Numer Algor (2019) 81:813–832
https://doi.org/10.1007/s11075-018-0572-5

ORIGINAL PAPER

A shrinking projection method for solving the split common null point problem in Banach spaces

Truong Minh Tuyen[1] · **Nguyen Song Ha**[1] · **Nguyen Thi Thu Thuy**[2]

Received: 16 December 2017 / Accepted: 2 July 2018 / Published online: 14 July 2018

Abstract In this paper, in order to solve the split common null point problem, we investigate a new explicit iteration method, base on the shrinking projection method and ε-enlargement of a maximal monotone operator. We also give some applications of our main results for the problem of split minimum point, multiple-sets split feasibility, and split variational inequality. Two numerical examples also are given to illustrate the effectiveness of the proposed algorithm.

Keywords Split common null point problem · Maximal monotone operator · Metric resolvent · ε-enlargement

Mathematics Subject Classification (2010) 47H05 · 47H09 · 49J53 · 90C25

✉ Truong Minh Tuyen
tuyentm@tnus.edu.vn

Nguyen Song Ha
hans@tnus.edu.vn

Nguyen Thi Thu Thuy
thuy.nguyenthithu2@huts.edu.vn

[1] Thai Nguyen University of Science, Thai Nguyen, Vietnam

[2] School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Hanoi, Vietnam

1 Introduction

Let C and D be non-empty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $T: H_1 \longrightarrow H_2$ be a bounded linear operator from H_1 into H_2. The split feasibility problem (SFP) is formulated as follow:

$$\text{Find an element} \quad x^* \in S = C \cap T^{-1}(D). \tag{1}$$

This problem was first introduced by Censor and Elfving [10] for modeling inverse problems. We also know that it plays an important role in medical image reconstruction and signal processing (see [4, 5]). In view of its applications, several iterative algorithms of solving (1) were presented in [4, 5, 11, 13, 24, 25, 29–32] and references therein.

There are some generalizations of the SFP, for example, the multiple-set SFP (MSSFP) (see [11, 17]), the split common fixed point problem (SCFPP) (see [12, 19]), the split variational inequality problem (SVIP) (see [13]), and the split common null point problem (SCNPP) (see [6, 14, 26, 27]).

Let $A: H_1 \longrightarrow 2^{H_1}$ and $B: H_2 \longrightarrow 2^{H_2}$ be two multi-valued operators. The SCNPP is stated as follow:

$$\text{Find an element} \quad x^* \in \left(A^{-1}0\right) \bigcap \left(T^{-1}(B^{-1}0)\right), \tag{2}$$

where $A^{-1}0 := \left\{x \in H_1 : 0 \in Ax\right\}$ and $B^{-1}0 := \left\{x \in H_2 : 0 \in Bx\right\}$.

In 2015, by using the metric resolvent of maximal monotone operator and the hybrid projection method, Takahashi et al. [27] proved a strong convergence theorem for finding a solution of SCNPP in Banach spaces.

Theorem 1.1 [27] *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the normalized duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T: E \longrightarrow F$ be a bounded linear operator such that $T \neq 0$, and let T^* be the adjoint operator of T. Suppose that $S = \left(A^{-1}0\right) \bigcap \left(T^{-1}(B^{-1}0)\right) \neq \emptyset$. Let $x_1 \in E$, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - J_E^{-1} T^* J_F(Tx_n - Q_{\mu_n} Tx_n), \\ C_n = \{z \in A^{-1}0 : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in A^{-1}0 : \langle x_n - z, J_E(x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \end{cases} \tag{3}$$

where $\mu_n \in (0, \infty)$ satisfies that for some $a, b \in \mathbb{R}$,

$$0 < a \leq \mu_n \leq b < \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in S$, where $z_0 = P_S x_1$.

We can see that in the iterative method (3), it is not easy to define C_n and Q_n, because we do not know the set of null points $A^{-1}0$ of A. Therefore, it is very difficult to find element $x_{n+1} = P_{C_n \cap Q_n} x_1$.

In 2017, Dadashi [14] introduced a shrinking projection method for split common null point problem in Hilbert space. He proved the following result:

Theorem 1.2 [14] *Let E be a uniformly convex and smooth Banach space with duality mapping J_E. Suppose that H is a Hilbert space and $T : H \longrightarrow E$ is a bounded linear operator such that $T \neq 0$ and T^* denote the adjoint operator of T. Let A and B be maximal monotone operators of H into 2^H and of E into 2^{E^*}, respectively, such that $S = \left(A^{-1}0\right) \bigcap \left(T^{-1}(B^{-1}0)\right) \neq \emptyset$. Let J_λ be the resolvent of A for $\lambda > 0$ and Q_μ the metric resolvent of B for $\mu > 0$. Generate the sequence $\{x_n\}$ by the algorithm*

$$\begin{cases} z_n = J_{\lambda_n}(x_n - \lambda_n T^* J_E(T x_n - Q_{\mu_n} T x_n)), \\ y_n = \alpha_n x_n + (1 - \alpha_n) z_n, \\ C_n = \{z \in C_{n-1} : \langle y_n - z, x_n - y_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n} x_1, \end{cases} \tag{4}$$

where $C_1 = H$ and $x_1 \in H$. If $0 < \|T\| \leq 2\alpha_n < 2$, $0 < b \leq \mu_n$ and $0 < c \leq \lambda_n < 1$, then $\{x_n\}$ converges strongly to a point $z_0 \in S$, where $z_0 = P_S x_1$.

From the result of Dadashi, there are two open questions which are posed as follows:

Question 1. Is it possible to remove the conditions about the boundedness of $\|T\|$ and the sequence $\{\lambda_n\} \subset [c, 1)$?

Question 2. Can we extend Theorem 1.2 for the case

$$S = \left(\bigcap_{i=1}^{N} A_i^{-1}0\right) \bigcap \left(T^{-1}\left(\bigcap_{j=1}^{M} B_j^{-1}0\right)\right) \neq \emptyset$$

where A_i and B_j are maximal monotone operators on the Banach spaces E and F, respectively?

The purpose of this paper is to introduce a new parallel iterative method to answers the above two open questions. The rest of this paper is organized as follows. In Section 2, we list some related facts that will be use in the proof of our result. In Section 3, we introduce a new parallel iterative algorithm and prove a strong convergence theorem for this algorithm. Some applications of the main result are presented in Section 4. Finally, in Section 5, we give two numerical examples for illustrating our method and showing its performance.

2 Preliminaries

In this section, we recall some definitions and results that will be used later. Let E be a real Banach space with the dual space E^*. For the sake of simplicity, the norms of E

and E^* are denote by the symbol $\|.\|$. We use $\langle x, f\rangle$ instead of $f(x)$ for $f \in E^*$ and $x \in E$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (respectively $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x. Let J_E denote the normalized duality mapping from E into 2^{E^*} defined by

$$J_E x = \left\{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\right\} \quad \forall x \in E.$$

We always use S_E to denote the unit sphere $S_E = \{x \in E : \|x\| = 1\}$. Recall that a Banach space E is said to be

(i) uniformly convex, if for any $\varepsilon, 0<\varepsilon\leq 2$, the inequalities $\|x\|\leq 1$, $\|y\|\leq 1$, and $\|x-y\|\geq\varepsilon$ imply that there exists a $\delta=\delta(\varepsilon)>0$ such that $\|(x+y)/2\|\leq 1-\delta$;

(ii) strictly convex, if for $x, y \in S_E$ with $x \neq y$, then

$$\|(1-\lambda)x + \lambda y\| < 1 \quad \forall \lambda \in (0, 1).$$

(iii) smooth, if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_E. (In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in S_E$, this limit attained uniformly for $x \in S_E$).

It is well known that each uniformly convex Banach space E is strictly convex and reflexive; E is uniformly convex if and only if E^* is uniformly smooth; If E is smooth, then duality mapping is single-valued (see [1, 15]).

Recall that a Banach space E has Kadec-Klee property if every sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. We know that if E is a uniformly convex Banach space, then E has Kadec-Klee property.

We have the following properties of the normalized duality mapping J_E (see [1, 15, 20]):

(i) E is reflexive if and only if J_E is surjective;

(ii) If E^* is strictly convex, then J_E is single-valued;

(iii) If E is a smooth, strictly convex and reflexive Banach space, then J_E is single-valued bijection;

(iv) If E^* is uniformly convex, then J_E is uniformly continuous on each bounded set of E.

Furthermore, if E is a smooth, strictly convex, and reflexive Banach space and C is a non-empty, closed, and convex subset of E, then for each $x \in E$, there exists unique $z \in C$ such that

$$\|x - z\| = \inf_{y\in C} \|x - y\|.$$

The mapping $P_C : E \longrightarrow C$ defined by $P_C x = z$ is called metric projection from E into C.

Let $A : E \longrightarrow 2^{E^*}$ be an operator. The effective domain of A is denoted by $D(A)$, that is, $D(A) = \{x \in E : Ax \neq \emptyset\}$. Recall that A is called monotone operator if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in D(A)$ and for all $u \in Ax$, $v \in A(y)$. A monotone operator A on E is called maximal monotone if its graph is not properly contained in the graph of any other monotone operator on E. It is well known that if A is a maximal monotone operator on E and E is a uniformly convex and smooth Banach space, then $R(J_E + rA) = E^*$ for all $r > 0$, where $R(J_E + rA)$ is the range of $J_E + rA$ (see [7, 22]). Thus, for all $x \in E$ and $r > 0$, there exists unique $x_r \in E$ such that

$$0 \in J_E(x_r - x) + rAx_r.$$

We define J_r by $x_r = J_r x$ and J_r is called metric resolvent of A.

The set of null point of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$ and we know that $A^{-1}0$ is a closed and convex subset of E (see [23]).

Let $A : E \longrightarrow 2^{E^*}$ be a maximal monotone operator. In [9], for each $\varepsilon \geq 0$, Burachik and Svaiter defined A^ε, an ε-enlargement of A, as follows

$$A^\varepsilon x = \big\{u \in E^* : \langle y - x, v - u\rangle \geq -\varepsilon, \ \forall y \in E, \ v \in Ay\big\}.$$

It is easy to see that, $A^0 x = Ax$ and if $0 \leq \varepsilon_1 \leq \varepsilon_2$, then $A^{\varepsilon_1}x \subseteq A^{\varepsilon_2}x$ for any $x \in E$. The use of element in A^ε instead of T allows an extra degree of freedom which is very useful in various applications.

Let $\{C_n\}$ be the sequence of closed, convex, and non-empty subsets of a reflexive Banach space E. We define the subsets s-Li$_n C_n$ and w-Ls$_n C_n$ of E as follows:

(i) $x \in$ s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset E$ converges strongly to x and that $x_n \in C_n$ for all $n \geq 1$.
(ii) $x \in$ w-Ls$_n C_n$ if and only if there exists a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and the sequence $\{y_k\} \subset E$ such that $y_k \rightharpoonup x$ and $y_k \in C_{n_k}$ for all $k \geq 1$.
(iii) If s-Li$_n C_n =$ w-Ls$_n C_n = \Omega_0$, then Ω_0 is called the limits of $\{C_n\}$ in the sense of Mosco in [18] and it is denoted by $\Omega_0 =$ M-$\lim\limits_{n\to\infty} C_n$.

Next, we list some lemmas that will be used in the sequel for the proof of our main result.

Lemma 2.1 (see [2, 3, 16]) *Let E be a smooth, strictly convex, and reflexive Banach space. Let C be a non-empty, closed, and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(i) $z = P_C x_1$;
(ii) $\langle z - y, J_E(x_1 - z)\rangle \geq 0$ *for all* $y \in C$.

Lemma 2.2 (see [28]) *Let E be a smooth, reflexive, and strictly convex Banach space having the Kadec-Klee property. Let $\{C_n\}$ be a sequence of non-empty, closed, and convex subsets of E. If $\Omega_0 = M$-$\lim\limits_{n\to\infty} C_n$ exists and is non-empty, then $\{P_{C_n}x\}$ converges strongly to $P_{\Omega_0}x$ for each $x \in E$.*

Lemma 2.3 (see [9]) *The graph of $A^\varepsilon : \mathbb{R}_+ \times E \longrightarrow 2^{E^*}$ is demiclosed, i.e., the conditions below hold:*

(i) *If $\{x_n\} \subset E$ converges strongly to x_0, $\{u_n \in A^{\varepsilon_n}x_n\}$ converges weak* to u_0 in E^* and $\{\varepsilon_n\} \subset \mathbb{R}_+$ converges to ε, then $u_0 \in A^\varepsilon x_0$;*
(ii) *If $\{x_n\} \subset E$ converges weak to x_0, $\{u_n \in A^{\varepsilon_n}x_n\}$ converges strongly to u_0 in E^* and $\{\varepsilon_n\} \subset \mathbb{R}_+$ converges to ε, then $u_0 \in A^\varepsilon x_0$.*

3 Main results

Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the normalized duality mappings on E and F, respectively. Let $A_i,\ i = 1, 2, ..., N$ and $B_j,\ j = 1, 2, ..., M$ be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*}, respectively. Let J_λ^i and Q_μ^j be the metric resolvents of A_i for $\lambda > 0$ and B_j for $\mu > 0$, respectively. Let $T : E \longrightarrow F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $S = \left(\bigcap_{i=1}^{N} A_i^{-1}0\right) \bigcap \left(T^{-1}\left(\bigcap_{j=1}^{M} B_j^{-1}0\right)\right) \neq \emptyset$.

We consider the following problem:

$$\text{Find an element } x^* \in S. \tag{5}$$

In order to solve the problem (5), we introduce the following algorithm:

Algorithm 1 Let $C_0 = E$, $x_0 \in E$ and let $\{x_n\}$ be a sequence generated by

$$J_F(y_{j,n} - Tx_n) + \mu_n B_j^{\varepsilon_n} y_{j,n} \ni 0,\ \ j = 1, 2, ..., M; \tag{6}$$

$$z_{j,n} = x_n - r_n J_E^{-1} T^*(J_F(Tx_n - y_{j,n})),\ \ j = 1, 2, ..., M; \tag{7}$$

$$\text{Choose } j_n \text{ such that } \|z_{j_n,n} - x_n\| = \max_{j=1,\ldots,M} \|z_{j,n} - x_n\|,\ \text{ let } z_n = z_{j_n,n},$$

$$D_n = \left\{z \in E : \langle x_n - z, J_E(x_n - z_n)\rangle \geq r_n\|Tx_n - y_{j_n,n}\|^2 - r_n\mu_n\varepsilon_n\right\};$$

$$J_E(t_{i,n} - z_n) + \lambda_n A_i^{\varepsilon_n} t_{i,n} \ni 0,\ \ i = 1, 2, ..., N; \tag{8}$$

$$\text{Choose } i_n \text{ such that } \|t_{i_n,n} - z_n\| = \max_{i=1,\ldots,N} \|t_{i,n} - z_n\|,\ \text{ let } t_n = t_{i_n,n};$$

$$C_{n+1} = \left\{z \in C_n : \langle t_n - z, J_E(z_n - t_n)\rangle \geq -\lambda_n\varepsilon_n\right\} \bigcap D_n;$$

$$x_{n+1} = P_{C_{n+1}} x_0,$$

where $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$, $\{r_n\} \subset (0, \infty)$ and $\{\varepsilon_n\} \subset (0, \infty)$.

We will prove strong convergence of the above sequence $\{x_n\}$ under the following conditions:

(C1) $\min\left\{\inf_n\{\lambda_n\}, \inf_n\{\mu_n\}, \inf_n\{r_n\}\right\} \geq a > 0$ and $\sup_n\{r_n\} < +\infty$;

(C2) $(\lambda_n + \mu_n)\varepsilon_n \to 0$, as $n \to \infty$.

First, we have the following lemma:

Lemma 3.1 *If $\{C_n\}$ is a decreasing sequence of closed and convex subsets of a reflexive Banach space E and $\Omega_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$, then $\Omega_0 = M\text{-}\lim_{n\to\infty} C_n$.*

Proof Clearly, if $x \in \Omega_0$, then the sequence $\{x_n\}$ with $x_n = x$ for all $n \geq 1$ converges strongly to x. Thus, we have $x \in$ s-Li$_n C_n$ and $x \in$ w-Ls$_n C_n$. It implies that $\Omega_0 \subseteq$ s-Li$_n C_n$ and $\Omega_0 \subseteq$ w-Ls$_n C_n$.

If we take $x \in$ s-Li$_n C_n$ then there exists sequence $\{x_n\} \subset E$ with $x_n \in C_n$ for all $n \geq 1$ such that $x_n \to x$ as $n \to \infty$. On the other hand, $x_{n+k} \in C_n$ for all $n \geq 1$ and $k \geq 0$ because $\{C_n\}$ is a decreasing sequence. So, letting $k \to \infty$ and using the closedness property of C_n, we get that $x \in C_n$ for all $n \geq 1$. Thus, $x \in \Omega_0$ and hence $\Omega_0 \supseteq$ s-Li$_n C_n$.

Next, let $y \in$ w-Ls$_n C_n$, from the definition of w-Ls$_n C_n$, there exists a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and the sequence $\{y_k\} \subset E$ such that $y_k \rightharpoonup x$ and $y_k \in C_{n_k}$ for all $k \geq 1$. From $\{C_n\}$ is a decreasing sequence, we have

$$y_{k+p} \in C_{n_k} \tag{9}$$

for all $k \geq 1$ and $p \geq 0$. Due to the closedness and convexity of C_{n_k}, we have C_{n_k} is weakly closed in E for all $k \geq 1$. So, in (9), letting $p \to \infty$, we obtain that $y \in C_{n_k}$ for all $k \geq 1$. Moreover, $C_k \supseteq C_{n_k}$, $y \in C_k$ for all $k \geq 1$. Therefore, $y \in \Omega_0$ and hence $\Omega_0 \supseteq$ w-Ls$_n C_n$.

Consequently, we obtain that s-Li$_n C_n =$ w-Ls$_n C_n = \Omega_0$. Thus, $\Omega_0 =$M-$\lim\limits_{n\to\infty} C_n$. □

Next, we have the following propositions:

Proposition 3.2 *The sequence $\{x_n\}$ generated by Algorithm 1 is well defined.*

Proof We will prove this proposition by several steps.

Step 1. C_n and D_n are the closed and convex subsets of E.

Indeed, we rewrite D_n and C_{n+1} in the forms

$$D_n = \{z \in E : \langle z, J_E(x_n - z_n)\rangle \leq \langle x_n, J_E(x_n - z_n)\rangle - r_n\|Tx_n - y_{j_n,n}\|^2 + r_n\mu_n\varepsilon_n\},$$
$$C_{n+1} = W_n \cap D_n,$$

for all $n \geq 0$, where $W_n := \{z \in C_n : \langle z, J_E(z_n - t_n)\rangle \leq \langle t_n, J_E(z_n - t_n)\rangle + \lambda_n\varepsilon_n\}$. We note that D_n and W_n are the closed half-spaces of E. Thus, C_n and D_n are the closed and convex subsets of E.

Step 2. $S \subset D_n$ for all $n \geq 0$.

Let $z \in S$, from (7), we have

$$J_E(x_n - z_n) = r_n T^*(J_F(Tx_n - y_{j_n,n})).$$

It implies that

$$\begin{aligned}\langle x_n - z, J_E(x_n - z_n)\rangle &= r_n\langle x_n - z, T^*(J_F(Tx_n - y_{j_n,n}))\rangle \\ &= r_n\langle Tx_n - Tz, J_F(Tx_n - y_{j_n,n})\rangle \\ &= r_n\langle Tx_n - y_{j_n,n} + y_{j_n,n} - Tz, J_F(Tx_n - y_{j_n,n})\rangle \\ &= r_n\|Tx_n - y_{j_n,n}\|^2 \\ &\quad + r_n\langle y_{j_n,n} - Tz, J_F(Tx_n - y_{j_n,n})\rangle.\end{aligned} \tag{10}$$

On the other hand, from (6) and $z \in S$, we have

$$\frac{1}{\mu_n} J_F(Tx_n - y_{j_n,n}) \in B_{j_n}^{\varepsilon_n} y_{j_n,n} \quad \text{and} \quad 0 \in B_{j_n} Tz.$$

Thus, from the definition of $B_{j_n}^{\varepsilon_n}$, we get that

$$r_n \langle y_{j_n,n} - Tz, J_F(Tx_n - y_{j_n,n}) \rangle \geq -r_n \mu_n \varepsilon_n. \tag{11}$$

From (10) and (11), we obtain

$$\langle x_n - z, J_E(x_n - z_n) \rangle \geq r_n \|Tx_n - y_{j_n,n}\|^2 - r_n \mu_n \varepsilon_n.$$

This follows that $z \in D_n$ and hence $S \subset D_n$ for all $n \geq 0$.

Step 3. $S \subset C_n$ for all $n \geq 0$.

Indeed, obviously, $S \subset C_0 = E$. Suppose that $S \subset C_n$ for some $n \geq 1$, we will prove that $S \subset C_{n+1}$. Now, from (8) and $z \in S$, we have

$$\frac{1}{\lambda_n} J_E(z_n - t_n) \in A_{i_n}^{\varepsilon_n} t_n \quad \text{and} \quad 0 \in A_{i_n} z.$$

Hence, from the definition of $A_{i_n}^{\varepsilon_n}$, we get that

$$\langle t_n - z, J_E(z_n - t_n) \rangle \geq -\lambda_n \varepsilon_n.$$

It implies that $z \in W_n$. From the definition of $C_{n+1} = W_n \cap D_n$ and step 2, we have $z \in C_{n+1}$. Thus, $S \subset C_{n+1}$. Finally, by mathematical induction, we obtain that $S \subset C_n$ for all $n \geq 0$. Hence, the sequence $\{x_n\}$ is well defined. □

Proposition 3.3 *If the conditions (C1) and (C2) are satisfied, then the sequences $\{x_n\}$, $\{z_{j,n}\}$, $j = 1, 2, ..., M$ and $\{t_{i,n}\}$, $i = 1, 2, ..., N$ in Algorithm 1 converge strongly to a same point $p_0 \in E$.*

Proof We will prove this proposition by several steps.

Step 1. $x_n \to p_0 = P_{\Omega_0} x_0$, where $\Omega_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

Indeed, from step 1 and step 3 in the proof of Proposition 3.2 and the definition of $\{C_n\}$, we have $\{C_n\}$ is the sequence of decreasing of closed convex subsets of E and $S \subset \Omega_0 \neq \emptyset$. Thus, from Lemma 3.1, there exists the limit $\Omega_0 =$M-$\lim_{n\to\infty} C_n$. By using Lemma 2.2, we have $x_n = P_{C_n} x_0 \to p_0 = P_{\Omega_0} x_0$ as $n \to \infty$.

Step 2. $\{y_{j,n}\}$ is bounded for all $j = 1, 2, ..., M$.

Indeed, fixing $z \in S$ and from (6), we have

$$\frac{1}{\mu_n} J_F(Tx_n - y_{j,n}) \in B_j^{\varepsilon_n} y_{j,n} \quad \text{and} \quad 0 \in B_j Tz.$$

Thus, from the definition of $B_j^{\varepsilon_n}$, we get that

$$\langle y_{j,n} - z, J_F(Tx_n - y_{j,n}) \rangle \geq -\mu_n \varepsilon_n,$$

for all $j = 1, 2, ..., M$. It follows that

$$\begin{aligned}\|y_{j,n} - Tx_n\|^2 &\leq \langle Tx_n - z, J_F(Tx_n - y_{j,n})\rangle + \mu_n\varepsilon_n \\ &\leq \frac{1}{2}(\|Tx_n - z\|^2 + \|Tx_n - y_{j,n}\|^2) + \mu_n\varepsilon_n,\end{aligned}$$

which implies that

$$\|y_{j,n} - Tx_n\|^2 \leq \|Tx_n - z\|^2 + 2\mu_n\varepsilon_n,$$

for all $j = 1, 2, ..., M$ and for all $n \geq 0$.

Because of the boundedness of $\{x_n\}$, the sequence $\{Tx_n\}$ also is bounded. From this and $\mu_n\varepsilon_n \to 0$, it ensures that there exists $K > 0$ such that

$$K = \max\left\{\sup_n\{\|Tx_n - z\|^2\}, \sup_n\{\mu_n\varepsilon_n\}\right\} < \infty.$$

So, we obtain that

$$\|y_{j,n} - Tx_n\|^2 \leq 3K,$$

which implies that $\{y_{j,n} - Tx_n\}$ is bounded and hence $\{y_{j,n}\}$ also is bounded for all $j = 1, 2, ..., M$.

Step 3. $\{z_{j,n}\}$ is bounded for all $j = 1, 2, ..., M$.

From (7), we have

$$J_E(x_n - z_{j,n}) = r_nT^*J_F(Tx_n - y_{j,n}). \tag{12}$$

Thus, by the boundedness of $\{Tx_n\}$, $\{y_{j,n}\}$ and $\{r_n\}$, we also get that $\{z_{j,n}\}$ is bounded for all $j = 1, 2, ..., M$.

Step 4. $\{t_{i,n}\}$ is bounded for all $i = 1, 2, ..., N$.

Indeed, fixing $z \in S$ and from (8), we have

$$\frac{1}{\lambda_n}J_E(z_n - t_{i,n}) \in A_i^{\varepsilon_n}t_{i,n} \quad \text{and} \quad 0 \in A_iz.$$

Thus, from the definition of $A_i^{\varepsilon_n}$, we get that

$$\langle t_{i,n} - z, J_E(z_n - t_{i,n})\rangle \geq -\lambda_n\varepsilon_n,$$

for all $i = 1, 2, ..., N$. It follows that

$$\begin{aligned}\|t_{i,n} - z_n\|^2 &\leq \langle z_n - z, J_E(z_n - t_{i,n})\rangle + \lambda_n\varepsilon_n \\ &\leq \frac{1}{2}(\|z_n - z\|^2 + \|z_n - t_{i,n}\|^2) + \lambda_n\varepsilon_n,\end{aligned}$$

which implies that

$$\|t_{i,n} - z_n\|^2 \leq \|z_n - z\|^2 + 2\lambda_n\varepsilon_n,$$

for all $i = 1, 2, ..., N$ and for all $n \geq 0$.

Because of the boundedness of $\{z_n\}$ and $\lambda_n\varepsilon_n \to 0$, it ensures that $\{t_{i,n}\}$ is bounded for all $i = 1, 2, ..., N$.

Step 5. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_{j,n} = \lim_{n\to\infty} t_{i,n} = p_0$.

From the definition of D_n, we have

$$\langle x_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq r_n(\|Tx_n - y_{j_n,n}\|^2 - \mu_n\varepsilon_n)$$

and hence,

$$\|Tx_n - y_{j_n,n}\|^2 \leq \frac{1}{a}(K_1\|x_n - x_{n+1}\| + \mu_n\varepsilon_n) \to 0,$$

where $K_1 = \sup_n\{\|x_n - z_n\|\} < \infty$. This implies that

$$\|Tx_n - y_{j_n,n}\| \to 0.$$

So, from (12), we obtain

$$\|x_n - z_{j_n,n}\| \to 0. \tag{13}$$

From the definition of $z_{j_n,n}$, we have

$$\|x_n - z_{j,n}\| \to 0, \tag{14}$$

for all $j = 1, 2, ..., M$. It follows from (12) that

$$\|J_F(Tx_n - y_{j,n})\| \to 0, \tag{15}$$

for all $j = 1, 2, ..., M$.

Next, from $x_{n+1} \in C_{n+1}$, we have

$$\langle t_n - x_{n+1}, J_E(z_n - t_n)\rangle \geq -\lambda_n\varepsilon_n.$$

Thus, we get that

$$\begin{aligned}\|z_n - t_n\|^2 &\leq \langle z_n - x_{n+1}, J_E(z_n - t_n)\rangle + \lambda_n\varepsilon_n\\ &\leq \frac{1}{2}(\|z_n - x_{n+1}\|^2 + \|z_n - t_n\|^2) + \lambda_n\varepsilon_n,\end{aligned}$$

which follows that

$$\|z_n - t_n\|^2 \leq \|z_n - x_{n+1}\|^2 + 2\lambda_n\varepsilon_n \to 0.$$

This implies that

$$\|z_n - t_n\| \to 0.$$

By the definition of t_n, we have

$$\|z_n - t_{i,n}\| \to 0, \tag{16}$$

for all $i = 1, 2, ..., N$.

Finally, from $x_n \to p_0$, (14) and (16), we obtain that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_{j,n} = \lim_{n\to\infty} t_{i,n} = p_0, \tag{17}$$

for all $j = 1, 2, ..., M$ and for all $i = 1, 2, ..., N$. □

Now, we are in position to prove our main result.

Theorem 3.4 *If the conditions (C1) and (C2) are satisfied, then the sequence* $\{x_n\}$ *generated by Algorithm 1 converges strongly to* $P_S x_0$ *as* $n \to \infty$.

Proof We proceed with the following steps.

Step 1. $p_0 \in S$.

Indeed, from (16) and the condition (C1), we have

$$A_i^{\varepsilon_n} t_{i,n} \ni \frac{1}{\lambda_n} J_E(z_n - t_{i,n}) \to 0,$$

for all $i = 1, 2, ..., N$. This combines with $t_{i,n} \to p_0$, $\varepsilon_n \to 0$ and using Lemma 2.3, we obtain that

$$p_0 \in A_i^{-1} 0, \tag{18}$$

for all $i = 1, 2, ..., N$.

Obviously, $Tx_n \to Tp_0$ and from (15), we get that

$$y_{j,n} \to Tp_0, \tag{19}$$

for all $j = 1, 2, ..., M$.

From (6), (15) and the condition (C1), we have

$$B_j^{\varepsilon_n} y_{j,n} \ni \frac{1}{\mu_n} J_E(Tx_n - y_{j,n}) \to 0, \tag{20}$$

for all $j = 1, 2, ..., M$.

From (19), (20) and Lemma 2.3 imply that $Tp_0 \in B_j^{-1} 0$. Hence,

$$p_0 \in T^{-1}(B_j^{-1} 0), \tag{21}$$

for all $j = 1, 2, ..., M$. Therefore, from (18) and (21), we obtain $p_0 \in S$.

Step 2. $p_0 = P_S x_0$.

Indeed, let $x^\dagger = P_S x_0$. From $p_0 = P_{\Omega_0} x_0$ and $x^\dagger \in S \subset \Omega_0$, we have

$$\|x_0 - p_0\| \le \|x_0 - x^\dagger\|.$$

On the other hand, from the fact that $p_0 \in S$, we have

$$\|x_0 - x^\dagger\| \le \|x_0 - p_0\|.$$

Thus, we get $\|x_0 - x^\dagger\| = \|x_0 - p_0\|$. By the uniqueness of $x^\dagger$, it ensures that $p_0 = x^\dagger = P_S x_0$. □

Remark 3.5 When $\varepsilon_n = 0$ for all n, then Algorithm 1 can be rewritten in the following form: Let $C_0 = E$, $x_0 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{aligned}
&z_{j,n} = x_n - r_n J_E^{-1} T^*(J_F(Tx_n - Q^j_{\mu_n} Tx_n)),\ j = 1, 2, ..., M;\\
&\text{Choose } j_n \text{ such that } \|z_{j_n,n} - x_n\| = \max_{j=1,\ldots,M} \|z_{j,n} - x_n\|, \text{ let } z_n = z_{j_n,n},\\
&D_n = \{z \in E : \langle x_n - z, J_E(x_n - z_n)\rangle \geq r_n \|Tx_n - Q^{j_n}_{\mu_n} Tx_n\|^2\};\\
&t_{i,n} = J^i_{\lambda_n} z_n,\ i = 1, 2, ..., N;\\
&\text{Choose } i_n \text{ such that } \|t_{i_n,n} - z_n\| = \max_{i=1,\ldots,N} \|t_{i,n} - z_n\|, \text{ let } t_n = t_{i_n,n};\\
&C_{n+1} = \left\{z \in C_n : \langle t_n - z, J_E(z_n - t_n)\rangle \geq 0\right\} \bigcap D_n;\\
&x_{n+1} = P_{C_{n+1}} x_0, \qquad (22)
\end{aligned}$$

where $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ and $\{r_n\} \subset (0, \infty)$.

The following result is a direct consequence of Theorem 3.4.

Corollary 3.6 *If the condition (C1) is satisfied, then the sequence $\{x_n\}$ generated by* (22) *converges strongly to $x^\dagger = P_S x_0$.*

4 Applications

4.1 Split minimum point problem

Let E be a Banach space and let $f : E \longrightarrow (-\infty, \infty]$ be a proper, lower semi-continuous and convex function. The subdifferentiable of f is multi-valued mapping $\partial f : E \longrightarrow 2^{E^*}$ which is defined by

$$\partial f(x) = \{g \in E^* : f(y) - f(x) \geq \langle y - x, g\rangle,\ \forall y \in E\}$$

for all $x \in E$. We know that ∂f is maximal monotone operator [21] and $x_0 \in \operatorname{argmin}_E f(x)$ if and only if $\partial f(x_0) \ni 0$.

The ε-subdifferential enlargement of ∂f is given by

$$\partial_\varepsilon f(x) = \{u \in E^* : f(y) - f(x) \geq \langle y - x, u\rangle - \varepsilon,\ \forall y \in E\},$$

for each $\varepsilon \geq 0$. It is well know that $\partial_\varepsilon f(x) \subset \partial^\varepsilon f(x)$, for any $x \in E$. Moreover, in some particular cases, we have that $\partial_\varepsilon f(x) \subsetneq \partial^\varepsilon f(x)$ (see, example 2 and example 3 in [8]).

From the Theorem 3.4, we have the following theorem:

Theorem 4.1 *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the normalized duality mappings on E and F, respectively. Let f_i, $i = 1, 2, ..., N$ and g_j, $j = 1, 2, ..., M$ be proper, lower semi-continuous, and convex functions of E into $(-\infty, \infty]$ and F into $(-\infty, \infty]$,* *respectively. Let $T : E \longrightarrow F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$S = \left(\bigcap_{i=1}^{N}(\partial f_i)^{-1}0\right)\bigcap\left(T^{-1}\left(\bigcap_{j=1}^{M}(\partial g_j)^{-1}0\right)\right) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by $C_0 = E$, $x_0 \in E$ and

$$\begin{aligned}
&J_F(y_{j,n} - Tx_n) + \mu_n \partial^{\varepsilon_n} g_j(y_{j,n}) \ni 0,\ j = 1, 2, ..., M;\\
&z_{j,n} = x_n - r_n J_E^{-1} T^*(J_F(Tx_n - y_{j,n})),\ j = 1, 2, ..., M;\\
&\textit{Choose } j_n \textit{ such that } \|z_{j_n,n} - x_n\| = \max_{j=1,...,M} \|z_{j,n} - x_n\|,\ \textit{let } z_n = z_{j_n,n},\\
&D_n = \{z \in E : \langle x_n - z, J_E(x_n - z_n)\rangle \geq r_n\|Tx_n - y_{j_n,n}\|^2 - r_n\mu_n\varepsilon_n\};\\
&J_E(t_{i,n} - z_n) + \lambda_n \partial^{\varepsilon_n} f_i(t_{i,n}) \ni 0,\ i = 1, 2, ..., N;\\
&\textit{Choose } i_n \textit{ such that } \|t_{i_n,n} - z_n\| = \max_{i=1,...,N} \|t_{i,n} - z_n\|,\ \textit{let } t_n = t_{i_n,n};\\
&C_{n+1} = \{z \in C_n : \langle t_n - z, J_E(z_n - t_n)\rangle \geq -\lambda_n\varepsilon_n\}\bigcap D_n;\\
&x_{n+1} = P_{C_{n+1}}x_0,
\end{aligned}$$

where $\{\lambda_n\}$, $\{\mu_n\} \subset (0, \infty)$, $\{r_n\} \subset (0, \infty)$ and $\{\varepsilon_n\} \subset (0, \infty)$. If the conditions (C1) and (C2) are satisfied, then the sequence $\{x_n\}$ converges strongly to $x^\dagger = P_S x_0$.

The following result is a direct consequence of the theorem above.

Corollary 4.2 *Let E, F, J_E, J_F, f_i, g_j, T, T^* be as in Theorem 4.1. Suppose that*

$$S = \left(\bigcap_{i=1}^{N}(\partial f_i)^{-1}0\right)\bigcap\left(T^{-1}\left(\bigcap_{j=1}^{M}(\partial g_j)^{-1}0\right)\right) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by $C_0 = E$, $x_0 \in E$ and

$$\begin{aligned}
&y_{j,n} = \underset{y\in F}{\textit{argmin}}\left\{g_j(y) + \frac{1}{2\mu_n}\|y - Tx_n\|^2\right\},\ j = 1, 2, ..., M;\\
&z_{j,n} = x_n - r_n J_E^{-1} T^*(J_F(Tx_n - y_{j,n})),\ j = 1, 2, ..., M;\\
&\textit{Choose } j_n \textit{ such that } \|z_{j_n,n} - x_n\| = \max_{j=1,...,M} \|z_{j,n} - x_n\|,\ \textit{let } z_n = z_{j_n,n},\\
&D_n = \{z \in E : \langle x_n - z, J_E(x_n - z_n)\rangle \geq r_n\|Tx_n - y_{j_n,n}\|^2\};\\
&t_{i,n} = \underset{y\in E}{\textit{argmin}}\{f_i(x) + \frac{1}{2\lambda_n}\|x - z_n\|^2\},\ i = 1, 2, ..., N;\\
&\textit{Choose } i_n \textit{ such that } \|t_{i_n,n} - z_n\| = \max_{i=1,...,N} \|t_{i,n} - z_n\|,\ \textit{let } t_n = t_{i_n,n};\\
&C_{n+1} = \left\{z \in C_n : \langle t_n - z, J_E(z_n - t_n)\rangle \geq 0\right\}\bigcap D_n;\\
&x_{n+1} = P_{C_{n+1}}x_0,
\end{aligned}$$

where $\{\lambda_n\}$, $\{\mu_n\} \subset (0, \infty)$ and $\{r_n\} \subset (0, \infty)$. If the condition (C1) is satisfied, then the sequence $\{x_n\}$ converges strongly to $x^\dagger = P_S x_0$.

4.2 Multiple-sets split feasibility problem

Let C be a non-empty, closed, and convex subset of E. Let i_C be the indicator function of C, that is,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

It is easy to see that i_C is the proper, semi-continuous, and convex function, so its subdifferentiable ∂i_C is a maximal monotone operator. We know that

$$\partial i_C(u) = N(u, C) = \{f \in E^* : \langle u - y, f\rangle \geq 0 \quad \forall y \in C\},$$

where $N(u, C)$ is the normal cone of C at u.

We denote metric resolvent of ∂i_C by J_r with $r > 0$. Suppose $u = J_r x$ for $x \in E$, that is

$$\frac{J_E(x - u)}{r} \in \partial i_C(u) = N(u, C).$$

Thus, we have

$$\langle u - y, J_E(x - u)\rangle \geq 0,$$

for all $y \in C$. From Lemma 2.2, we get that $u = P_C x$.

Thus, from Theorem 3.4, we have the following theorem:

Theorem 4.3 *Let E, F, J_E, J_F, T, T^* be as in Theorem 4.1. Let L_i, $i = 1, 2, ..., N$ and K_j, $j = 1, 2, ..., M$ be nonempty, closed and convex subsets of E and F, respectively. Suppose that $S = \left(\bigcap_{i=1}^{N} L_i\right) \bigcap \left(T^{-1}\left(\bigcap_{j=1}^{M} K_j\right)\right) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by $C_0 = E$, $x_0 \in E$ and*

$$\begin{aligned}
&z_{j,n} = x_n - r_n J_E^{-1} T^*(J_F(Tx_n - P_{K_j} Tx_n)), \; j = 1, 2, ..., M; \\
&\textit{Choose } j_n \textit{ such that } \|z_{j_n,n} - x_n\| = \max_{j=1,\dots,M} \|z_{j,n} - x_n\|, \; \textit{let } z_n = z_{j_n,n}, \\
&D_n = \{z \in E : \langle x_n - z, J_E(x_n - z_n)\rangle \geq r_n \|Tx_n - P_{K_{j_n}} Tx_n\|^2\}; \\
&t_{i,n} = P_{L_i} z_n, \; i = 1, 2, ..., N; \\
&\textit{Choose } i_n \textit{ such that } \|t_{i_n,n} - z_n\| = \max_{i=1,\dots,N} \|t_{i,n} - z_n\|, \; \textit{let } t_n = t_{i_n,n}; \\
&C_{n+1} = \left\{z \in C_n : \langle t_n - z, J_E(z_n - t_n)\rangle \geq 0\right\} \bigcap D_n; \\
&x_{n+1} = P_{C_{n+1}} x_0,
\end{aligned}$$

where $\{r_n\} \subset (0, \infty)$. If the condition (C1) is satisfied, then the sequence $\{x_n\}$ converges strongly to $x^\dagger = P_S x_0$.

4.3 The split variational inequality problem

Let C be a non-empty, closed, and convex subset of E and let $A : C \longrightarrow E^*$ be a monotone operator which is hemi-continuous (that is for any $c \in C$ and $t_n \to 0^+$ we have $A(x + t_n y) \rightharpoonup Ax$ for all $y \in E$ such that $x + t_n y \in C$). Then, a point $u \in C$ is called a solution of the variational inequality for A, if

$$\langle y - u, Au \rangle \geq 0 \quad \forall y \in C.$$

We denote by $\mathrm{VI}(C, A)$ the set of all solutions of the variational inequality for A.

Define a mapping T_A by

$$T_A x = \begin{cases} Ax + N(x, C), & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

By Rockafellar [22], we know that T_A is maximal monotone, and $T_A^{-1}0 = \mathrm{VI}(C, A)$.

For any $y \in E$ and $r > 0$, we know that the variational inequality $\mathrm{VI}(C, rA + J_E(\bullet - y))$ has a unique solution. Suppose that $x = \mathrm{VI}(C, rAx + J_E(x - y))$, that is,

$$\langle z - x, rA(x) + J_E(x - y)\rangle \geq 0 \quad \forall z \in C.$$

From the definition of $N(x, C)$, we have

$$-rAx - J_E(x - y) \in N(x, C) = rN(x, C),$$

which implies that

$$\frac{J_E(y - x)}{r} \in Ax + N(x, C) = T_A x.$$

Thus, we obtain that $x = J_r y$, where J_r is metric resolvent of T_A.

Now, let E and F be two uniformly convex and smooth Banach spaces and let K_i, $i = 1, 2, ..., N$ and L_j, $j = 1, 2, ..., M$ be closed and convex subsets of E and F, respectively. Let $A_i : K_i \longrightarrow E^*$ and $B_j : L_j \longrightarrow F^*$ be monotone operators which are hemi-continuous. Let $T : E \longrightarrow F$ be a bounded linear operator such that $T \neq 0$. Suppose that $S = \left(\bigcap_{i=1}^{N} \mathrm{VI}(K_i, A_i)\right) \bigcap \left(T^{-1}\left(\bigcap_{j=1}^{M} \mathrm{VI}(B_j, L_j)\right)\right)$ $\neq \emptyset$.

We consider the following split variational inequality problem:

$$\text{Find an element } x^* \in S. \tag{23}$$

To solve the problem (23), we define the operators T_{A_i} and T_{B_j} as following

$$T_{A_i} x = \begin{cases} A_i x + N(x, K_i) \text{ if } x \in K_i, \\ \emptyset \text{ if } x \notin K_i, \end{cases} \quad \text{and } T_{B_j} x = \begin{cases} B_j x + N(x, L_j) \text{ if } x \in L_j, \\ \emptyset \text{ if } x \notin L_j, \end{cases}$$

for all $i = 1, 2, ..., N$ and $j = 1, 2, ..., M$. For any $r > 0$, we denote by J_r^i and Q_r^j the metric resolvents of T_{A_i} and T_{B_j}, respectively.

From the above argument, the problem (23) is equivalent to the split common null point problem for maximal monotone operators T_{A_i} and T_{B_j}. Thus, from Theorem 3.4, we have the following result:

Theorem 4.4 *Let $C_0 = E$, $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&t_{j,n} = VI(L_j, \mu_n B_j + J_F(\bullet - Tx_n)),\ \ j = 1, 2, ..., M;\\
&z_{j,n} = x_n - r_n J_E^{-1} T^*(J_F(Tx_n - t_{j,n})),\ \ j = 1, 2, ..., M;\\
&\textit{Choose } j_n \textit{ such that } \|z_{j_n,n} - x_n\| = \max_{j=1,...,M} \|z_{j,n} - x_n\|,\ \textit{let } z_n = z_{j_n,n},\\
&D_n = \{z \in E :\ \langle x_n - z, J_E(x_n - z_n)\rangle \geq r_n \|Tx_n - Q_{\mu_n}^{j_n} Tx_n\|^2\};\\
&y_{i,n} = VI(K_i, \lambda_n A_i + J_E(\bullet - z_n)),\ \ i = 1, 2, ..., N;\\
&\textit{Choose } i_n \textit{ such that } \|y_{i_n,n} - z_n\| = \max_{i=1,...,N} \|y_{i,n} - z_n\|,\ \textit{let } y_n = y_{i_n,n};\\
&C_{n+1} = \Big\{z \in C_n :\ \langle y_n - z, J_E(z_n - y_n)\rangle \geq 0\Big\} \bigcap D_n;\\
&x_{n+1} = P_{C_{n+1}} x_0,
\end{aligned} \tag{24}$$

where $\{\lambda_n\}$, $\{\mu_n\}$ and $\{r_n\}$ satisfy the following the condition (C1). Then, the sequence $\{x_n\}$ converges strongly to a point $x^\dagger \in S$, where $x^\dagger = P_S x_1$.

5 Numerical test

In this section, we apply our result in Theorem 4.3 to solve the multiple-set feasibility problem.

Example 5.1 Consider the following problem:

$$\text{Find an element} \quad x_* \in S = \left(\bigcap_{i=1}^{100} L_i\right) \bigcap \left(T^{-1}\left(\bigcap_{j=1}^{50} K_j\right)\right), \tag{25}$$

where

$$\begin{aligned}
&L_i = [0, i] \times [-i, 2i - 1] \times [1 - i, 1 + i] \subset \mathbb{R}^3, \quad i = 1, 2, \ldots, 100,\\
&K_j = [1 - j, j] \times [0, 1 + j] \subset \mathbb{R}^2, \quad j = 1, 2, \ldots, 50,
\end{aligned}$$

and $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T(x_1, x_2, x_3) = (4x_1, 4x_3) \quad \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

It is easy to see that

$$S = [0, 1/4] \times [-1, 1] \times [0, 1/2].$$

Next, the Xu' algorithms [29] can be applied to the same above problem. They are formulated as follows:

$$x_{n+1} = T_{100} T_{99} \ldots T_1 x_n \quad n \geq 1, \tag{26}$$

where

$$T_i := P_{L_i}\Big(x_n - \gamma \sum_{j=1}^{50} \beta_j T^*(I - P_{K_j})Tx_n\Big), \quad i = 1, 2, \ldots, 100.$$

$$x_{n+1} = \sum_{i=1}^{100} \lambda_i P_{L_i}\Big(x_n - \gamma \sum_{j=1}^{50} \beta_j T^*(I - P_{K_j})Tx_n\Big), \quad n \geq 1 \tag{27}$$

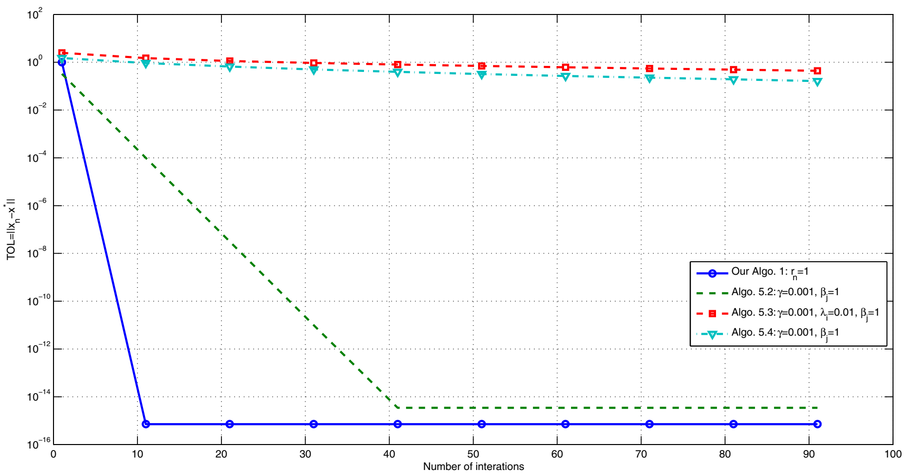
and

$$x_{n+1} = P_{L_{[n+1]}}\Big(x_n - \gamma \sum_{j=1}^{50} \beta_j T^*(I - P_{K_j})Tx_n\Big), \quad n \geq 1 \tag{28}$$

where $\lambda_i > 0$ for all i such that $\sum_{i=1}^{N} \lambda_i = 1$, $\beta_j > 0$ for all j, $0 < \gamma < 2/L$ with $L = \|T\|^2 \sum_{i=1}^{50} \beta_j$ and $C_{[n]} = C_{n \bmod N}$ and the mod function takes values in $\{1, 2, \ldots, N\}$.

Remark 5.2 In [29], Xu showed that the sequence $\{x_n\}$ is defined by (26), (27) and (28) converges weakly to an element $x^\dagger \in S$. Moreover, it is clear to see that if the sequence $\{x_n\}$ converges strongly to $x^\dagger \in S$, then $x^\dagger = P_S x_0$.

If we choose the starting point $x_0 = (-1, -2, 3)$, then $x^\dagger = (0, -1, 1/2) = P_S x_0$. We test our algorithm with $r_n = 1$ for all $n \geq 1$. It is not hard to show that $\|T\| = 4$. Hence, if we take $\beta_j = 1$ for all j, then $L = \|T\|^2 \sum_{i=1}^{50} \beta_j = 800$. Thus, we can apply Xu's algorithm with the collections of the parameters $\lambda_i = 1/100$ for all i and $\gamma = 1/1000$. In these cases, we obtain the following figure result:



where, the function TOL is defined by $\mathrm{TOL}_n := \|x_n - x^\dagger\|$ is the error between the approximation solution x_n and exactly solution $x^\dagger$ of considered problem.

Example 5.3 Now, we consider problem (25) with

$$L_i = [0, i] \times [-i, 2i - 1] \times [1 - i, 1 + i] \subset \mathbb{R}^3, \quad i = 1, 2, \ldots, 100,$$

$$K_j = [1 - j, j] \times [1 - j, 2j] \times [0, 1 + j] \subset \mathbb{R}^3, \quad j = 1, 2, \ldots, 50,$$

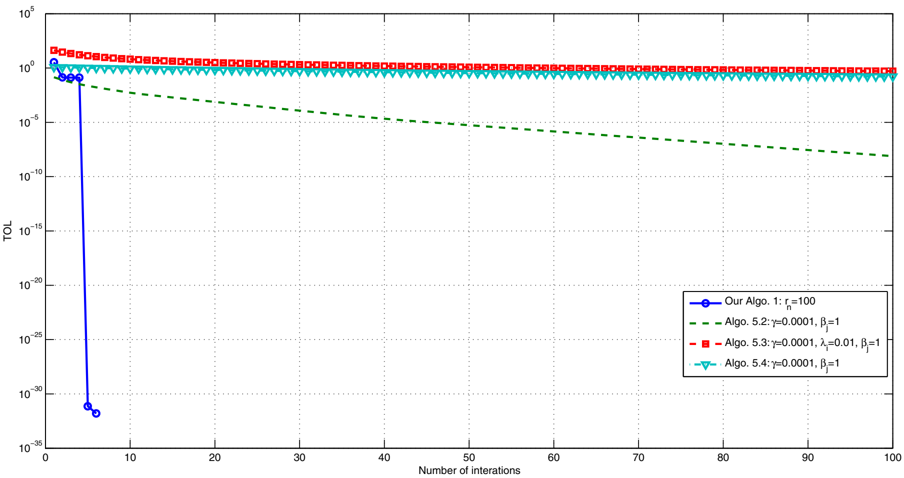
and the bounded linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$Tx = \begin{pmatrix} 5 & 10 & 5 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

It is not difficult to verify that $\|T\| = 15$.

Obviously, $S \neq \emptyset$ because of the point $(0, 0, 0) \in S$. We test our algorithm with $r_n = 100$ for all $n \geq 1$. We use Xu's algorithm with the collections of the parameters $\beta_j = 1$ for all j, $L = 11250$, $\lambda_i = 1/100$ for all i and $\gamma = 1/10,000$. With the starting point $x_0 = (3, -2, 5)$, we obtain the following figure result:



where, the function TOL is defined by

$$\mathrm{TOL}_n := \frac{1}{100} \sum_{i=1}^{100} \|x_n - P_{L_i} x_n\|^2 + \frac{1}{50} \sum_{j=1}^{50} \|Tx_n - P_{K_j} Tx_n\|^2.$$

The above figures of numerical examples show that our new algorithm enjoys a faster rate of convergence and needs less computation time than algorithms (26), (27), and (28).

References

1. Agarwal, R.P., O'Regan, D., Sahu, D.R.: Fixed point theory for Lipschitzian-type mappings with applications. Springer (2009)
2. Alber, Y.I.: Metric and generalized projections in Banach spaces: properties and applications. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15–50 (1996)
3. Alber, Y.I., Reich, S.: An iterative method for solving a class of nonlinear operator in Banach spaces. Panamer. Math. J. **4**, 39–54 (1994)
4. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**(2), 441–453 (2002)
5. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **18**, 103–120 (2004)
6. Byrne, C., Censor, Y., Gibali, A., Reich, S.: The split common null point problem. J. Nonlinear Convex Anal. **13**, 759–775 (2012)
7. Browder, F.E.: Nonlinear maximal monotone operators in Banach spaces. Math. Ann. **175**, 89–113 (1968)
8. Burachik, R.S., Iusem, A.N., Svaiter, B.F.: Enlargement of monotone operators with applications to variational inequalities. Set-Valued Anal. **5**, 159–180 (1997)
9. Burachik, R.S., Svaiter, B.F.: ε-Enlargements of maximal monotone operators in Banach spaces. Set-Valued Anal. **7**, 117–132 (1999)
10. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms. **8**(2-4), 221–239 (1994)
11. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its application. Inverse Probl. **21**, 2071–2084 (2005)
12. Censor, Y., Segal, A.: The split common fixed point problem for directed operators. J. Convex Anal. **16**, 587–600 (2009)
13. Censor, Y., Gibali, A., Reich, R.: Algorithms for the split variational inequality problems. Numer. Algo. **59**, 301–323 (2012)
14. Dadashi, V.: Shrinking projection algorithms for the split common null point problem. Bull. Aust. Math. Soc. **99**(2), 299–306 (2017)
15. Goebel, K., Kirk, W.A.: Topics in metric fixed point theory. Cambridge Stud. Adv Math., vol. 28. Cambridge Univ. Press, Cambridge (1990)
16. Kamimura, S., Takahashi, W.: Strong convergence of a proximal-type algorithm in a Banach space. SIAM J. Optim. **13**, 938–945 (2002)
17. Masad, E., Reich, S.: A note on the multiple-set split convex feasibility problem in Hilbert space. J. Nonlinear Convex Anal. **8**, 367–371 (2007)
18. Mosco, U.: Convergence of convex sets and of solutions of variational inequalities. Adv. Math. **3**, 510–585 (1969)
19. Moudafi, A.: The split common fixed point problem for demicontractive mappings. Inverse Probl. **26**(5), 055007 (2010)
20. Reich, S.: Book review: Geometry of Banach spaces, duality mappings and nonlinear problems. Bull. Amer. Math. Soc. **26**, 367–370 (1992)
21. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. Pacific J. Math. **33**(1), 209–216 (1970)
22. Rockafellar, R.T.: On the maximality of sums of nonlinear monotone operators. Trans. Amer. Math. Soc. **149**, 75–88 (1970)
23. Takahashi, W.: Convex analysis and approximation of fixed points. Yokohama Publishers Yokohama (2000)
24. Takahashi, W.: The split feasibility problem in Banach spaces. J. Nonlinear Convex Anal. **15**, 1349–1355 (2014)
25. Takahashi, W.: The split feasibility problem and the shrinking projection method in Banach spaces. J. Nonlinear Convex Anal. **16**(7), 1449–1459 (2015)
26. Takahashi, S., Takahashi, W.: The split common null point problem and the shrinking projection method in Banach spaces. Optimization **65**(2), 281–287 (2016)
27. Takahashi, W.: The split common null point problem in Banach spaces. Arch. Math. **104**, 357–365 (2015)

28. Tsukada, M.: Convergence of best approximations in a smooth Banach space. J. Approx. Theory. **40**, 301–309 (1984)
29. Xu, H.K.: A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem. Inverse Probl. **22**, 2021–2034 (2006)
30. Xu, H.K.: Iterative methods for the split feasibility problem in infinite dimensional Hilbert spaces. Inverse Probl. **105018**, 26 (2010)
31. Yang, Q.: The relaxed CQ algorithm for solving the problem split feasibility problem. Inverse Probl. **20**, 1261–1266 (2004)
32. Wang, F., Xu, H.K.: Cyclic algorithms for split feasibility problems in Hilbert spaces. Nonlinear Anal. **74**, 4105–4111 (2011)