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ORIGINAL PAPER

The quasi-boundary value method for identifying the initial value of heat equation on a columnar symmetric domain

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Abstract

In this paper, we consider an inverse problem for determining the initial value of heat equation with inhomogeneous source on a columnar symmetric domain. The quasi-boundary value regularization method is applied to solve this inverse problem. Under the a *priori* and a *posteriori* regularization parameter choice rules, the convergence estimates between the regularization solution and the exact solution are given. The numerical examples show this regularization method is effective and stable for dealing with this problem.

Keywords Inverse problem · Initial value · Quasi-boundary value · Columnar symmetric domain · Ill-posed

Mathematics Subject Classification (2010) 35R25 · 47A52 · 35R30

1 Introduction

The initial value problem is one of the backward heat conduction problem (BHCP). The solution which satisfies the heat conduction equation with final data and the boundary conditions does not exist. Even if a solution exists, it will not be continuously dependent on the final data, i.e., any small perturbation in the input data may cause large change to the solution. To overcome these difficulties, some regularization techniques are required. Several regularization methods have been proposed for the BHCP, such as the kernel-based method [2], the Fourier regularization method [8], optimal filtering method [22], the iterative method [10], the quasi-reversibility method [1, 11, 20, 23, 26], the central difference method [26], the filter regularization method [21], the method of fundamental solutions [18], the boundary element

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method [9, 19], the group preserving scheme [13, 19], the Tikhonov regularization method [15], quasi-boundary value method [7, 12, 14], and so on. But above these references about BHCP, there are some drawbacks as follows: firstly, the regularization parameter is a priori choice rule, according to this choice rule, the parameter depends on the priori bound of the exact solution. But in practice, we can not obtain the exact solution, and the inaccurate priori bound may lead to the bad regularized solution. Secondly, they only considered the one-dimensional BHCP; however, about high-dimensional BHCP, there is little research results. Thirdly, the equation is homogenous and the measurement data is only one. In this paper, we consider an inhomogeneous heat equation on a symmetric domain as follows:

$$\begin{cases} u_t - \frac{1}{r}u_r - u_{rr} = f(r,t), & 0 < t < T, \quad 0 < r < r_0, \\ u(r,0) = \varphi(r), & 0 \le r \le r_0, \\ u(r_0,t) = 0, & 0 \le t \le T, \\ \lim\limits_{r\to 0} u(r,t) \text{ is bounded}, & 0 < t < T, \quad 0 < r < r_0, \\ u(r,T) = g(r), & 0 \le r \le r_0, \end{cases} \tag{1}$$

where r_0 is the radius, $\varphi(r)$ is the initial value. We use the additional condition $u(r,T) = g(r)$ and $f(r,t)$ to determine the initial value $\varphi(r)$. The measured data of $g(r)$ and $f(r,t)$ are $g^\delta(r)$ and $f^\delta(r,t)$, which satisfy the following:

$$\|g^\delta(\cdot) - g(\cdot)\|_{L^2[0,r_0;r]} \le \delta; \quad \|f^\delta(\cdot,t) - f(\cdot,t)\|_{L^\infty(0,T;L^2[0,r_0;r])} \le \delta. \tag{2}$$

This problem is ill-posed, we use the quasi-boundary value regularization to solve this problem. We not only give the *priori* choice of the regularization parameter, but also we give the *a posteriori* choice of the regularization parameter, which depends only on the measurable data. Moreover, we give different type examples to show the effectiveness of this method. We also compare the effectiveness between the *posteriori* choice rule and the *priori* choice rule. The quasi-boundary value method, also called nonlocal boundary value method, is a regularization technique by replacing the final condition or boundary condition by a new approximate condition. This method has been used to solve some inverse problems for parabolic equation [4], hype-parabolic equations [24], and elliptic equations [5, 6].

Using the separation of variables, we obtain the solution of the problem (1) as follows:

$$u(r,t) = \sum_{n=1}^{\infty} \left(e^{-(\frac{\mu_n}{r_0})^2 t}\varphi_n + \int_0^t e^{-(\frac{\mu_n}{r_0})^2(t-\tau)} f_n(\tau)d\tau \right)\omega_n(r), \tag{3}$$

where

$$\omega_n(r) = \frac{\sqrt{2}}{r_0 J_1(\mu_n)} J_0(\frac{\mu_n r}{r_0}) \tag{4}$$

is the eigenfunction system and is orthonormal with weight r on $[0, r_0]$. It is also a complete system in $L^2[0, r_0; r]$, $J_0(x)$, and $J_1(x)$ denote the zero order and first-order Bessel functions, respectively [3], and $\{\mu_n\}_{n=1}^{\infty}$ are the sequence of roots of the equation $J_0(x) = 0$ which satisfy the following:

$$0 < \mu_1 < \mu_2 < \mu_3 < \cdots < \mu_n < \cdots, \lim_{n\to\infty} \mu_n = \infty. \tag{5}$$

Now let $\varphi_n = (\varphi(r), \omega_n(r))$, $f_n(t) = (f(r,t), \omega_n(r))$ and $g_n = (g(r), \omega_n(r))$, $h_n = \varphi_n + \int_0^T e^{(\frac{\mu_n}{r_0})^2\tau} f_n(\tau)d\tau$ and $h(r) = \sum_{n=1}^{\infty} h_n\omega_n(r)$. Using $u(r, T) = g(r)$, we have the following:

$$g(r) = \sum_{n=1}^{\infty} \left(e^{-(\frac{\mu_n}{r_0})^2 T} \varphi_n + \int_0^T e^{-(\frac{\mu_n}{r_0})^2 (T-\tau)} f_n(\tau)d\tau \right) \omega_n(r), \tag{6}$$

$$g_n = e^{-(\frac{\mu_n}{r_0})^2 T} \varphi_n + \int_0^T e^{-(\frac{\mu_n}{r_0})^2 (T-\tau)} f_n(\tau)d\tau. \tag{7}$$

Define operator $K : h(r) \to g(r)$, then

$$g(r) = \mathrm{Kh(r)} = \sum_{\mathrm{n}=1}^{\infty} \mathrm{e}^{-(\frac{\mu_\mathrm{n}}{\mathrm{r}_0})^2 \mathrm{T}} \left(\varphi_\mathrm{n} + \int_0^\mathrm{T} \mathrm{e}^{(\frac{\mu_\mathrm{n}}{\mathrm{r}_0})^2 \tau} \mathrm{f}_\mathrm{n}(\tau)\mathrm{d}\tau \right) \omega_\mathrm{n}(\mathrm{r}). \tag{8}$$

The operator K is a linear self adjoint compact operator [17]. Using (4) and (7), (8) can be rewritten as follows:

$$(g(r), \omega_n(r)) = (h(r), \omega_n(r)) e^{-(\frac{\mu_n}{r_0})^2 T}. \tag{9}$$

So

$$\varphi(r) = \sum_{n=1}^{\infty} \frac{g_n - \int_0^T e^{-(\frac{\mu_n}{r_0})^2 (T-\tau)} f_n(\tau)d\tau}{e^{-(\frac{\mu_n}{r_0})^2 T}} \omega_n(r). \tag{10}$$

When $n \to \infty$, $\mu_n \to \infty$, therefore $(e^{-(\frac{\mu_n}{r_0} T)})^{-1} \to \infty$. Thus, problem (1) is ill-posed.

We give a priori bound on the initial value, i.e.,

$$\| \varphi(\cdot) \|_p \leq E, \quad p > 0, \tag{11}$$

where $E > 0$ is a constant and $\| \cdot \|_p$ denotes the norm in Sobolev space which is defined as follows:

$$\|\varphi(\cdot)\|_p := \left(\sum_{n=1}^{\infty} (\frac{\mu_n}{r_0})^p |(\varphi(\cdot), \omega_n(\cdot))|^2 \right)^{\frac{1}{2}}. \tag{12}$$

This article is organized as follows. In Section 2, we present some preliminary results. Section 3 presents the convergence estimates under the two parameter choice rules. In Section 4, some numerical examples are proposed to show the effectiveness of this method. In Section 5, at the end of the article, the brief conclusion is given.

2 Some auxiliary results

Throughout this paper, we use the following definition and lemmas. $L^2[0, r_0; r]$ denotes the Hilbert space of Lebesgue measurable function φ with weight r on $[0, r_0]$. $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner and norm on $L^2[0, r_0; r]$, respectively. The norm of φ is defined as follows:

$$\|\varphi\| = \left(\int_0^{r_0} r|\varphi(r)|^2 \mathrm{dr}\right)^{\frac{1}{2}}. \tag{13}$$

Lemma 2.1 $\{e^{-(\frac{\mu_n}{r_0})^2 T}\}_{n=1}^{\infty}$ *mentioned from* [16, 25], μ_n *is the infinite number real root of the equation* $J_0(r) = 0$, *then*

$$\frac{C_1}{\mu_n} \le e^{-(\frac{\mu_n}{r_0})^2 T} \le \frac{C_2}{\mu_n}, \tag{14}$$

where C_1, C_2 *are constants.*

Lemma 2.2 *For any positive constant* $p > 0, \quad 0 < \mu < 1, \quad s \ge \mu_1 > 0$, *we have the following:*

$$F(s) = \frac{\mu s^{\frac{2-p}{4}}}{\mu s + C_1} \le \begin{cases} C_3 \mu^{\frac{p+2}{4}}, & 0 < p < 2, \\ C_4 \mu, & p \ge 2. \end{cases} \tag{15}$$

where $C_3 = C_3(p, C_1)$, $C_4 = C_4(p, \mu_1, C_1)$.

Proof (1) If $p \ge 2$, it is clear to see that

$$F(s) = \frac{\mu s^{\frac{2-p}{4}}}{\mu s + C_1} = \frac{\mu}{(\mu s + C_1)s^{\frac{p-2}{4}}} \le \frac{\mu}{C_1 s^{\frac{p-2}{4}}} \le \frac{\mu}{C_1 \mu_1^{\frac{p-2}{4}}}.$$

(2) If $0 < p < 2$, it is clear to see that $\lim\limits_{s\to 0} F(s) = \lim\limits_{s\to\infty} F(s) = 0$. $F(s)$ attains maximum value at $s = s_0$ which satisfies $F'(s_0) = 0$ for $F''(s_0) < 0$. Solving $F'(s_0) = 0$, we get $s_0 = \frac{(2-p)C_1}{(2+p)\mu}$.

Hence

$$F(s) \le F(s_0) = F\left(\frac{(2-p)C_1}{(2+p)\mu}\right) = \frac{\left(\frac{(2-p)C_1}{(2+p)}\right)^{\frac{2-p}{4}} \mu^{\frac{p+2}{4}}}{(\frac{4}{2+p})C_1}.$$

□

Lemma 2.3 *For any positive constant* $p > 0, \quad 0 < \mu < 1, \quad s \ge \mu_1 > 0$, *we have the following:*

$$F(s) = \frac{\mu s^{1-\frac{p}{2}}}{\mu s + C_1} \le \begin{cases} C_5 \mu^{\frac{p}{2}}, & 0 < p < 2, \\ C_6 \mu, & p \ge 2, \end{cases} \tag{16}$$

where $C_5 = C_5(p, C_1)$, $C_6 = C_6(p, \mu_1, C_1)$.

The proof is similar to proof of Lemma (2.2), we omit it.

Lemma 2.4 *Let $f \in L^{\infty}(0, T; L^2[0, r_0; r])$ and give $g_n - \int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f_n(\tau)d\tau$ in (10). Then, there exists a positive M such that*

$$\| g_n - \int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f_n(\tau)d\tau \| \leq \sqrt{2\Big(\| g \|^2_{L^2[0,r_0;r]} + M \| f \|^2_{L^\infty(0,T;L^2[0,r_0;r])}\Big)}, \tag{17}$$

where $M := \sum_{n=1}^{\infty} (\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} d\tau)^2$.

Proof For $t \in [0, T]$, there holds

$$| f_n(t) |^2 \leq \sum_{n=1}^{\infty} | (f(\cdot, t), \omega_n) |^2 \leq \| f \|^2_{L^\infty(0,T;L^2[0,r_0;r])} .$$

Thus,

$$\begin{aligned} \|g_n - \int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f_n(\tau)\, d\tau\|^2 &= \sum_{n=1}^{\infty} \Big(g_n - \int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f_n(\tau)\, d\tau\Big)^2 \\ &\leq 2\sum_{n=1}^{\infty} g_n^2 + 2\sum_{n=1}^{\infty} \Big|\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f_n(\tau)\, d\tau\Big|^2 \\ &\leq 2\Big(\|g\|^2_{L^2[0,r_0;r]} + M\|f\|^2_{L^\infty(0,T;L^2[0,r_0;r])}\Big). \end{aligned}$$

□

3 Regularization method and convergence estimate

In this section, through modifying $u(r, T) = g(r)$ as $u(r, T) + \mu u(r, 0) = g^\delta(r)$, we use quasi-boundary value method to solve the following problem:

$$\begin{cases} u_t^{\mu,\delta} - \frac{1}{r}u_r^{\mu,\delta} - u_{rr}^{\mu,\delta} = f^\delta(r, t), & 0 < t < T, \quad 0 < r < r_0, \\ u^{\mu,\delta}(r, 0) = \varphi^{\mu,\delta}(r), & 0 \leq r \leq r_0, \\ u^{\mu,\delta}(r_0, t) = 0, & 0 \leq t \leq T, \\ \lim_{r\to 0} u^{\mu,\delta}(r, t) \text{ is bounded}, & 0 < t < T, \quad 0 < r < r_0, \\ u^{\mu,\delta}(r, T) + \mu u^{\mu,\delta}(r, 0) = g^\delta(r), & 0 \leq r \leq r_0, \end{cases} \tag{18}$$

where μ is regularization parameter.

By the separation of variables, we obtain the solution of problem (18) as follows:

$$u^{\mu,\delta}(r, t) = \sum_{n=1}^{\infty} \Big(e^{-(\frac{\mu_n}{r_0})^2 t} \varphi_n^{\mu,\delta} + \int_0^t e^{-(\frac{\mu_n}{r_0})^2(t-\tau)} f_n^\delta(\tau)d\tau\Big) \omega_n(r). \tag{19}$$

Using $u^{\mu,\delta}(r,T)+\mu u^{\mu,\delta}(r,0)=g^{\delta}(r)$, we obtain the following:

$$g^{\delta}(r)=\sum_{n=1}^{\infty}\left((\mu+e^{-(\frac{\mu_n}{r_0})^2T})\varphi_n^{\mu,\delta}+\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}f_n^{\delta}(\tau)d\tau\right)\omega_n(r). \tag{20}$$

Hence,

$$\varphi^{\mu,\delta}(r)=\sum_{n=1}^{\infty}\frac{g_n^{\delta}-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}f_n^{\delta}(\tau)d\tau}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\omega_n(r). \tag{21}$$

3.1 Error estimate under an a prior parameter choice rule

Theorem 3.1.1 *Let* $\varphi(r)$ *given by* (11) *be the exact solution of problem* (1)*. Let* $\varphi^{\mu,\delta}(r)$ *given by* (21) *be the regularization solution.*

(a) *When* $0<p<2$*, choosing the regularization parameter* $\mu=(\frac{\delta}{E})^{\frac{2}{p+2}}$*, then we obtain the following estimate:*

$$\|\ \varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\ \|\le(\sqrt{2(M+1)}+C_5r_0^{\frac{p}{2}})E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{22}$$

(b) *As* $p\ge 2$*, choosing the regularization parameter* $\mu=(\frac{\delta}{E})^{\frac{1}{2}}$*, then we obtain the following estimate:*

$$\|\ \varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\ \|\le(\sqrt{2(M+1)}+C_6r_0^{\frac{p}{2}})E^{\frac{1}{2}}\delta^{\frac{1}{2}}. \tag{23}$$

Proof By the triangle inequality, we have the following:

$$\|\ \varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\ \|\le\|\ \varphi^{\mu,\delta}(\cdot)-\varphi^{\mu}(\cdot)\ \|+\|\ \varphi^{\mu}(\cdot)-\varphi(\cdot)\ \|. \tag{24}$$

Due to (2), we can get the following:

$$\begin{aligned}\|\varphi^{\mu,\delta}(\cdot)-\varphi^{\mu}(\cdot)\|^2&=\left\|\sum_{n=1}^{\infty}\left(\frac{g_n^{\delta}-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}f_n^{\delta}(\tau)d\tau}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}-\frac{g_n-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}f_n(\tau)d\tau}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)\omega_n(r)\right\|^2\\
&\le 2\sum_{n=1}^{\infty}\left(\frac{g_n^{\delta}-g_n}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2+2\sum_{n=1}^{\infty}\frac{|\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}(f_n^{\delta}(\tau)-f_n(\tau))d\tau\ |^2}{(\mu+e^{-(\frac{\mu_n}{r_0})^2T})^2}\\
&\le 2\sum_{n=1}^{\infty}\left(\frac{g_n^{\delta}-g_n}{\mu}\right)^2+2M\sum_{n=1}^{\infty}\left(\frac{f_n^{\delta}(\tau)-f_n(\tau)}{\mu}\right)^2\\
&\le 2\frac{\delta^2}{\mu^2}+2M\frac{\delta^2}{\mu^2}\\
&=2(M+1)\frac{\delta^2}{\mu^2}.\end{aligned}$$

Thus,

$$\|\ \varphi^{\mu,\delta}(\cdot)-\varphi^{\mu}(\cdot)\ \|\le\sqrt{2(M+1)}\frac{\delta}{\mu}. \tag{25}$$

Now, we estimate the second term of (24) as follows:

$$\begin{aligned}\| \varphi^{\mu}(\cdot)-\varphi(\cdot) \| &= \| \sum_{n=1}^{\infty}\left(\frac{g_n-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f_n(\tau)d\tau}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}} - \frac{g_n-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f_n(\tau)d\tau}{e^{-(\frac{\mu_n}{r_0})^2T}}\right)\omega_n(r)\| \\ &= \left(\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\left(\frac{g_n-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f_n(\tau)d\tau}{e^{-(\frac{\mu_n}{r_0})^2T}}\right)\right)^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=1}^{\infty}\left(\frac{\mu(\frac{\mu_n}{r_0})^{-\frac{p}{2}}}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}(\frac{\mu_n}{r_0})^{\frac{p}{2}}\left(\frac{g_n-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f_n(\tau)d\tau}{e^{-(\frac{\mu_n}{r_0})^2T}}\right)\right)^2\right)^{\frac{1}{2}} \\ &\le \sup(A_n)E,\end{aligned}$$

where $A_n = \frac{\mu(\frac{\mu_n}{r_0})^{-\frac{p}{2}}}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}$.

Using Lemmas 2.1 and 2.3, we obtain the following:

$$A_n \le \frac{r_0^{\frac{p}{2}}\mu\mu_n^{1-\frac{p}{2}}}{\mu\mu_n+C_1} \le \begin{cases} C_5 r_0^{\frac{p}{2}}\mu^{\frac{p}{2}}, & 0<p<2, \\ C_6 r_0^{\frac{p}{2}}\mu, & p\ge 2. \end{cases}$$

Hence,

$$\| \varphi^{\mu,\delta}(\cdot)-\varphi(\cdot) \| \le \sqrt{2(M+1)}\frac{\delta}{\mu} + \begin{cases} C_5 r_0^{\frac{p}{2}}\mu^{\frac{p}{2}}E, & 0<p<2, \\ C_6 r_0^{\frac{p}{2}}\mu E, & p\ge 2. \end{cases} \quad (26)$$

Choosing regularization parameter $\mu = (\frac{\delta}{E})^{\frac{2}{p+2}}$ $(0<p<2)$ and $\mu = (\frac{\delta}{E})^{\frac{1}{2}}$ $(p\ge 2)$, we obtain the following:

$$\| \varphi^{\mu,\delta}(\cdot)-\varphi(\cdot) \| \le \begin{cases} (\sqrt{2(M+1)}+C_5 r_0^{\frac{p}{2}})E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, & 0<p<2, \\ (\sqrt{2(M+1)}+C_6 r_0^{\frac{p}{2}})E^{\frac{1}{2}}\delta^{\frac{1}{2}}, & p\ge 2. \end{cases} \quad (27)$$

□

3.2 Error estimate under an a posteriori parameter choice rule

Applying a discrepancy principle, we choose the solution of the following equation as a posteriori regularization parameter:

$$\|\mu(K+\mu I)^{-1}(K\varphi_{\mu}^{\delta}(r)-(g^{\delta}-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)} f^{\delta}(\tau)d\tau))\| = \tau\delta, \quad (28)$$

where $\tau > \sqrt{2(M+1)}$ is constant.

Lemma 3.2.1 *Let* $\rho(\mu) = \|\mu(K+\mu I)^{-1}(K\varphi_\mu^\delta(r) - (g^\delta - \int_0^T e^{-(\frac{\mu n}{r_0})^2(T-\tau)} f^\delta(\tau)d\tau))\|$, *if* $\|g^\delta - \int_0^T e^{-(\frac{\mu n}{r_0})^2(T-\tau)} f^\delta(\tau)d\tau)\| > \delta$, *then we have the following conclusions:*

(a) $\rho(\mu)$ *is a continuous function;*
(b) $\lim_{\mu\to 0}\rho(\mu) = 0$;
(c) $\lim_{\mu\to +\infty}\rho(\mu) = \|g^\delta - \int_0^T e^{-(\frac{\mu n}{r_0})^2(T-\tau)} f^\delta(\tau)d\tau)\|$;
(d) $\rho(\mu)$ *is a strictly increasing function, for any* $\mu \in (0,\infty)$.

Proof It can be proven by the following:

$$\rho(\mu) = \left(\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}}\right)^4 \left(g^\delta - \int_0^T e^{-(\frac{\mu n}{r_0})^2(T-\tau)} f^\delta(\tau)d\tau\right)^2\right)^{\frac{1}{2}}.$$

here, we skip it. □

Theorem 3.2.2 *Let* $\varphi(r)$ *given by* (11) *be the exact solution of problem* (1). *Let* $\varphi^{\mu,\delta}(r)$ *given by* (21) *be the regularization solution, the regularization parameter* μ *is chosen in* (28). *Then we obtain the following:*

(a) *If* $0 < p < 2$, *the following estimate holds as follows:*

$$\|\varphi^{\mu,\delta}(\cdot) - \varphi(\cdot)\| \le \left[r_0^{\frac{p}{p+2}}\left(\frac{\tau + \sqrt{2(M+1)}}{C_1}\right)^{\frac{p}{p+2}} + \sqrt{2(M+1)}\left(r_0^{\frac{p}{2}}\frac{C_2C_3^2}{\tau - \sqrt{2(M+1)}}\right)^{\frac{p}{p+2}}\right] E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \quad (29)$$

(b) *If* $p \ge 2$, *the following estimate holds as follows:*

$$\|\varphi^{\mu,\delta}(\cdot) - \varphi(\cdot)\| \le \left[r_0^{\frac{1}{2}}\left(\frac{\tau + \sqrt{2(M+1)}}{C_1}\right)^{\frac{1}{2}} + \sqrt{2(M+1)}\left(r_0^{\frac{p}{2}}\frac{C_2C_4^2}{\tau - \sqrt{2(M+1)}}\right)^{\frac{1}{2}}\right] E^{\frac{1}{2}}\delta^{\frac{1}{2}}. \quad (30)$$

where C_1, C_2, C_3, *and* C_4 *are positive constants.*

Proof According to (24), we have the following:

$$\| \varphi^{\mu,\delta}(\cdot) - \varphi(\cdot) \| \leq \| \varphi^{\mu,\delta}(\cdot) - \varphi^{\mu}(\cdot) \| + \| \varphi^{\mu}(\cdot) - \varphi(\cdot) \| . \quad (31)$$

Now, we estimate the second term of (31). As $0 < p < 2$, using (19), Lemma 2.1 and Hölder inequality, we have the following:

$$\begin{aligned}
\|\varphi^{\mu}(\cdot)-\varphi(\cdot)\|^2 &= \| \sum_{n=1}^{\infty} \left(\frac{g_n - \int_0^T e^{-(\frac{\mu n}{r_0})^2(T-\tau)} f_n(\tau)d\tau}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} - \frac{g_n - \int_0^T e^{-(\frac{\mu n}{r_0})^2(T-\tau)} f_n(\tau)d\tau}{e^{-(\frac{\mu n}{r_0})^2 T}} \right) \omega_n(r) \|^2 \\
&= \| \sum_{n=1}^{\infty} \frac{-\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \varphi_n \omega_n(r) \|^2 \\
&= \sum_{n=1}^{\infty} \left(\frac{-\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \varphi_n \right)^2 \\
&= \sum_{n=1}^{\infty} \left(\left(\frac{\mu e^{-(\frac{\mu n}{r_0})^2 T}}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{\frac{p}{2}} \left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{1-\frac{p}{2}} \frac{\varphi_n}{(e^{-(\frac{\mu n}{r_0})^2 T})^{\frac{p}{2}}} \right)^2 \\
&= \sum_{n=1}^{\infty} \left(\frac{\mu e^{-(\frac{\mu n}{r_0})^2 T}}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{p} \left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{\frac{2p-p^2}{p+2}} \left(\frac{\varphi_n}{(e^{-(\frac{\mu n}{r_0})^2 T})^{\frac{p}{2}}} \right)^{\frac{2p}{p+2}} \\
&\quad \cdot \left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{\frac{4-2p}{p+2}} \left(\frac{\varphi_n}{(e^{-(\frac{\mu n}{r_0})^2 T})^{\frac{p}{2}}} \right)^{\frac{4}{p+2}} \\
&\leq \left(\sum_{n=1}^{\infty} \left(\left(\frac{\mu e^{-(\frac{\mu n}{r_0})^2 T}}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{p} \left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{\frac{2p-p^2}{p+2}} \left(\frac{\varphi_n}{(e^{-(\frac{\mu n}{r_0})^2 T})^{\frac{p}{2}}} \right)^{\frac{2p}{p+2}} \right)^{\frac{p+2}{2p}} \right)^{\frac{2p}{p+2}} \\
&\quad \cdot \left(\sum_{n=1}^{\infty} \left(\left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{\frac{4-2p}{p+2}} \left(\frac{\varphi_n}{(e^{-(\frac{\mu n}{r_0})^2 T})^{\frac{p}{2}}} \right)^{\frac{4}{p+2}} \right)^{\frac{p+2}{4}} \right)^{\frac{4}{p+2}} \\
&= \left(\sum_{n=1}^{\infty} \left(\frac{\mu e^{-(\frac{\mu n}{r_0})^2 T}}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{\frac{p+2}{2}} \left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{\frac{2-p}{2}} \frac{\varphi_n}{(e^{-(\frac{\mu n}{r_0})^2 T})^{\frac{p}{2}}} \right)^{\frac{2p}{p+2}} \\
&\quad \cdot \left(\sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{\frac{2-p}{2}} \frac{\varphi_n}{(e^{-(\frac{\mu n}{r_0})^2 T})^{\frac{p}{2}}} \right)^{\frac{4}{p+2}} \\
&= \| \sum_{n=1}^{\infty} \left(\frac{\mu e^{-(\frac{\mu n}{r_0})^2 T}}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{\frac{p+2}{2}} \left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{1-\frac{p}{2}} \frac{\varphi_n}{(e^{-(\frac{\mu n}{r_0})^2 T})^{\frac{p}{2}}} \omega_n(r) \|^{\frac{2p}{p+2}} \\
&\quad \cdot \| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-(\frac{\mu n}{r_0})^2 T}} \right)^{1-\frac{p}{2}} \frac{\varphi_n}{(e^{-(\frac{\mu n}{r_0})^2 T})^{\frac{p}{2}}} \omega_n(r) \|^{\frac{4}{p+2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2\varphi_n e^{-(\frac{\mu_n}{r_0})^2T}\omega_n(r)\|^{\frac{2p}{p+2}}\cdot\|\sum_{n=1}^{\infty}\frac{\varphi_n}{(e^{-(\frac{\mu_n}{r_0})^2T})^{\frac{p}{2}}}\omega_n(r)\|^{\frac{4}{p+2}}\\
&\leq \|\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2(g_n-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}f_n(\tau)d\tau)\omega_n(r)\|^{\frac{2p}{p+2}}\\
&\cdot\|\sum_{n=1}^{\infty}\left(\frac{\mu_n}{r_0}\right)^{\frac{p}{2}}\varphi_n\omega_n(r)\|^{\frac{4}{p+2}}r_0^{\frac{2p}{p+2}}C_1^{-\frac{2p}{p+2}}\\
&\leq\left(\|\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2((g_n(r)-g_n^{\delta}(r)\right)\\
&-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}(f_n(\tau)-f_n^{\delta}(\tau))d\tau)\omega_n(r)\|^{\frac{2p}{p+2}}\\
&+\|\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2(g_n^{\delta}(r)\\
&-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}f_n^{\delta}(\tau)d\tau)\omega_n(r)\|^{\frac{2p}{p+2}}\Big)E^{\frac{4}{p+2}}C_1^{-\frac{2p}{p+2}}r_0^{\frac{2p}{p+2}}\\
&\leq\left(\frac{\tau+\sqrt{2(M+1)}}{C_1}\right)^{\frac{2p}{p+2}}r_0^{\frac{2p}{p+2}}E^{\frac{4}{p+2}}\delta^{\frac{2p}{p+2}}.
\end{aligned}
$$

Thus, we obtain the following:

$$
\|\varphi^{\mu}(\cdot)-\varphi(\cdot)\|\leq\left(\frac{\tau+\sqrt{2(M+1)}}{C_1}\right)^{\frac{p}{p+2}}r_0^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{32}
$$

When $p\geq 2$, we estimate the second term of (31). H^p compacts into H^1, then there exists an a such that $\|f(\cdot)\|_{H^1}\leq a\|f(\cdot)\|_{H^p}\leq aE$, where a is a constant.

$$
\begin{aligned}
\|\varphi^{\mu}(\cdot)-\varphi(\cdot)\|^2 &= \|\sum_{n=1}^{\infty}\frac{-\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\varphi_n\omega_n(r)\|^2\\
&=\sum_{n=1}^{\infty}\left(\frac{-\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\varphi_n\right)^2\\
&=\sum_{n=1}^{\infty}\left(\frac{\mu e^{-(\frac{\mu_n}{r_0})^2T}}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\frac{\varphi_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty}\left(\frac{\mu e^{-(\frac{\mu_n}{r_0})^2T}}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\left(\frac{\varphi_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\right)^{\frac{1}{2}}\left(\frac{\varphi_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\right)^{\frac{1}{2}}\right)^2\\
&= \sum_{n=1}^{\infty}\left(\frac{\mu e^{-(\frac{\mu_n}{r_0})^2T}}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2\frac{\varphi_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\frac{\varphi_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\\
&\le \left(\sum_{n=1}^{\infty}\left(\left(\frac{\mu e^{-(\frac{\mu_n}{r_0})^2T}}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2\frac{\varphi_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\left(\frac{\varphi_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2\right)^{\frac{1}{2}}\\
&= \left(\sum_{n=1}^{\infty}\left(\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2\varphi_n e^{-(\frac{\mu_n}{r_0})^2T}\right)^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\left(\frac{\varphi_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2\right)^{\frac{1}{2}}\\
&= \|\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2\varphi_n e^{-(\frac{\mu_n}{r_0})^2T}\omega_n(r)\|\|\sum_{n=1}^{\infty}\frac{\varphi_n}{e^{-(\frac{\mu_n}{r_0})^2T}}\omega_n(r)\|\\
&\le \|\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2\left(g_n-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}f_n(\tau)d\tau\right)\omega_n(r)\|\\
&\cdot\ \|\sum_{n=1}^{\infty}\frac{\mu_n}{r_0}\varphi_n\omega_n(r)\|r_0C_1^{-1}\\
&\le \left(\|\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2((g_n(r)-g_n^{\delta}(r))\right.\\
&- \int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}(f_n(\tau)-f_n^{\delta}(\tau))d\tau)\omega_n(r)\|\\
&+ \left.\|\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2(g_n^{\delta}(r)-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}f_n^{\delta}(\tau)d\tau)\omega_n(r)\|\right)(aE)C_1^{-1}r_0\\
&\le \left(\|\sum_{n=1}^{\infty}(g_n(r)-g_n^{\delta}(r))-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}(f_n(\tau)-f_n^{\delta}(\tau))d\tau\omega_n(r)\|\right.\\
&+ \|\sum_{n=1}^{\infty}\left(\frac{\mu}{\mu+e^{-(\frac{\mu_n}{r_0})^2T}}\right)^2(g_n^{\delta}(r)-\int_0^T e^{-(\frac{\mu_n}{r_0})^2(T-\tau)}f_n^{\delta}(\tau)d\tau)\omega_n(r)\|(aE)C_1^{-1}r_0\\
&\le \left(\frac{r_0a\tau+r_0a\sqrt{2(M+1)}}{C_1}\right)E\delta.
\end{aligned}
$$

So

$$
\|\varphi^{\mu}(\cdot)-\varphi(\cdot)\| \le \left(\frac{r_0a\tau+r_0a\sqrt{2(M+1)}}{C_1}\right)^{\frac{1}{2}}E^{\frac{1}{2}}\delta^{\frac{1}{2}}. \tag{33}
$$

From (28), we obtain the following:

$$
\begin{aligned}
\tau\delta &= \| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-(\frac{\mu_n}{r_0})^T}} \right)^2 \left(g_n^\delta - \int_0^T e^{-(\frac{\mu_n}{r_0})^2 (T-\tau)} f_n^\delta(\tau) d\tau \right) \omega_n(r) \| \\
&\le \| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-(\frac{\mu_n}{r_0})^2 T}} \right)^2 \left((g_n - g_n^\delta) - \int_0^T e^{-(\frac{\mu_n}{r_0})^2 (T-\tau)} (f_n(\tau) - f_n^\delta(\tau)) d\tau \right) \omega_n(r) \| \\
&+ \| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-(\frac{\mu_n}{r_0})^2 T}} \right)^2 \left(g_n - \int_0^T e^{-(\frac{\mu_n}{r_0})^2 (T-\tau)} f_n(\tau) d\tau \right) \omega_n(r) \| \\
&\le \sqrt{2(M+1)}\delta + \left(\sum_{n=1}^{\infty} \left(\left(\frac{\mu}{\mu + e^{-(\frac{\mu_n}{r_0})^2 T}} \right)^2 e^{-(\frac{\mu_n}{r_0})^2 T} (\frac{\mu_n}{r_0})^{-\frac{p}{2}} (\frac{\mu_n}{r_0})^{\frac{p}{2}} \varphi_n \right)^2 \right)^{\frac{1}{2}} \\
&\le \sqrt{2(M+1)}\delta + E \sup \left\{ \left(\frac{\mu}{\mu + e^{-(\frac{\mu_n}{r_0})^2 T}} \right)^2 e^{-(\frac{\mu_n}{r_0})^2 T} (\frac{\mu_n}{r_0})^{-\frac{p}{2}} \right\} \\
&\le \sqrt{2(M+1)}\delta + E \sup \left(\frac{\mu \mu_n^{\frac{2-p}{4}}}{C_1 + \mu\mu_n} \right)^2 r_0^{\frac{p}{2}} C_2 \\
&\le \sqrt{2(M+1)}\delta + E \sup (B_n)^2 r_0^{\frac{p}{2}} C_2.
\end{aligned} \tag{34}
$$

According to Lemma 2.2, we obtain the following:

$$
B_n = \frac{\mu \mu_n^{\frac{2-p}{4}}}{C_1 + \mu\mu_n} \le \begin{cases} C_3 \mu^{\frac{p+2}{4}}, & 0 < p < 2, \\ C_4 \mu, & p \ge 2. \end{cases}
$$

Hence,

$$
\frac{1}{\mu} \le \begin{cases} \left(\frac{C_2 C_3^2 r_0^{\frac{p}{2}}}{\tau - \sqrt{2(M+1)}} \right)^{\frac{2}{p+2}} \left(\frac{E}{\delta}\right)^{\frac{2}{p+2}}, & 0 < p < 2. \\ \left(\frac{C_2 C_4^2 r_0^{\frac{p}{2}}}{\tau - \sqrt{2(M+1)}} \right)^{\frac{1}{2}} \left(\frac{E}{\delta}\right)^{\frac{1}{2}}, & p \ge 2. \end{cases} \tag{35}
$$

Substituting (35) into (34), we can get the following:

$$
\|\varphi^{\mu,\delta}(\cdot) - \varphi^{\mu}(\cdot)\| \le \sqrt{2(M+1)} \frac{\delta}{\mu} \le \begin{cases} \sqrt{2(M+1)} \left(r_0^{\frac{p}{2}} \frac{C_2 C_3^2}{\tau - \sqrt{2(M+1)}} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2, \\ \sqrt{2(M+1)} \left(r_0^{\frac{p}{2}} \left(\frac{C_2 C_4^2}{\tau - \sqrt{2(M+1)}} \right)^{\frac{1}{2}} \right) E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p \ge 2. \end{cases} \tag{36}
$$

Combining (32) and (36), we obtain the following:

$$\|\varphi^{\mu,\delta}(\cdot)-\varphi(\cdot)\| \leq \begin{cases} \left(\left(\frac{\tau+\sqrt{2(M+1)}}{C_1}\right)^{\frac{p}{p+2}} r_0^{\frac{p}{p+2}} + \sqrt{2(M+1)}\left(r_0^{\frac{p}{2}} \frac{C_2 C_3^2}{\tau-\sqrt{2(M+1)}}\right)^{\frac{2}{p+2}}\right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, 0<p<2, \\ \left(\left(\frac{a\tau+a\sqrt{2(M+1)}}{C_1}\right)^{\frac{1}{2}} r_0^{\frac{1}{2}} + \sqrt{2(M+1)}\left(r_0^{\frac{p}{2}} \frac{C_2 C_4^2}{\tau-\sqrt{2(M+1)}}\right)^{\frac{1}{2}}\right) E^{\frac{1}{2}} \delta^{\frac{1}{2}}, \quad p \geq 2. \end{cases} \tag{37}$$

Theorem 3.2.2 is proved. □

4 Numerical examples

In this section, we present numerical experiments for above regularization method. The exact solution of problem (18) is difficult to obtain. So we give $\varphi(r)$ to solve the direct problem.

$$\begin{cases} u_t - \frac{1}{r}u_r - u_{rr} = f(r,t), & 0 < t < T, \quad 0 < r < r_0, \\ u(r,0) = \varphi(r), & 0 \leq r \leq r_0, \\ u(r_0,t) = 0, & 0 \leq t \leq T, \\ \lim\limits_{r\to 0} u(r,t) \text{ is bounded}, & 0 < t < T, \quad 0 < r < r_0. \end{cases} \tag{38}$$

Time and space of grid step size are $\Delta t = \frac{T}{N}$ and $\Delta r = \frac{r_0}{M}$. The grid point on the time interval $[0, T]$ is $t_n = n\Delta t$, $(n = 0, 1, 2\cdots, N)$. $r_i = i\Delta r$, $(i = 1, 2, \cdots, M)$ is grid points on the space interval.

Using the finite difference scheme, we discrete the equation of problem (1.1) as follows:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{1}{r_j}\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta r} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta r)^2} = f(r_j, t_n). \tag{39}$$

Denote $U^{n+1} = (u_1^{n+1}, u_2^{n+1}, \cdots, u_{M-1}^{n+1})^T$, $\varphi = (\varphi(r_1), \varphi(r_2), \cdots, \varphi(r_{M-1}))^T$ and source term function $f^{n+1} = (f^{n+1}(r_1), f^{n+1}(r_2), \cdots, f^{n+1}(r_{M-1}))^T$, then we get following iterative scheme as follows:

$$\begin{aligned} U^1 &= A\varphi + \Delta t f^0, \\ U^n &= A^n\varphi + \Delta t A^{n-1} f^0 + \Delta t A^{n-2} f^1 + \cdots + \Delta t f^{n-1}, \end{aligned} \tag{40}$$

where the matrix $A = (a_{ij})$ is non-symmetric tridiagonal, and $a_{i,j}$ is defined as follows:

$$a_{i,j} = \begin{cases} \frac{\lambda}{2r_j} + \kappa, & i = j-1, \\ 1 - 2\kappa, & i = j, \\ \kappa - \frac{\lambda}{2r_{j+1}}, & i = j+1. \end{cases} \tag{41}$$

With $\lambda := \frac{\Delta t}{\Delta r}$ and $\kappa := \frac{\Delta t}{(\Delta r)^2}$, we get $g(r) = U^{n+1}$ by (4.3). Similar to forward problem that above mention, we use the finite difference scheme mentioned to discrete equation of problem (3.1). Denote

$$V^{n+1} = (u^{\mu,\delta}(r_1, t_{n+1}), u^{\mu,\delta}(r_2, t_{n+1}), \cdots, u^{\mu,\delta}(r_{M-1}, t_{n+1}))^T,$$

we get scheme

$$V^{n+1} = AV^n + \Delta t f^n. \tag{42}$$

From above, we can get the following:

$$\begin{aligned} V^1 &= A\varphi^\delta + \Delta t f^{0,\delta}, \\ U^n &= A^n\varphi^\delta + \Delta t A^{n-1} f^{0,\delta} + \Delta t A^{n-2} f^{1,\delta} + \cdots + \Delta t f^{n-1,\delta}. \end{aligned} \tag{43}$$

By using boundary condition of regularization problem (18), we have the following:

$$V^N = G^\delta - \mu\varphi^\delta. \tag{44}$$

Noise data is generated by adding random perturbation, that is as follows:

$$\begin{aligned} g^\delta(\cdot) &= g(\cdot) + \varepsilon \cdot g(\cdot) \cdot (2\text{rand}(\cdot) - 1), \\ f^\delta(\cdot,\cdot) &= f(\cdot,\cdot) + \varepsilon \cdot f(\cdot,\cdot) \cdot (2\text{rand}(\cdot,\cdot) - 1), \end{aligned}$$

where ε is relative error level.

Let $T = 1$, $r_0 = \pi$. In the computational procedure, we take source function $f(r,t) = \text{rt}$.

Example 1 Take initial function $\varphi(r) = \text{rsin}(r)$.

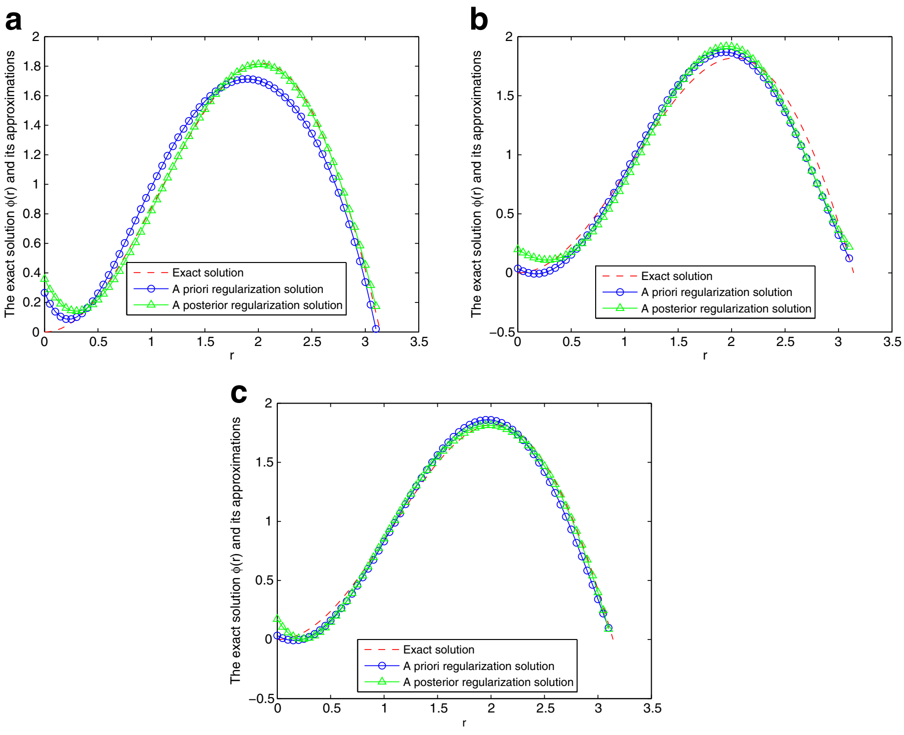

Fig. 1 The comparison of numerical effects between the exact solution and its regularization solution for Example 1: **a** $p = 2, \varepsilon = 0.01$, **b** $p = 2, \varepsilon = 0.001$, **c** $p = 2, \varepsilon = 0.0001$

Example 2 Consider a piecewise smooth function

$$\varphi(r) = \begin{cases} 0, & 0 \le r \le \frac{\pi}{4}, \\ \frac{4}{\pi}r - 1, & \frac{\pi}{4} < r \le \frac{\pi}{2}, \\ -\frac{4}{\pi}r + 3, & \frac{\pi}{2} < r \le \frac{3}{4}\pi, \\ 0, & \frac{3}{4}\pi < r \le \pi. \end{cases} \tag{45}$$

Example 3 Consider the following discontinuous function

$$\varphi(r) = \begin{cases} 0, & 0 \le r \le \frac{\pi}{3}, \\ 1, & \frac{\pi}{3} < r \le \frac{2}{3}\pi, \\ 0, & \frac{2}{3}\pi < r \le \pi. \end{cases} \tag{46}$$

Figures 1, 2, and 3 show the comparisons of the numerical effects between the exact solution and its regularization solution for the *a priori* and *a posteriori* regularization parameter choose rule. We can find that the smaller ε, the better the computed approximation is. Moreover, we can also easily find that the *a posteriori* parameter choice rule works better than the *a priori* parameter choice rule well. This is consistent with our theoretical analysis.

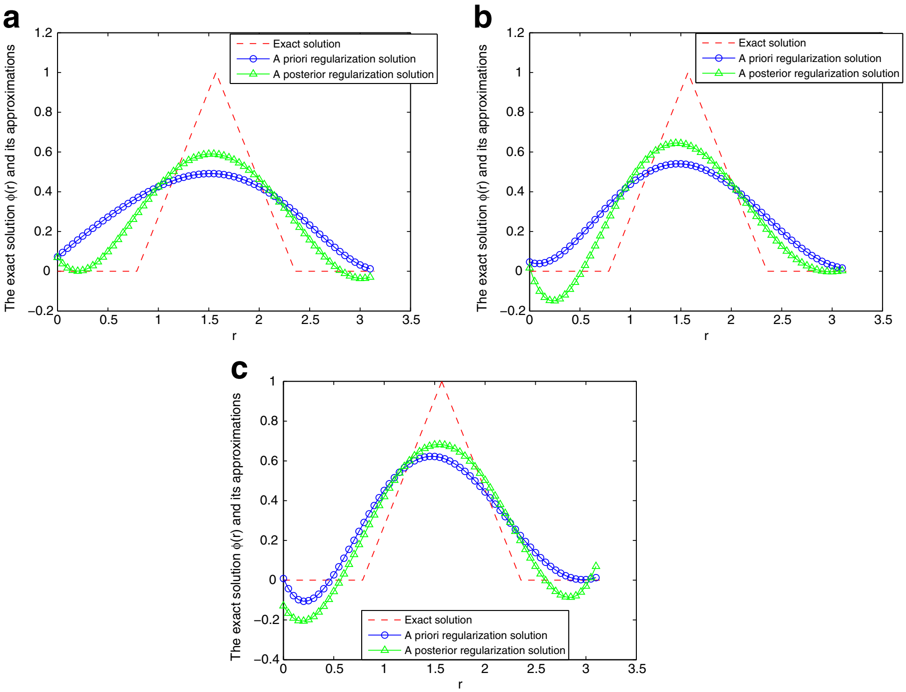

Fig. 2 The comparison of numerical effects between the exact solution and its regularization solution for Example 2: **a** $p = 2$, $\varepsilon = 0.001$, **b** $p = 2$, $\varepsilon = 0.0001$, **c** $p = 2$, $\varepsilon = 0.00001$

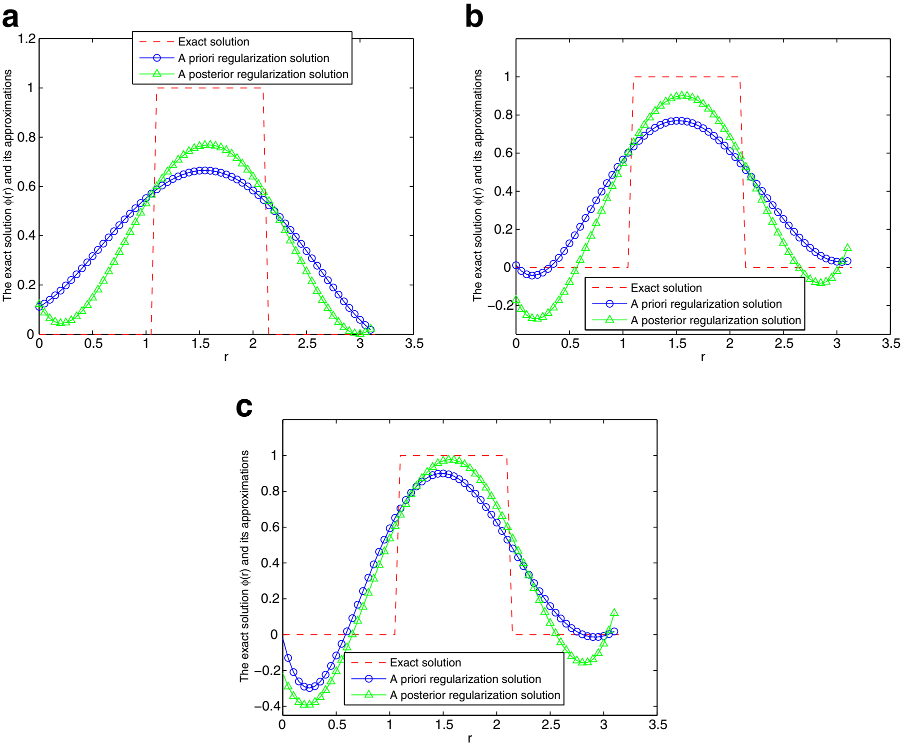

Fig. 3 The comparison of numerical effects between the exact solution and its regularization solution for Example 3: **a** $p = 2, \varepsilon = 0.001$, **b** $p = 2, \varepsilon = 0.0001$, **c** $p = 2, \varepsilon = 0.00001$

5 Conclusion

We consider an inverse problem to determine an initial value for heat equation with inhomogeneous source on a columnar symmetric domain. We construct the quasi-boundary value method to solve this inverse problem and obtain regularization solution. Moreover, we obtain the Hölder type error estimate under *a priori* and *a posteriori* parameter choice rules. Finally, several examples are given to show the effectiveness of quasi-boundary value method.

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