| &\leq CN^{-m}\|\partial_v^m \mathbf{K}(t, s(t,\cdot))\|_{L^\infty(-1,1)}\|\mathbf{y}^N(s(t,\cdot))\|_{L^\infty(-1,1)} \\
&\leq CN^{-m}\|\partial_s^m \mathbf{K}(t,\cdot)\|_{L^\infty(0,t)}\|\mathbf{y}^N\|_{L^\infty(0,t)} \\
&\leq CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{y}^N\|_{L^\infty(0,1)} \\
&\leq CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{e}_0\|_{L^\infty(0,1)} + \|\mathbf{y}\|_{L^\infty(0,1)}) \\
&\leq CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{e}_0\|_{L^\infty(0,1)} + \|\mathbf{f}\|_{L^\infty(\Omega)}).
\end{aligned} \tag{58}$$

Therefore,

$$\|\mathbf{E}_2\|_{L^\infty(0,1)} \leq CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{e}_0\|_{L^\infty(0,1)} + \|\mathbf{f}\|_{L^\infty(\Omega)}). \tag{59}$$

Combine (57) with (59):

$$\|I_N(\mathbf{E}_2\|_{L^\infty(0,1)} \leq C\log(N)N^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{e}_0\|_{L^\infty(0,1)} + \|\mathbf{f}\|_{L^\infty(\Omega)}). \tag{60}$$

Now, we begin to estimate $\mathbf{E}_1(t)$ in (52). By Lemma 1:

$$\|\mathbf{E}_1\|_{L^\infty(0,1)} \leq C\|\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{e}_0\|_{L^\infty(0,1)}. \tag{61}$$

Combining (52) with (53), (56), (60), and (61) yields:

$$\begin{aligned}
&\|\mathbf{Y} - \mathbf{Y}^N\|_{L^\infty(0,1)} \\
\leq\ & N^{(1/2)-m}\|\partial_t^m \mathbf{Y}\|_{L^\infty(0,1)} + C\log(N)N^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{f}\|_{L^\infty(\Omega)} \\
&+ C\|\mathbf{e}_0\|_{L^\infty(0,1)}(\|\mathbf{K}\|_{L^\infty(\Delta)} + N^{-1/2}\|\partial_t^1 \mathbf{K}\|_{L^\infty(\Delta)} \\
&+ \log(N)N^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}),
\end{aligned} \tag{62}$$

which shows that the estimation of $\|\mathbf{Y} - \mathbf{Y}^N\|_{L^\infty(0,1)}$ relates to $\|\mathbf{e}_0\|_{L^\infty(0,1)}$.

Now, we begin to estimate $\|\mathbf{e}_0\|_{L^\infty(0,1)}$.

Subtract (6) from (4):

$$\mathbf{y}(t_i) - \mathbf{y}_i = \mathbf{e}_1(t_i) + \mathbf{e}_2(t_i), \tag{63}$$

where,

$$\overline{\mathbf{Y}}(t) := \mathbf{g}(t) + \int_0^t \mathbf{K}(t,s)\mathbf{y}^N(s)ds,$$

$$\mathbf{e}_1(t) := \mathbf{f}(t, \mathbf{Y}(t)) - \mathbf{f}(t, \overline{\mathbf{Y}}(t)),$$

$$\widetilde{\mathbf{Y}}(t) := \mathbf{g}(t) + \frac{t}{2}\sum_{k=0}^{N} \mathbf{K}(t, s(t, v_k))\mathbf{y}^N(s(t, v(k)))w_k,$$

$$\mathbf{e}_2(t) := \mathbf{f}(t, \overline{\mathbf{Y}}(t)) - \mathbf{f}(t, \widetilde{\mathbf{Y}}(t)).$$

Multiply $L_i(t)$ to both side of (63) and sum up from $i = 0$ to N:

$$I_N(\mathbf{y})(t) - \mathbf{y}^N(t) = I_N(\mathbf{e}_1)(t) + I_N(\mathbf{e}_2)(t), \tag{64}$$

which can be written as:

$$\mathbf{e}_0(t) = (I - I_N)(\mathbf{y})(t) + (I_N - I)(\mathbf{e}_1)(t) + I_N(\mathbf{e}_2)(t) + \mathbf{e}_1(t). \tag{65}$$

Next, we estimate each term in the right-hand side of (65).

We first estimate $(I - I_N)(\mathbf{y})(t)$ in (65). By Lemma 4:

$$\begin{aligned}\|(I - I_N)(\mathbf{y})\|_{L^\infty(0,1)} &\le CN^{(1/2)-m}\|\partial_t^m \mathbf{y}\|_{L^\infty(0,1)} \\ &= CN^{(1/2)-m}\|\partial_t^m \mathbf{f}(\cdot, Y(\cdot))\|_{L^\infty(0,1)}.\end{aligned} \tag{66}$$

Now, we begin to estimate $(I_N - I)(\mathbf{e}_1)(t)$ in (65). By Lemma 4 with $m = 1$:

$$\|(I_N - I)(\mathbf{e}_1)\|_{L^\infty(0,1)} \le CN^{-1/2}\|\partial_t^1 \mathbf{e}_1\|_{L^\infty(0,1)}. \tag{67}$$

Note that:

$$\begin{aligned}\partial_t^1 \mathbf{e}_1(t) &= \partial_{z_1}^1 \mathbf{f}(t, \mathbf{Y}(t)) - \partial_{z_1}^1 \mathbf{f}(t, \overline{\mathbf{Y}}(t)) \\ &\quad + \partial_{\mathbf{z}_2}^1 \mathbf{f}(t, \mathbf{Y}(t))\mathbf{Y}'(t) - \partial_{\mathbf{z}_2}^1 \mathbf{f}(t, \overline{\mathbf{Y}}(t))\overline{\mathbf{Y}}'(t) \\ &= \partial_{z_1}^1 \mathbf{f}(t, \mathbf{Y}(t)) - \partial_{z_1}^1 \mathbf{f}(t, \overline{\mathbf{Y}}(t)) \\ &\quad + (\partial_{\mathbf{z}_2}^1 \mathbf{f}(t, \mathbf{Y}(t)) - \partial_{\mathbf{z}_2}^1 \mathbf{f}(t, \overline{\mathbf{Y}}(t)))\mathbf{Y}'(t) \\ &\quad + \partial_{\mathbf{z}_2}^1 \mathbf{f}(t, \overline{\mathbf{Y}}(t))(\mathbf{Y}'(t) - \overline{\mathbf{Y}}'(t)).\end{aligned} \tag{68}$$

Next, we estimate each term in the right-hand side of (68).

We first estimate $\partial_{z_1}^1 \mathbf{f}(t, \mathbf{Y}(t)) - \partial_{z_1}^1 \mathbf{f}(t, \overline{\mathbf{Y}}(t))$ in (68). By Lemma 2:

$$\begin{aligned}&|\partial_{z_1}^1 \mathbf{f}(t, \mathbf{Y}(t)) - \partial_{z_1}^1 \mathbf{f}(t, \overline{\mathbf{Y}}(t))| \\ &\le C\|\partial_{z_1 \mathbf{z}_2}^2 \mathbf{f}\|_{L^\infty(\Omega)}|\mathbf{Y}(t) - \overline{\mathbf{Y}}(t)|.\end{aligned} \tag{69}$$

By Lemma 1:

$$|\mathbf{Y}(t) - \overline{\mathbf{Y}}(t)| = |\textstyle\int_0^t \mathbf{K}(t,s)\mathbf{e}_0(s)ds| \le C\|\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{e}_0\|_{L^\infty(0,1)}. \tag{70}$$

Combine (69) with (70):

$$\begin{aligned}&|\partial_{z_1}^1 \mathbf{f}(t, \mathbf{Y}(t)) - \partial_{z_1}^1 \mathbf{f}(t, \overline{\mathbf{Y}}(t))| \\ &\le C\|\partial_{z_1 \mathbf{z}_2}^2 \mathbf{f}\|_{L^\infty(\Omega)}\|\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{e}_0\|_{L^\infty(0,1)}.\end{aligned} \tag{71}$$

Now, we begin to estimate $(\partial^1_{z_2}\mathbf{f}(t,\mathbf{Y}(t))-\partial^1_{z_2}\mathbf{f}(t,\overline{\mathbf{Y}}(t)))\mathbf{Y}'(t)$ in (68). By Lemma 1:

$$\begin{aligned}&|(\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\mathbf{Y}(t))-\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\overline{\mathbf{Y}}(t)))\mathbf{Y}'(t)|\\ \leq\ & C|(\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\mathbf{Y}(t))-\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\overline{\mathbf{Y}}(t)))||\mathbf{Y}'(t)|\\ \leq\ & C\|\mathbf{Y}'\|_{L^\infty(0,1)}|(\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\mathbf{Y}(t))-\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\overline{\mathbf{Y}}(t)))|.\end{aligned}\tag{72}$$

By Lemma 2 and (70):

$$\begin{aligned}&|(\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\mathbf{Y}(t))-\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\overline{\mathbf{Y}}(t)))|\\ \leq\ & C\|\partial^2_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}|\mathbf{Y}(t)-\overline{\mathbf{Y}}(t))|\\ \leq\ & C\|\partial^2_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}\|\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{e}_0\|_{L^\infty(0,1)}.\end{aligned}\tag{73}$$

Combine (72) with (73):

$$\begin{aligned}&|(\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\mathbf{Y}(t))-\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\overline{\mathbf{Y}}(t)))\mathbf{Y}'(t)|\\ \leq\ & C\|\mathbf{Y}'\|_{L^\infty(0,1)}\|\partial^2_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}\|\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{e}_0\|_{L^\infty(0,1)}.\end{aligned}\tag{74}$$

Now, we begin to estimate $\partial^1_{z_2}\mathbf{f}(t,\overline{\mathbf{Y}}(t))(\mathbf{Y}'(t)-\overline{\mathbf{Y}}'(t))$ in (68). By Lemma 1:

$$\begin{aligned}&|\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\overline{\mathbf{Y}}(t))(\mathbf{Y}'(t)-\overline{\mathbf{Y}}'(t))|\\ \leq\ & C|\partial^1_{\mathbf{z}_2}\mathbf{f}(t,\overline{\mathbf{Y}}(t))||\mathbf{Y}'(t)-\overline{\mathbf{Y}}'(t)|\\ =\ & C\|\partial^1_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}|\partial^1_t\int_0^t\mathbf{K}(t,s)\mathbf{e}_0(s)ds|\\ \leq\ & C\|\partial^1_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}\|\mathbf{e}_0\|_{L^\infty(0,1)}(\|\mathbf{K}\|_{L^\infty(\Delta)}+\|\partial^1_t\mathbf{K}\|_{L^\infty(\Delta)}).\end{aligned}\tag{75}$$

Combining (68) with (71), (74), and (75) yields the estimation of $\partial^1_t\mathbf{e}_1$ in (68):

$$\begin{aligned}\|\partial^1_t\mathbf{e}_1\|_{L^\infty(0,1)}\ \leq\ & C\|\mathbf{e}_0\|_{L^\infty(0,1)}(\|\mathbf{K}\|_{L^\infty(\Delta)}(\|\partial^2_{z_1\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}\\ &+\|\partial^2_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}\|\mathbf{Y}'\|_{L^\infty(0,1)}+\|\partial^1_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)})\\ &+\|\partial^1_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}\|\partial^1_t\mathbf{K}\|_{L^\infty(\Delta)}).\end{aligned}\tag{76}$$

Combine (67) with (76):

$$\begin{aligned}\|(I_N-I)(\mathbf{e}_1)\|_{L^\infty(0,1)}\ \leq\ & CN^{-1/2}\|\mathbf{e}_0\|_{L^\infty(0,1)}(\|\mathbf{K}\|_{L^\infty(\Delta)}(\|\partial^2_{z_1\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}\\ &+\|\partial^2_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}\|\mathbf{Y}'\|_{L^\infty(0,1)}+\|\partial^1_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)})\\ &+\|\partial^1_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}\|\partial^1_t\mathbf{K}\|_{L^\infty(\Delta)}).\end{aligned}\tag{77}$$

Now, we begin to estimate $I_N(\mathbf{e}_2)(t)$ in (65). By Lemma 4:

$$\|I_N(\mathbf{e}_2)\|_{L^\infty(0,1)}\leq C\log(N)\|\mathbf{e}_2\|_{L^\infty(0,1)}.\tag{78}$$

By Lemma 2:

$$|\mathbf{e}_2(t)|=|\mathbf{f}(t,\overline{\mathbf{Y}}(t))-\mathbf{f}(t,\widetilde{\mathbf{Y}}(t))|\leq C\|\partial^1_{\mathbf{z}_2}\mathbf{f}\|_{L^\infty(\Omega)}|\overline{\mathbf{Y}}(t)-\widetilde{\mathbf{Y}}(t)|.\tag{79}$$

By Lemma 6:

$$
\begin{aligned}
|\overline{\mathbf{Y}}(t)-\widetilde{\mathbf{Y}}(t)| &\le CN^{-m}\|\partial_s^m \mathbf{K}(t,s(t,\cdot))\|_{L^\infty(-1,1)}\|\mathbf{y}^N(s(t,\cdot))\|_{L^\infty(-1,1)} \\
&\le CN^{-m}\|\partial_s^m \mathbf{K}(t,\cdot)\|_{L^\infty(0,t)}\|\mathbf{y}^N\|_{L^\infty(0,t)}. \qquad (80)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\overline{\mathbf{Y}}-\widetilde{\mathbf{Y}}\|_{L^\infty(0,1)} &\le CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{y}^N\|_{L^\infty(0,1)} \\
&\le CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{e}_0\|_{L^\infty(0,1)}+\|\mathbf{y}\|_{L^\infty(0,1)}) \\
&\le CN^{-m}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{e}_0\|_{L^\infty(0,1)}+\|\mathbf{f}\|_{L^\infty(\Omega)}). \qquad (81)
\end{aligned}
$$

Combine (78) with (79) and (81):

$$
\begin{aligned}
&\|I_N(\mathbf{e}_2)\|_{L^\infty(0,1)} \\
&\le C\log(N)N^{-m}\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|_{L^\infty(\Omega)}\|\partial_s^m \mathbf{K}\|_{L^\infty(\Delta)}(\|\mathbf{e}_0\|_{L^\infty(0,1)}+\|\mathbf{f}\|_{L^\infty(\Omega)}). \qquad (82)
\end{aligned}
$$

Now, we begin to estimate $\mathbf{e}_1(t)$ in (65). By Lemma 2:

$$
\begin{aligned}
|\mathbf{e}_1(t)| &\le C\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|\|_{L^\infty(\Omega)}|\mathbf{Y}(t)-\overline{\mathbf{Y}}(t)| \\
&\le C\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|\|_{L^\infty(\Omega)}\int_0^t|\mathbf{K}(t,s)||\mathbf{e}_0(s)|ds \\
&\le C\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|\|_{L^\infty(\Omega)}\|\mathbf{K}\|_{L^\infty(\Delta)}\int_0^t|\mathbf{e}_0(s)|ds. \qquad (83)
\end{aligned}
$$

Combining (65) with (66), (77), (82), and (83) yields the Gronwall inequality of $|\mathbf{e}_0(t)|$:

$$
|\mathbf{e}_0(t)| \le v(t)+C\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|\|_{L^\infty(\Omega)}\|\mathbf{K}\|_{L^\infty(\Delta)}\textstyle\int_0^t|\mathbf{e}_0(s)|ds, \qquad (84)
$$

where,

$$
\begin{aligned}
v(t) := {}& CN^{(1/2)-m}\|\partial_t^m\mathbf{f}(\cdot,Y(\cdot))\|_{L^\infty(0,1)} \\
&+C\log(N)N^{-m}\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|_{L^\infty(\Omega)}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{f}\|_{L^\infty(\Omega)} \\
&+CN^{-1/2}\|\mathbf{e}_0\|_{L^\infty(0,1)}(\|\mathbf{K}\|_{L^\infty(\Delta)}(\|\partial_{z_1\mathbf{z}_2}^2\mathbf{f}\|_{L^\infty(\Omega)} \\
&+\|\partial_{\mathbf{z}_2}^2\mathbf{f}\|_{L^\infty(\Omega)}\|\mathbf{Y}'\|_{L^\infty(0,1)}+\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|_{L^\infty(\Omega)})+\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|_{L^\infty(\Omega)}\|\partial_t^1\mathbf{K}\|_{L^\infty(\Delta)} \\
&+C\log(N)N^{-m}\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|_{L^\infty(\Omega)}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}). \qquad (85)
\end{aligned}
$$

By Lemma 7:

$$
\|\mathbf{e}_0\|_{L^\infty(0,1)} \le C\|v\|_{L^\infty(0,1)}, \qquad (86)
$$

which can be written as the following form if N is sufficient large,

$$
\begin{aligned}
\|\mathbf{e}_0\|_{L^\infty(0,1)} &\le CN^{(1/2)-m}\|\partial_t^m\mathbf{f}(\cdot,\mathbf{Y}(\cdot))\|_{L^\infty(0,1)} \\
&\quad +C\log(N)N^{-m}\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|_{L^\infty(\Omega)}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}.\|\mathbf{f}\|_{L^\infty(\Omega)} \qquad (87)
\end{aligned}
$$

Table 1 Errors versus N (example 1)

N	2	5	10	15	20
$\epsilon(N)$	6.77e−07	1.78e−15	5.33e−15	5.33e−15	3.55e−15
$\xi(N)$	3.87e−07	1.89e−15	2.89e−15	2.89e−15	1.33e−15

Combine (62) with (87):

$$\begin{aligned}&\|\mathbf{Y}-\mathbf{Y}^N\|_{L^\infty(0,1)}\\ \leq\ & CN^{(1/2)-m}(\|\partial_t^m\mathbf{Y}\|_{L^\infty(0,1)}+\log(N)N^{-1/2}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{f}\|_{L^\infty(\Omega)}\\ &+(\|\partial_t^m\mathbf{f}(\cdot,Y(\cdot))\|_{L^\infty(0,1)}+C\log(N)N^{-1/2}\|\partial_{\mathbf{z}_2}^1\mathbf{f}\|_{L^\infty(\Omega)}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)}\|\mathbf{f}\|_{L^\infty(\Omega)})\\ &(\|\mathbf{K}\|_{L^\infty(\Delta)}+N^{-1/2}\|\partial_t^1\mathbf{K}\|_{L^\infty(\Delta)}+\log(N)N^{-m}\|\partial_s^m\mathbf{K}\|_{L^\infty(\Delta)})),\end{aligned}\tag{88}$$

which lead to (48). □

5 Numerical experiments

If the expression of exact solution $\mathbf{Y}(t)$ to VIEs (1) is given, we investigate the following numerical errors:

$$\xi(N) := \max\{|\mathbf{Y}(t)-\mathbf{Y}^N(t)| : t=i/100, i=0,1,\cdots,100\},$$

otherwise, we investigate following numerical errors:

$$\epsilon(N) := \max\{|\mathbf{Y}^N(t)-\mathbf{g}(t)-\mathbf{S}(t)| : t=i/100, i=0,1,\cdots,100\},$$

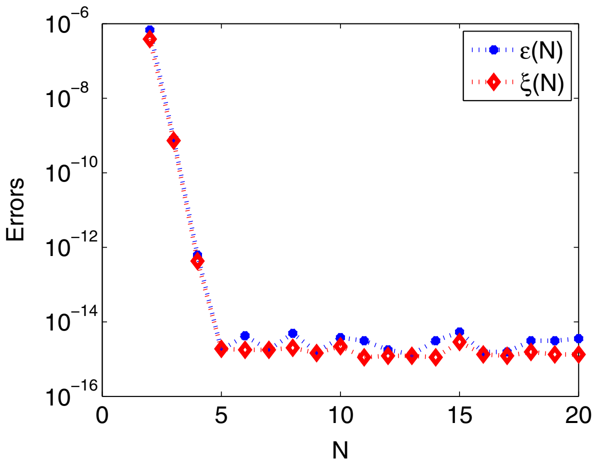

Fig. 1 Example 1: Errors versus N

Table 2 Errors versus N (Example 2)

N	2	10	20	30	40
$\epsilon(N)$	3.08e−01	4.75e−04	1.19e−06	4.12e−09	1.34e−11

where,

$$\mathbf{S}(t) := \frac{t}{2}\sum_{k=0}^{50}\mathbf{K}(t, s(t, v(k)))\mathbf{f}(s(t, \hat{v}_k), \mathbf{Y}^N(s(t, \hat{v}_k)))\hat{w}_k,$$

and $\hat{v}_k, \hat{w}_k, k = 0, 1, \cdots, 50$ are Legendre Gauss points of order 50 on the interval $[-1, 1]$.

Example 1 Consider the following system of VIEs:

$$\begin{aligned}
y_1(t) &= 1 - \sin(1)(\cos t - \cos 2t) + \int_0^t \sin(t+s)\sin(y_1(s) + y_2(s) - y_3(s))ds,\\
y_2(t) &= 1 - \cos(1)(\sin 2t - \sin t) + \int_0^t \cos(t+s)\cos(y_1(s) + y_2(s) - y_3(s))ds,\\
y_3(t) &= 1 - e^1(1 - e^{-t}) + \int_0^t e^{s-t+y_1(s)+y_2(s)-y_3(s)}ds. \qquad (89)
\end{aligned}$$

Exact solutions are $y_1(t) = y_2(t) = y_3(t) = 1$.

This example is provided to investigate the difference between $\xi(N)$ and $\epsilon(N)$. Numerical errors are recorded in Table 1 and plotted in Fig. 1. Numerical error results show that $\xi(N)$ and $\epsilon(N)$ almost have the same performance. Based on this, it is reliable to investigate performance of numerical errors by $\epsilon(N)$ if expressions of exact solutions to VIEs (1) are not given.

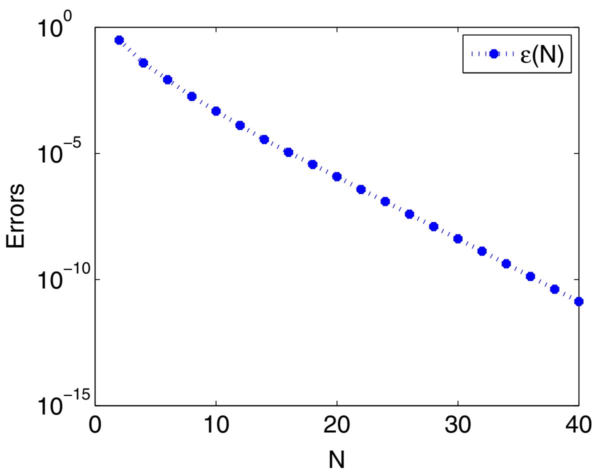

Fig. 2 Example 2: errors versus N

Table 3 Given functions (example 3)

Case	$\mathbf{g}(t)$	$\mathbf{K}(t,s)$	$\mathbf{f}(t,\mathbf{Y}(t))$
Case I	$t^{1/2}\mathbf{g}_0(t)$	$\mathbf{K}_0(t,s)$	$\mathbf{f}_0(t,\mathbf{Y}(t))$
Case II	$\mathbf{g}_0(t)$	$t^{1/2}\mathbf{K}_0(t,s)$	$\mathbf{f}_0(t,\mathbf{Y}(t))$
Case III	$\mathbf{g}_0(t)$	$s^{1/2}\mathbf{K}_0(t,s)$	$\mathbf{f}_0(t,\mathbf{Y}(t))$
Case IV	$\mathbf{g}_0(t)$	$\mathbf{K}_0(t,s)$	$\mathbf{f}_0(t^{1/2},\mathbf{Y}(t))$
Case V	$\mathbf{g}_0(t)$	$\mathbf{K}_0(t,s)$	$\mathbf{f}_0(t,\mathbf{Y}(t))$

Example 2 Consider VIEs (1) with the given functions as follows:

$$\mathbf{g}(t) = \begin{bmatrix} \sin t \\ \cos t \\ e^t \end{bmatrix}, \quad \mathbf{f}(t,\mathbf{Y}(t)) = \begin{bmatrix} \sin(t + Y_1(t) + Y_2(t) - Y_3(t)) \\ \cos(tY_1(t)Y_2(t)Y_3(t)) \\ e^{t^2 Y_1(t) - tY_2(t) - Y_3(t)} \end{bmatrix}, \tag{90}$$

$$\mathbf{K}(t,s) = \begin{bmatrix} \sin(t+s) & \sin(t-s) & \sin(ts) \\ \cos(t+s) & \cos(t-s) & \cos(ts) \\ e^{t+s} & e^{t-s} & e^{ts} \end{bmatrix}. \tag{91}$$

This example is provided to underline the role of N on the performance of numerical errors decaying. Numerical errors are recorded in Table 2 and plotted in Fig. 2. Errors decay exponentially. This is consistent with the theoretical result.

Example 3 Assume that:

$$\mathbf{g}_0(t) = [1; 1; 1],$$

$$\mathbf{K}_0(t,s) = [1, 0, 0; 0, 1, 0; 0, 0, 1],$$

$$\mathbf{f}_0(t,\mathbf{Y}(t)) = [\sin(t + \sum_{p=1}^{3} y_p(t)); \cos(t \prod_{p=1}^{3} y_p(t)); e^{t(y_1(t)+y_2(t))-y_3(t)}].$$

Consider VIEs (1) with the given functions listed in Table 3. We investigate the role of m, the index of regularity of given functions, on errors decaying. It is worth noting that in each case listed in Table 3, one of the given functions is not smooth enough. Numerical errors are recorded in Table 4 and plotted in Fig. 3. Numerical results show that given functions possess more regularity, and errors decay faster. This is consistent with the theoretical result.

Table 4 Errors versus N (example 3)

N	2	10	20	30	40
$\epsilon(N)$ for case I	1.82e−01	4.56e−02	2.38e−02	1.61e−02	1.23e−02
$\epsilon(N)$ for case II	9.15e−02	2.33e−04	3.35e−05	1.04e−05	4.48e−06
$\epsilon(N)$ for case III	4.72e−02	1.56e−04	2.23e−05	6.92e−06	2.99e−06
$\epsilon(N)$ for case IV	3.86e−02	6.37e−05	7.51e−06	1.86e−06	8.86e−07
$\epsilon(N)$ for case V	5.05e−02	9.79e−06	5.84e−10	4.40e−14	3.41e−15

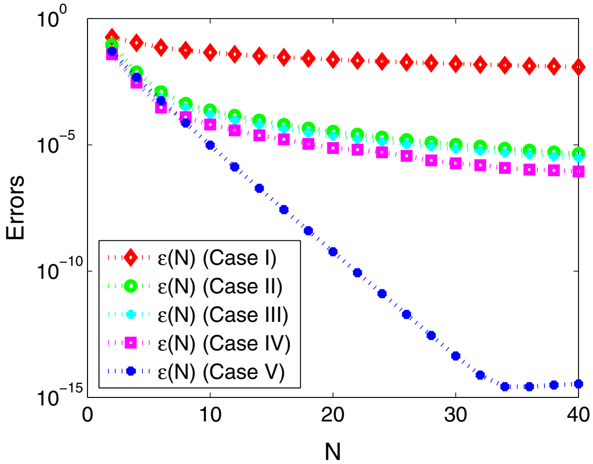

Fig. 3 Example 3: Errors versus N

6 Conclusion and future work

We investigate the Chebyshev spectral collocation method for the system of nonlinear VIEs. In the present method, we choose Chebyshev Gauss points as collocation points, approximate integral term by Legendre Gauss quadrature formula, and finally obtain the numerical solution. We provide convergence analysis for the present method to show that numerical errors decay exponentially. We carry out numerical experiments to confirm the theoretical result.

Our future work will focus on nonlinear VIEs with weakly singular kernels.

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