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Linesearch methods for bilevel split pseudomonotone variational inequality problems

Tran Viet Anh[1]

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Abstract

In this paper, we propose Linesearch methods for solving a bilevel split variational inequality problem (BSVIP) involving a strongly monotone mapping in the upper-level problem and pseudomonotone mappings in the lower-level one. A strongly convergent algorithm for such a BSVIP is proposed and analyzed.

Keywords Bilevel split variational inequality problem · Linesearch methods · Pseudomonotone mapping · Strong convergence

Mathematics Subject Classification (2010) 47J25 · 47N10 · 90C25

1 Introduction

Let C and Q be two nonempty closed convex subsets of two real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator. Given mappings $F_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ and $F_2 : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$. The split variational inequality problem (in short, SVIP) introduced first by Censor et al. [10] can be formulated as

$$\text{Find } x^* \in C : \langle F_1(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C \tag{1}$$

such that

$$y^* = Ax^* \in Q : \langle F_2(y^*), y - y^* \rangle \geq 0 \quad \forall y \in Q. \tag{2}$$

If the solution sets of variational inequality problems (1) and (2) are denoted by $\mathrm{Sol}(C, F_1)$ and $\mathrm{Sol}(Q, F_2)$, respectively, then the SVIP becomes the problem of finding $x^* \in \mathrm{Sol}(C, F_1)$ such that $Ax^* \in \mathrm{Sol}(Q, F_2)$. If we consider only the problem (1) then (1) is a classical variational inequality problem, which was studied by many authors, for example [2, 4, 14, 17, 19, 21, 29]. A special case of the SVIP, when

✉ Tran Viet Anh
tranvietanh@outlook.com

[1] Department of Scientific Fundamentals, Posts and Telecommunications Institute of Technology, Hanoi, Vietnam

$F_1 = F_2 = 0$, is the split feasibility problem (SFP), which has been studied intensively and used to model the intensity-modulated radiation therapy [11–13, 24] and further development of this topic [5–9, 24, 26].

The SVIP was introduced and investigated by Censor et al. [10] in the case when F_1 and F_2 are inverse strongly monotone mappings. Specifically, they proposed the following iteration method

$$\begin{cases} x^0 \in \mathcal{H}_1 \text{ chosen arbitrarily}, \\ x^{k+1} = P_C^{F_1,\lambda}(x^k + \gamma A^*(P_Q^{F_2,\lambda} - I)(Ax^k)) \quad \forall k \geq 0, \end{cases}$$

where F_1 is α_1-inverse strongly monotone on $\mathcal{H}_1$, F_2 is α_2-inverse strongly monotone on $\mathcal{H}_2$, $\gamma \in \left(0, \dfrac{1}{\|A\|^2}\right)$, $0 \leq \lambda \leq 2\min\{\alpha_1, \alpha_2\}$ and $P_C^{F_1,\lambda}$ and $P_Q^{F_2,\lambda}$ stand for $P_C(I - \lambda F_1)$ and $P_Q(I - \lambda F_2)$, respectively. They proved that the sequence $\{x^k\}$ converges weakly to a solution of the split variational inequality problem, provided that the solution set of the SVIP is nonempty.

In this paper, we suppose that $\Phi : C \longrightarrow \mathcal{H}_1$ is β-strongly monotone and L-Lipschitz continuous on C, $F : C \longrightarrow \mathcal{H}_1$ and $G : Q \longrightarrow \mathcal{H}_2$ be pseudomonotone mappings. Our main purpose is to investigate the following bilevel split variational inequality problem (BSVIP)

$$\text{Find } x^* \in \Omega \text{ such that } \langle \Phi(x^*), x - x^* \rangle \geq 0 \quad \forall x \in \Omega, \qquad (BSVIP)$$

where $\Omega = \{x^* \in \text{Sol}(C, F) : Ax^* \in \text{Sol}(Q, G)\}$. Here, A is a bounded linear operator between $\mathcal{H}_1$ and $\mathcal{H}_2$.

The remaining part of the paper is organized as follows. In Section 2, we collect some basic definitions and preliminary results that are needed. Section 3 deals with the algorithm and its convergence analysis. Finally, in Section 4, we illustrate the proposed algorithm by considering some preliminary computational results and experiments.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. We denote the strong convergence and the weak convergence of a sequence $\{x^k\}$ to x in $\mathcal{H}$ by $x^k \longrightarrow x$ and $x^k \rightharpoonup x$, respectively. By P_C, we denote the metric projection onto C. Namely, for each $x \in \mathcal{H}$, $P_C(x)$ is the unique element in C such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \forall y \in C.$$

Some important properties of the projection operator P_C are gathered in the following lemma.

Lemma 1 ([18])

(i) *For given $x \in \mathcal{H}$ and $y \in C$, $y = P_C(x)$ if and only if*

$$\langle x - y, z - y \rangle \leq 0 \quad \forall z \in C.$$

(ii) P_C *is firmly nonexpansive, that is,*

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle \ \ \forall x, y \in \mathcal{H}.$$

Consequently, P_C *is nonexpansive, i.e.,*

$$\|P_C(x) - P_C(y)\| \leq \|x - y\| \ \ \forall x, y \in \mathcal{H}.$$

(iii) *For all* $x \in \mathcal{H}$ *and* $y \in C$*, we have*

$$\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|P_C(x) - x\|^2.$$

Let us also recall some well-known definitions, which will be used in this paper.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator. The linear operator $A^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ with the property

$$\langle A(x), y\rangle = \langle x, A^*(y)\rangle$$

for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, is called the adjoint operator.

The adjoint operator of a bounded linear operator A on a Hilbert space always exists and is uniquely determined. Futhermore, A^* is a bounded linear operator and $\|A^*\| = \|A\|$.

The following definitions are commonly used in the variational inequality theory

Definition 1 ([15, 20, 23])

A mapping $\phi : C \longrightarrow \mathcal{H}$ is said to be

(i) β-strongly monotone on C if there exists $\beta > 0$ such that

$$\langle \phi(x) - \phi(y), x - y\rangle \geq \beta\|x - y\|^2 \quad \forall x, y \in C;$$

(ii) L-Lipschitz continuous on C if

$$\|\phi(x) - \phi(y)\| \leq L\|x - y\| \, \forall x, y \in C;$$

(iii) monotone on C if

$$\langle \phi(x) - \phi(y), x - y\rangle \geq 0 \quad \forall x, y \in C;$$

(iv) pseudomonotone on C if

$$\langle \phi(y), x - y\rangle \geq 0 \Longrightarrow \langle \phi(x), x - y\rangle \geq 0 \quad \forall x, y \in C.$$

The next lemmas will be used for proving the convergence of the proposed algorithm described below.

Lemma 2 ([22, Remark 4.4]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers. Suppose that for any integer* m*, there exists an integer* p *such that* $p \geq m$ *and* $a_p \leq a_{p+1}$*. Let* n_0 *be an integer such that* $a_{n_0} \leq a_{n_0+1}$ *and define, for all integer* $n \geq n_0$*, by*

$$\tau(n) = \max\{k \in \mathbb{N} : n_0 \leq k \leq n, a_k \leq a_{k+1}\}.$$

Then, $\{\tau(n)\}_{n\geq n_0}$ *is a nondecreasing sequence satisfying* $\lim_{n\longrightarrow\infty}\tau(n)=\infty$ *and the following inequalities hold true:*

$$a_{\tau(n)}\leq a_{\tau(n)+1},\ a_n\leq a_{\tau(n)+1}\ \forall n\geq n_0.$$

Lemma 3 ([27, Lemma 2.5]) *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers satisfying the condition*

$$a_{n+1}\leq(1-\alpha_n)a_n+\alpha_n\xi_n,\quad \forall n\geq 0,$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\xi_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=0}^{\infty}\alpha_n=\infty$*;*
(ii) $\limsup_{n\longrightarrow\infty}\xi_n\leq 0.$

Then, $\lim_{n\longrightarrow\infty}a_n=0.$

3 The algorithm and convergence analysis

In this section, we propose a strong convergence algorithm for solving BSVIP by using the linesearch technique for an equilibrium problem [25]. The linesearch technique has been used widely in descent methods for equilibrium problems as well as for variational inequalities in order to avoid the Lipschitz continuity assumption [3, 16, 17, 20, 25]. We impose the following assumptions on the mappings F and G associated with the problem $(BSVIP)$.

(A_1): $F:C\longrightarrow\mathcal{H}_1$ be pseudomonotone on C.
(A_2): $\lim_{k\longrightarrow\infty}F(x^k)=F(\overline{x})$ for every sequence $\{x^k\}$ converging weakly to $\overline{x}$.
(A_3): $G:Q\longrightarrow\mathcal{H}_2$ be pseudomonotone on Q.
(A_4): $\lim_{k\longrightarrow\infty}G(u^k)=G(\overline{u})$ for every sequence $\{u^k\}$ converging weakly to $\overline{u}$.

Let us make some remarks on the above assumptions.

i) Assumptions $(A_1)-(A_4)$ are widely used in the theory of VIPs.
ii) In finite dimensional spaces, conditions (A_3) and (A_5) become the conditions for the continuity of F_1, F_2.
iii) If F and G satisfy the properties (A_1), (A_2) and (A_3), (A_4) respectively, then by [1, Lemma 6], the solution sets $\mathrm{Sol}(C,F)$ and $\mathrm{Sol}(Q,G)$ of the variational inequalities $\mathrm{VIP}(C,F)$ and $\mathrm{VIP}(Q,G)$ are closed and convex. Therefore, the solution set $\Omega=\{x^*\in\mathrm{Sol}(C,F):Ax^*\in\mathrm{Sol}(Q,G)\}$ of the SVIP is also closed and convex.
iii) If $\{x^k\}\subset C$ is bounded then $\{F(x^k)\}$ is bounded. Indeed, suppose that $\{F(x^k)\}$ is unbounded, that is, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $\lim_{i\longrightarrow\infty}\|F(x^{k_i})\|=+\infty$. Since $\{x^{k_i}\}$ is bounded then there exists a subsequence $\{x^{k_{i_j}}\}$ of $\{x^{k_i}\}$ such that $x^{k_{i_j}}\rightharpoonup\overline{x}$. Therefore, $\lim_{k\longrightarrow\infty}F(x^{k_{i_j}})=F(\overline{x})$.

Thus, $\lim_{k\longrightarrow\infty}\|F(x^{k_{i_j}})\| = \|F(\overline{x})\|$. Since $\lim_{i\longrightarrow\infty}\|F(x^{k_i})\| = +\infty$, we have $\lim_{j\longrightarrow\infty}\|F(x^{k_{i_j}})\| = +\infty$, a contradiction. Therefore, $\{F(x^k)\}$ is bounded.

Our algorithm can be expressed as follows.

The following theorem shows validity and convergence of the algorithm.

Theorem 1 *Suppose that the assumptions* $(A_1) - (A_5)$ *and* $\Omega \neq \emptyset$ *hold. Then, the sequence* $\{x^k\}$ *in Algorithm 1 converges strongly to the unique solution of the bilevel split variational inequality problem* $(BSVIP)$.

Proof Since $\Omega \neq \emptyset$, problem $(BSVIP)$ has a unique solution, denoted by x^*. In particular, $x^* \in \Omega$, i.e., $x^* \in \mathrm{Sol}(C, F) \subset C$, $Ax^* \in \mathrm{Sol}(Q, G) \subset Q$. We will prove that $\{x^k\}$ converges in norm to x^*. We divide the proof into several steps.

Step 1. The linesearchs corresponding to u^k, v^k (Step 3) and $\overline{u}^k, \overline{v}^k$ (Step 6) are well defined.

If $v^k \neq u^k$ and suppose, to get a contradiction, that the following inequality holds for every nonnegative integer n

$$\langle G(w^{k,n}), u^k - v^k\rangle < \frac{1}{2}\|u^k - v^k\|^2,$$

where $w^{k,n} = (1-\gamma^n)u^k + \gamma^n v^k$.
Taking the limit as $n \longrightarrow \infty$, from $w^{k,n} \longrightarrow u^k$ as $n \longrightarrow \infty$, it follows that

$$\langle G(u^k), u^k - v^k\rangle \leq \frac{1}{2}\|u^k - v^k\|^2. \tag{3}$$

Since $v^k = P_Q(u^k - G(u^k))$, we have

$$\langle u^k - G(u^k) - v^k, u - v^k\rangle \leq 0 \quad \forall u \in Q.$$

Choose $u = u^k \in Q$, we get

$$\langle G(u^k), u^k - v^k\rangle \geq \|u^k - v^k\|^2.$$

Combining with (3) yields

$$\|u^k - v^k\|^2 \leq \frac{\|u^k - v^k\|^2}{2}$$

which contradicts to the fact that $u^k \neq v^k$.

Therefore, the linesearch corresponding to u^k and v^k (Step 3) is well defined.

By proving in the same way, we find that the linesearch corresponding to $\overline{u}^k$ and $\overline{v}^k$ (Step 6) is also well defined.

Step 2. (a) If $v^k \neq u^k$ for some $k \geq 0$, then $G(w^k) \neq 0$, $\sigma_k > 0$ and

$$\|t^k - Ax^*\|^2 \leq \|u^k - Ax^*\|^2 - (\sigma_k\|G(w^k)\|)^2.$$

(b) If $\overline{v}^k \neq \overline{u}^k$ for some $k \geq 0$, then $F(\overline{w}^k) \neq 0$, $\overline{\sigma}_k > 0$ and

$$\|y^k - x^*\|^2 \leq \|\overline{u}^k - x^*\|^2 - (\overline{\sigma}_k\|F(\overline{w}^k)\|)^2.$$

Algorithm 1

Step 0. Choose $\eta \in (0,1), \gamma \in (0,1), 0 < \mu < \frac{2\beta}{L^2}, \{\delta_k\} \subset [\underline{\delta}, \overline{\delta}] \subset \left(0, \frac{1}{\|A\|^2+1}\right)$, $\{\lambda_k\} \subset (0,1), \lim_{k\longrightarrow\infty} \lambda_k = \lambda \in (0,1), \{\alpha_k\} \subset (0,1), \lim_{k\longrightarrow\infty} \alpha_k = 0, \sum_{k=0}^{\infty} \alpha_k = \infty.$

Step 1. Let $x^0 \in C$. Set $k := 0$.

Step 2. Compute $u^k = P_Q(Ax^k)$ and

$$v^k = P_Q(u^k - G(u^k)).$$

If $v^k = u^k$, then set $t^k = u^k$ and go to **Step 5**. Otherwise, go to **Step 3**.

Step 3 Find n_k as the smallest nonnegative integer n such that

$$w^{k,n} = (1-\gamma^n)u^k + \gamma^n v^k,$$
$$\langle G(w^{k,n}), u^k - v^k\rangle \geq \frac{1}{2}\|u^k - v^k\|^2.$$

Set $\gamma_k = \gamma^{n_k}$, $w^k = w^{k,n_k}$.

Step 4 Compute

$$t^k = P_Q(u^k - \sigma_k G(w^k))$$

where

$$\sigma_k = \frac{\langle G(w^k), u^k - w^k\rangle}{\|G(w^k)\|^2}.$$

Step 5 Compute

$$\overline{u}^k = P_C(x^k + \delta_k A^*(t^k - Ax^k))$$

and

$$\overline{v}^k = P_C(\overline{u}^k - F(\overline{u}^k)).$$

If $\overline{v}^k = \overline{u}^k$, then set $y^k = \overline{u}^k$ and go to **Step 8**. Otherwise, go to **Step 6**.

Step 6 Find m_k as the smallest nonnegative integer m such that

$$\overline{w}^{k,m} = (1-\eta^m)\overline{u}^k + \eta^m \overline{v}^k,$$
$$\langle F(\overline{w}^{k,m}), \overline{u}^k - \overline{v}^k\rangle \geq \frac{1}{2}\|\overline{u}^k - \overline{v}^k\|^2.$$

Set $\eta_k = \eta^{m_k}$, $\overline{w}^k = \overline{w}^{k,m_k}$.

Step 7 Compute

$$y^k = P_C(\overline{u}^k - \overline{\sigma}_k F(\overline{w}^k))$$

where

$$\overline{\sigma}_k = \frac{\langle F(\overline{w}^k), \overline{u}^k - \overline{w}^k\rangle}{\|F(\overline{w}^k)\|^2}.$$

Step 8 Compute $z^k = (1-\lambda_k)\overline{u}^k + \lambda_k y^k$ and $x^{k+1} = P_C(z^k - \alpha_k \mu \Phi(z^k))$.

Step 9 Set $k := k+1$, and go to **Step 2**.

Indeed, since $v^k \neq u^k$ then

$$\langle G(w^k), u^k - w^k\rangle = \gamma^n \langle G(w^k), u^k - v^k\rangle \geq \frac{\gamma^n \|u^k - v^k\|^2}{2} > 0,$$

which implies

$$G(w^k) \neq 0, \quad \sigma_k = \frac{\langle G(w^k), u^k - w^k \rangle}{\|G(w^k)\|^2} > 0.$$

From $Ax^* \in \mathrm{Sol}(Q, G)$ and $w^k \in Q$, we have $\langle G(Ax^*), w^k - Ax^* \rangle \geq 0$. Using the pseudomonotonicity of G, we get

$$\langle G(w^k), w^k - Ax^* \rangle \geq 0. \tag{4}$$

It follows from (4) that

$$\begin{aligned}
\|t^k - Ax^*\|^2 &= \|P_Q(u^k - \sigma_k G(w^k)) - P_Q(Ax^*)\|^2 \\
&\leq \|u^k - \sigma_k G(w^k) - Ax^*\|^2 \\
&= \|u^k - Ax^*\|^2 + \sigma_k^2 \|G(w^k)\|^2 - 2\sigma_k \langle G(w^k), u^k - Ax^* \rangle \\
&\leq \|u^k - Ax^*\|^2 + \sigma_k^2 \|G(w^k)\|^2 - 2\sigma_k \langle G(w^k), u^k - w^k \rangle \\
&= \|u^k - Ax^*\|^2 + \sigma_k^2 \|G(w^k)\|^2 - 2\sigma_k^2 \|G(w^k)\|^2 \\
&= \|u^k - Ax^*\|^2 - (\sigma_k \|G(w^k)\|)^2.
\end{aligned}$$

By using the same argument as in the proof of Step 2 (a), we get Step 2 (b).

Step 3. For all $k \geq 0$, we have

$$\|\overline{u}^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \delta_k(1 - \delta_k \|A\|^2)\|t^k - Ax^k\|^2 - \delta_k \|u^k - Ax^k\|^2.$$

From Step 2 (a) and the fact that $t^k = u^k$ if $v^k = u^k$, we have

$$\|t^k - Ax^*\| \leq \|u^k - Ax^*\|. \tag{5}$$

From the property of the projection mapping (Lemma 1 (iii)), we get

$$\begin{aligned}
\|u^k - Ax^*\|^2 &= \|P_Q(Ax^k) - Ax^*\|^2 \\
&\leq \|Ax^k - Ax^*\|^2 - \|P_Q(Ax^k) - Ax^k\|^2 \\
&= \|Ax^k - Ax^*\|^2 - \|u^k - Ax^k\|^2.
\end{aligned} \tag{6}$$

It follows from (5) and (6) that

$$\|t^k - Ax^*\|^2 - \|Ax^k - Ax^*\|^2 \leq -\|u^k - Ax^k\|^2. \tag{7}$$

Then, from (7), it follows that

$$\begin{aligned}
\langle A(x^k - x^*), t^k - Ax^k \rangle &= \langle t^k - Ax^*, t^k - Ax^k \rangle - \|t^k - Ax^k\|^2 \\
&= \frac{1}{2}\Big[(\|t^k - Ax^*\|^2 - \|Ax^k - Ax^*\|^2) - \|t^k - Ax^k\|^2\Big] \\
&\leq \frac{1}{2}(-\|u^k - Ax^k\|^2 - \|t^k - Ax^k\|^2).
\end{aligned}$$

Since $\delta_k > 0$, from the above inequality, we have

$$2\delta_k \langle A(x^k - x^*), t^k - Ax^k \rangle \leq -\delta_k \|u^k - Ax^k\|^2 - \delta_k \|t^k - Ax^k\|^2. \tag{8}$$

Using the nonexpansiveness of P_C and (8), we get

$$\begin{aligned}\|\overline{u}^k - x^*\|^2 &= \|P_C(x^k + \delta_k A^*(t^k - Ax^k)) - P_C(x^*)\|^2\\ &\le \|(x^k - x^*) + \delta_k A^*(t^k - Ax^k)\|^2\\ &= \|x^k - x^*\|^2 + \delta_k^2\|A^*(t^k - Ax^k)\|^2 + 2\delta_k\langle x^k - x^*, A^*(t^k - Ax^k)\rangle\\ &\le \|x^k - x^*\|^2 + \delta_k^2\|A^*\|^2\|t^k - Ax^k\|^2 + 2\delta_k\langle A(x^k - x^*), t^k - Ax^k\rangle\\ &\le \|x^k - x^*\|^2 + \delta_k^2\|A\|^2\|t^k - Ax^k\|^2 - \delta_k\|u^k - Ax^k\|^2 - \delta_k\|t^k - Ax^k\|^2\\ &= \|x^k - x^*\|^2 - \delta_k(1 - \delta_k\|A\|^2)\|t^k - Ax^k\|^2 - \delta_k\|u^k - Ax^k\|^2.\end{aligned}$$

Step 4. For all $k \ge 0$, we have

$$\|z^k - x^*\|^2 \le \|\overline{u}^k - x^*\|^2 - \lambda_k(1 - \lambda_k)\|y^k - \overline{u}^k\|^2.$$

Indeed, from $y^k = \overline{u}^k$ if $\overline{v}^k = \overline{u}^k$ and Step 2 (b), we have

$$\|y^k - x^*\| \le \|\overline{u}^k - x^*\|. \tag{9}$$

From (9), we get

$$\begin{aligned}\|z^k - x^*\|^2 &= \|(1 - \lambda_k)\overline{u}^k + \lambda_k y^k - x^*\|^2\\ &= \|(1 - \lambda_k)(\overline{u}^k - x^*) + \lambda_k(y^k - x^*)\|^2\\ &= (1 - \lambda_k)\|\overline{u}^k - x^*\|^2 + \lambda_k\|y^k - x^*\|^2 - \lambda_k(1 - \lambda_k)\|y^k - \overline{u}^k\|^2\\ &\le (1 - \lambda_k)\|\overline{u}^k - x^*\|^2 + \lambda_k\|\overline{u}^k - x^*\|^2 - \lambda_k(1 - \lambda_k)\|y^k - \overline{u}^k\|^2\\ &= \|\overline{u}^k - x^*\|^2 - \lambda_k(1 - \lambda_k)\|y^k - \overline{u}^k\|^2.\end{aligned}$$

Step 5. For all $k \ge 0$, we have

$$\|z^k - \alpha_k\mu\Phi(z^k) - (x^* - \alpha_k\mu\Phi(x^*))\| \le (1 - \alpha_k\tau)\|z^k - x^*\|,$$

where

$$\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0, 1].$$

It is clear that

$$\begin{aligned}\|z^k - \alpha_k\mu\Phi(z^k) - (x^* - \alpha_k\mu\Phi(x^*))\| &= \|(1 - \alpha_k)(z^k - x^*)\\ &\quad +\alpha_k[z^k - x^* - \mu(\Phi(z^k) - \Phi(x^*))]\|\\ &\le (1 - \alpha_k)\|z^k - x^*\| + \alpha_k\|z^k\\ &\quad -x^* - \mu(\Phi(z^k) - \Phi(x^*))\|.\end{aligned} \tag{10}$$

Using β-strongly monotonicity and L-Lipschitz continuity on C of Φ, we get

$$\begin{aligned}\|z^k - x^* - \mu(\Phi(z^k) - \Phi(x^*))\|^2 &= \|z^k - x^*\|^2 - 2\mu\langle z^k - x^*, \Phi(z^k) - \Phi(x^*)\rangle\\ &\quad +\mu^2\|\Phi(z^k) - \Phi(x^*)\|^2\\ &\le \|z^k - x^*\|^2 - 2\mu\beta\|z^k - x^*\|^2 + \mu^2L^2\|z^k - x^*\|^2\\ &= (1 - 2\mu\beta + \mu^2L^2)\|z^k - x^*\|^2.\end{aligned} \tag{11}$$

Combining (10) and (11), we obtain

$$\begin{aligned}\|z^k - \alpha_k\mu\Phi(z^k) - (x^* - \alpha_k\mu\Phi(x^*))\| &\le (1-\alpha_k)\|z^k - x^*\| + \alpha_k\sqrt{1-\mu(2\beta - \mu L^2)}\|z^k - x^*\| \\ &= (1-\alpha_k\tau)\|z^k - x^*\|.\end{aligned}$$

Step 6. We show that the sequence $\{x^k\}$, $\{\overline{u}^k\}$, $\{z^k\}$ and $\{\Phi(z^k)\}$ are bounded. Since $\{\delta_k\} \subset [\underline{\delta}, \overline{\delta}] \subset \left(0, \dfrac{1}{\|A\|^2+1}\right)$ and $\lambda_k \in (0,1)$, by combining these with Step 3 and Step 4, we obtain

$$\|z^k - x^*\| \le \|\overline{u}^k - x^*\| \le \|x^k - x^*\|. \tag{12}$$

Using the nonexpansiveness property of P_C, Step 5 and (12), we find

$$\begin{aligned}\|x^{k+1} - x^*\| &= \|P_C(z^k - \alpha_k\mu\Phi(z^k)) - P_C(x^*)\| \\ &\le \|z^k - \alpha_k\mu\Phi(z^k) - x^*\| \\ &= \|z^k - \alpha_k\mu\Phi(z^k) - (x^* - \alpha_k\mu\Phi(x^*)) - \alpha_k\mu\Phi(x^*)\| \\ &\le \|z^k - \alpha_k\mu\Phi(z^k) - (x^* - \alpha_k\mu\Phi(x^*))\| + \alpha_k\mu\|\Phi(x^*)\| \\ &\le (1-\alpha_k\tau)\|z^k - x^*\| + \alpha_k\mu\|\Phi(x^*)\| \\ &\le (1-\alpha_k\tau)\|x^k - x^*\| + \alpha_k\mu\|\Phi(x^*)\|. \end{aligned}\tag{13}$$

So, by induction, we obtain, for every $k \ge 0$, that

$$\|x^k - x^*\| \le \max\left\{\|x^0 - x^*\|, \frac{\mu\|\Phi(x^*)\|}{\tau}\right\}.$$

Hence, the sequence $\{x^k\}$ is bounded and so are the sequences $\{\overline{u}^k\}$, $\{z^k\}$ and $\{\Phi(z^k)\}$ thank to (12) and the Lipschitz continuity of Φ.

Step 7. For all $k \ge 0$, we have

$$\|x^{k+1} - x^*\|^2 \le (1-\alpha_k\tau)\|z^k - x^*\|^2 + 2\alpha_k\mu\langle\Phi(x^*), x^* - z^k + \alpha_k\mu\Phi(z^k)\rangle.$$

Using the inequality

$$\|x-y\|^2 \le \|x\|^2 - 2\langle y, x-y\rangle \quad \forall x, y \in \mathcal{H}_1,$$

and Step 5, we obtain successively

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &= \|P_C(z^k - \alpha_k\mu\Phi(z^k)) - P_C(x^*)\|^2 \\ &\le \|z^k - \alpha_k\mu\Phi(z^k) - x^*\|^2 \\ &= \|z^k - \alpha_k\mu\Phi(z^k) - (x^* - \alpha_k\mu\Phi(x^*)) - \alpha_k\mu\Phi(x^*)\|^2 \\ &\le \|z^k - \alpha_k\mu\Phi(z^k) - (x^* - \alpha_k\mu\Phi(x^*))\|^2 - 2\alpha_k\mu\langle\Phi(x^*), z^k - \alpha_k\mu\Phi(z^k) - x^*\rangle \\ &\le (1-\alpha_k\tau)^2\|z^k - x^*\|^2 + 2\alpha_k\mu\langle\Phi(x^*), x^* - z^k + \alpha_k\mu\Phi(z^k)\rangle \\ &\le (1-\alpha_k\tau)\|z^k - x^*\|^2 + 2\alpha_k\mu\langle\Phi(x^*), x^* - z^k + \alpha_k\mu\Phi(z^k)\rangle.\end{aligned}$$

Step 8. (a) Suppose that $\{u^{k_i}\}$ is a subsequence of $\{u^k\}$ converging weakly to some $\overline{u}$ and $\|t^{k_i} - u^{k_i}\| \longrightarrow 0$ as $i \longrightarrow \infty$ then $\overline{u} \in \mathrm{Sol}(Q, G)$.

(b) Suppose that $\{\overline{u}^{k_i}\}$ is a subsequence of $\{\overline{u}^k\}$ converging weakly to some $\widetilde{u}$ and $\|y^{k_i} - \overline{u}^{k_i}\| \longrightarrow 0$ as $i \longrightarrow \infty$ then $\widetilde{u} \in \mathrm{Sol}(C, F)$.

First, we see that since $\{u^k\} \subset Q$, $u^{k_i} \rightharpoonup \overline{u}$ and Q is closed and convex, it is also weakly closed, and thus $\overline{u} \in Q$. Since $u^{k_i} \rightharpoonup \overline{u}$, we obtain $\{u^{k_i}\}$ is bounded. From (5), we get $\{t^{k_i}\}$ is also bounded.

Case A. Suppose that there exists a subsequence of $\{u^{k_i}\}$, denoted again by $\{u^{k_i}\}$ such that $t^{k_i} = u^{k_i}$ for all i. If $v^{k_i} \neq u^{k_i}$ then from Step 2, we have $\|t^{k_i} - Ax^*\| < \|u^{k_i} - Ax^*\|$. This contradicts the fact that $t^{k_i} = u^{k_i}$. Thus, $v^{k_i} = u^{k_i}$ or $P_Q(u^{k_i} - G(u^{k_i})) = u^{k_i}$ for all i. Then, by Lemma 1, we have

$$\langle G(u^{k_i}), v - u^{k_i}\rangle \geq 0 \ \ \forall v \in Q. \tag{14}$$

From the weak convergence of the sequence $\{u^{k_i}\}$ to $\overline{u}$, we get $\lim_{i\longrightarrow\infty} G(u^{k_i}) = G(\overline{u})$. Thus, from (14), we have

$$\langle G(\overline{u}), v - \overline{u}\rangle \geq 0 \ \ \forall v \in Q,$$

i.e. $\overline{u} \in \mathrm{Sol}(Q, G)$.

Case B. Suppose that there exists a subsequence of $\{u^{k_i}\}$, denoted again by $\{u^{k_i}\}$ such that $t^{k_i} \neq u^{k_i}$ for all i. Let $\{n_i\}$ be the sequence of the smallest nonnegative integers such that, for all i

$$\langle G((1-\gamma^{n_i})u^{k_i} + \gamma^{n_i} v^{k_i}), u^{k_i} - v^{k_i}\rangle \geq \frac{\|u^{k_i} - v^{k_i}\|^2}{2},$$

where $v^{k_i} = P_Q(u^{k_i} - G(u^{k_i}))$, $w^{k_i} = (1-\gamma_{k_i})u^{k_i} + \gamma_{k_i} v^{k_i}$, $\gamma_{k_i} = \gamma^{n_i}$.

From Step 2 (a), we get

$$\|t^{k_i} - Ax^*\|^2 \leq \|u^{k_i} - Ax^*\|^2 - (\sigma_{k_i}\|G(w^{k_i})\|)^2,$$

where

$$\sigma_{k_i} = \frac{\langle G(w^{k_i}), u^{k_i} - w^{k_i}\rangle}{\|G(w^{k_i})\|^2}.$$

Therefore,

$$\begin{aligned}(\sigma_{k_i}\|G(w^{k_i})\|)^2 &\leq \|u^{k_i} - Ax^*\|^2 - \|t^{k_i} - Ax^*\|^2\\ &\leq (\|u^{k_i} - Ax^*\| + \|t^{k_i} - Ax^*\|)\|u^{k_i} - t^{k_i}\|.\end{aligned}$$

This inequality together with the boundedness of two sequences $\{u^{k_i}\}$, $\{t^{k_i}\}$ and $\|t^{k_i} - u^{k_i}\| \longrightarrow 0$ imply

$$\lim_{i\longrightarrow\infty} \sigma_{k_i}\|G(w^{k_i})\| = 0. \tag{15}$$

Since $\{u^{k_i}\} \subset Q$, then

$$\begin{aligned}\|v^{k_i} - u^{k_i}\| &= \|P_Q(u^{k_i} - G(u^{k_i})) - P_Q(u^{k_i})\|\\ &\leq \|G(u^{k_i})\|.\end{aligned}$$

The above inequality together with the boundedness of $\{u^{k_i}\}$ and $\{G(u^{k_i})\}$ imply that $\{v^{k_i}\}$ is bounded. We also imply that $\{w^{k_i}\}$ is bounded. Therefore, $\{G(w^{k_i})\}$ is bounded. So from (15) and $\langle G(w^{k_i}), u^{k_i} - w^{k_i}\rangle = \sigma_{k_i}\|G(w^{k_i})\|^2$, we get

$$\lim_{i\longrightarrow\infty} \langle G(w^{k_i}), u^{k_i} - w^{k_i}\rangle = 0. \tag{16}$$

Since $w^{k_i} = (1-\gamma_{k_i})u^{k_i} + \gamma_{k_i} v^{k_i}$, we have

$$\langle G(w^{k_i}), u^{k_i} - w^{k_i}\rangle = \gamma_{k_i}\langle G(w^{k_i}), u^{k_i} - v^{k_i}\rangle \geq \frac{\gamma_{k_i}\|u^{k_i} - v^{k_i}\|^2}{2}. \tag{17}$$

From (16) and (17), we have

$$\lim_{i\longrightarrow\infty} \gamma_{k_i}\|u^{k_i} - v^{k_i}\|^2 = 0. \tag{18}$$

From

$$v^{k_i} = P_Q(u^{k_i} - G(u^{k_i})),$$

we have

$$\langle u^{k_i} - G(u^{k_i}) - v^{k_i}, v - v^{k_i}\rangle \leq 0 \ \ \forall v \in Q.$$

Therefore

$$\langle G(u^{k_i}), v - v^{k_i}\rangle \geq \langle u^{k_i} - v^{k_i}, v - v^{k_i}\rangle \ \ \forall v \in Q. \tag{19}$$

We now consider two distinct cases:

Case $B.1$. $\limsup_{i\longrightarrow\infty} \gamma_{k_i} > 0$. In this case, there exist $\overline{\gamma}$ and a subsequence of $\{\gamma_{k_i}\}$, denoted again by $\{\gamma_{k_i}\}$ such that $\gamma_{k_i} \longrightarrow \overline{\gamma}$. Then, by (18), we obtain that $\lim_{i\longrightarrow\infty} \|u^{k_i} - v^{k_i}\| = 0$. Since $u^{k_i} \rightharpoonup \overline{u}$, then $v^{k_i} \rightharpoonup \overline{u}$.
Applying the Cauchy-Schwarz inequality, we get

$$|\langle u^{k_i} - v^{k_i}, v - v^{k_i}\rangle| \leq \|u^{k_i} - v^{k_i}\|.\|v - v^{k_i}\|. \tag{20}$$

Since $\|u^{k_i} - v^{k_i}\| \longrightarrow 0$ and the sequence $\{v^{k_i}\}$ is bounded, from (20), it ensures that $\lim_{i\longrightarrow\infty} \langle u^{k_i} - v^{k_i}, v - v^{k_i}\rangle = 0$. So, using (19), the weak convergence of two sequences $\{u^{k_i}\}$, $\{v^{k_i}\}$ to $\overline{u}$, we get

$$\langle G(\overline{u}), v - \overline{u}\rangle \geq 0 \ \ \forall v \in Q,$$

i.e. $\overline{u} \in \mathrm{Sol}(Q, G)$.
Case $B.2$. $\lim_{i\longrightarrow\infty} \gamma_{k_i} = 0$. From the boundedness of $\{v^{k_i}\}$, without loss of generality, we may assume that $v^{k_i} \rightharpoonup \overline{v}$ as $i \longrightarrow \infty$. Since $\gamma^{m_i} = \gamma_{k_i} \longrightarrow 0$ as $i \longrightarrow \infty$, it follows that $m_i > 1$ for i sufficiently large and consequently, that

$$\langle G(s^{k_i}), u^{k_i} - v^{k_i}\rangle < \frac{\|u^{k_i} - v^{k_i}\|^2}{2}, \tag{21}$$

where

$$s^{k_i} = (1-\gamma^{n_i-1})u^{k_i} + \gamma^{n_i-1}v^{k_i}.$$

Choose $v = u^{k_i}$ in (19), we have

$$\langle G(u^{k_i}), u^{k_i} - v^{k_i}\rangle \geq \|u^{k_i} - v^{k_i}\|^2. \tag{22}$$

From (21) and (22), we have

$$\langle G(s^{k_i}), u^{k_i} - v^{k_i}\rangle < \frac{1}{2}\langle G(u^{k_i}), u^{k_i} - v^{k_i}\rangle, \tag{23}$$

Since $\{u^{k_i}\}$ and $\{v^{k_i}\}$ are bounded and $\gamma_{k_i} \longrightarrow 0$, then

$$\|s^{k_i} - u^{k_i}\| = \frac{\gamma_{k_i}}{\gamma}\|u^{k_i} - v^{k_i}\| \longrightarrow 0.$$

From $u^{k_i} \rightharpoonup \overline{u}$ and $\|s^{k_i} - u^{k_i}\| \longrightarrow 0$, then we have $s^{k_i} \rightharpoonup \overline{u}$. Since $v^{k_i} \rightharpoonup \overline{v}$, then from (23), we have

$$\langle G(\overline{u}), \overline{u} - \overline{v}\rangle \leq \frac{1}{2}\langle G(\overline{u}), \overline{u} - \overline{v}\rangle.$$

Thus,

$$\langle G(\overline{u}), \overline{u} - \overline{v}\rangle \leq 0. \tag{24}$$

From (22), we have

$$\langle G(u^{k_i}), u^{k_i} - v^{k_i}\rangle \geq 0. \tag{25}$$

Since $u^{k_i} \rightharpoonup \overline{u}$, $v^{k_i} \rightharpoonup \overline{v}$, from (25), we have

$$\langle G(\overline{u}), \overline{u} - \overline{v}\rangle \geq 0.$$

Combine with (24), we have

$$\langle G(\overline{u}), \overline{u} - \overline{v}\rangle = 0. \tag{26}$$

From (26), we get

$$\lim_{i \longrightarrow \infty} \langle G(u^{k_i}), u^{k_i} - v^{k_i}\rangle = \langle G(\overline{u}), \overline{u} - \overline{v}\rangle = 0.$$

Thus from (22), we get

$$\lim_{i \longrightarrow \infty} \|u^{k_i} - v^{k_i}\| = 0$$

Since $u^{k_i} \rightharpoonup \overline{u}$, then $v^{k_i} \rightharpoonup \overline{u}$. From $\|u^{k_i} - v^{k_i}\| \longrightarrow 0$ and the boundedness of sequence $\{v^{k_i}\}$, we get $\lim_{i \longrightarrow \infty} \langle u^{k_i} - v^{k_i}, v - v^{k_i}\rangle = 0$. So, using (19) and the weak convergence of two sequences $\{u^{k_i}\}$, $\{v^{k_i}\}$ to $\overline{u}$, we get

$$\langle G(\overline{u}), v - \overline{u}\rangle \geq 0 \ \ \forall v \in Q.$$

i.e. $\overline{u} \in \mathrm{Sol}(Q, G)$.

By using the same argument as in the proof of Step 8 (a), we get Step 8 (b).

Step 9. We prove that $\{x^k\}$ converges strongly to the unique solution x^* of the problem $BSVIP$.

Let us consider two cases.

Case 1: There exists k_0 such that the sequence $\{\|x^k - x^*\|\}$ is decreasing for $k \geq k_0$. In this case the limit of $\{\|x^k - x^*\|\}$ exists. So, it follows from Step 7 and (12) that

$$\begin{aligned}
&(\|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2) - 2\alpha_k\mu\langle\Phi(x^*), x^* - z^k + \alpha_k\mu\Phi(z^k)\rangle \\
&\leq \|z^k - x^*\|^2 - \|x^k - x^*\|^2 \\
&\leq \|\overline{u}^k - x^*\|^2 - \|x^k - x^*\|^2 \\
&\leq 0.
\end{aligned} \tag{27}$$

Since the limit of $\{\|x^k - x^*\|\}$ exists, $\lim_{k\longrightarrow\infty} \alpha_k = 0$ and $\{z^k\}$, $\{\Phi(z^k)\}$ are bounded, it follows from (27) that

$$\lim_{k\longrightarrow\infty} (\|z^k - x^*\|^2 - \|x^k - x^*\|^2) = 0, \tag{28}$$

$$\lim_{k\longrightarrow\infty} (\|\overline{u}^k - x^*\|^2 - \|x^k - x^*\|^2) = 0. \tag{29}$$

Thus, from (28) and (29), we conclude that

$$\lim_{k\longrightarrow\infty} (\|\overline{u}^k - x^*\|^2 - \|z^k - x^*\|^2) = 0. \tag{30}$$

From $\{\delta_k\} \subset [\underline{\delta}, \overline{\delta}] \subset \left(0, \frac{1}{\|A\|^2+1}\right)$ and Step 3, we obtain

$$\underline{\delta}(1 - \overline{\delta}\|A\|^2)\|t^k - Ax^k\|^2 + \underline{\delta}\|u^k - Ax^k\|^2 \leq \|x^k - x^*\|^2 - \|\overline{u}^k - x^*\|^2. \tag{31}$$

From (29) and (31), it follows that

$$\lim_{k\longrightarrow\infty} \|t^k - Ax^k\| = 0, \quad \lim_{k\longrightarrow\infty} \|u^k - Ax^k\| = 0. \tag{32}$$

From (32), we have

$$\lim_{k\longrightarrow\infty} \|t^k - u^k\| = 0. \tag{33}$$

Since the projection operator P_C is nonexpansive and $\{x^k\} \subset C$, we can write

$$\begin{aligned}
\|x^k - \overline{u}^k\| &= \|P_C(x^k) - P_C(x^k + \delta_k A^*(t^k - Ax^k))\| \\
&\leq \|x^k - x^k - \delta_k A^*(t^k - Ax^k)\| \\
&= \|\delta_k A^*(t^k - Ax^k)\| \\
&\leq \delta_k \|A^*\| \|t^k - Ax^k\| \\
&\leq \overline{\delta}\|A\| \|t^k - Ax^k\|.
\end{aligned}$$

It follows from the above inequality and $\lim_{k\longrightarrow\infty} \|t^k - Ax^k\| = 0$ that

$$\lim_{k\longrightarrow\infty} \|x^k - \overline{u}^k\| = 0. \tag{34}$$

From Step 4 and (30) and $\lim_{k\longrightarrow\infty} \lambda_k = \lambda \in (0, 1)$, we obtain

$$\lim_{k\longrightarrow\infty} \|y^k - \overline{u}^k\| = 0. \tag{35}$$

Note that, for any $k \geq 0$,

$$\begin{aligned}
\|z^k - \overline{u}^k\| &= \lambda_k \|y^k - \overline{u}^k\| \\
&\leq \|y^k - \overline{u}^k\|.
\end{aligned}$$

Taking into account the last inequality together with (35), we have

$$\lim_{k\longrightarrow\infty} \|z^k - \overline{u}^k\| = 0. \tag{36}$$

Note that

$$\|x^k - z^k\| \leq \|x^k - \overline{u}^k\| + \|\overline{u}^k - z^k\| \quad \forall k,$$

which together with (34) and (36) implies that

$$\lim_{k\longrightarrow\infty} \|x^k - z^k\| = 0. \tag{37}$$

Take a subsequence $\{z^{k_i}\}$ of $\{z^k\}$ such that

$$\limsup_{k\longrightarrow\infty} \langle \Phi(x^*), x^* - z^k\rangle = \lim_{i\longrightarrow\infty} \langle \Phi(x^*), x^* - z^{k_i}\rangle.$$

Since $\{z^{k_i}\}$ ce that z^{k_i} converges weakly to some $\overline{z} \in \mathcal{H}_1$.
Therefore,

$$\begin{aligned}\limsup_{k\longrightarrow\infty} \langle \Phi(x^*), x^* - z^k\rangle &= \lim_{i\longrightarrow\infty} \langle \Phi(x^*), x^* - z^{k_i}\rangle \\ &= \langle \Phi(x^*), x^* - \overline{z}\rangle. \end{aligned} \tag{38}$$

From (36), (37) and $z^{k_i} \rightharpoonup \overline{z}$, we conclude that $\overline{u}^{k_i}$ and x^{k_i} converge weakly to $\overline{z}$. It follows from (35) that $\lim_{i\longrightarrow\infty} \|y^{k_i} - \overline{u}^{k_i}\| = 0$. So, using $\overline{u}^{k_i} \rightharpoonup \overline{z}$ and Step 8 (b), we get $\overline{z} \in \mathrm{Sol}(C, F)$.

Next, we prove that $A\overline{z} \in \mathrm{Sol}(Q, G)$.

From $x^{k_i} \rightharpoonup \overline{z}$, we get $Ax^{k_i} \rightharpoonup A\overline{z}$. This together with (32) implies that $u^{k_i} \rightharpoonup A\overline{z}$. From (33), we obtain $\lim_{i\longrightarrow\infty} \|t^{k_i} - u^{k_i}\| = 0$. Thus, using the weak convergence of sequence $\{u^{k_i}\}$ to $A\overline{z}$ and Step 8 (a), we get $A\overline{z} \in \mathrm{Sol}(Q, G)$.
From $\overline{z} \in \mathrm{Sol}(C, F)$ and $A\overline{z} \in \mathrm{Sol}(Q, G)$, we have $\overline{z} \in \Omega$. Since x^* is the solution of Problem $(BSVIP)$, we have $\langle \Phi(x^*), \overline{z} - x^*\rangle \geq 0$. So, from (38), we get

$$\limsup_{k\longrightarrow\infty} \langle \Phi(x^*), x^* - z^k\rangle \leq 0.$$

From (12) and Step 7, we obtain

$$\|x^{k+1} - x^*\|^2 \leq (1 - \alpha_k\tau)\|x^k - x^*\|^2 + \alpha_k\tau\xi_k,$$

where

$$\xi_k = \frac{2\mu\langle \Phi(x^*), x^* - z^k + \alpha_k\mu\Phi(z^k)\rangle}{\tau}.$$

Using $\lim_{k\longrightarrow\infty} \alpha_k = 0$, the boundedness of $\{\Phi(z^k)\}$ and $\limsup_{k\longrightarrow\infty} \langle \Phi(x^*), x^* - z^k\rangle \leq 0$, we get

$$\limsup_{k\longrightarrow\infty} \xi_k \leq 0.$$

By Lemma 3, we have $\lim_{k\longrightarrow\infty} \|x^k - x^*\|^2 = 0$, i.e., $x^k \longrightarrow x^*$ as $k \longrightarrow \infty$.

Case 2: Suppose that for any integer m, there exists an integer k such that $k \geq m$ and $\|x^k - x^*\| \leq \|x^{k+1} - x^*\|$. According to Lemma 2, there exists a nondecreasing sequence $\{\tau(k)\}$ of $\mathbb{N}$ such that $\lim_{k\longrightarrow\infty} \tau(k) = \infty$ and the following inequalities hold for all (sufficiently large) $k \in \mathbb{N}$.

$$\|x^{\tau(k)} - x^*\| \leq \|x^{\tau(k)+1} - x^*\|, \quad \|x^k - x^*\| \leq \|x^{\tau(k)+1} - x^*\|. \tag{39}$$

From (13), we get

$$\begin{aligned}\|x^{\tau(k)} - x^*\| &\le \|x^{\tau(k)+1} - x^*\| \\ &\le (1 - \alpha_{\tau(k)}\tau)\|z^{\tau(k)} - x^*\| + \alpha_{\tau(k)}\mu\|\Phi(x^*)\|. \end{aligned} \tag{40}$$

From (40) and (12), we obtain

$$\begin{aligned}\alpha_{\tau(k)}\tau\|z^{\tau(k)} - x^*\| - \alpha_{\tau(k)}\mu\|\Phi(x^*)\| &\le \|z^{\tau(k)} - x^*\| - \|x^{\tau(k)} - x^*\| \\ &\le \|\overline{u}^{\tau(k)} - x^*\| - \|x^{\tau(k)} - x^*\| \\ &\le 0.\end{aligned}$$

Then, it follows from the boundedness of $\{z^k\}$ and $\lim_{k\longrightarrow\infty} \alpha_k = 0$ that

$$\begin{aligned}&\lim_{k\longrightarrow\infty} (\|z^{\tau(k)} - x^*\| - \|x^{\tau(k)} - x^*\|) = 0, \\ &\lim_{k\longrightarrow\infty} (\|\overline{u}^{\tau(k)} - x^*\| - \|x^{\tau(k)} - x^*\|) = 0.\end{aligned} \tag{41}$$

From (41) and the boundedness of $\{x^k\}$, $\{\overline{u}^k\}$, $\{z^k\}$, we obtain

$$\lim_{k\longrightarrow\infty} (\|z^{\tau(k)} - x^*\|^2 - \|x^{\tau(k)} - x^*\|^2) = 0,$$

$$\lim_{k\longrightarrow\infty} (\|\overline{u}^{\tau(k)} - x^*\|^2 - \|x^{\tau(k)} - x^*\|^2) = 0.$$

As proved in the first case, we obtain

$$\limsup_{k\longrightarrow\infty}\langle\Phi(x^*), x^* - z^{\tau(k)}\rangle \le 0.$$

Then, the boundedness of $\{\Phi(z^k)\}$ and $\lim_{k\longrightarrow\infty} \alpha_k = 0$ yield

$$\begin{aligned}&\limsup_{k\longrightarrow\infty}\langle\Phi(x^*), x^* - z^{\tau(k)} + \alpha_{\tau(k)}\mu\Phi(z^{\tau(k)})\rangle \\ &= \limsup_{k\longrightarrow\infty}[\langle\Phi(x^*), x^* - z^{\tau(k)}\rangle + \alpha_{\tau(k)}\mu\langle\Phi(x^*), \Phi(z^{\tau(k)})\rangle] \\ &= \limsup_{k\longrightarrow\infty}\langle\Phi(x^*), x^* - z^{\tau(k)}\rangle \\ &\le 0.\end{aligned} \tag{42}$$

From (12), Step 7 and (39), we get

$$\begin{aligned}\|x^{\tau(k)+1} - x^*\|^2 &\le (1 - \alpha_{\tau(k)}\tau)\|x^{\tau(k)} - x^*\|^2 + 2\alpha_{\tau(k)}\mu\langle\Phi(x^*), x^* - z^{\tau(k)} + \alpha_{\tau(k)}\mu\Phi(z^{\tau(k)})\rangle \\ &\le (1 - \alpha_{\tau(k)}\tau)\|x^{\tau(k)+1} - x^*\|^2 + 2\alpha_{\tau(k)}\mu\langle\Phi(x^*), x^* - z^{\tau(k)} + \alpha_{\tau(k)}\mu\Phi(z^{\tau(k)})\rangle.\end{aligned}$$

In particular, since $\alpha_{\tau(k)} > 0$

$$\|x^{\tau(k)+1} - x^*\|^2 \le \frac{2\mu}{\tau}\langle\Phi(x^*), x^* - z^{\tau(k)} + \alpha_{\tau(k)}\mu\Phi(z^{\tau(k)})\rangle. \tag{43}$$

From (39) and (43), we have

$$\|x^k - x^*\|^2 \le \frac{2\mu}{\tau}\langle\Phi(x^*), x^* - z^k + \alpha_k\mu\Phi(z^k)\rangle. \tag{44}$$

Taking the limit in (44) as $k \longrightarrow \infty$, and using (42), we obtain that

$$\limsup_{k\longrightarrow\infty} \|x^k - x^*\|^2 \leq 0.$$

Therefore, $x^k \longrightarrow x^*$ as $k \longrightarrow \infty$. This completes the proof of Theorem 1. □

Let us analyze the condition $\sum_{k=0}^{\infty} \alpha_k = \infty$, which was given in Algorithm 1.

Example 1 Choose Φ is the identical mapping, $F = G = 0$, $C = \mathbb{R}$, $Q = \mathbb{R}$. In this case, the bilevel split variational inequality problem becomes the problem of finding the minimum-norm solution of the split feasibility problem. One can find the solution set of the split split feasibility problem $\Omega = \mathbb{R}$ and, therefore, the minimum-norm solution x^* of the split feasibility problem is $x^* = 0$.

Choose $\alpha_k = \dfrac{1}{(k+2)^2}$ for all $k \geq 0$. An elementary computation shows that $\{\alpha_k\} \subset (0, 1)$, $\lim_{k\to\infty} \alpha_k = 0$. Since $\sum_{k=0}^{\infty} \alpha_k < \infty$, condition $\sum_{k=0}^{\infty} \alpha_k = \infty$ is violated.

The iterative sequence $\{x^k\}$ produced by Algorithm 1 for $\mu = 1$ and $x^0 = 1$ is given by

$$x^{k+1} = (1 - \alpha_k)x^k \quad \forall k \geq 0.$$

Thus, by induction, for every $k \geq 1$, we have

$$\begin{aligned} x^k &= \prod_{j=0}^{k-1}(1-\alpha_j) \\ &= \prod_{j=0}^{k-1}\left(1 - \frac{1}{(j+2)^2}\right) \\ &= \prod_{j=0}^{k-1}\frac{(j+1)(j+3)}{(j+2)^2} \\ &= \frac{k+2}{2(k+1)}. \end{aligned}$$

Therefore, $\lim_{k\to\infty} x^k = \dfrac{1}{2}$. This means that $\{x^k\}$ does not converge to the the minimum-norm solution $x^* = 0$ of the split feasibility problem. Hence, condition $\sum_{k=0}^{\infty} \alpha_k = \infty$ cannot be dropped.

Remark In Example 1, conditions $\{\alpha_k\} \subset (0,1)$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$ guarantee the strong convergence of $\{x^k\}$ to the the minimum-norm solution $x^* = 0$. In other words, condition $\lim_{k\to\infty} \alpha_k = 0$ can be dropped.

Indeed, using the inequality

$$1 + x \le e^x \quad \forall x \in \mathbb{R},$$

we have

$$\begin{aligned} x^k &= \prod_{j=0}^{k-1}(1-\alpha_j) \\ &\le \prod_{j=0}^{k-1} e^{-\alpha_j} \\ &= e^{-\sum_{j=0}^{k-1}\alpha_j}. \end{aligned}$$

It follows from the above inequality and $\{\alpha_k\} \subset (0,1)$ that

$$0 < x^k \le e^{-\sum_{j=0}^{k-1}\alpha_j} \quad \forall k. \tag{45}$$

From $\sum_{k=0}^{\infty} \alpha_k = \infty$, we have $\lim_{k\to\infty} \sum_{j=0}^{k-1} \alpha_j = \infty$. Consequently, (45) implies that $\lim_{k\to\infty} x^k = 0$.

4 Numerical Results

To illustrate Theorem 1, we consider the following example:

Example 2 Let $\mathcal{H}_1 = \mathbb{R}^4$ with the norm $\|x\| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}$ for $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ and $\mathcal{H}_2 = \mathbb{R}^2$ with the standard norm $\|y\| = (y_1^2 + y_2^2)^{\frac{1}{2}}$. Let $A(x) = (x_1 + x_3 + x_4, x_2 + x_3 - x_4)^T$ for all $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ then A is a bounded linear operator from $\mathbb{R}^4$ into $\mathbb{R}^2$ with $\|A\| = \sqrt{3}$. For $y = (y_1, y_2)^T \in \mathbb{R}^2$, let $B(y) = (y_1, y_2, y_1 + y_2, y_1 - y_2)^T$, then B is a bounded linear operator from $\mathbb{R}^2$ into $\mathbb{R}^4$ with $\|B\| = \sqrt{3}$. Moreover, for any $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ and $y = (y_1, y_2)^T \in \mathbb{R}^2$, $\langle A(x), y\rangle = \langle x, B(y)\rangle$, so $B = A^*$ is an adjoint operator of A.

Let

$$C = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - x_2 - x_3 \ge 1\}$$

and $F : C \longrightarrow \mathbb{R}^4$ be defined by $F(x) = (\|x\|^2 + 2)a$ for all $x \in C$, where $a = (1, -1, -1, 0)^T \in \mathbb{R}^4$. It is easy to verify that F is pseudomonotone on C.

Choose $x = (3, 1, 0, 4)^T \in C$, $y = (4, 1, 0, 0)^T \in C$. It is easy to see that

$$\langle F(x) - F(y), x - y\rangle = -9 < 0.$$

Hence, F is not monotone on C.

Suppose that there exists $L > 0$ such that

$$\|F(x) - F(y)\| \le L\|x - y\| \quad \forall x, y \in C. \tag{46}$$

Choose $x = (1, 0, 0, k)^T \in C$ $(k > 0)$ and $y = (1, 0, 0, 0)^T \in C$. From (46), we obtain

$$\sqrt{3}k \le L \ \ \forall k > 0,$$

which is a contradiction.

It is easy to see that the solution set $\mathrm{Sol}(C, F)$ of $VIP(C, F)$ is given by

$$\mathrm{Sol}(C, F) = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - x_2 - x_3 = 1\}.$$

Now let $Q = \{(u_1, u_2)^T \in \mathbb{R}^2 : u_1 - u_2 \ge 2\}$ and define another mapping $G : Q \longrightarrow \mathbb{R}^2$ as follows:

$$G(u) = (\|u\|^2 + 3)b$$

for all $u \in Q$, where $b = (1, -1)^T \in \mathbb{R}^2$.

It is easy to see that G is pseudomonotone on Q, not monotone on Q, not Lipschitz on Q and that the solution set $\mathrm{Sol}(Q, G)$ of $VIP(Q, G)$ is given by

$$\mathrm{Sol}(Q, G) = \{(u_1, u_2)^T \in \mathbb{R}^2 : u_1 - u_2 = 2\}.$$

We consider the case when $\Phi(x) = x$ for all $x \in C$. This mapping Φ is 1-Lipschitz continuous and 1-strongly monotone on C, and in this situation, by choosing $\mu = 1$, the Problem $(BSVIP)$ becomes the problem of finding the minimum-norm solution of the SVIP.

The solution set Ω of the SVIP is given by

$$\begin{aligned}\Omega &= \{x = (x_1, x_2, x_3, x_4)^T \in \mathrm{Sol}(C, F) : A(x) \in \mathrm{Sol}(Q, G)\}\\ &= \{x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - x_2 - x_3 = 1, (x_1 + x_3 + x_4) - (x_2 + x_3 - x_4) = 2\}\\ &= \{x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - x_2 - x_3 = 1, x_1 - x_2 + 2x_4 = 2\}\\ &= \{(a + 2 - 2b, a, 1 - 2b, b)^T : a, b \in \mathbb{R}\}.\end{aligned}$$

Suppose $x = (a + 2 - 2b, a, 1 - 2b, b)^T \in \Omega$ then

$$\begin{aligned}\|x\| &= \sqrt{(a + 2 - 2b)^2 + a^2 + (1 - 2b)^2 + b^2}\\ &= \sqrt{2(a + 1 - b)^2 + 7\Big(b - \frac{4}{7}\Big)^2 + \frac{5}{7}}\\ &\ge \sqrt{\frac{5}{7}}.\end{aligned}$$

The above equality holds if and only if $b = \frac{4}{7}$ and $a = -\frac{3}{7}$. So the minimum-norm solution x^* of the SVIP is $x^* = \Big(\frac{3}{7}, -\frac{3}{7}, -\frac{1}{7}, \frac{4}{7}\Big)^T$.

Select a random starting point $x^0 = (-3, 2, -7, -4)^T \in C$ for the Algorithm 1. We choose $\alpha_k = \frac{1}{k+2}$, $\lambda_k = \frac{k+1}{3k+2}$, $\delta_k = \frac{k+1}{5k+6}$. An elementary computation

Table 1 Algorithm 1 for Example 2, with $\alpha_k = \frac{1}{k+2}$, $\lambda_k = \frac{k+1}{3k+2}$, $\delta_k = \frac{k+1}{5k+6}$, $\varepsilon = 10^{-7}$ and starting point $x^0 = (-3, 2, -7, -4)^T$

Iter(k)	x_1^k	x_2^k	x_3^k	x_4^k
0	−3.00000	2.00000	−7.00000	−4.00000
1	−1.16667	0.66667	−3.20833	−0.75000
2	−0.57407	0.24074	−2.01936	0.14646
3	−0.31674	0.06674	−1.45101	0.46449
4	−0.16780	−0.03220	−1.13559	0.59323
5	−0.05892	−0.10774	−0.95118	0.64646
...	...	...	...	...
4936	0.42799	−0.42820	−0.14381	0.57161
4937	0.42799	−0.42820	−0.14381	0.57161
4938	0.42800	−0.42820	−0.14381	0.57161
4939	0.42800	−0.42820	−0.14381	0.57161
4940	0.42800	−0.42820	−0.14381	0.57161

shows that $\{\alpha_k\} \subset (0, 1)$, $\lim_{k\longrightarrow\infty} \alpha_k = \lim_{k\longrightarrow\infty} \frac{1}{k+2} = 0$, $\sum_{k=0}^{\infty} \alpha_k = \sum_{k=0}^{\infty} \frac{1}{k+2} = \infty$, $\{\lambda_k\} \subset (0, 1)$, $\lim_{k\longrightarrow\infty} \lambda_k = \frac{1}{3} \in (0, 1)$, $\{\delta_k\} \subset \left[\frac{1}{6}, \frac{1}{5}\right] \subset \left(0, \frac{1}{4}\right) = \left(0, \frac{1}{\|A\|^2 + 1}\right)$.
We have computational results in Table 1

The approximate solution obtained after 4940 iterations (with elapsed time 9.9318 s) is (see Table 1)

$$x^{4940} = (0.42800, -0.42820, -0.14381, 0.57161)^T,$$

which is a good approximation to the minimum-norm solution $x^* = \left(\frac{3}{7}, -\frac{3}{7}, -\frac{1}{7}, \frac{4}{7}\right)^T$.

We perform the iterative schemes in MATLAB R2012a running on a laptop with Intel(R) Core(TM) i3-3217U CPU @ 1.80GHz, 2 GB RAM.

5 Conclusion

In this paper, we have presented an iterative algorithm for solving strongly variational inequality problems with the split variational inequality problem constraints. The proposed algorithm is a combination of the linesearch method and the hybrid steepest descent method for the variational inequality problem [28]. The strong convergence of the iterative sequence generated by the proposed iterative algorithm to the unique solution of BSVIP is obtained.

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