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ORIGINAL PAPER

Two alternating direction implicit spectral methods for two-dimensional distributed-order differential equation

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Abstract

In this paper, two alternating direction implicit Galerkin-Legendre spectral methods for distributed-order differential equation in two-dimensional space are developed. It is proved that the schemes are unconditionally stable and convergent with the convergence orders $O(\Delta t + \sigma^2 + N^{-m})$ and $O(\Delta t^2 + \sigma^2 + N^{-m})$, respectively, where Δt, σ, N, and m are the time step size, step size in distributed-order variable, polynomial degree, and regularity in the space variable of the exact solution, respectively. Moreover, the applicability and accuracy of the two schemes are demonstrated by numerical experiments to support our theoretical analysis.

Keywords Alternating direction implicit · Spectral method · Distributed-order differential equation

Mathematics Subject Classification (2010) 26A33 · 65M70 · 65M12

1 Introductions

Nowadays, the fractional calculus has been greatly developed and applied to the problems in system biology, physics, biochemistry, medicine, finance, and so on. One of the model equations is the time-fractional diffusion equation. It can be obtained by replacing the first-order time-derivative in the classical diffusion equation with a single-order time fractional derivative of order α $(0 < \alpha < 1)$. Fractional derivatives

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are used to demonstrate various materials and processes with memory and hereditary properties, which are nonlocal with long memory and weak singular kernels. The numerical approximation of a single-order temporal derivative has been well studied in literature. In [12], Gao et al. demonstrated a new fractional numerical differentiation formula to approximate the Caputo fractional derivative. Alikhanov [1] presented a new numerical differentiation formula of order $O(\Delta t^{3-\alpha})$ to approximate the Caputo fractional derivative.

However, the single-order time-fractional diffusion equation can not characterize the process which is nonself-similar and exhibits a continuous distribution of time-scales. Thus, it must be generalized, among which the distributed-order time-fractional diffusion equation is a good alternative choice. An extensive application of distributed-order equations is to describe ultraslow diffusion phenomenon where a plume of particles spreads at a logarithmic rate [3, 14, 20]. Recent researches demonstrate that many other applications of the distributed-order differential equation have appeared in modeling, for example, the stress behavior of an elastic medium, the torsional phenomenon of anelastic or dielectric spherical shells and infinite planes, the rheological properties of composite materials, dielectric induction, and diffusion and so on [6, 10]. A numerical scheme for the solution of a distributed-order fractional differential equation was developed by Diethelm and Ford in [4]. Gao and Sun [11] demonstrated two unconditionally stable and convergent difference schemes with the extrapolation method for the one-dimensional distributed-order differential equations. Besides, they also developed two alternating direction implicit difference schemes for two-dimensional distributed-order fractional diffusion equations in [10]. An implicit finite difference approximation for the solution of the diffusion equation with distributed order in time was demonstrated in [7]. But the most work is based on finite difference method, and published papers on spectral method are sparse. This motivates us to consider effective spectral methods for two-dimensional distributed-order differential equation.

In this work, we consider two alternating direction implicit Galerkin-Legendre spectral methods for two-dimensional distributed-order differential equation. Although other kinds of numerical methods can also be considered, the spectral method would be found preferable in some practical applications, especially when the problem under consideration admits smooth solutions. As we know, the spectral method has the significant merit of the high accuracy. For this reason, we will focus on the spectral method for the numerical approximation of the distributed-order differential equation. When considering the high-dimensional models, alternating direction implicit method is also taken into consideration to reduce the computational time efficiently. As far as we known, the works on numerical methods for fractional differential equations of distributed order dealt with the fractional ordinary differential problems, or fractional partial differential equations in one-dimensional case only and the numerical accuracy of most related difference methods. Our attention in this paper will be paid on the effective alternating direction implicit Galerkin-Legendre spectral methods for solving a class of time-fractional diffusion equations of distributed order in two-dimensional case. Compared with the previous work in [8], the major difficulty is the estimates of the distributed-order interpolation in projection operator by using Galerkin-Legendre spectral methods, where we do not encounter when using finite difference method. Moreover, in order to obtain estimates with optimal order of convergence for the unknown, we should reconsider the properties of coefficients in the distributed-order numerical differentiation formula. Finally, some numerical experiments are demonstrated to show the accuracy of alternating direction implicit Galerkin-Legendre spectral methods.

The paper is organized as follows. In Section 2, we give the problem and some preliminaries. In Section 3, we present an alternating direction implicit spectral method with the first-order accuracy in time and demonstrate the analysis of stability and error estimates for the presented method. Then in Section 4, we present an alternating direction implicit spectral method with the second-order accuracy in time and give the rigorous analysis of stability and error estimates. Some numerical experiments using the alternating direction implicit spectral methods are carried out in Section 5.

Throughout the paper, we use C, with or without subscript, to denote a positive constant, which could have different values at different appearances.

2 The problem and some preliminaries

In this section, we first describe the problem of two dimension distributed-order differential equation (see [9, 10]) in this paper, and present some notations which will be found helpful in the following analysis.

Find $p = p(x, y, t)$ such that

$$\mathcal{D}_t^{\omega} p(x, y, t) - K\Delta p(x, y, t) = f(x, y, t), \quad (x, y, t) \in \Omega \times J, \tag{1}$$

with Dirichlet boundary condition

$$p(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times J,$$

and initial condition

$$p\mid_{t=0} = p_0(x, y), \quad (x, y) \in \Omega,$$

where $\Omega = I_x \times I_y = (a, b) \times (c, d)$, $J = (0, T]$, and

$$\mathcal{D}_t^{\omega}\mathbf{p}(t) = \int_0^1 \omega(\alpha)\, {}_0^C D_t^{\alpha}\mathbf{p}(t)d\alpha, \tag{2}$$

$${}_0^C D_t^{\alpha}\mathbf{p}(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-\xi)^{-\alpha}\frac{\partial \mathbf{p}}{\partial \xi}(\xi)d\xi, & 0 \le \alpha < 1, \\ \mathbf{p}_t(t), \quad \alpha = 1, \end{cases}$$

$\omega(\alpha) \ge 0$, $\int_0^1 \omega(\alpha)d\alpha = c_0 > 0$ and $\omega(\alpha) \in C^0(0, 1)$.

Let $N_T > 0$ be a positive integer. Set

$$\Delta t = T/N_T; \quad t_n = n\Delta t, \quad for \quad n \le N_T.$$

We define $(\cdot, \cdot)$ as the inner product on the interval I_x or I_y. $H^m(\Omega)$, $H_0^m(\Omega)$ and $\|\cdot\|_m (m = 0, 1, \cdot)$ are used to denote the standard Sobolev spaces and their norms, respectively. In particular, the norm and inner product of $L^2(\Omega) = H^0(\Omega)$ are denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively. We also denote the maximum norm of $L^{\infty}(\Omega)$ by $\|\cdot\|_{\infty}$. Denote by $P_N(I_k)$ the space of polynomials defined on the domain I_k

with the degree no greater than $N \in Z^+,\ k = x, y$. The approximation space V_N^0 is defined as

$$V_N^0 = (P_N(I_x) \otimes P_N(I_y)) \cap H_0^1(\Omega).$$

As in [17, 21, 23], the function spaces $V_N^{0,x}$ and $V_N^{0,y}$ can be expressed as

$$V_N^{0,x} = span\{\phi_i(x) : i = 0, 1, \ldots, N-2\},$$
$$V_N^{0,y} = span\{\varphi_j(y) : j = 0, 1, \ldots, N-2\},$$

where $\phi_i(x)$ and $\varphi_j(y)$ are defined as in [17, 23]:

$$\phi_i(x) = L_i(\xi) - L_{i+2}(\xi),\ \ \xi \in [-1, 1],\ \ x = \frac{(b-a)\xi + a + b}{2} \in [a, b],$$
$$\varphi_j(y) = L_j(\eta) - L_{j+2}(\eta),\ \ \eta \in [-1, 1],\ \ y = \frac{(d-c)\eta + c + d}{2} \in [c, d],$$

where $L_i(\xi)$ is the Legendre polynomial defined by the following recurrence relation [18]:

$$\begin{cases} L_0(\xi) = 1,\ \ L_1(\xi) = \xi, \\ L_{i+1}(\xi) = \dfrac{2i+1}{i+1}\xi L_i(\xi) - \dfrac{i}{i+1}L_{i-1}(\xi),\ \ i \geq 1, \end{cases} \tag{3}$$

and $L_j(\eta)$ is defined similarly. Then the function space $V_N^0 = V_N^{0,x} \otimes V_N^{0,y}$ can be defined by

$$V_N^0 = span\{\phi_i(x)\varphi_j(y),\ i, j = 0, 1, \ldots, N-2\}.$$

The Legendre-Gauss (LG) interpolation operator $I_N : C(\Omega) \to V_N^0$ is introduced as

$$I_N v(x_i, y_j) = v(x_i, y_j),\ \ i, j = 0, 1, \ldots, N,$$

where x_i and y_j are Legendre-Gauss points on the intervals I_x and I_y respectively, that is, $\{x_i\}_{i=0}^N$ are the zeros of $L_{N+1}(x)$ and $\{y_j\}_{j=0}^N$ are the zeros of $L_{N+1}(y)$. Besides, the following lemma holds:

Lemma 1 [2] *If $v \in H^m(\Omega)$, then we have*

$$\|I_N v - v\| \leq CN^{-m}\|v\|_m. \tag{4}$$

We denote by Π_N the usual L^2 projection operator in V_N^0, namely,

$$(\Pi_N v - v, \omega) = 0,\ \ \forall \omega \in V_N^0,$$

and we define a projection operator $\Pi_N^1 : H_0^1(\Omega) \to V_N^0$ by

$$(\nabla(\Pi_N^1 v - v), \nabla\omega) = 0,\ \ \forall \omega \in V_N^0. \tag{5}$$

It is well known (cf. [13, 19]) that the following estimates hold:

$$\|u - \Pi_N u\| \leq N^{-m}\|u\|_m,\ \ \forall u \in H^m(\Omega),\ \ m \geq 0, \tag{6}$$

and

$$\|u - \Pi_N^1 u\|_k \leq N^{k-m}\|u\|_m,\ \ k = 0, 1,\ \ \forall u \in H^m(\Omega),\ \ m \geq 1. \tag{7}$$

Next, we give the lemma which will be used in the following estimates.

Lemma 2 [5, 22] *Let* $s(\alpha) \in C^2(0,1)$, $\Delta\alpha = \dfrac{1}{q} = \sigma$ $(q \in \mathbb{N})$, *then we have*

$$\int_0^1 s(\alpha)d\alpha = \sum_{k=1}^{q} s\left(\frac{2k-1}{2q}\right)\frac{1}{q} + O(\sigma^2).$$

3 An alternating direction implicit spectral method with the first-order accuracy in time

The objective of this section is to consider the alternating direction implicit spectral scheme with the first-order accuracy in time for (1). Besides, the stability and error analysis are considered rigorously.

3.1 Fully discrete scheme

In this subsection, we present the fully discrete scheme for (1).

Firstly, we discretize the integral term in the distributed-order equation. Let us discretize the interval [0,1] in which the order α is changing. Using the grid $0 = \eta_0 < \eta_1 < \cdots < \eta_q = 1$ $(q \in \mathbb{N})$, where $\Delta\eta_l$ is not necessarily equidistant. But for simplicity of the presentation, we take $\Delta\eta_l = \frac{1}{q} = \sigma$. The midpoint quadrature rule is applied for approximating the integral (2). Let $\alpha_l = \frac{\eta_{l-1}+\eta_l}{2} = \frac{2l-1}{2q}$ and $d_l = \frac{\omega(\alpha_l)}{q}$, $l = 1, 2, \cdots, q$, then we have that

$$\mathcal{D}_t^{\omega} p(x,y,t) = \sum_{l=1}^{q} d_l\, {}_0^C D_t^{\alpha_l} p(x,y,t) + R_1. \tag{8}$$

Suppose $\omega(\alpha) \in C^2(0,1)$, then from Lemma 2, we have that $R_1 = O(\sigma^2)$. Now the (1) can be transformed into the following multi-term fractional equations

$$\sum_{l=1}^{q} d_l\, {}_0^C D_t^{\alpha_l} p(x,y,t) + R_1 + K\Delta p(x,y,t) = f(x,y,t), \quad (x,y,t) \in \Omega \times J. \tag{9}$$

Next, we need to approximate the multi-term fractional derivatives in time. Firstly, for the convenience of theoretical analysis, we now denote

$$G_k^{\alpha_l} = (k+1)^{1-\alpha_l} - k^{1-\alpha_l}. \tag{10}$$

Then from the literature [15], it is not difficult to verify that for $0 < \alpha_l < 1$,

$$1 = G_0^{\alpha_l} > G_1^{\alpha_l} > G_2^{\alpha_l} > \cdots > G_n^{\alpha_l} > \cdots \to \Delta t^{\alpha_l} \to 0. \tag{11}$$

For $0 < \alpha_l < 1$, by the simple calculation at $t = t_n$, we can get the equality easily.

$$\begin{aligned} {}_0^C D_t^{\alpha_l} \mathbf{p}(t_n) = {} & \frac{\Delta t^{-\alpha_l}}{\Gamma(2-\alpha_l)}\left[G_0^{\alpha_l}\mathbf{p}(t_n) - \sum_{k=1}^{n-1}(G_{n-k-1}^{\alpha_l} - G_{n-k}^{\alpha_l})\mathbf{p}(t_k) - G_{n-1}^{\alpha_l}\mathbf{p}(t_0)\right] \\ & + R_2^{l,n}, \end{aligned} \tag{12}$$

where

$$R_2^{l,n} = \frac{1}{\Gamma(1-\alpha_l)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left[\left(\tau - \frac{t_{k+1}+t_k}{2} \right) \frac{\partial^2 \mathbf{p}(t_{k+\frac{1}{2}})}{\partial t^2} \right.$$
$$\left. + O\left((\tau - t_{k+\frac{1}{2}})^2 \right) + O\left(\Delta t^2 \right) \right] \frac{d\tau}{(t_n - \tau)^{\alpha_l}}. \tag{13}$$

From [16], we get

$$|R_2^{l,n}| \leqslant \frac{C}{\Gamma(2-\alpha_l)} \left[\max_{t\in[0,T]} \left| \frac{\partial^2 p(x,y,t)}{\partial t^2} \right| \Delta t^{2-\alpha_l} + T\Delta t^2 \right]. \tag{14}$$

Let $L_x p = \frac{\partial^2 p}{\partial x^2}$ and $L_y p = \frac{\partial^2 p}{\partial y^2}$, then (9) can be transformed into

$$\sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \left[G_0^{\alpha_l} p^n - \sum_{k=1}^{n-1} (G_{n-k-1}^{\alpha_l} - G_{n-k}^{\alpha_l}) p^k - G_{n-1}^{\alpha_l} p^0 \right] - K(L_x + L_y) p^n$$
$$= f^n - R_1^n - \sum_{l=1}^{q} d_l R_2^{l,n}. \tag{15}$$

Next, we give the following lemma.

Lemma 3 *Denote*

$$\mu = \sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} G_0^{\alpha_l}, \tag{16}$$

then we have

$$\mu = \frac{1}{O(\Delta t|ln(\Delta t)|)}. \tag{17}$$

Proof: A direct calculation gives

$$\begin{aligned}
\mu &= \sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} G_0^{\alpha_l} \sim \int_0^1 \frac{\omega(\alpha)\Delta t^{-\alpha}}{\Gamma(2-\alpha)} d\alpha \\
&= \frac{\omega(\alpha^*)}{\Gamma(2-\alpha^*)} \int_0^1 (\frac{1}{\Delta t})^{\alpha} d\alpha \\
&= \frac{\omega(\alpha^*)}{\Gamma(2-\alpha^*)} \left. \frac{(\frac{1}{\Delta t})^{\alpha}}{ln\frac{1}{\Delta t}} \right|_{\alpha=0}^{1} \\
&= \frac{\omega(\alpha^*)}{\Gamma(2-\alpha^*)} \frac{\frac{1}{\Delta t} - 1}{ln\frac{1}{\Delta t}},
\end{aligned}$$

where $\alpha^* \in (0, 1)$. We can easily obtain that $\mu = \dfrac{1}{O(\Delta t|ln(\Delta t)|)}$, this completes the proof.

Adding the perturbation term $\frac{K^2}{\mu}L_xL_yp^n$ to the both sides of (15) gives

$$\sum_{l=1}^{q}\frac{\Delta t^{-\alpha_l}d_l}{\Gamma(2-\alpha_l)}\left[G_0^{\alpha_l}p^n-\sum_{k=1}^{n-1}(G_{n-k-1}^{\alpha_l}-G_{n-k}^{\alpha_l})p^k-G_{n-1}^{\alpha_l}p^0\right]-K(L_x+L_y)p^n$$
$$+\frac{K^2}{\mu}L_xL_yp^n=f^n+R^n, \tag{18}$$

where $R^n=-R_1^n-\sum\limits_{l=1}^{q}d_lR_2^{l,n}+\frac{K^2}{\mu}L_xL_yp^n$.

We can also rewrite (18) into the following equivalent form:

$$\begin{aligned}&(\sqrt{\mu}-\frac{K}{\sqrt{\mu}}L_x)(\sqrt{\mu}-\frac{K}{\sqrt{\mu}}L_y)p^n\\ &=\sum_{l=1}^{q}\frac{\Delta t^{-\alpha_l}d_l}{\Gamma(2-\alpha_l)}\left[\sum_{k=1}^{n-1}(G_{n-k-1}^{\alpha_l}-G_{n-k}^{\alpha_l})p^k+G_{n-1}^{\alpha_l}p^0\right]+f^n+R^n.\end{aligned} \tag{19}$$

Then we can obtain the fully discrete alternating direction implicit spectral method with the first-order accuracy in time for (1) as follows: Find $p_N^n\in V_N^0$ for $n=1,2,\ldots,N_T$ such that

$$\begin{cases}\left(\left(\sqrt{\mu}-\frac{K}{\sqrt{\mu}}L_x\right)\left(\sqrt{\mu}-\frac{K}{\sqrt{\mu}}L_y\right)p_N^n,v\right)\\ =\sum\limits_{l=1}^{q}\frac{\Delta t^{-\alpha_l}d_l}{\Gamma(2-\alpha_l)}\left(\sum\limits_{k=1}^{n-1}(G_{n-k-1}^{\alpha_l}-G_{n-k}^{\alpha_l})p_N^k+G_{n-1}^{\alpha_l}p_N^0,v\right)+(I_Nf^n,v),\quad\forall v\in V_N^0,\\ p_N^0=\Pi_Np_0.\end{cases} \tag{20}$$

3.2 Implementation of the alternating direction implicit spectral method with the first-order accuracy in time

In this subsection, A detailed description of the implementation of the alternating direction implicit spectral method with the first-order accuracy in time is given. The unknown function $p_N^n\in V_N^0$ has the following form:

$$p_N^n=\sum_{i=0}^{N-2}\sum_{j=0}^{N-2}u_{i,j}^n\phi_i(x)\varphi_j(y). \tag{21}$$

Denote the matrices $M_x, M_y, S_x, S_y\in\mathbb{R}^{(N-1)\times(N-1)}$ that satisfy

$$\begin{aligned}(M_x)_{i,j}&=(\phi_i,\phi_j),\quad &(S_x)_{i,j}&=\left(\frac{\partial\phi_i}{\partial x},\frac{\partial\phi_j}{\partial x}\right),\\ (M_y)_{i,j}&=(\varphi_i,\varphi_j),\quad &(S_y)_{i,j}&=\left(\frac{\partial\varphi_i}{\partial y},\frac{\partial\varphi_j}{\partial y}\right).\end{aligned}$$

Taking notice of (21) and setting $v = \phi_i\varphi_j (i, j = 0, 1, \ldots, N-2)$, the matrix representation of the alternating direction implicit spectral method with the first-order accuracy in time (20) can be given as follows:

$$\left(\sqrt{\mu}M_x + \frac{K}{\sqrt{\mu}}S_x\right) U^n \left(\sqrt{\mu}M_y + \frac{K}{\sqrt{\mu}}S_y\right)^T = A^n + F^n, \tag{22}$$

where $U^n,\ A^n,\ F^n \in \mathbb{R}^{(N-1)\times(N-1)}$ that satisfying

$$\begin{aligned}
&(U^n)_{i,j} = u^n_{i,j},\quad i, j = 0, 1, \ldots, N-2,\\
&A^n = M_x \left(\sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \sum_{k=1}^{n-1} (G^{\alpha_l}_{n-k-1} - G^{\alpha_l}_{n-k})U^k + G^{\alpha_l}_{n-1}U^0\right) M_y^T,\\
&(F^n)_{i,j} = (I_N f^n, \phi_i\varphi_j),\quad i, j = 0, 1, \ldots, N-2.
\end{aligned} \tag{23}$$

Then we can solve the system (22) by the following algorithm:
For $n = 1, 2, \ldots, N_T$
  solve $(\sqrt{\mu}M_x + \frac{K}{\sqrt{\mu}}S_x)U^* = A^n + F^n$ to obtain U^*;
  solve $(\sqrt{\mu}M_y + \frac{K}{\sqrt{\mu}}S_y)(U^n)^T = (U^*)^T$ to obtain U^n.
end

We can easily obtain that the coefficient matrices of the algebraic system derived from the alternating direction implicit spectral method have the same size as those of the coefficient matrix derived from the corresponding one-dimensional system.

3.3 Stability and convergence

In this subsection, we give an illustration of the stability and convergence of the alternating direction implicit spectral scheme with (20). We consider the following lemma first.

Lemma 4 *Suppose $\{p_N^n\} \in V_N^0$ is the solution of*

$$\begin{cases}
\left((\sqrt{\mu} - \frac{K}{\sqrt{\mu}}L_x)(\sqrt{\mu} - \frac{K}{\sqrt{\mu}}L_y)p_N^n, v\right)\\
= \sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \left(\sum_{k=1}^{n-1} (G^{\alpha_l}_{n-k-1} - G^{\alpha_l}_{n-k})p_N^k + G^{\alpha_l}_{n-1}p_N^0, v\right) + (Q^n, v),\quad \forall v \in V_N^0,\\
p_N^0 = \Pi_N p_0.
\end{cases} \tag{24}$$

Then we have

$$\sum_{n=1}^{M} \Delta t\, \|p_N^n\|^2 + \sum_{n=1}^{M} \Delta t \|\nabla p_N^n\|^2 \leqslant C\,\|p_0\|^2 + C\sum_{n=1}^{M} \Delta t \|Q^n\|^2,\ \ 1 \le n \le N_T. \tag{25}$$

Proof Rewrite the scheme (24) into the following equivalent form:

$$\sum_{l=1}^{q}\frac{\Delta t^{-\alpha_l}d_l}{\Gamma(2-\alpha_l)}\left[G_0^{\alpha_l}(p_N^n,v)-\sum_{k=1}^{n-1}(G_{n-k-1}^{\alpha_l}-G_{n-k}^{\alpha_l})(p_N^k,v)-G_{n-1}^{\alpha_l}(p_N^0,v)\right]$$
$$+K(\nabla p_N^n,\nabla v)+\frac{K^2}{\mu}(L_xL_yp_N^n,v)=(Q^n,v),\ \ \forall v\in V_N^0. \tag{26}$$

Set $v=p_N^n$ in (26) to obtain that

$$\sum_{l=1}^{q}\frac{\Delta t^{-\alpha_l}d_l}{\Gamma(2-\alpha_l)}\left[G_0^{\alpha_l}\|p_N^n\|-\sum_{k=1}^{n-1}(G_{n-k-1}^{\alpha_l}-G_{n-k}^{\alpha_l})(p_N^k,p_N^n)-G_{n-1}^{\alpha_l}(p_N^0,p_N^n)\right]$$
$$+K\|\nabla p_N^n\|^2+\frac{K^2}{\mu}(L_xL_yp_N^n,p_N^n)=(Q^n,p_N^n). \tag{27}$$

By using Green formula. the last term on the left hand side of (27) can be estimated as

$$\frac{K^2}{\mu}(L_xL_yp_N^n,p_N^n)=\frac{K^2}{\mu}\|\frac{\partial^2p_N^n}{\partial x\partial y}\|^2. \tag{28}$$

In particular, observing that $G_k^{\alpha_l}-G_{k+1}^{\alpha_l}>0$ and using Cauchy-Schwarz inequality, we have

$$\sum_{l=1}^{q}\frac{\Delta t^{-\alpha_l}d_l}{2\Gamma(2-\alpha_l)}\left(\left\|p_N^n\right\|^2+\sum_{k=0}^{n-2}G_{k+1}^{\alpha_l}\left\|p_N^{n-1-k}\right\|^2\right)+K\|\nabla p_N^n\|^2+\frac{K^2}{\mu}\|\frac{\partial^2p_N^n}{\partial x\partial y}\|^2$$
$$\leqslant\sum_{l=1}^{q}\frac{\Delta t^{-\alpha_l}d_l}{2\Gamma(2-\alpha_l)}\sum_{k=0}^{n-2}G_k^{\alpha_l}\left\|p_N^{n-1-k}\right\|^2+\sum_{l=1}^{q}\frac{\Delta t^{-\alpha_l}d_l}{2\Gamma(2-\alpha_l)}G_{n-1}^{\alpha_l}\left\|p_N^0\right\|^2$$
$$+(Q^n,p_N^n). \tag{29}$$

Multiplying both sides of (29) by $2\Delta t$ and summing on n, $n=1,\cdots M$ ($M\leqslant N_T$) give that

$$\sum_{l=1}^{q}\frac{\Delta t^{1-\alpha_l}d_l}{\Gamma(2-\alpha_l)}\sum_{k=0}^{M-1}G_k^{\alpha_l}\left\|p_N^{M-k}\right\|_m^2+2\sum_{n=1}^{M}\Delta t\left(K\|\nabla p_N^n\|^2+\frac{K^2}{\mu}\|\frac{\partial^2p_N^n}{\partial x\partial y}\|^2\right)$$
$$\leqslant\sum_{l=1}^{q}\frac{\Delta t^{1-\alpha_l}d_l}{\Gamma(2-\alpha_l)}\sum_{n=1}^{M}G_{n-1}^{\alpha_l}\left\|p_N^0\right\|_m^2+2\sum_{n=1}^{M}\Delta t(Q^n,p_N^n). \tag{30}$$

Next, we estimate the first term on the left hand side of (30). Noting that for $l=1,2,\cdots,q$, we have

$$G_{M-n}^{\alpha_l}\geq G_{N_T-1}^{\alpha_l}\geq(1-\alpha_l)\Delta t^{\alpha_l}, \tag{31}$$

and noting that if $q \to +\infty$, we have

$$\sum_{l=1}^{q} \frac{d_l}{\Gamma(1-\alpha_l)} = \sum_{l=1}^{q} \frac{\omega(\alpha_l)}{q\Gamma(1-\alpha_l)} \to \int_0^1 \frac{\omega(\alpha)}{\Gamma(1-\alpha)} d\alpha = C_1 > 0. \tag{32}$$

Then the first term on the left hand side of (30) can be transformed into the following.

$$\begin{aligned} \sum_{l=1}^{q} \frac{\Delta t^{1-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \sum_{k=0}^{M-1} G_k^{\alpha_l} \left\| p_N^{M-k} \right\|^2 &= \sum_{l=1}^{q} \frac{\Delta t^{1-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \sum_{n=1}^{M} G_{M-n}^{\alpha_l} \left\| p_N^n \right\|^2 \\ &\geq \sum_{l=1}^{q} \frac{d_l}{\Gamma(1-\alpha_l)} \sum_{n=1}^{M} \Delta t \left\| p_N^n \right\|^2 \\ &\geq \frac{C_1}{2} \sum_{n=1}^{M} \Delta t \left\| p_N^n \right\|^2 . \end{aligned} \tag{33}$$

Next, we estimate the first term on the right hand side of (30). Noting (6), we can obtain

$$\begin{aligned} &\sum_{l=1}^{q} \frac{\Delta t^{1-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \sum_{n=1}^{M} G_{n-1}^{\alpha_l} \left\| p_N^0 \right\|^2 = \sum_{l=1}^{q} \frac{t_M^{1-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \left\| \Pi_N p_0 \right\|^2 \\ &\leq \max\{t_M^{1-\frac{1}{2q}}, t_M^{\frac{1}{2q}}\} \sum_{l=1}^{q} \frac{d_l}{\Gamma(2-\alpha_l)} \left\| \Pi_N p_0 \right\|^2 \\ &\leq C \left\| p_0 \right\|^2 . \end{aligned} \tag{34}$$

Using Cauchy-Schwarz inequality, the last term on the right hand side of (30) can be estimated by

$$2\sum_{n=1}^{M} \Delta t (Q^n, p_N^n) \leq C \sum_{n=1}^{M} \Delta t \| Q^n \|^2 + \frac{C_1}{4} \sum_{n=1}^{M} \Delta t \| p_N^n \|^2. \tag{35}$$

Then combining (30) with (33)–(35) gives that

$$\frac{C_1}{4} \sum_{n=1}^{M} \Delta t \left\| p_N^n \right\|^2 + 2K \sum_{n=1}^{M} \Delta t \| \nabla p_N^n \|^2 \leqslant C \left\| p_0 \right\|^2 + C \sum_{n=1}^{M} \Delta t \| Q^n \|^2. \tag{36}$$

This completes the proof. □

Theorem 5 *For the scheme* (20), *we have the following stability inequality:*

$$\sum_{n=1}^{M} \Delta t \left\| p_N^n \right\|^2 + \sum_{n=1}^{M} \Delta t \| \nabla p_N^n \|^2 \leqslant C \left\| p_0 \right\|^2 + C \sum_{n=1}^{M} \Delta t \| f^n \|^2, \ \ 1 \leq n \leq N_T. \tag{37}$$

Proof Recalling Lemmas 1 and 4, the result needed can be easily obtained.

Next, we give the error estimates for the alternating direction implicit spectral scheme (20). In order to simplify the notations, denote

$$p^n - p_N^n = p^n - \Pi_N^1 p^n + \Pi_N^1 p^n - p_N^n = \tilde{e}_N^n + \hat{e}_N^n.$$

□

Theorem 6 *Suppose that $p \in C^2(J; H^m(\Omega))$, then for the scheme (20), there exists a positive constant C independent of $\Delta t,\ \sigma,\ N$ such that*

$$\left(\sum_{n=1}^{M} \Delta t \left\| p^n - p_N^n \right\|^2\right)^{1/2} \leqslant C(\Delta t + \sigma^2 + N^{-m}), \quad 1 \le n \le N_T,$$

and

$$\left(\sum_{n=1}^{M} \Delta t \left\| \nabla(p^n - p_N^n) \right\|^2\right)^{1/2} \leqslant C(\Delta t + \sigma^2 + N^{1-m}), \quad 1 \le n \le N_T.$$

Proof Recalling (18) and (20) and using the property of the projector Π_N^1 defined by (5), we can obtain the error equation

$$\begin{aligned}
&\sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \left[G_0^{\alpha_l}(\hat{e}_N^n, v) - \sum_{k=1}^{n-1} (G_{n-k-1}^{\alpha_l} - G_{n-k}^{\alpha_l})(\hat{e}_N^k, v) - G_{n-1}^{\alpha_l}(\hat{e}_N^0, v) \right] \\
&+ K(\nabla \hat{e}_N^n, \nabla v) + \frac{K^2}{\mu}(L_x L_y \hat{e}_N^n, v) \\
&= -\sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \left[G_0^{\alpha_l}(\tilde{e}_N^n, v) - \sum_{k=1}^{n-1} (G_{n-k-1}^{\alpha_l} - G_{n-k}^{\alpha_l})(\tilde{e}_N^k, v) - G_{n-1}^{\alpha_l}(\tilde{e}_N^0, v) \right] \\
&+ (f^n - I_N f^n, v) + (R^n - \frac{1}{\mu} L_x L_y \tilde{e}_N^n, v).
\end{aligned} \tag{38}$$

Set $v = \hat{e}_N^n$ in (38) to obtain

$$\begin{aligned}
&\sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \left[G_0^{\alpha_l} \|\hat{e}_N^n\|^2 - \sum_{k=1}^{n-1} (G_{n-k-1}^{\alpha_l} - G_{n-k}^{\alpha_l})(\hat{e}_N^k, \hat{e}_N^n) - G_{n-1}^{\alpha_l}(\hat{e}_N^0, \hat{e}_N^n) \right] \\
&+ K\|\nabla \hat{e}_N^n\|^2 + \frac{K^2}{\mu} \|\frac{\partial^2 \hat{e}_N^n}{\partial x \partial y}\|^2 \\
&= -\sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} \left[G_0^{\alpha_l}(\tilde{e}_N^n, \hat{e}_N^n) - \sum_{k=1}^{n-1} (G_{n-k-1}^{\alpha_l} - G_{n-k}^{\alpha_l})(\tilde{e}_N^k, \hat{e}_N^n) - G_{n-1}^{\alpha_l}(\tilde{e}_N^0, \hat{e}_N^n) \right] \\
&+ (f^n - I_N f^n, \hat{e}_N^n) + (R^n - \frac{K^2}{\mu} L_x L_y \tilde{e}_N^n, \hat{e}_N^n).
\end{aligned} \tag{39}$$

Supposing that $p \in C^2(J, H^m(\Omega))$, the first term on the right hand side of (39) can be estimated as

$$
\begin{aligned}
&-\sum_{l=1}^{q}\frac{\Delta t^{-\alpha_l}d_l}{\Gamma(2-\alpha_l)}\left[G_0^{\alpha_l}(\tilde{e}_N^n,\hat{e}_N^n)-\sum_{k=1}^{n-1}(G_{n-k-1}^{\alpha_l}-G_{n-k}^{\alpha_l})(\tilde{e}_N^k,\hat{e}_N^n)-G_{n-1}^{\alpha_l}(\tilde{e}_N^0,\hat{e}_N^n)\right]\\
\leq& C\|\int_0^1\omega(\alpha){}_0^C D_t^\alpha\tilde{e}_N(x,y,t)d\alpha\|^2+\frac{C_1}{24}\|\hat{e}_N^n\|^2\\
\leq&\frac{C_1}{24}\|\hat{e}_N^n\|^2+O(N^{-2m}).
\end{aligned}
\tag{40}
$$

Taking notice of Lemma 1, the second term on the right hand side of (39) can be estimated as

$$
(f^n - I_N f^n, \hat{e}_N^n) \leq \frac{C_1}{24}\|\hat{e}_N^n\|^2 + O(N^{-2m}). \tag{41}
$$

Using (8), (14) and Lemma 3, we can transform the last term on the right hand side of (39) into the following:

$$
\begin{aligned}
(R^n-\frac{K^2}{\mu}L_xL_y\tilde{e}_N^n,\hat{e}_N^n)\leq& C\|\frac{K^2}{\mu}L_xL_y\Pi_N^1p^n\|^2+\frac{C_1}{24}\|\hat{e}_N^n\|^2+O(\Delta t^2+\sigma^4)\\
\leq&\frac{C_1}{24}\|\hat{e}_N^n\|^2+O(\Delta t^2+\sigma^4).
\end{aligned}
\tag{42}
$$

Using Lemma 4, we can easily obtain that

$$
\frac{C_1}{4}\sum_{n=1}^{M}\Delta t\left\|\hat{e}_N^n\right\|^2+2K\sum_{n=1}^{M}\Delta t\|\nabla\hat{e}_N^n\|^2\leqslant C\left\|\hat{e}_N^0\right\|^2+O(\Delta t^2+\sigma^4+N^{-2m}). \tag{43}
$$

By (6) and (7), we have that

$$
\left(\sum_{n=1}^{M}\Delta t\left\|p^n-p_N^n\right\|^2\right)^{1/2}\leqslant C(\Delta t+\sigma^2+N^{-m}), \tag{44}
$$

and

$$
\left(\sum_{n=1}^{M}\Delta t\left\|\nabla(p^n-p_N^n)\right\|^2\right)^{1/2}\leqslant C(\Delta t+\sigma^2+N^{1-m}). \tag{45}
$$

This completes the proof. □

4 An alternating direction implicit spectral method with the second-order accuracy in time

The objective of this section is to consider the alternating direction implicit spectral scheme with the second-order accuracy in time for (1). Besides, the stability and error analysis are considered rigorously.

4.1 Fully discrete scheme

In this subsection, we present the fully discrete scheme for (1).

Recalling (9), we need to find higher accurate interpolation to approximate the multi-term fractional derivatives in time. A similar argument in Gao and Sun [8] can be applied to obtain second-order accuracy in time. Denote

$$a = \min_{1\le l\le q}\left\{1-\frac{\alpha_l}{2}\right\}, \quad b = \max_{1\le l\le q}\left\{1-\frac{\alpha_l}{2}\right\},$$

and

$$G(\tau) = \sum_{l=1}^{q}\frac{d_l}{\Gamma(3-\alpha_l)}\tau^{1-\alpha_l}[\tau-(1-\frac{\alpha_l}{2})]\Delta t^{2-\alpha_l}, \quad \tau\ge 0.$$

Easily we have that

$$\begin{aligned} a_1 &= 1-\frac{1}{2}\max_{1\le l\le q}\alpha_l = 1-\frac{\alpha_q}{2} = \frac{1}{2}+\frac{1}{4q}, \quad b_1 = 1-\frac{1}{2}\min_{1\le l\le q}\alpha_l \\ &= 1-\frac{\alpha_1}{2} = 1-\frac{1}{4q}. \end{aligned}$$

Following the similar line as in [8] allows us to get Lemmas 7 and 8.

Lemma 7 *The equation $G(\tau)=0$ has a unique positive root $\tau^*\in[a_1,b_1]$.*

Lemma 8 *For $q\ge 2$, the Newton iteration sequence $\{\tau_k\}_{k=0}^{\infty}$, generated by*

$$\begin{cases} \tau_{k+1} = \tau_k - \frac{G(\tau_k)}{G'(\tau_k)}, & k=0,1,2,\ldots, \\ \tau_0 = b, \end{cases}$$

is monotonically decreasing and convergent to τ^*.

From now on, we denote $\tau=\tau^*$ for simplicity, which means that $\tau\in(\frac{1}{2},1)$ such that $G(\tau)=0$. Denote $t_{n-1+\tau}=(n-1+\tau)\Delta t$ for simplicity. The next Lemma gives a numerical differentiation formula to approximate the multi-term fractional derivatives in time at the point $t=t_{n-1+\tau}$ and reveals its numerical accuracy.

Lemma 9 [8] *Suppose $g\in C^3([t_0,t_n])$. Let*

$$\begin{aligned} D_t^{\omega}g(t_{n-1+\tau}) &\equiv \sum_{l=1}^{q} d_l\, {}_0^C D_t^{\alpha_l} g(t_{n-1+\tau}) \\ &= \sum_{l=1}^{q}\frac{d_l}{\Gamma(1-\alpha_l)}\left(\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} g'(s)(t_{n-1+\tau}-s)^{-\alpha_l}ds \right.\\ &\quad \left. + \int_{t_{n-1}}^{t_{n-1+\tau}} g'(s)(t_{n-1+\tau}-s)^{-\alpha_l}ds\right), \end{aligned}$$

and

$$\Delta_t^{\omega} g(t_{n-1+\tau}) \equiv \sum_{l=1}^{q} \frac{d_l}{\Gamma(1-\alpha_l)} \left(\sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} L'_{2,k}(s)(t_{n-1+\tau}-s)^{-\alpha_l} ds \right.$$
$$\left. + \int_{t_{n-1}}^{t_{n-1+\tau}} L'_{1,n}(s)(t_{n-1+\tau}-s)^{-\alpha_l} ds \right),$$

where $L_{1,n}(s)$ *is a linear polynomial on* $[t_{n-1}, t_n]$ *satisfying*

$$L_{1,n}(t_{n-1}) = g(t_{n-1}), \quad L_{1,n}(t_n) = g(t_n),$$

and $L_{2,k}(s)$ *is a quadratic polynomial on* $[t_{k-1}, t_{k+1}]$ *satisfying*

$$L_{2,k}(t_{k-1}) = g(t_{k-1}), \quad L_{1,k}(t_k) = g(t_k), \quad L_{1,k}(t_{k+1}) = g(t_{k+1}).$$

Then we have

$$|D_t^{\omega} g(t_{n-1+\tau}) - \Delta_t^{\omega} g(t_{n-1+\tau})| \le \max_{t_0 \le s \le t_n} |g'''(s)| \sum_{l=1}^{q} \frac{d_l}{\Gamma(2-\alpha_l)}$$
$$\cdot \left(\frac{1-\alpha_l}{12} + \frac{\tau}{6} \right) \tau^{-\alpha_l} \Delta t^{3-\alpha_l}. \qquad (46)$$

For $\alpha \in [0, 1]$, denote

$$a_0^{(\alpha)} = \tau^{1-\alpha}, \quad a_l^{(\alpha)} = (l+\tau)^{1-\alpha} - (l-1+\tau)^{1-\alpha}, \quad l \ge 1,$$
$$b_l^{(\alpha)} = \frac{1}{2-\alpha} \left((l+\tau)^{2-\alpha} - (l-1+\tau)^{2-\alpha} \right)$$
$$- \frac{1}{2} \left((l+\tau)^{1-\alpha} + (l-1+\tau)^{1-\alpha} \right), \quad l \ge 1.$$

when $n = 1$, denote

$$c_0^{(n,\alpha)} = a_0^{(\alpha)};$$

when $n \ge 2$, denote

$$c_k^{(n,\alpha)} = \begin{cases} a_0^{(\alpha)} + b_1^{(\alpha)}, & k = 0, \\ a_k^{(\alpha)} + b_{k+1}^{(\alpha)} - b_k^{(\alpha)}, & 1 \le k \le n-2, \\ a_k^{(\alpha)} - b_k^{(\alpha)}, & k = n-1. \end{cases}$$

Then the numerical differentiation formula $\mathcal{D}g(t_{n-1+\tau})$ can be transformed into the following:

$$\mathcal{D}g(t_{n-1+\tau}) = \sum_{k=0}^{n-1} \tilde{c}_k^n \left(g(t_{n-k}) - g(t_{n-k-1}) \right), \qquad (47)$$

where

$$\tilde{c}_k^n = \sum_{l=1}^{q} \frac{d_l \Delta t^{-\alpha_l}}{\Gamma(2-\alpha_l)} c_k^{(n,\alpha_l)}.$$

Next, we give some properties of the coefficients $\{\tilde{c}_k^n\}$ which will be useful in the following stability and error estimates.

Lemma 10 [8] *For* $\alpha_l \in [0,1],\ l = 1, 2, \ldots, q$, *we have*

$$\tilde{c}_1^n > \tilde{c}_2^n > \cdots > \tilde{c}_{n-2}^n > \tilde{c}_{n-1}^n > \sum_{l=1}^{q} \frac{d_l \Delta t^{-\alpha_l}}{\Gamma(2-\alpha_l)} \cdot \frac{1-\alpha_l}{2} (n-1+\tau)^{-\alpha_l}. \tag{48}$$

Besides, there exists a positive constant Δt_0 *such that*

$$(2\tau - 1)\tilde{c}_0^n - \tau \tilde{c}_1^n > 0, \quad for\ \Delta t \le \Delta t_0,\ n \ge 2. \tag{49}$$

Considering (9) at the point $(x, y, t_{n-1+\tau})$, we have

$$\begin{aligned} \sum_{k=0}^{n-1} \tilde{c}_k^n \left(p^{n-k} - p^{n-k-1}\right) - K(L_x + L_y) p(x, y, t_{n-1+\tau}) = f^{n-1+\tau} \\ -R_1^{n-1+\tau} + E_1^{n-1+\tau}, \end{aligned} \tag{50}$$

where $E_1^{n-1+\tau} = \Delta_t^{\omega} p(t_{n-1+\tau}) - D_t^{\omega} p(t_{n-1+\tau})$.
Besides, it holds that

$$\begin{aligned} (L_x + L_y) p(x, y, t_{n-1+\tau}) = \tau (L_x + L_y) p(x, y, t_n) \\ + (1-\tau)(L_x + L_y) p(x, y, t_{n-1}) + E_2^{n-1+\tau}, \end{aligned} \tag{51}$$

where $E_2^{n-1+\tau} = O(\Delta t^2)$.
Substituting (51) into (50), we have

$$\begin{aligned} &\sum_{k=0}^{n-1} \tilde{c}_k^n \left(p^{n-k} - p^{n-k-1}\right) - K\tau(L_x + L_y) p(x, y, t_n) \\ &- K(1-\tau)(L_x + L_y) p(x, y, t_{n-1}) \\ =&f^{n-1+\tau} - R_1^{n-1+\tau} + E_1^{n-1+\tau} + E_2^{n-1+\tau}. \end{aligned} \tag{52}$$

Next, we give the following lemma:

Lemma 11 *Denote*

$$\nu = \sum_{l=1}^{q} \frac{d_l \Delta t^{-\alpha_l}}{\Gamma(2-\alpha_l)} c_0^{(n,\alpha_l)}, \tag{53}$$

then we have

$$\nu = \frac{1}{O(\Delta t|ln(\Delta t)|)}. \tag{54}$$

Proof A direct calculation gives

$$\begin{aligned}
\nu =& \sum_{l=1}^{q} \frac{\Delta t^{-\alpha_l} d_l}{\Gamma(2-\alpha_l)} c_0^{(n,\alpha_l)} \sim \int_0^1 \frac{\omega(\alpha)\Delta t^{-\alpha}}{\Gamma(2-\alpha)} (a_0^{(\alpha)} + b_1^{(\alpha)}) d\alpha \\
=& \frac{\omega(\alpha^*)}{\Gamma(2-\alpha^*)} (a_0^{(\alpha_*)} + b_1^{(\alpha_*)}) \int_0^1 (\frac{1}{\Delta t})^{\alpha} d\alpha \\
=& \frac{\omega(\alpha^*)}{\Gamma(2-\alpha^*)} (a_0^{(\alpha_*)} + b_1^{(\alpha_*)}) \frac{(\frac{1}{\Delta t})^{\alpha}}{ln\frac{1}{\Delta t}} \Bigg|_{\alpha=0}^{1} \\
=& \frac{\omega(\alpha^*)}{\Gamma(2-\alpha^*)} (a_0^{(\alpha_*)} + b_1^{(\alpha_*)}) \frac{\frac{1}{\Delta t} - 1}{ln\frac{1}{\Delta t}},
\end{aligned}$$

where $\alpha^* \in (0, 1)$. We can easily obtain that $\nu = \dfrac{1}{O(\Delta t|ln(\Delta t)|)}$, this completes the proof. □

Adding the perturbation term $\frac{K^2\tau^2\Delta t}{\nu} L_x L_y \frac{p^n - p^{n-1}}{\Delta t}$ to the both sides of (52) gives

$$\begin{aligned}
&\sum_{k=0}^{n-1} \tilde{c}_k^n \left(p^{n-k} - p^{n-k-1}\right) - K\tau(L_x + L_y)p(x, y, t_n) + \frac{K^2\tau^2\Delta t}{\nu} L_x L_y \frac{p^n - p^{n-1}}{\Delta t} \\
&- K(1-\tau)(L_x + L_y)p(x, y, t_{n-1}) \\
=& f^{n-1+\tau} + E^{n-1+\tau},
\end{aligned} \tag{55}$$

where $E^{n-1+\tau} = -R_1^{n-1+\tau} + E_1^{n-1+\tau} + E_2^{n-1+\tau} + \frac{K^2\tau^2\Delta t}{\nu} L_x L_y \frac{p^n - p^{n-1}}{\Delta t}$.

We can also rewrite (55) into the following equivalent form:

$$\begin{aligned}
&(\sqrt{\nu} - \frac{K\tau}{\sqrt{\nu}} L_x)(\sqrt{\nu} - \frac{K\tau}{\sqrt{\nu}} L_y) p^n \\
=& (\sqrt{\nu} - \frac{K\tau}{\sqrt{\nu}} L_x)(\sqrt{\nu} - \frac{K\tau}{\sqrt{\nu}} L_y) p^{n-1} \\
&- \sum_{k=1}^{n-1} \tilde{c}_k^n \left(p^{n-k} - p^{n-k-1}\right) + K(L_x + L_y)p^{n-1} \\
&+ f^{n-1+\tau} + E^{n-1+\tau}.
\end{aligned} \tag{56}$$

Then we can obtain the fully discrete alternating direction implicit spectral method with the second-order accuracy in time for (1) as follows: Find $p_N^n \in V_N^0$ for $n = 1, 2, \ldots, N_T$ such that

$$\begin{cases} \left((\sqrt{\nu} - \frac{K\tau}{\sqrt{\nu}}L_x)(\sqrt{\nu} - \frac{K\tau}{\sqrt{\nu}}L_y)p_N^n, v\right) \\ = \left((\sqrt{\nu} - \frac{K\tau}{\sqrt{\nu}}L_x)(\sqrt{\nu} - \frac{K\tau}{\sqrt{\nu}}L_y)p_N^{n-1}, v\right) \\ \quad - \sum_{k=1}^{n-1} \tilde{c}_k^n \left((p_N^{n-k} - p_N^{n-k-1}), v\right) + K\left((L_x + L_y)p_N^{n-1}, v\right) \\ \quad + (I_N f^{n-1+\tau}, v), \quad \forall v \in V_N^0, \\ p_N^0 = \Pi_N p_0. \end{cases} \tag{57}$$

4.2 Implementation of the alternating direction implicit spectral method with the second-order accuracy in time

In this subsection, a detailed description of the implementation of the alternating direction implicit spectral method with the second-order accuracy in time is given.

Taking notice of (21) and setting $v = \phi_i\varphi_j (i, j = 0, 1, \ldots, N-2)$, the matrix representation of the alternating direction implicit spectral method with the second-order accuracy in time (57) can be given as follows:

$$(\sqrt{\nu}M_x + \frac{K\tau}{\sqrt{\nu}}S_x)U^n(\sqrt{\nu}M_y + \frac{K\tau}{\sqrt{\nu}}S_y)^T = N^{n-1} + B^n + F^{n-1+\tau}, \tag{58}$$

where $U^n,\ N^{n-1},\ B^n,\ F^{n-1+\tau} \in \mathbb{R}^{(N-1)\times(N-1)}$ that satisfying

$$\begin{aligned} &(U^n)_{i,j} = u_{i,j}^n, \quad i, j = 0, 1, \ldots, N-2, \\ &N^{n-1} = (\sqrt{\nu}M_x + \frac{K\tau}{\sqrt{\nu}}S_x)U^{n-1}(\sqrt{\nu}M_y + \frac{K\tau}{\sqrt{\nu}}S_y)^T - K(S_xU^{n-1}M_y^T + M_xU^{n-1}S_y^T), \\ &B^n = -M_x\left(\sum_{k=1}^{n-1}\tilde{c}_k^n(U^{n-k} - U^{n-k-1})\right)M_y^T, \\ &(F^n)_{i,j} = (I_N f^n, \phi_i\varphi_j), \quad i, j = 0, 1, \ldots, N-2. \end{aligned} \tag{59}$$

Then we can solve the system (58) by the following algorithm:
For $n = 1, 2, \ldots, N_T$
solve $(\sqrt{\nu}M_x + \frac{K\tau}{\sqrt{\nu}}S_x)U^* = N^{n-1} + B^n + F^{n-1+\tau}$ to obtain U^*;
solve $(\sqrt{\nu}M_y + \frac{K\tau}{\sqrt{\nu}}S_y)(U^n)^T = (U^*)^T$ to obtain U^n.
end

We can easily obtain that the coefficient matrices of the algebraic system derived from the alternating direction implicit spectral method have the same size as those of the coefficient matrix derived from the corresponding one-dimensional system.

4.3 Stability and convergence

In this subsection, we give an illustration of the stability and convergence of the alternating direction implicit spectral scheme with (57). We consider the following lemma first.

Lemma 12 *Suppose* $\{p_N^n\} \in V_N^0$ *is the solution of*

$$\begin{cases} \left((\sqrt{\nu}-\frac{K\tau}{\sqrt{\nu}}L_x)(\sqrt{\nu}-\frac{K\tau}{\sqrt{\nu}}L_y)p_N^n, v\right) \\ =\left((\sqrt{\nu}-\frac{K\tau}{\sqrt{\nu}}L_x)(\sqrt{\nu}-\frac{K\tau}{\sqrt{\nu}}L_y)p_N^{n-1}, v\right) \\ \quad -\sum_{k=1}^{n-1}\tilde{c}_k^n\left((p_N^{n-k}-p_N^{n-k-1}), v\right)+K\left((L_x+L_y)p_N^{n-1}, v\right) \\ \quad +(Q^{n-1+\tau}, v), \quad \forall v\in V_N^0, \\ p_N^0=\Pi_N p_0. \end{cases} \tag{60}$$

Then we have

$$\begin{aligned} &\sum_{n=1}^{M}\Delta t\|p_N^n\|^2+\sum_{n=1}^{M}\Delta t\|\tau\nabla p_N^n+(1-\tau)\nabla p_N^{n-1}\|^2 \\ &\leq C\frac{K^2\tau^2}{2\nu}\|\frac{\partial^2 p_0}{\partial x\partial y}\|^2+C\|p_0\|^2+C\|Q^\tau\|^2+C\sum_{n=2}^{M}\Delta t\|Q^{n-1+\tau}\|^2, \quad 1\leq n\leq N_T. \end{aligned} \tag{61}$$

Proof Rewrite the scheme (60) into the following equivalent form:

$$\begin{aligned} &\sum_{k=0}^{n-1}\tilde{c}_k^n\left((p_N^{n-k}-p_N^{n-k-1}), v\right)+K\left(\tau\nabla p_N^n+(1-\tau)\nabla p_N^{n-1}, \nabla v\right) \\ &+\frac{K^2\tau^2\Delta t}{\nu}(L_xL_y\frac{p_N^n-p_N^{n-1}}{\Delta t}, v)=(Q^{n-1+\tau}, v), \quad \forall v\in V_N^0. \end{aligned} \tag{62}$$

Set $v=\tau p_N^n+(1-\tau)p_N^{n-1}$ in (62) to obtain that

$$\begin{aligned} &\sum_{k=0}^{n-1}\tilde{c}_k^n\left((p_N^{n-k}-p_N^{n-k-1}), \tau p_N^n+(1-\tau)p_N^{n-1}\right)+K\|\tau\nabla p_N^n+(1-\tau)\nabla p_N^{n-1}\|^2 \\ &+\frac{K^2\tau^2\Delta t}{\nu}(L_xL_y\frac{p_N^n-p_N^{n-1}}{\Delta t}, \tau p_N^n+(1-\tau)p_N^{n-1}) \\ &=(Q^{n-1+\tau}, \tau p_N^n+(1-\tau)p_N^{n-1}). \end{aligned} \tag{63}$$

Similar to the estimates in [1, 8], recalling Lemma 10, the first term on the left hand side of (63) can be estimated as follows:

$$\sum_{k=0}^{n-1} \tilde{c}_k^n \left((p_N^{n-k} - p_N^{n-k-1}), \tau p_N^n + (1-\tau) p_N^{n-1} \right) \geq \frac{1}{2} \sum_{k=0}^{n-1} \tilde{c}_k^n (\| p_N^{n-k} \|^2 - \| p_N^{n-k-1} \|^2). \tag{64}$$

By using Green formula. the last term on the left hand side of (63) can be estimated as

$$\begin{aligned}
&\frac{K^2 \tau^2 \Delta t}{\nu} (L_x L_y \frac{p_N^n - p_N^{n-1}}{\Delta t}, \tau p_N^n + (1-\tau) p_N^{n-1}) \\
=&\frac{K^2 \tau^2 \Delta t}{\nu} (L_x L_y \frac{p_N^n - p_N^{n-1}}{\Delta t}, (1-\tau)(p_N^n + p_N^{n-1}) + (2\tau - 1) p_N^n) \\
=&\frac{K^2 \tau^2 (1-\tau)}{\nu} (\| \frac{\partial^2 p_N^n}{\partial x \partial y} \|^2 - \| \frac{\partial^2 p_N^{n-1}}{\partial x \partial y} \|^2) + \frac{K^2 \tau^2 (2\tau - 1)}{2\nu} (\| \frac{\partial^2 p_N^n}{\partial x \partial y} \|^2 - \| \frac{\partial^2 p_N^{n-1}}{\partial x \partial y} \|^2) \\
&+ \frac{K^2 \tau^2 (2\tau - 1)}{2\nu} \| \frac{\partial^2 (p_N^n - p_N^{n-1})}{\partial x \partial y} \|^2 \\
=&\frac{K^2 \tau^2}{2\nu} (\| \frac{\partial^2 p_N^n}{\partial x \partial y} \|^2 - \| \frac{\partial^2 p_N^{n-1}}{\partial x \partial y} \|^2) + \frac{K^2 \tau^2 (2\tau - 1)}{2\nu} \| \frac{\partial^2 (p_N^n - p_N^{n-1})}{\partial x \partial y} \|^2.
\end{aligned} \tag{65}$$

Besides, we have

$$\begin{aligned}
&(Q^{n-1+\tau}, \tau p_N^n + (1-\tau) p_N^{n-1}) \\
\leq&\frac{1}{16} \sum_{l=1}^{q} \frac{d_l}{T^{\alpha_l} \Gamma(1-\alpha_l)} \left(\| p_N^n \|^2 + \| p_N^{n-1} \|^2 \right) + C \| Q^{n-1+\tau} \|^2.
\end{aligned} \tag{66}$$

Combining (63) with (64)–(66) leads to

$$\begin{aligned}
&\frac{1}{2} \sum_{k=0}^{n-1} \tilde{c}_k^n \| p_N^{n-k} \|^2 + K \| \tau \nabla p_N^n + (1-\tau) \nabla p_N^{n-1} \|^2 \\
&+ \frac{K^2 \tau^2}{2\nu} (\| \frac{\partial^2 p_N^n}{\partial x \partial y} \|^2 - \| \frac{\partial^2 p_N^{n-1}}{\partial x \partial y} \|^2) + \frac{K^2 \tau^2 (2\tau - 1)}{2\nu} \| \frac{\partial^2 (p_N^n - p_N^{n-1})}{\partial x \partial y} \|^2 \\
\leq&\frac{1}{2} \sum_{k=0}^{n-1} \tilde{c}_k^n \| p_N^{n-k-1} \|^2 + \frac{1}{16} \sum_{l=1}^{q} \frac{d_l}{T^{\alpha_l} \Gamma(1-\alpha_l)} \left(\| p_N^n \|^2 + \| p_N^{n-1} \|^2 \right) \\
&+ C \| Q^{n-1+\tau} \|^2.
\end{aligned} \tag{67}$$

The first term on the right hand side of (67) can be transformed into

$$\begin{aligned}
&\frac{1}{2} \sum_{k=0}^{n-1} \tilde{c}_k^n \| p_N^{n-k-1} \|^2 = \frac{1}{2} \sum_{k=0}^{n-2} \tilde{c}_k^n \| p_N^{n-k-1} \|^2 \\
=&\frac{1}{2} \sum_{k=0}^{n-2} \tilde{c}_k^{n-1} \| p_N^{n-1-k} \|^2 + \frac{1}{2} \sum_{l=1}^{q} \frac{d_l \Delta t^{-\alpha_l}}{\Gamma(2-\alpha_l)} b_{n-1}^{(\alpha_l)} \| p_N^1 \|^2.
\end{aligned} \tag{68}$$

Substituting (68) into (67) gives that

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n-1}\tilde{c}_k^n\|p_N^{n-k}\|^2+K\|\tau\nabla p_N^n+(1-\tau)\nabla p_N^{n-1}\|^2\\
&+\frac{K^2\tau^2}{2\nu}(\|\frac{\partial^2 p_N^n}{\partial x\partial y}\|^2-\|\frac{\partial^2 p_N^{n-1}}{\partial x\partial y}\|^2)+\frac{K^2\tau^2(2\tau-1)}{2\nu}\|\frac{\partial^2(p_N^n-p_N^{n-1})}{\partial x\partial y}\|^2\\
\leq&\frac{1}{2}\sum_{k=0}^{n-2}\tilde{c}_k^{n-1}\|p_N^{n-1-k}\|^2+\frac{1}{2}\sum_{l=1}^{q}\frac{d_l\Delta t^{-\alpha_l}}{\Gamma(2-\alpha_l)}b_{n-1}^{(\alpha_l)}\|p_N^1\|^2\\
&+\frac{1}{16}\sum_{l=1}^{q}\frac{d_l}{T^{\alpha_l}\Gamma(1-\alpha_l)}\left(\|p_N^n\|^2+\|p_N^{n-1}\|^2\right)\\
&+C\|Q^{n-1+\tau}\|^2.
\end{aligned}
\tag{69}
$$

Multiplying both sides of (69) by $2\Delta t$ and summing on n, $n=2,\cdots M$ $(M\leqslant N_T)$ lead to

$$
\begin{aligned}
&\Delta t\sum_{k=0}^{M-1}\tilde{c}_k^M\|p_N^{M-k}\|^2+2K\sum_{n=2}^{M}\Delta t\|\tau\nabla p_N^n+(1-\tau)\nabla p_N^{n-1}\|^2\\
&+\frac{K^2\tau^2\Delta t}{\nu}\|\frac{\partial^2 p_N^M}{\partial x\partial y}\|^2\\
\leq&\sum_{n=1}^{M-1}\Delta t\sum_{l=1}^{q}\frac{d_l\Delta t^{-\alpha_l}}{\Gamma(2-\alpha_l)}b_n^{(\alpha_l)}\|p_N^1\|^2+\frac{K^2\tau^2\Delta t}{\nu}\|\frac{\partial^2 p_N^1}{\partial x\partial y}\|^2\\
&+\frac{1}{8}\sum_{n=2}^{M}\Delta t\sum_{l=1}^{q}\frac{d_l}{T^{\alpha_l}\Gamma(1-\alpha_l)}\left(\|p_N^n\|^2+\|p_N^{n-1}\|^2\right)\\
&+C\sum_{n=2}^{M}\Delta t\|Q^{n-1+\tau}\|^2.
\end{aligned}
\tag{70}
$$

For the case of $n=1$, (67) can be transformed into

$$
\begin{aligned}
&\frac{1}{2}\tilde{c}_0^1\|p_N^1\|^2+K\|\tau\nabla p_N^1+(1-\tau)\nabla p_N^0\|^2\\
&+\frac{K^2\tau^2}{2\nu}(\|\frac{\partial^2 p_N^1}{\partial x\partial y}\|^2-\|\frac{\partial^2 p_N^0}{\partial x\partial y}\|^2)\\
\leq&\frac{1}{2}\tilde{c}_0^1\|p_N^0\|^2+\frac{1}{16}\sum_{l=1}^{q}\frac{d_l}{T^{\alpha_l}\Gamma(1-\alpha_l)}\left(\|p_N^1\|^2+\|p_N^0\|^2\right)\\
&+C\|Q^{\tau}\|^2.
\end{aligned}
\tag{71}
$$

Since

$$
\tilde{c}_{n-1}^n\geq\sum_{l=1}^{q}\frac{d_l\Delta t^{-\alpha_l}}{\Gamma(2-\alpha_l)}\cdot\frac{1-\alpha_l}{2}(n-1+\sigma)^{-\alpha_l}\geq\frac{1}{2}\sum_{l=1}^{q}\frac{d_l}{T^{\alpha_l}\Gamma(1-\alpha_l)}.
\tag{72}
$$

Then easily we can obtain

$$\begin{aligned}&\|p_N^1\|^2+K\|\tau\nabla p_N^1+(1-\tau)\nabla p_N^0\|^2+\frac{K^2\tau^2}{2\nu}\|\frac{\partial^2 p_N^1}{\partial x\partial y}\|^2\\ \le& C\|p_N^0\|^2+C\frac{K^2\tau^2}{2\nu}\|\frac{\partial^2 p_N^0}{\partial x\partial y}\|^2+C\|Q^\tau\|^2.\end{aligned} \tag{73}$$

Using Taylor's expansion, we have

$$\begin{aligned}b_n^{(\alpha_l)}=&\frac{1}{2-\alpha_l}\left((n+\tau)^{2-\alpha_l}-(n-1+\tau)^{2-\alpha_l}\right)-\frac{1}{2}\left((n+\tau)^{1-\alpha_l}+(n-1+\tau)^{1-\alpha_l}\right)\\ =&\frac{1}{2-\alpha_l}\Big[(n+\tau)^{2-\alpha_l}-\Big((n+\tau)^{2-\alpha_l}-(2-\alpha_l)(n+\tau)^{1-\alpha_l}\\ &+\frac{(2-\alpha_l)(1-\alpha_l)}{2}\zeta_{1n}^{-\alpha_l}\Big)\Big]\\ &-\frac{1}{2}\left(2(n+\tau)^{1-\alpha_l}-(1-\alpha_l)\zeta_{2n}^{-\alpha_l}\right)\\ =&\frac{1-\alpha_l}{2}(\zeta_{2n}^{-\alpha_l}-\zeta_{1n}^{-\alpha_l}),\qquad n-1+\tau<\zeta_{1n},\ \zeta_{2n}<n+\tau.\end{aligned} \tag{74}$$

Then using (73) and (74), the first term on the right hand side of (70) can be estimated as follows:

$$\begin{aligned}&\sum_{n=1}^{M-1}\Delta t\sum_{l=1}^{q}\frac{d_l\Delta t^{-\alpha_l}}{\Gamma(2-\alpha_l)}b_n^{(\alpha_l)}\|p_N^1\|^2\\ \le&\sum_{l=1}^{q}\frac{d_l}{\Gamma(2-\alpha_l)}\|p_N^1\|^2\Delta t\sum_{n=1}^{M-1}t_{n-1+\tau}^{-\alpha_l}\\ \le&2\sum_{l=1}^{q}\frac{(1-\alpha_l)T^{-\alpha_l}}{\Gamma(2-\alpha_l)}d_l\|p_N^1\|^2\\ \le&C\|p_N^0\|^2+C\frac{K^2\tau^2}{2\nu}\|\frac{\partial^2 p_N^0}{\partial x\partial y}\|^2+C\|Q^\tau\|^2.\end{aligned} \tag{75}$$

Helpful in establishing above result is the following inequality holds:

$$\Delta t\sum_{n=1}^{M-1}t_{n-1+\tau}^{-\alpha_l}\to\int_{t_\sigma}^{t_{M-1+\tau}}t^{-\alpha_l}dt\le 2(1-\alpha_l)T^{1-\alpha_l}. \tag{76}$$

Combining (70) with (73) and (75), we can obtain

$$
\begin{aligned}
&\Delta t\sum_{k=0}^{M-1}\tilde{c}_k^M\|p_N^{M-k}\|^2+2K\sum_{n=2}^{M}\Delta t\|\tau\nabla p_N^n+(1-\tau)\nabla p_N^{n-1}\|^2\\
\le&\frac{1}{4}\sum_{n=1}^{M}\Delta t\sum_{l=1}^{q}\frac{d_l}{T^{\alpha_l}\Gamma(1-\alpha_l)}\|p_N^n\|^2+C\frac{K^2\tau^2}{2\nu}\|\frac{\partial^2 p_0}{\partial x\partial y}\|^2\\
&+C\|p_N^0\|^2+C\|Q^\tau\|^2+C\sum_{n=2}^{M}\Delta t\|Q^{n-1+\tau}\|^2.
\end{aligned}
\tag{77}
$$

Recalling (73), easily we have

$$
\begin{aligned}
&\sum_{n=1}^{M}\Delta t\|p_N^n\|^2+\sum_{n=1}^{M}\Delta t\|\tau\nabla p_N^n+(1-\tau)\nabla p_N^{n-1}\|^2\\
\le& C\frac{K^2\tau^2}{2\nu}\|\frac{\partial^2 p_0}{\partial x\partial y}\|^2+C\|p_0\|^2+C\|Q^\tau\|^2+C\sum_{n=2}^{M}\Delta t\|Q^{n-1+\tau}\|^2.
\end{aligned}
\tag{78}
$$

This completes the proof. □

Theorem 13 *For the scheme (57), we have the following stability inequality:*

$$
\begin{aligned}
&\sum_{n=1}^{M}\Delta t\|p_N^n\|^2+\sum_{n=1}^{M}\Delta t\|\tau\nabla p_N^n+(1-\tau)\nabla p_N^{n-1}\|^2\\
\le& C\frac{K^2\tau^2}{2\nu}\|\frac{\partial^2 p_0}{\partial x\partial y}\|^2+C\|p_0\|^2+C\|f^\tau\|^2+C\sum_{n=2}^{M}\Delta t\|f^{n-1+\tau}\|^2,\quad 1\le n\le N_T.
\end{aligned}
$$

Proof Recalling Lemma 12, the result needed can be easily obtained. □

Theorem 14 *Suppose that $p\in C^3(J;H^m(\Omega))$, then for the scheme (57), there exists a positive constant C independent of $\Delta t,\ \sigma, \tau,\ N$ such that*

$$
\left(\sum_{n=1}^{M}\Delta t\left\|p^n-p_N^n\right\|^2\right)^{1/2}\leqslant C(\Delta t^2+\sigma^2+N^{-m}),\quad 1\le n\le N_T,
$$

and

$$
\begin{aligned}
&\left(\sum_{n=1}^{M}\Delta t\left\|\tau\nabla(p^n-p_N^n)+(1-\tau)\nabla(p^{n-1}-p_N^{n-1})\right\|^2\right)^{1/2}\\
\leqslant& C(\Delta t^2+\sigma^2+N^{1-m}),\quad 1\le n\le N_T.
\end{aligned}
$$

Proof Recalling Lemma 12, we have

$$\begin{aligned}
&\sum_{n=1}^{M}\Delta t\|\hat{e}_N^n\|^2+\sum_{n=1}^{M}\Delta t\|\tau\nabla\hat{e}_N^n+(1-\tau)\nabla\hat{e}_N^{n-1}\|^2\\
\leq &C\frac{K^2\tau^2}{2\nu}\|\frac{\partial^2\hat{e}_0}{\partial x\partial y}\|^2+C\|\hat{e}_0\|^2+O(\Delta t^4+\sigma^4+N^{-2m})\\
\leq &C\|\Pi_N p_0-p_0\|^2+\|p_0-\Pi_N^1 p_0\|^2+O(\Delta t^4+\sigma^4+N^{-2m})\\
\leq &O(\Delta t^4+\sigma^4+N^{-2m}).
\end{aligned}\tag{79}$$

Taking notice of (6) and (7), the results needed can be easily obtained. □

5 Numerical examples

In this section, we are devoted to some numerical illustration on the theoretical results in the previous sections. We test examples 1 $\sim$ 3 to verify the convergence rates of the presented schemes, where $T=3/4$ and $\Omega=(-1,1)\times(-1,1)$. Let Scheme I and Scheme II denote the alternating direction implicit spectral schemes (20) and (57) respectively. For simplicity, we define

$$\|p-p_N\|_{l^2(J;L^2(\Omega))}=\left(\sum_{n=1}^{M}\Delta t\left\|p^n-p_N^n\right\|^2\right)^{1/2}.$$

To confirm the theoretical prediction, a numerical experiment is carried out by considering the problem with the following example 1:

$$\begin{cases}
p=16t^4(x+1)^2(x-1)(y+1)^2(y-1),\ \omega(\alpha)=\Gamma(5-\alpha),\ p_0(x,y)=0,\\
f=384t^3(t-1)/log(t)(x+1)^2(x-1)(y+1)^2(y-1)+16\left(-2t^4(x-1)(x+1)^2(y-1)\right.\\
\quad\left.-2t^4(x-1)(y-1)(y+1)^2-2t^4(2y+2)(x-1)(x+1)^2-2t^4(2x+2)(y-1)(y+1)^2\right).
\end{cases}$$

The main purpose is to check the convergence behavior of numerical solution for Scheme I. Tables 1 and 2 present the results of Scheme I for example 1, where to show the time and the distributed-order variables convergence rates, we take several different steps $\Delta t\approx\sigma^2$ (i.e., an optimal step size ratio) and fix the polynomial degree $N=20$ which is sufficiently small to avoid contamination of the spatial error.

Next, we investigate the convergence behavior for Scheme II by the following example 2:

$$\begin{cases}
p=t^2 sin(\pi x)sin(\pi y),\ \omega(\alpha)=\Gamma(3-\alpha),\ p_0(x,y)=0,\\
f=2t(t-1)/(log(t))sin(\pi x)sin(\pi y)+2t^2\pi^2 sin(\pi x)sin(\pi y).
\end{cases}$$

Tables 3 and 4 present the results of example 2 for Scheme II. These clearly indicate the second-order convergence rate with respect to the time step size and distributed order, respectively. In Fig. 1 (Left), we demonstrate the errors as a function of the

Table 1 Errors and convergence rates in time of example 1 with $N = 20$

Δt	$\|p - p_N\|_{l^2(J;L^2(\Omega))}$	Rate
1/40	2.68E-2	—
1/80	1.13E-2	1.25
1/160	4.83E-3	1.23
1/320	2.14E-3	1.17
1/640	9.93E-4	1.11

Table 2 Errors and convergence rates in distributed order of example 1 with $N = 20$

σ	$\|p - p_N\|_{l^2(J;L^2(\Omega))}$	Rate
1/5	2.20E-1	—
1/10	4.50E-2	2.29
1/20	8.37E-3	2.43
1/40	1.63E-3	2.36
1/80	3.87E-4	2.07

Table 3 Errors and convergence rates in time of example 2 with $N = 20$

Δt	τ	$\|p - p_N\|_{l^2(J;L^2(\Omega))}$	Rate
1/40	0.61	2.82E-3	—
1/80	0.60	7.95E-4	1.83
1/160	0.59	2.21E-4	1.85
1/320	0.58	6.10E-5	1.86
1/640	0.57	1.67E-5	1.87

Table 4 Errors and convergence rates in distributed order of example 2 with $N = 20$

σ	τ	$\|p - p_N\|_{l^2(J;L^2(\Omega))}$	Rate
1/10	0.65	3.58E-2	—
1/20	0.63	9.80E-3	1.87
1/40	0.61	2.82E-3	1.80
1/80	0.60	7.95E-4	1.83
1/160	0.59	2.21E-4	1.85

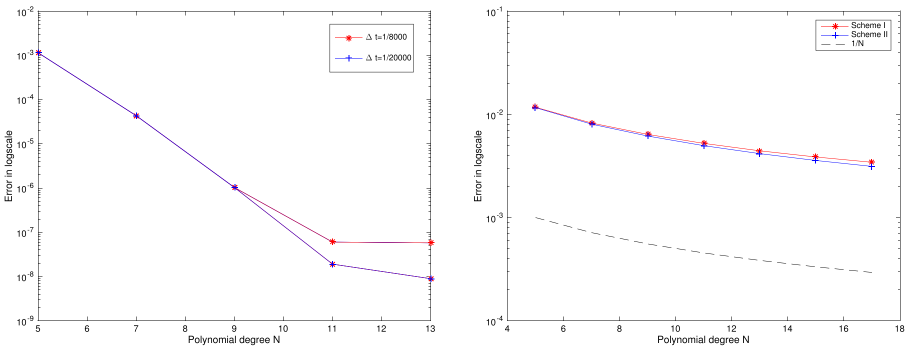

Fig. 1 *Left:* Errors in $l^2(J; L^2(\Omega))$-norm as a function of N for example 2 (*Left*) and example 3 (*Right*)

polynomial degree N for two time steps of $\Delta t = 1/8000$ and $\Delta t = 1/20000$. A logarithmic scale is used for the error-axis. Clearly, it is observed that the error variations are linear versus the degrees of polynomial N, which means that the convergence is exponential since it is a semi-log plot. Besides, it seems that the error flattens when a certain polynomial degree is reached in Fig. 1. The main reason is that the error is saturated with the other components. In order to vividly demonstrate the accuracy of the alternating direction implicit spectral method, we present exact and approximate solutions for unknown function in Fig. 2 for example 2 with $\sigma = \Delta t = 1/160$ and $N = 20$.

As for all other spectral methods, the accuracy of the present spectral method depends on the regularity of the solution. To this end, we consider the following example 3 with limited regularity to examine the sharpness of the estimates given in Theorems 6 and 14:

$$\begin{cases} p = t^2(1-x^2)^{1/2} sin(\pi y), \ \omega(\alpha) = \Gamma(3-\alpha), \ p_0(x,y) = 0, \\ f = 2t(t-1)/(log(t))(1-x^2)^{1/2} sin(\pi y) + t^2 sin(\pi y)(1-x^2)^{-1/2} \\ \qquad + t^2\pi^2 sin(\pi y)(1-x^2)^{1/2} + t^2x^2 sin(\pi y)(1-x^2)^{-3/2}. \end{cases}$$

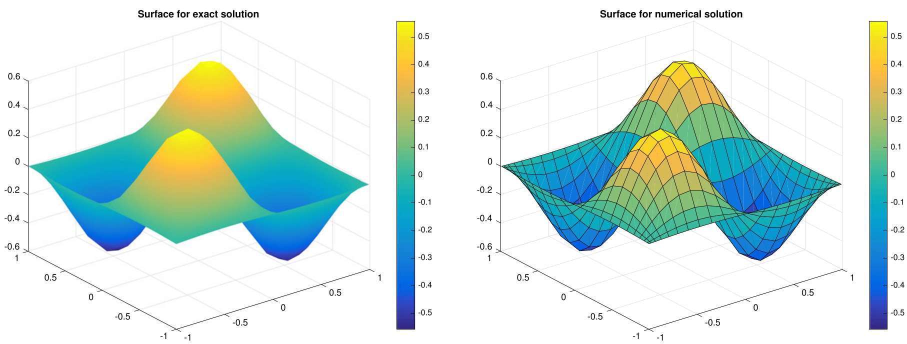

Fig. 2 Exact and numerical solutions for example 2 with $\sigma = \Delta t = 1/160$ and $N = 20$

In this example, we take the exact solution with limit regularity in Ω and investigate the spatial accuracy of the proposed Schemes I and II. It can be verified that this solution of example 3 belongs to $H^{1-\epsilon}(\Omega)$ for any small $\epsilon > 0$. In Fig. 1 (*Right*), we demonstrate the error decay rates with respect to polynomials degree for Schemes I and II. The N^{-1} decay rate is also shown in order to make a close comparison. A closer look at the two different error curves corresponding to two schemes respectively predicts that the decay rate of the $l^2(J; L^2(\Omega))$-norm is close to N^{-1}, which is consistent with the estimates given in Theorems 6 and 14.

6 Conclusion

We have developed two alternating direction implicit Galerkin-Legendre spectral methods for two-dimensional distributed-order differential equation. Two schemes employ the spectral approximation using Legendre functions in space. For the discretization of the time fractional derivative, we demonstrate two methods. One is the classical $L1$ formula and a special point for the interpolation approximation of the linear combination of the distributed-order fractional derivatives is used in another scheme to achieve at second order accuracy in time. Two proposed schemes have been proved to be unconditionally stable and convergent rigorously. We have presented some numerical experiments to verify the theoretical analysis.

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