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New explicit stabilized stochastic Runge-Kutta methods with weak second order for stiff Itô stochastic differential equations

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Abstract
This paper introduces a new class of weak second-order explicit stabilized stochastic Runge-Kutta methods for stiff Itô stochastic differential equations. The convergence and mean-square stability properties of our new methods are analyzed. The numerical results of two examples are presented to confirm our theoretical results.

Keywords Stiff stochastic differential equations · Explicit stabilized methods · Weak second order

Mathematics Subject Classification (2010) 65C30 · 60H35 · 65L20

1 Introduction

Consider the autonomous d-dimensional Itô stochastic differential equation (SDE) system

$$dy(t) = f(y(t))dt + \sum_{j=1}^{m} g^j(y(t))dW^j(t),\ y(0) = y_0,\ t \in [0, T], \tag{1}$$

where $W(t)$ is an m-dimensional Wiener process, the vector functions $f, g^j \in \mathbb{R}^d$, $j = 1, 2, \ldots, m$ are assumed to satisfy the standard conditions to guarantee the existence of unique solution of the SDE (1). The inequality $E(\|y(0)\|^2) < \infty$ holds for any given initial value $y(0)$. Let $C_P^L(\mathbb{R}^d, \mathbb{R})$ be the family of L times continuously

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differentiable real-valued functions on $\mathbb{R}^d$, all of whose partial derivatives of order up to and including L have polynomial growth.

The standard explicit methods will face a severe step size restriction when the SDE (1) is stiff (see, e.g., [9, 11]). This restriction can be effectively avoided by using the (semi)implicit methods (see, e.g., [5, 8, 10, 15]). But the (semi)implicit methods will be difficult to implement for complex problems and be expensive for large systems. In recent years, many explicit methods with extended stability regions have been studied in [1–3, 6, 12, 13]. These methods are well suited for some stiff problems. In particular, Abdulle, Vilmart, and Zygalakis [6] and Komori and Burrage [12] have studied the weak second-order explicit stabilized methods for stiff Itô and Stratonovitch SDEs, respectively.

In this paper, we will introduce a new class of weak second-order explicit stabilized stochastic Runge-Kutta (SRK) methods. Our new methods can be seen as a combination of the second-order orthogonal Runge-Kutta-Chebyshev (ROCK2) methods introduced in [4, 6] and the weak second-order SRK methods introduced in [14]. As shown in the sequel, our new methods have advantages in terms of both stability and computational cost.

2 The ROCK2 methods and the weak second-order SRK methods

Before introducing the new weak second-order explicit stabilized SRK methods, let's review briefly the ROCK2 methods introduced in [4, 6] and the weak second-order SRK methods introduced in [14].

For the ordinary differential equation $dy(t) = f(y(t))dt$, the ROCK2 method with the damping parameter α can be written as the form

$$\begin{aligned}
H_0 &= y_n, H_1 = H_0 + \alpha\mu_1 hf(H_0),\\
H_i &= \alpha\mu_i hf(H_{i-1}) - \nu_i H_{i-1} - \kappa_i H_{i-2}, i = 2, 3, \ldots, \hat{s}-2,\\
\hat{H}_1 &= H_{\hat{s}-2} + 2\tau_\alpha hf(H_{\hat{s}-2}),\\
y_{n+1} &= H_{\hat{s}-2} + \left(2\sigma_\alpha - \frac{1}{2}\right) hf(H_{\hat{s}-2}) + \frac{1}{2}hf(\hat{H}_1),
\end{aligned} \tag{2}$$

where $\hat{s} \geq 2$, $\hat{s} \in \mathbb{Z}^+$, $h = \frac{T}{N}$ denotes stepsize, $n = 0, 1, \ldots, N-1$, $y_0 = y(0)$ and $\sigma_\alpha = \dfrac{1-\alpha}{2} + \alpha\sigma$, $\tau_\alpha = \dfrac{(\alpha-1)^2}{2} + 2\alpha(1-\alpha)\sigma + \alpha^2\tau$. All the parameters $\mu_i, \nu_i, \kappa_i, \sigma, \tau$ depend on $\hat{s}$ (see the literature [6] for more details).

In [14], for SDE (1), we proposed the efficient weak second-order SRK method

$$\begin{aligned}
y_{n+1} = y_n &+ \sum_{i=1}^{s} \alpha_i^{(0)} f(H_i^{(0)})h + \sum_{i=1}^{s}\sum_{k=1}^{m} \beta_i^{(0)} g^k(H_i^{(k)})\hat{I}_k\\
&+ \sum_{i=1}^{s}\sum_{k=1}^{m} \beta_i^{(1)} g^k(H_i^{(k)})\hat{I}_{(k,k)},
\end{aligned} \tag{3}$$

where

$$
\begin{aligned}
H_i^{(0)} &= y_n + \sum_{j=1}^{s} a_{ij}^{(0)} f(H_j^{(0)})h + \sum_{i=1}^{s}\sum_{k=1}^{m} b_{ij}^{(0)} g^k(H_j^{(k)})\hat{I}_k,\\
H_i^{(k)} &= y_n + \sum_{j=1}^{s} a_{ij}^{(1)} f(H_j^{(0)})h + \sum_{j=1}^{s} b_{ij}^{(1)} g^k(H_j^{(k)})\xi\\
&\quad + \sum_{i=1}^{s}\sum_{\substack{l=1\\ l\neq k}}^{m} b_{ij}^{(2)} g^l(H_i^{(l)})\hat{I}_{(k,l)}.
\end{aligned}
$$

The random variables $\hat{I}_k, \xi, \hat{I}_{(k,l)}$ are defined by

$$
P(\hat{I}_k = \pm\sqrt{3h}) = \frac{1}{6}, P(\hat{I}_k = 0) = \frac{2}{3}, \xi = \eta_1\sqrt{h}, \tag{4}
$$

$$
\hat{I}_{(k,l)} = \begin{cases} \frac{1}{2}(\hat{I}_l - \eta_2\hat{I}_l), & k < l,\\ \frac{1}{2}(\hat{I}_l + \eta_2\hat{I}_l), & k > l,\\ \frac{1}{2}(\frac{\hat{I}_k^2}{\xi} - \xi), & k = l, \end{cases} \tag{5}
$$

where $k, l \in \{1, 2, \ldots, m\}$, and η_1, η_2 are independent two-point distributed random variables with $P(\eta_i = \pm 1) = \frac{1}{2}, i \in \{1, 2\}$. Based on [14], we have the following result.

Theorem 1 *Let $f, g^j \in C_P^6(\mathbb{R}^d, \mathbb{R}^d)$, $j = 1, 2, \ldots, m$. Then, the SRK method (3) converges weakly to solution of the SDE (1) with second order if the coefficients of the SRK method (3) satisfy the system of the following equations*

1. $\alpha^{(0)^T} e = 1$,
2. $\beta^{(0)^T} e = 1$,
3. $\beta^{(1)^T} e = 0$,
4. $\alpha^{(0)^T} A^{(0)} e = \frac{1}{2}$,
5. $\alpha^{(0)^T} B^{(0)} e = \frac{1}{2}$,
6. $\beta^{(0)^T} A^{(1)} e = \frac{1}{2}$,
7. $\alpha^{(0)^T} (B^{(0)} e)^2 = \frac{1}{2}$,
8. $\beta^{(0)^T} (A^{(1)}(B^{(0)} e)) = 0$,
9. $\beta^{(0)^T} B^{(1)} e = 0$,
10. $\beta^{(1)^T} B^{(1)} e = 1$,
11. $\beta^{(0)^T} B^{(2)} e = 1$,
12. $\beta^{(1)^T} B^{(2)} e = 0$,
13. $\beta^{(0)^T} (B^{(1)} e)^2 = \frac{1}{2}$,
14. $\beta^{(0)^T} (B^{(2)} e)^2 = 1$,
15. $\beta^{(0)^T} (B^{(1)}(B^{(1)} e)) = 0$,

where

$$e=(1,1,\ldots,1)_s^T,\ A^{(0)}=(a_{ij}^{(0)})_{s\times s},\ A^{(1)}=(a_{ij}^{(1)})_{s\times s},$$
$$B^{(0)}=(b_{ij}^{(0)})_{s\times s},\ B^{(1)}=(b_{ij}^{(1)})_{s\times s},\ B^{(2)}=(b_{ij}^{(2)})_{s\times s},$$
$$\alpha^{(0)^T}=(\alpha_i^{(0)},\ldots,\alpha_s^{(0)})^T,\ \beta^{(0)}=(\beta_1^{(0)},\ldots,\beta_s^{(0)})^T,\ \beta^{(1)}=(\beta_1^{(1)},\ldots,\beta_s^{(1)})^T.$$

3 New explicit stabilized SRK methods

Combining the methods (2) and (3), we can construct a new class of explicit stabilized SRK methods for SDE (1). New explicit stabilized SRK methods take the form

$$\begin{aligned}
H_0 &= y_n,\ H_1=H_0+\alpha\mu_1 hf(H_0),\\
H_i &= \alpha\mu_i hf(H_{i-1})-\nu_i H_{i-1}-\kappa_i H_{i-2},\ i=2,3,\ldots,\hat{s}-1,\\
\hat{H}_1 &= \sum_{i=1}^{\hat{n}} c_{1i}H_{\hat{s}-\hat{n}+i-1},\ \sum_{i=1}^{\hat{n}} c_{1i}=1,\ \hat{n}\in\mathbb{Z}^+,\ \hat{n}\le\hat{s},\\
\hat{H}_2^{(k)} &= \sum_{i=1}^{\hat{n}} c_{2i}H_{\hat{s}-\hat{n}+i-1}+\frac{1}{2}g^k(\hat{H}_1)\xi+\sum_{\substack{l=1\\l\ne k}}^{m} g^l(\hat{H}_1)\hat{I}_{(k,l)},\ \sum_{i=1}^{\hat{n}} c_{2i}=1,\\
\hat{H}_3^{(k)} &= \sum_{i=1}^{\hat{n}} c_{3i}H_{\hat{s}-\hat{n}+i-1}-\frac{1}{2}g^k(\hat{H}_1)\xi,\ \sum_{i=1}^{\hat{n}} c_{3i}=1,\ k=1,2,\ldots,m,\\
\hat{H}_4 &= H_{\hat{s}-2}+2\tau_\alpha hf(H_{\hat{s}-2})+\sum_{k=1}^{m} g^k(\hat{H}_1)\hat{I}_k,\\
y_{n+1} &= H_{\hat{s}-2}+(2\sigma_\alpha-\frac{1}{2})hf(H_{\hat{s}-2})+\frac{1}{2}hf(\hat{H}_4)\\
&\quad+\sum_{k=1}^{m}\Big(-g^k(\hat{H}_1)+g^k(\hat{H}_2^{(k)})+g^k(\hat{H}_3^{(k)})\Big)\hat{I}_k\\
&\quad+2\sum_{k=1}^{m}\Big(g^k(\hat{H}_1)-g^k(\hat{H}_3^{(k)})\Big)\hat{I}_{(k,k)}.
\end{aligned}\tag{6}$$

The method (6) computes an additional stages $H_{\hat{s}-1}$ compared to the ROCK2 method (2). But it needs to be emphasized that the number of evaluations of the drift coefficient f is still $\hat{s}$ for the method (6) because we do not need to estimate $f(H_{\hat{s}-1})$. In addition, the method (6) only needs three evaluations of each diffusion function $g^k, k=1,2,\ldots,m$ and $m+2$ simulations of independent random variables at each step. This implies that the computational cost of our new method (6) is only 60% of the S-ROCK2 method proposed in [6] when m is large.

4 Convergence analysis

If A is a $s \times s$ matrix ($s = \hat{s} + 4$), then $A(i, :)$ denotes the ith row of matrix A, and $A(i, j)$ denotes the element in ith row and jth column. Let

$$A^{(0)}(i,:) = \begin{cases} \boldsymbol{0}, & i = 1, \hat{s}+1, \hat{s}+2, \hat{s}+3, \\ \alpha\mu_1 \boldsymbol{I}(1,:), & i = 2, \\ \alpha\mu_{i-1}\boldsymbol{I}(i-1,:) - \nu_{i-1}A^{(0)}(i-1,:) & \\ -\kappa_{i-1}A^{(0)}(i-2,:), & i = 3, 4, \ldots, \hat{s}, \\ A^{(0)}(\hat{s}-1,:) + 2\tau_\alpha \boldsymbol{I}(\hat{s}-1,:), & i = \hat{s}+4, \end{cases} \tag{7}$$

$$\begin{aligned}
A^{(1)}(i,:) &= \begin{cases} \sum_{l=1}^{\hat{n}} c_{jl}A^{(0)}(\hat{s}-\hat{n}+l,:), & i = \hat{s}+j, j = 1, 2, 3, \\ \boldsymbol{0}, & else, \end{cases} \\
B^{(0)}(i,:) &= \begin{cases} \boldsymbol{I}(\hat{s}+1,:), & i = \hat{s}+4, \\ \boldsymbol{0}, & else, \end{cases} \\
B^{(1)}(i,:) &= \begin{cases} \frac{1}{2}\boldsymbol{I}(\hat{s}+1,:), & i = \hat{s}+2, \\ -\frac{1}{2}\boldsymbol{I}(\hat{s}+1,:), & i = \hat{s}+3, \\ \boldsymbol{0}, & else, \end{cases} \\
B^{(2)}(i,:) &= \begin{cases} \boldsymbol{I}(\hat{s}+1,:), & i = \hat{s}+2, \\ \boldsymbol{0}, & else, \end{cases} \\
\alpha^{(0)^T} &= A^{(0)}(\hat{s}-1,:) + \left(2\sigma_\alpha - \frac{1}{2}\right)\boldsymbol{I}(\hat{s}-1,:) + \frac{1}{2}\boldsymbol{I}(\hat{s}+4,:), \\
\beta^{(0)^T} &= -\boldsymbol{I}(\hat{s}+1,:) + \boldsymbol{I}(\hat{s}+2,:) + \boldsymbol{I}(\hat{s}+3,:), \\
\beta^{(1)^T} &= 2\boldsymbol{I}(\hat{s}+1,:) - 2\boldsymbol{I}(\hat{s}+3,:),
\end{aligned} \tag{8}$$

where $\boldsymbol{0}$ is a $1 \times s$ zero vector and $\boldsymbol{I}$ denotes the $s \times s$ unit matrix. Then, the SRK method (6) can be seen as a special case of the SRK method (3) with the coefficients defined by (7) and (8).

Theorem 2 *Let $f, g^j \in C_P^6(\mathbb{R}^d, \mathbb{R}^d)$, $j = 1, 2, \ldots, m$. Then, the SRK method (6) converges weakly to solution of the SDE (1) with second order if*

$$\sum_{i=1}^{\hat{n}}\sum_{j=1}^{s}(-c_{1i} + c_{2i} + c_{3i})A^{(0)}(\hat{s}-\hat{n}+i, j) = \frac{1}{2}.$$

Proof We need only to show that all the order conditions in Theorem 1 can be fulfilled. Firstly, based on (7) and (8) and some simple calculations, it is easy to verify that all the conditions are fulfilled except for conditions 1, 4, and 6. Secondly, the SRK method (6) degenerates to the ROCK2 method (2) when $g^j = 0,\ j = 1, 2, ..., m$. This implies that the conditions 1 and 4 are fulfilled. Finally, we need only to verify $\beta^{(0)^T} A^{(1)} e = \frac{1}{2}$. A direct calculation shows that

$$\beta^{(0)^T} A^{(1)} e = \sum_{i=1}^{\hat{n}} \sum_{j=1}^{s} (-c_{1i} + c_{2i} + c_{3i}) A^{(0)}(\hat{s} - \hat{n} + i, j). \tag{9}$$

Thus, the conclusion follows from the (9). □

5 Stability analysis

In addition to the order of convergence, the long-term ($t \to \infty$) behavior of numerical solutions is equally important in many practice applications. The stability theory is important to understand this behavior. The mean-square stability is a widely used characterization of stability for an SDE. To give insight into the mean-square stability, we consider the linear scalar test equation [7, 9]

$$dy(t) = \lambda_1 y(t)dt + \lambda_2 y(t)dW(t), y(0) = 1, \tag{10}$$

where λ_1, λ_2 are fixed complex scalar parameters. The zero solution of SDE (10) is called mean-square stable if

$$\lim_{t\to\infty} E(|y(t)|^2) = 0. \tag{11}$$

Because the exact solution of SDE (10) is

$$y(t) = exp\left(\left(\lambda_1 - \frac{1}{2}\lambda_2^2\right) t + \lambda_2 W(t)\right), \tag{12}$$

the (10) is mean-square stable if

$$(\lambda_1, \lambda_2) \in \mathbb{S}_{SDE}^{MS}, \mathbb{S}_{SDE}^{MS} := \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \middle| \mathcal{R}(\lambda_1) + \frac{1}{2} \mid \lambda_2 \mid^2 < 0\right\}. \tag{13}$$

Similarly, a numerical solution of SDE (10) is called mean-square stable if

$$\lim_{n\to\infty} E(|y_n|^2) = 0. \tag{14}$$

Applying a one-step method to test (10) can yield the following recurrence formula

$$y_{n+1} = R(p, q, \zeta_1, \zeta_2, \ldots, \zeta_k) y_n, \tag{15}$$

where $p = \lambda_1 h, q = \lambda_2\sqrt{h}$, $\zeta_1, \zeta_2, \ldots, \zeta_k$ are some random variables. By [9], we have

$$\lim_{n\to\infty} E(|y_n|^2) = 0 \Leftrightarrow (p, q) \in \mathbb{S}_{num}^{MS},$$
$$\mathbb{S}_{num}^{MS} := \{(p, q) \in \mathbb{C}^2 \big| E(|R(p, q, \zeta_1, \zeta_2, \ldots, \zeta_k)|^2) < 1\}. \tag{16}$$

In particular, let us apply the SRK method (6) to test (10). A direct calculation shows that

$$\begin{aligned} y_{n+1} &= R(p,q,r,r_1,r_2)y_n, \\ R(p,q,r,r_1,r_2) &= (1+2\sigma_\alpha p+\tau_\alpha p^2)P_{\hat{s}-2}(\alpha p) \\ &\quad +\Big(\frac{1}{2}p\hat{P}_1(\alpha p)-\hat{P}_1(\alpha p)+\hat{P}_2(\alpha p)+\hat{P}_3(\alpha p)\Big)qr \\ &\quad +\Big(2\hat{P}_1(\alpha p)-2\hat{P}_3(\alpha p)\Big)qr_2 \\ &\quad +\hat{P}_1(\alpha p)q^2r_1r_2, \end{aligned} \tag{17}$$

where $r=\frac{\hat{I}_1}{\sqrt{h}}, r_1=\frac{\xi}{\sqrt{h}}, r_2=\frac{\hat{I}_{(1,1)}}{\sqrt{h}}$, the polynomials $P_i(\alpha p)$, $\hat{P}_j(\alpha p), i=0,1,\ldots,\hat{s}-1, j=1,2,3$ satisfy

$$\begin{aligned} P_0(\alpha p) &= 1,\ P_1(\alpha p)=1+\alpha\mu_1 p, \\ P_i(\alpha p) &= \alpha\mu_i pP_{i-1}(\alpha p)-\nu_i P_{i-1}(\alpha p)-\kappa_i P_{i-2}(\alpha p),\ i=2,3,\ldots,\hat{s}-1, \\ \hat{P}_j(\alpha p) &= \sum_{l=1}^{\hat{n}} c_{jl}P_{\hat{s}-\hat{n}+l-1}(\alpha p),\ j=1,2,3. \end{aligned} \tag{18}$$

Theorem 3 *The numerical mean-square stability domain* $\mathbb{S}_{num}^{MS}$ *can be given by*

$$\begin{aligned} \mathbb{S}_{num}^{MS} &= \{(p,q)\in\mathbb{C}^2\Big|A(p)+B(p)|q^2|+\frac{1}{2}C(p)|q^4|<1\}, \\ A(p) &= |(1+2\sigma_\alpha p+\tau_\alpha p^2)P_{\hat{s}-2}(\alpha p)|^2, \\ B(p) &= |\frac{1}{2}p\hat{P}_1(\alpha p)-\hat{P}_1(\alpha p)+\hat{P}_2(\alpha p)+\hat{P}_3(\alpha p)|^2+2|\hat{P}_1(\alpha p)-\hat{P}_3(\alpha p)|^2, \\ C(p) &= |\hat{P}_1(\alpha p)|^2 \end{aligned} \tag{19}$$

if we apply the SRK method (6) *to test equation* (10).

Proof It is not difficult to prove that

$$\begin{aligned} E(r) &= E(r_2)=E(r_1r_2)=E(rr_2)=E(rr_1r_2)=E(r_1r_2^2)=0, \\ E(r^2) &= 1,\ E(r_2^2)=\frac{1}{2},\ E(r_1^2r_2^2)=\frac{1}{2} \end{aligned} \tag{20}$$

based on the definitions of $\hat{I}_1, \xi, \hat{I}_{(1,1)}$. Combining the (17) and (20) yields

$$E(|R(p,q,r,r_1,r_2)|^2)=A(p)+B(p)|q^2|+\frac{1}{2}C(p)|q^4|. \tag{21}$$

Thus, the conclusion follows from the (21). □

Some new weak second-order explicit stabilized SRK methods are proposed based on Theorem 2, and they are displayed in Table 1, where $c_i=(c_{i1},,c_{i\hat{n}}), i=1,2,3$.

Based on Theorem 3 and Table 1, we can obtain Fig. 1, where the method W2Ito1 is a standard weak second-order explicit SRK method proposed in [14], the green

Table 1 New weak second-order explicit stabilized SRK methods

Method	$\hat{s}$	α	$\hat{n}$	c_1, c_2, c_3
ROCK2W2Ito1	5	1.0	2	$c_1 = c_3 = (0, 1)$, $c_2 = (-0.7538, 1.7538)$
ROCK2W2Ito2	10	1.0	2	$c_1 = c_3 = (0, 1)$, $c_2 = (-2.7962, 3.7962)$
ROCK2W2Ito3	5	1.25	2	$c_1 = c_3 = (-0.5000, 1.5000)$, $c_2 = (-0.0817, 1.0817)$
ROCK2W2Ito4	10	1.29	3	$c_1 = c_3 = (0, -1.8000, 2.8000)$, $c_2 = (-2.0400, 2.7066, 0.3334)$
ROCK2W2Ito5	20	1.33	4	$c_1 = c_3 = (0, 0, -4.3000, 5.3000)$, $c_2 = (-4.7462, 5.2462, 0.2500, 0.2500)$

region denotes the corresponding numerical mean-square stability domain $\mathbb{S}_{num}^{MS}$ of each method, and the red dotted line denotes the boundary of domain $\mathbb{S}_{SDE}^{MS}$.

Figure 1 shows that our new weak second-order explicit SRK methods inherit the good stability of the deterministic ROCK2 method. The second and third subfigures show that our new methods with $\alpha \equiv 1.0$ have the same optimal size as the ROCK2 method along the deterministic p-axis. But there is a small flaw for these methods because these methods have gaps in the mean-square stability regions close to the origin. Fortunately, we can avoid these gaps by adjusting the value of the parameters α, $\hat{n}$ and c_i, $i = 1, 2, 3$ (see the last three subfigures).

Define

$$d_{\tilde{s}} = sup\{a > 0 \big| (-a, 0) \times \{0\} \subset \mathbb{S}_{num}^{MS}\}, l_{\tilde{s}} = sup\{a > 0 \big| \mathbb{S}_{a}^{MS} \subset \mathbb{S}_{num}^{MS}\},$$

$$\mathbb{S}_{a}^{MS} = \{(p, q) \in (-a, 0) \times \mathbb{R} \big| p + \frac{1}{2}|q|^2 < 0\}, \quad (22)$$

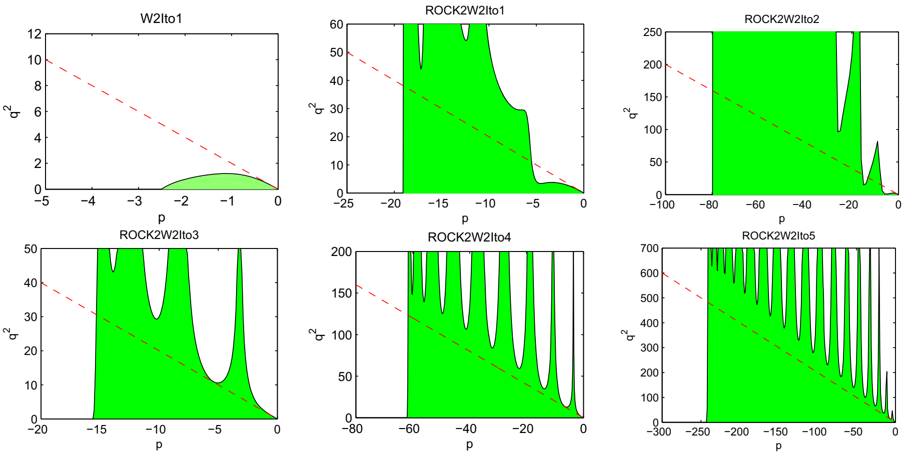

Fig. 1 Mean-square stability domains of W2Ito1 and the methods listed in Table 1

where $\tilde{s}$ denotes the number of evaluations of the drift function at each step. $d_{\tilde{s}}$ and $l_{\tilde{s}}$ can be seen as two metric sizes to characterize the numerical mean-square stability domain $\mathbb{S}_{num}^{MS}$. Based on (22), we can compare our new methods with other explicit stabilized methods. The corresponding sizes $d_{\tilde{s}}, l_{\tilde{s}}$ of each method are listed in Table 2, where SROCK2 is a weak second-order explicit stabilized method for stiff Stratonovitch SDEs introduced in [12], and S-ROCK2 is a weak second-order explicit stabilized method for stiff Itô SDEs introduced in [6]. Since the first two methods in Table 1 have similar stability characteristics, we only listed the sizes of ROCK2W2Ito2 in Table 2. Similarly, we only listed the sizes of ROCK2W2Ito5 for the last three methods in Table 1.

6 Numerical results

We take the Monte Carlo method to calculate expectation of error and choose $M = 10^6$ independent trajectories for each example. We use Err to denote expectation error, i.e.,

$$Err = \left| \frac{1}{M} \sum_{k=1}^{M} G(y_N^k) - E(G(y(t_N))) \right|, \tag{23}$$

where $G \in C_P^6(\mathbb{R}^d, \mathbb{R})$.

Example 1 We first consider the non-stiff non-linear SDE system with $d = 1, m = 10$ and non-commutative noise

$$dy(t) = y(t)dt + \sum_{j=1}^{10} \sigma_j^{-1} \sqrt{y(t) + k_j^{-1}} dW^j(t), t \in [0, 1], y(0) = 1, \tag{24}$$

where σ_j, $j = 1, 2, \ldots, 10$ equal to 10, 15, 20, 25, 40, 25, 20, 15, 20, 25 in turn, and k_j, $j = 1, 2, \ldots, 10$ equal to 2, 4, 5, 10, 20, 2, 4, 5, 10, 20 in turn. Take $G(y(t)) = y^2(t)$, then we can obtain

$$E(G(y(t))) = E(y^2(t)) = (-68013 - 458120e^t + 14926133e^{2t})/14400000$$

from [6]. The methods W2Ito1, S-ROCK2 ($\tilde{s} = 7$) and the five methods in Table 1 will be applied to this example. The detailed numerical results are presented in Fig. 2. Figure 2 shows that the weak convergence order of our new methods can reach to 2.0. This confirms the theoretical results. At the same time, Fig. 2 shows also that the error accuracy of SRK method (6) is nearly independent of $\hat{s}$.

Table 2 The size of mean-square stability regions for some explicit stabilized methods

Method	SROCK2	S-ROCK2	ROCK2W2Ito2	ROCK2W2Ito5
$d_{\tilde{s}}$	$0.54\tilde{s}^2$	$0.42\tilde{s}^2$	$0.81\tilde{s}^2$ (optimal)	$0.61\tilde{s}^2$
$l_{\tilde{s}}$	$< \sqrt{2}$	$0.42\tilde{s}^2$	0	$0.61\tilde{s}^2$

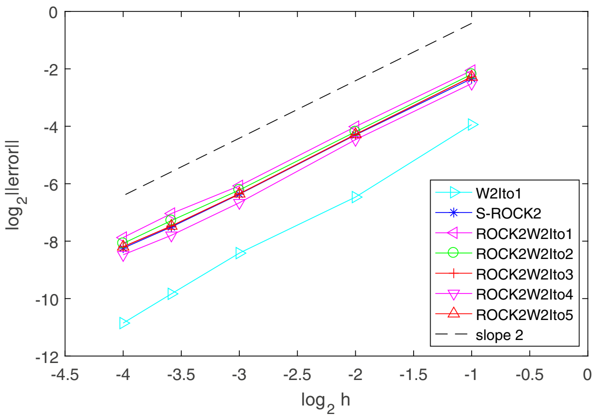

Fig. 2 Stepsize h vs. errors Err

Example 2 We consider the stiff non-linear SDE with $d = m = 1$

$$dy(t) = -\lambda_1 y(t)(1 - y(t))dt - \lambda_2 y(t)(1 - y(t))dW(t), t \in [0, 10], y(0) = 0.95, \tag{25}$$

where $\lambda_1 < 0, \lambda_2 = \sqrt{-\lambda_1(2-\varepsilon)}, 0 < \varepsilon < 2$. The SDE (25) leads to the linear test problem (10) if we linearize it close to the steady solution $y(t) = 1$. For the first case, we take $\lambda_1 = -15, \varepsilon = 1$. For the second case, we fix $\varepsilon = 1$ and take $\lambda_1 = -300, -350, -400$ in turn. Take $G(y(t)) = y^2(t)$, then the detailed numerical results can be given by Tables 3 and 4, where N_f, N_g denote the number of evaluations for the drift coefficient f and the diffusion coefficient g, respectively, and N_r denotes the number of simulations of independent random variables.

Table 3 shows that instability will occur for the methods ROCK2W2Ito1 and ROCK2W2Ito2 when $\lambda_1 = -15$. This is consistent with the result of Fig. 1 because p $(p = h\lambda_1 = -\frac{5}{2})$ is close to the origin at this time. The last two lines of Table 4 show that the method ROCK2W2Ito2 perform better than ROCK2W2Ito4

Table 3 The errors Err for the SDE (25) with $\lambda_1 = -15, \varepsilon = 1$

Method	h	Cost	Err
S-ROCK2 ($\tilde{s} = 12$)	$\frac{1}{6}$	$N_f = 720, N_g = 300, N_r = 120$	0
ROCK2W2Ito1	$\frac{1}{6}$	$N_f = 300, N_g = 180, N_r = 120$	∞
ROCK2W2Ito2	$\frac{1}{6}$	$N_f = 600, N_g = 180, N_r = 120$	∞
ROCK2W2Ito3	$\frac{1}{6}$	$N_f = 300, N_g = 180, N_r = 120$	0
ROCK2W2Ito4	$\frac{1}{6}$	$N_f = 600, N_g = 180, N_r = 120$	0
ROCK2W2Ito5	$\frac{1}{6}$	$N_f = 1200, N_g = 180, N_r = 120$	2.1e−16

Table 4 The errors Err for the SDE (25) with $\varepsilon = 1$ and $\lambda_1 = -300, -350, -400$

Method	h	cost: N_f, N_g, N_r	$Err : \lambda_1 = -300, -350, -400$
W2Ito1	$\frac{1}{320}$	9600, 9600, 6400	∞, ∞, ∞
W2Ito1	$\frac{1}{340}$	10200, 10200, 6800	$0, \infty, \infty$
W2Ito1	$\frac{1}{360}$	10800, 10800, 7200	$0, \infty, \infty$
W2Ito1	$\frac{1}{380}$	11400, 11400, 7600	$0, 0, \infty$
W2Ito1	$\frac{1}{440}$	13200, 13200, 8800	$0, 0, \infty$
W2Ito1	$\frac{1}{460}$	13800, 13800, 9200	0, 0, 0
S-ROCK2 ($\tilde{s} = 12$)	$\frac{1}{6}$	720, 300, 120	1.2e−16, ∞, ∞
ROCK2W2Ito2	$\frac{1}{6}$	600, 180, 120	1.5e−4, 1.2e−16, 7.3e−16
ROCK2W2Ito4	$\frac{1}{6}$	600, 180, 120	2.9e−14, 2.6e−13, ∞

when p is far from the origin. This mainly benefits from the fact that the method ROCK2W2Ito2 has the optimal size along the deterministic p-axis. From the above two points, we can see that the methods ROCK2W2Ito2 and ROCK2W2Ito4 have their own advantages.

In addition, Table 4 shows that the method S-ROCK2 and our new explicit stabilized methods perform better than the standard explicit method W2Ito1 for the stiff SDEs. $\hat{s} = 10$ for all the three methods S-ROCK2($\tilde{s} = 12$), ROCK2W2Ito2 and ROCK2W2Ito4. By observing the last three lines of Table 4, we can find that our new explicit stabilized methods ROCK2W2Ito2 and ROCK2W2Ito4 have better performance than the well-known explicit stabilized method S-ROCK2 not only in the computational cost, but also in the stability.

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