Numer Algor (2019) 81:269–291
https://doi.org/10.1007/s11075-018-0547-6

CrossMark

ORIGINAL PAPER

Convergence of an extragradient-type method for variational inequality with applications to optimal control problems

Phan Tu Vuong[1,2] **· Yekini Shehu**[3,4]

Received: 28 December 2017 / Accepted: 9 May 2018 / Published online: 28 May 2018

Abstract Our aim in this paper is to introduce an extragradient-type method for solving variational inequality with uniformly continuous pseudomonotone operator. The strong convergence of the iterative sequence generated by our method is established in real Hilbert spaces. Our method uses computationally inexpensive Armijo-type linesearch procedure to compute the stepsize under reasonable assumptions. Finally, we give numerical implementations of our results for optimal control problems governed by ordinary differential equations.

Keywords Variational inequality · Pseudomonotone operator · Strong convergence · Hilbert spaces · Optimal control problem

The research of the second author is supported by the Alexander von Humboldt-Foundation.

✉ Yekini Shehu
yekini.shehu@unn.edu.ng

Phan Tu Vuong
phantuvuong@tdt.edu.vn

1 Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam

2 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

3 Department of Mathematics, University of Nigeria, Nsukka, Nigeria

4 Institute of Mathematics, Campus Hubland Nord, University of Würzburg, Emil-Fischer-Str. 30, 97074 Würzburg, Germany

1 Introduction

The theory of variational inequality has served as an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework (see, for example, [2, 3, 15, 28, 29]). Several numerical methods have been developed for solving variational inequalities and related optimization problems, see the monographs [14, 29] and references therein.

Pseudomonotone operators in the sense of Karamardian were introduced back in 1976 as a generalization of monotone operators. The notion of pseudomonotone operator has been studied for 40 years and has found many applications in variational inequalities and economics. In case of gradient maps, this generalized monotonicity characterizes generalized convexity of the underlying function [22].

In finite dimensional spaces, it was proven that the extragradient method (6), introduced by Korpelevich [30], is globally convergent if the operator in the variational inequality is monotone and Lipschitz continuous on feasible set provided stepsize is sufficiently small. It is a known fact [14, Theorem 12.2.11] that the extragradient method can be successfully applied for solving pseudo-monotone variational inequalities. For more details on variational inequalities in finite dimensional spaces, please see [18, 20, 25, 37, 46, 50, 51, 53, 54].

The extragradient method has been recently extended for solving variational inequalities in infinite dimensional Hilbert spaces. Some results regarding weak convergence [23, 39, 52] as well as strong convergence [40, 47–49, 58] are obtained. A typical assumptions for proving the convergence of these results is that the operator in the variational inequality is monotone (and Lipschitz continuous). Exceptions are the papers [9, 56] where the operator is supposed to be pseudomonotone. However, to obtain the convergence, beside the monotonicity and Lipschitz continuity, an additional condition on the operator has to be posed: *The operator maps a weak convergence sequence to a strong convergence sequence.* As admitted by the authors in [9], this assumption is very strong, which restricts the applications of the method, for example, even the identity operator does not satisfy such assumption.

Kraikaew and Saejung [31] recently proved the strong convergence of the iterative sequence generated by a combination of subgradient extragradient method of Censor et al. in [11, 12] and Halpern method [16] for the problem of finding a solution (see (4) below) when the operator in the variational inequality is *monotone and Lipschitz continuous* in real Hilbert spaces.

An obvious disadvantage of algorithms [12, 31], which impedes their wide use, is the assumption that the *Lipschitz constant of the monotone operator is known* or admits a simple estimate. Moreover, in many problems, operators may not satisfy the Lipschitz condition and the operator may not even be monotone as our Example 5.4 shows in Section 5.

In this present paper, we propose an algorithm which is a modification of the subgradient extragradient algorithm and Halpern method with a new line search rule generating the step size for variational inequalities with pseudomonotone non-Lipschitz operator. We weaken the assumption *weak-strong* continuity proposed in [9, 56] by *weak-weak* continuity, i.e., the operator *maps a weak convergence sequence* *to a weak convergence sequence*. This assumption is known as sequentially weakly continuous in the literature. Clearly, this assumption is weaker and more reasonable. We prove strong convergence without any further assumptions either on the operator or on the iterative sequence. An obvious difference between our method and that of the existing ones in [25] is the line search rule generating the stepsize with pseudomonotone operator. Simply put, our contributions in this paper are the following:

- The operator involved needs not be monotone. Our result extends on many recent results (see, e.g., [23, 35, 36, 40]) where the operator is assumed to be monotone.
- The operator involved in the variational inequality needs not be Lipschitz continuous. This extends many recent results (see, e.g., [11, 12, 31, 40, 41]) on variational inequality where the involved operator has to be Lipschitz continuous.
- The strong convergence is obtained under reasonable assumptions. Our result complement results (see, e.g., [9, 32, 39]) where weak convergence results are obtained.

The paper is organized as follows: We first recall some basic definitions and results in Section 2. Some discussions about our projection-type method used in this paper are given in Section 3. The strong convergence of our projection type algorithm is then investigated in Section 4. Some numerical experiments can be found in Section 5. We conclude with some final remarks in Section 6.

2 Definitions and preliminaries

In this section, we give some definitions and basic results that will be used in our subsequent analysis. Throughout the paper, H always denotes a real Hilbert space.

Definition 2.1 Let $X \subseteq H$ be a nonempty subset. Then, a mapping $A : X \to H$ is called

(a) *monotone* on X if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in X$;

(b) *η-strongly monotone* on X if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2,\ \forall x, y \in X;$$

(c) *pseudomonotone* on X if, for all $x, y \in X$,

$$\langle Ax, y - x\rangle \geq 0 \Longrightarrow \langle Ay, y - x\rangle \geq 0;$$

(d) *Lipschitz continuous* on X if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|,\ \forall x, y \in X.$$

(e) *sequentially weakly continuous* if for each sequence $\{x_n\}$ we have: $\{x_n\}$ converges weakly to x implies $\{Ax_n\}$ converges weakly to Ax.

We note that every monotone operator is pseudomonotone but the converse is not true as seen in this example.

Example 2.2 The mapping $A : (0, \infty) \to (0, \infty)$, defined by $Ax = a/(a + x)$ with $a > 0$. It is clear that A is pseudomonotone but not monotone.

We next recall some properties of the projection, cf. [4] for more details. To this end, let C be a nonempty, closed, and convex subset of H. For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\|, \ \forall y \in C.$$

P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \ \ \forall x, y \in H. \tag{1}$$

In particular, we get from (1) that

$$\langle x - y, x - P_C y\rangle \geq \|x - P_C y\|^2, \ \ \forall x \in C, y \in H. \tag{2}$$

Furthermore, $P_C x$ is characterized by the properties

$$P_C x \in C \quad \text{and} \quad \langle x - P_C x, P_C x - y\rangle \geq 0, \ \forall y \in C. \tag{3}$$

Let us recall the definition of variational inequality problem and some iterative methods for solving this problem. Let C be a nonempty, closed and convex subset of H and $A : C \to H$ be a continuous mapping. The *variational inequality problem* (for short, VI(A, C)) is defined as find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \ \forall y \in C. \tag{4}$$

Let SOL denote the solution set of VI(A, C). It is well-known that x solves the VI(A, C) if and only if x solves the fixed point equation

$$x = P_C(x - \gamma Ax)$$

or, equivalently, x solves the residual equation

$$r_\gamma(x) = 0, \quad \text{where} \quad r_\gamma(x) := x - P_C(x - \gamma Ax) \tag{5}$$

for an arbitrary positive constant γ, see [15] for details. Therefore, the knowledge of fixed-point algorithms (see, for example, [14, 42]) can be used to solve (4).

The following projection type iterative method for solving (4) was introduced by Korpelevich in [30], which is of the form:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n) \\ x_{n+1} = P_C(x_n - \lambda_n Ay_n), \ \ n \geq 1, \end{cases} \tag{6}$$

where $\lambda_n > 0$ is a fixed number. The extragradient algorithm (6) can be incorporated with the Armijo-like stepsize rule, which is shown by Marcotte [37], Sun [50], and Iusem [18] in the form:

Algorithm 2.3

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \end{cases} \tag{7}$$

where $\lambda_n = \gamma \ell^{m_n} (\gamma > 0, \ell \in (0, 1))$ *and* m_n *is the smallest nonnegative integer* m *such that*

$$\|Ax_n - Ay_n\| \leq \mu \frac{\|x_n - y_n\|}{\lambda_n}, \quad \mu \in (0, 1).$$

Set

$$x_{n+1} = P_C(x_n - \lambda_n Ay_n).$$

The forthcoming method was introduced by Censor et al. [12], which is called the subgradient extragradient method:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda Ax_n), \\ T_n := \{w \in H : \langle x_n - \lambda Ax_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda Ay_n). \end{cases} \tag{8}$$

Inspired by this method, when A is monotone and Lipschitz continuous, Kraikaew and Saejung [31] recently proved the strong convergence of the iterative sequence generated by the following algorithm:

Algorithm 2.4

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda Ax_n), \\ T_n := \{w \in H : \langle x_n - \lambda Ax_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) P_{T_n}(x_n - \lambda Ay_n), \end{cases} \tag{9}$$

where $\lambda \in (0, \frac{1}{L})$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The following elementary lemma will be used in our convergence analysis.

Lemma 2.5 *The following statements hold in any real Hilbert space* H*:*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$*;*
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H$*;*
(iii) $2\langle x - y, x - z\rangle = \|x - y\|^2 + \|x - z\|^2 - \|y - z\|^2$ *for all* $x, y, z \in H$.

We next give some existing results from the literature which will be used in our proof of strong convergence.

Lemma 2.6 *([55]) Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 1,$$

where

(a) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\limsup_{n\to\infty} \sigma_n \leq 0$*;*
(c) $\gamma_n \geq 0$ $(n \geq 1)$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then, $a_n \to 0$ *as* $n \to \infty$.

Lemma 2.7 *([38]) Consider VI*(A, C) *in (4). If the mapping* $h : [0, 1] \to H$ *defined as* $h(t) := A(tx + (1-t)y)$ *is continuous for all* $x, y \in C$ *(i.e., h is hemicontinuous), then* $M(A, C) := \{x \in C : \langle Ay, y - x\rangle \geq 0, \forall y \in C\} \subset SOL$*. Moreover, if A is pseudomonotone, then SOL is closed, convex and* $M(A, C) = SOL$*.*

Lemma 2.8 *([19, 38]) Let* H_1 *and* H_2 *be two real Hilbert spaces. Suppose* $A : H_1 \to H_2$ *is uniformly continuous on bounded subsets of* H_1 *and M is a bounded subset of* H_1*. Then* $A(M)$ *is bounded.*

The following lemmas were given in $\mathbb{R}^n$ in [17]. The proof of the lemmas is the same if given in infinite dimensional real Hilbert spaces. Hence, we state the lemmas and omit the proof in real Hilbert spaces.

Lemma 2.9 *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let h be a real-valued function on H and define* $K := \{x \in C : h(x) \leq 0\}$*. If K is nonempty and h is Lipschitz continuous on C with modulus* $\theta > 0$*, then*

$$dist(x, K) \geq \theta^{-1} \max\{h(x), 0\}, \ \forall x \in C,$$

where $dist(x, K)$ *denotes the distance function from x to K.*

Lemma 2.10 *Let C be a nonempty closed and convex subset of a real Hilbert space H,* $y := P_C(x)$ *and* $x^* \in C$*. Then*

$$\|y - x^*\|^2 \leq \|x - x^*\|^2 - \|x - y\|^2. \tag{10}$$

3 Approximation method

In this section, we state our proposed projection-type method for solving VI(A, C) (4) and give some its associated properties. We start by giving these assumptions that we will assume to hold for the rest of this paper.

Assumption 3.1 Suppose that the following hold:

(a) The feasible set C is a nonempty, closed, and convex subset of the real Hilbert space H.
(b) $A : H \to H$ is a pseudomonotone, uniformly continuous and sequentially weakly continuous on bounded subsets of C.
(c) The solution set SOL of VI(A, C) is nonempty.

Observe that the assumption (a) implies that projections onto C are well-defined. Assumption (b) is new and weaker than the condition imposed in [9, 24, 56]. Assumption (c) is standard and has been used in several papers on variational inequality problems (see, e.g., [5, 19, 21])

In our convergence analysis, we assume that the following conditions hold for our iterative parameter sequence $\{\alpha_n\}$.

Assumption 3.2 (a) $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

These conditions are satisfied, e.g., for $\alpha_n = 1/(n+1)$ for all $n \in \mathbb{N}$.

We next give a precise statement of our projection-type method. To this end, we use the abbreviation

$$r(x) := r_1(x) = x - P_C(x - Ax)$$

for the residual from (5) with $\gamma = 1$.

Observe that if we take $y = x - Ax$ in (2), then we have

$$\langle Ax, r(x)\rangle \geq \|r(x)\|^2, \ \forall x \in C. \tag{11}$$

Algorithm 3.3
- ***Initialization:*** *Choose sequence* $\{\alpha_n\} \subset (0, 1)$ *such that the conditions from Assumption 3.2 hold,* $\sigma \in (0, 1)$, $\gamma \in (0, 1)$. *Let* $x_1 \in C$ *be a given starting point. Set* $n := 1$.
- ***Step 1:*** *Compute* $z_n := P_C(x_n - Ax_n)$. *If* $r(x_n) = x_n - z_n = 0$*: STOP. Otherwise*
- ***Step 2:*** *Compute* $y_n = x_n - \gamma^{k_n} r(x_n)$, *where* k_n *is the smallest nonnegative integer satisfying*

$$\langle Ay_n, r(x_n)\rangle \geq \frac{\sigma}{2}\|r(x_n)\|^2. \tag{12}$$

Set $\eta_n := \gamma^{k_n}$.
- ***Step 3:*** *Compute*

$$x_{n+1} = \alpha_n x_1 + (1 - \alpha_n)P_{C_n}(x_n), \tag{13}$$

where $C_n = \{x \in C : h_n(x) \leq 0\}$ *and*

$$h_n(x) := \langle Ay_n, x - y_n\rangle. \tag{14}$$

- ***Step 4:*** *Let* $n := n + 1$ *and go to Step 1.*

Observe that we are at a solution of VI(A, C) (4) if $x_n - z_n = 0$. In our convergence analysis, we will implicitly assume that this does not occur after finitely many iterations, so that Algorithm 3.3 generates an infinite sequence satisfying, in particular, $x_n - z_n \neq 0$ for all $n \in \mathbb{N}$.

Remark 3.4 (a) It is easy to see by a simple induction argument from Algorithm 3.3 that $x_n, y_n, z_n \in C$.
(b) By the uniform continuity (hence continuity) of A and (11), we see that Step 2 in Algorithm 3.3 is well-defined. Furthermore, if SOL $\neq \emptyset$, the Step 3 is well-defined since SOL $\subset C_n$ by the lemma below and hence $C_n \neq \emptyset$ for all $n \in \mathbb{N}$.

Lemma 3.5 *Let* $x^* \in$ *SOL and the function* h_n *be defined by (14). Then*

$$h_n(x_n) \geq \frac{\sigma\eta_n}{2}\|x_n - z_n\|^2$$

and $h_n(x^*) \leq 0$. *In particular, if* $x_n \neq z_n$, *then* $h_n(x_n) > 0$.

Proof Since $y_n = x_n - \eta_n(x_n - z_n)$, using (12), we have

$$\begin{aligned} h_n(x_n) &= \langle Ay_n, x_n - y_n\rangle \\ &= \eta_n\langle Ay_n, x_n - z_n\rangle \geq \eta_n\frac{\sigma}{2}\|x_n - z_n\|^2 \geq 0. \end{aligned}$$

If $x_n \neq z_n$, then $h_n(x_n) > 0$. Since $x^* \in \mathrm{SOL}$, we have

$$\langle Ax^*, y - x^*\rangle \geq 0, \ \forall y \in C,$$

and thus implies (by the fact $M(A, C) = \mathrm{SOL}$) that $h_n(x^*) = \langle Ay_n, x^* - y_n\rangle \leq 0$. □

4 Convergence analysis

For the rest of this paper, we define

$$w_n := P_{C_n}(x_n), \ \forall n \geq 1.$$

and

$$z := P_{\mathrm{SOL}}x_1.$$

In the next lemma, we prove a result that shows that the sequence $\{x_n\}$ generated by Algorithm 3.3 is bounded.

Lemma 4.1 *We have from Algorithm 3.3 that* $\{x_n\}$ *is bounded and*

$$\|w_n - z\|^2 \leq \|x_n - z\|^2 - \left(\frac{1}{M}\sigma\eta_n\|x_n - z_n\|^2\right)^2, \tag{15}$$

for some $M > 0$.

Proof By Lemma 2.10, we get (since $z \in C_n$) that

$$\|w_n - z\|^2 = \|P_{C_n}(x_n) - z\|^2 \leq \|x_n - z\|^2 - \|w_n - x_n\|^2. \tag{16}$$

Furthermore,

$$\begin{aligned} \|w_n - z\|^2 &= \|P_{C_n}(x_n) - z\|^2 \\ &\leq \|x_n - z\|^2 - \|P_{C_n}(x_n) - x_n\|^2 \\ &= \|x_n - z\|^2 - \mathrm{dist}^2(x_n, C_n). \end{aligned} \tag{17}$$

From (13) and (17), we have

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n\|x_1 - z\| + (1 - \alpha_n)\|w_n - z\| \\ &\leq \alpha_n\|x_1 - z\| + (1 - \alpha_n)\|x_n - z\| \\ &\leq \max\left\{\|x_n - z\|, \|x_1 - z\|\right\} \\ &\ \ \vdots \\ &\leq \max\left\{\|x_1 - z\|, \|x_1 - z\|\right\} \\ &= \|x_1 - z\|. \end{aligned}$$

This shows that $\{x_n\}$ is bounded. Since A is uniformly continuous on bounded subsets of C, then $\{Ax_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{Ay_n\}$ are bounded. In particular, there exists $M > 0$ such that $\|Ay_n\| \leq M/2$ for all $n \in \mathbb{N}$. Combining (17), Lemmas 2.9 and 3.5, we get

$$\begin{aligned} \|w_n - z\|^2 &\leq \|x_n - z\|^2 - \left(\frac{2}{M} h_n(x_n)\right)^2 \\ &\leq \|x_n - z\|^2 - \left(\frac{1}{M}\sigma\eta_n \|r(x_n)\|^2\right)^2 \\ &= \|x_n - z\|^2 - \left(\frac{1}{M}\sigma\eta_n \|x_n - z_n\|^2\right)^2, \end{aligned} \tag{18}$$

which yields (15). □

In the next lemma, we show that the weak cluster points of $\{x_n\}$ belong to the set SOL.

Lemma 4.2 *If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $p \in H$ and $\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0$, then $p \in SOL$.*

Proof By the definition of z_{n_k} together with (3), we have

$$\langle x_{n_k} - Ax_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq 0, \ \forall x \in C,$$

which implies that

$$\langle x_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq \langle Ax_{n_k}, x - z_{n_k}\rangle, \ \forall x \in C.$$

Hence,

$$\langle x_{n_k} - z_{n_k}, x - z_{n_k}\rangle + \langle Ax_{n_k}, z_{n_k} - x_{n_k}\rangle \leq \langle Ax_{n_k}, x - x_{n_k}\rangle, \ \forall x \in C. \tag{19}$$

Fix $x \in C$ and let $k \to \infty$ in (19). Since $\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0$, we have

$$0 \leq \liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k}\rangle \tag{20}$$

for all $x \in C$.

Now we choose a sequence $\{\epsilon_k\}_k$ of positive numbers decreasing and tending to 0. For each ϵ_k, we denote by N_k the smallest positive integer such that

$$\left\langle Ax_{n_j}, x - x_{n_j}\right\rangle + \epsilon_k \geq 0 \quad \forall j \geq N_k, \tag{21}$$

where the existence of N_k follows from (20). Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, $Ax_{N_k} \neq 0$ and, setting

$$v_{N_k} = \frac{Ax_{N_k}}{\|Ax_{N_k}\|^2},$$

we have $\left\langle Ax_{N_k}, v_{N_k}\right\rangle = 1$ for each k. Now, we can deduce from (21) that for each k

$$\left\langle Ax_{N_k}, x + \epsilon_k v_{N_k} - x_{N_k}\right\rangle \geq 0,$$

and, since A is pseudo-monotone, that

$$\left\langle A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - x_{N_k}\right\rangle \geq 0. \tag{22}$$

On the other hand, we have that $\{x_{n_k}\}$ converges weakly to p when $k \to \infty$. Since A is sequentially weakly continuous on C, $\{Ax_{n_k}\}$ converges weakly to Ap. We can suppose that $Ap \neq 0$ (otherwise, p is a solution). Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 < \|Ap\| \leq \lim \inf_{k\to\infty} \|Ax_{n_k}\|.$$

Since $\{x_{N_k}\} \subset \{x_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$\begin{aligned} 0 \leq \lim \sup_{k\to\infty} \|\epsilon_k v_{N_k}\| &= \lim \sup_{k\to\infty} \left(\frac{\epsilon_k}{\|Ax_{n_k}\|}\right) \\ &\leq \frac{\lim\sup_{k\to\infty} \epsilon_k}{\lim\inf_{k\to\infty} \|Ax_{n_k}\|} \leq \frac{0}{\|Ap\|} = 0, \end{aligned}$$

which implies that $\lim_{k\to\infty} \|\epsilon_k v_{N_k}\| = 0$. Hence, taking the limit as $k \to \infty$ in (22), we obtain

$$\langle Ax, x - p\rangle \geq 0.$$

In view of Lemma 2.7, this implies $p \in \mathrm{SOL}$. □

Remark 4.3 When the function A is monotone, it is not necessary to impose the sequential weak continuity on A. Indeed, in that case, it follows from (19) and the monotonicity of A that

$$\begin{aligned} \langle x_{n_k} - z_{n_k}, x - z_{n_k}\rangle + \langle Ax_{n_k}, z_{n_k} - x_{n_k}\rangle &\leq \langle Ax_{n_k}, x - x_{n_k}\rangle \\ &\leq \langle Ax, x - x_{n_k}\rangle \quad \forall x \in C. \end{aligned}$$

Letting $k \to +\infty$ in the last inequality, remembering that $\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0$ for all k, we have

$$\langle Ax, x - p\rangle \geq 0 \quad \forall x \in C.$$

We are now in position to prove our main strong convergence result.

Theorem 4.4 *Let Assumptions 3.1 and 3.2 hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.3 strongly converges to a solution z.*

Proof Then, from Algorithm 3.3, we have by Lemma 2.5 (ii) that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(x_1 - z) + (1 - \alpha_n)(w_n - z)\|^2 \\ &\leq (1 - \alpha_n)\|w_n - z\|^2 + 2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle. \end{aligned} \tag{23}$$

This implies from (18) that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle \\ &\quad - \left(\frac{(1 - \alpha_n)\sigma\eta_n}{M}\|x_n - z_n\|^2\right)^2 \\ &= (1 - \alpha_n)\|x_n - z\|^2 \\ &\quad + \alpha_n\left(2\langle x_1 - z, x_{n+1} - z\rangle - \left(\frac{(1 - \alpha_n)\sigma\eta_n}{\alpha_n M}\|x_n - z_n\|^2\right)^2\right) \\ &= (1 - \alpha_n)a_n + \alpha_n b_n, \end{aligned} \tag{24}$$

where $a_n := \|x_n - z\|^2$ and $b_n := 2\langle x_1 - z, x_{n+1} - z\rangle - \left(\frac{(1-\alpha_n)\sigma\eta_n}{\alpha_n M}\|x_n - z_n\|^2\right)^2$.

We next show that $\limsup_{n\to\infty} b_n < \infty$. Since $\{x_n\}$ is bounded, we have

$$\begin{aligned} b_n &\leq 2\langle x_1 - z, x_{n+1} - z\rangle \\ &\leq 2\|x_1 - z\|\|x_{n+1} - z\| < \infty, \end{aligned}$$

and this implies that $\limsup_{n\to\infty} b_n < \infty$. We now show that $\limsup_{n\to\infty} b_n \geq -1$. Assume the contrary that $\limsup_{n\to\infty} b_n < -1$. Then, there exists $n_0 \in \mathbb{N}$ such that $b_n < -1$ for all $n \geq n_0$. For all $n \geq n_0$, we get from (24) that

$$\begin{aligned} a_{n+1} &\leq (1-\alpha_n)a_n + \alpha_n b_n < (1-\alpha_n)a_n - \alpha_n \\ &= a_n - \alpha_n(a_n + 1) \leq a_n - \alpha_n. \end{aligned}$$

By induction, we get

$$a_{n+1} \leq a_{n_0} - \sum_{i=n_0}^{n} \alpha_i.$$

Taking limit superior of both sides of the last inequality, we have

$$\limsup_{n\to\infty} a_n \leq a_{n_0} - \lim_{n\to\infty} \sum_{i=n_0}^{n} \alpha_i = -\infty.$$

This contradicts the fact that $\{a_n\}$ is a nonnegative real sequence. Therefore, $\limsup_{n\to\infty} b_n \geq -1$.

Using (16) and Lemma 2.5 (ii) in (13), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(x_1 - z) + (1-\alpha_n)(w_n - z)\|^2 \\ &\leq (1-\alpha_n)\|w_n - z\|^2 + 2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle \\ &\quad - (1-\alpha_n)\|w_n - x_n\|^2. \end{aligned} \tag{25}$$

We next consider two cases:

Case 1: Assume that there exists $n_0 \in \mathbb{N}$ such that $\|x_{n+1} - z\| \leq \|x_n - z\|, \forall n \geq n_0$. Then $\lim_{k\to\infty} \|x_n - z\|$ exists. From (25), we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle \\ &\quad - (1-\alpha_n)\|w_n - x_n\|^2 \\ &\leq \|x_n - z\|^2 + \alpha_n M_1 - (1-\alpha_n)\|w_n - x_n\|^2, \end{aligned} \tag{26}$$

for some $M_1 > 0$. Thus,

$$(1-\alpha_n)\|w_n - x_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M_1 \to 0, n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} \|w_n - x_n\| = 0.$$

Now, since $\limsup_{n\to\infty} b_n$ is finite and $\{x_n\}$ is bounded, we can take a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightharpoonup p \in C$ and

$$\limsup_{n\to\infty} b_n = \lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty}\left(2\langle x_1 - z, x_{n_k+1} - z\rangle - \left(\frac{(1-\alpha_{n_k})\sigma\eta_{n_k}}{\alpha_{n_k}M}\|x_{n_k} - z_{n_k}\|^2\right)^2\right). \tag{27}$$

Since $\{\langle x_1 - z, x_{n_k+1} - z\rangle\}$ is a bounded real sequence, without loss of generality, we may assume there exists the limit

$$\lim_{k\to\infty}\langle x_1 - z, x_{n_k+1} - z\rangle.$$

We obtain from (27) that

$$\lim_{k\to\infty}\frac{(1-\alpha_{n_k})\sigma\eta_{n_k}}{\alpha_{n_k}M}\|x_{n_k} - z_{n_k}\|^2$$

exists. This implies that the sequence $\left\{\frac{\sigma\eta_{n_k}}{\alpha_{n_k}M}\|x_{n_k} - z_{n_k}\|^2\right\}$ is bounded. By Assumption 3.2, we get

$$\lim_{k\to\infty}\eta_{n_k}\|x_{n_k} - z_{n_k}\|^2 = 0.$$

We now claim that

$$\lim_{k\to\infty}\|x_{n_k} - z_{n_k}\| = 0.$$

Indeed, let us distinguish two cases depending on the behavior of (the bounded) sequence of stepsizes $\{\eta_{n_k}\}$.

(i): If $\liminf_{k\to\infty}\eta_{n_k} > 0$, then

$$0 \le \|r(x_{n_k})\|^2 = \frac{\eta_{n_k}\|r(x_{n_k})\|^2}{\eta_{n_k}}$$

and this implies that

$$\begin{aligned}\limsup_{k\to\infty}\|r(x_{n_k})\|^2 &\le \limsup_{k\to\infty}\left(\eta_{n_k}\|r(x_{n_k})\|^2\right)\left(\limsup_{k\to\infty}\frac{1}{\eta_{n_k}}\right)\\ &= \left(\limsup_{k\to\infty}\eta_{n_k}\|r(x_{n_k})\|^2\right)\frac{1}{\liminf_{k\to\infty}\eta_{n_k}}\\ &= 0.\end{aligned}$$

Hence, $\limsup_{k\to\infty}\|r(x_{n_k})\|^2 = 0$. Therefore,

$$\lim_{k\to\infty}\|x_{n_k} - z_{n_k}\| = \lim_{k\to\infty}\|r(x_{n_k})\| = 0.$$

(ii): If $\liminf_{k\to\infty}\eta_{n_k} = 0$. Subsequencing if necessary, we may assume without loss of generality that $\lim_{k\to\infty}\eta_{n_k} = 0$ and $\lim_{k\to\infty}\|x_{n_k} - z_{n_k}\| = a \ge 0$.

Define $\bar{y}_k := \frac{1}{\gamma}\eta_{n_k} z_{n_k} + \left(1 - \frac{1}{\gamma}\eta_{n_k}\right) x_{n_k}$ or, equivalently, $\bar{y}_k - x_{n_k} = \frac{1}{\gamma}\eta_{n_k}(z_{n_k} - x_{n_k})$. Since $\{z_{n_k} - x_{n_k}\}$ is bounded and since $\lim_{k\to\infty} \eta_{n_k} = 0$ holds, it follows that

$$\lim_{k\to\infty} \|\bar{y}_k - x_{n_k}\| = 0. \tag{28}$$

From the stepsize rule and the definition of $\bar{y}_k$, we have

$$\langle A\bar{y}_k, x_{n_k} - z_{n_k}\rangle < \frac{\sigma}{2}\|x_{n_k} - z_{n_k}\|^2, \ \forall k \in \mathbb{N},$$

or equivalently

$$2\langle Ax_{n_k}, x_{n_k} - z_{n_k}\rangle + 2\langle A\bar{y}_k - Ax_{n_k}, x_{n_k} - z_{n_k}\rangle < \sigma\|x_{n_k} - z_{n_k}\|^2, \ \forall k \in \mathbb{N}.$$

Setting $t_{n_k} := x_{n_k} - Ax_{n_k}$, we obtain form the last inequality that

$$2\langle x_{n_k} - t_{n_k}, x_{n_k} - z_{n_k}\rangle + 2\langle A\bar{y}_k - Ax_{n_k}, x_{n_k} - z_{n_k}\rangle < \sigma\|x_{n_k} - z_{n_k}\|^2, \ \forall k \in \mathbb{N}.$$

Using Lemma 2.5 (iii), we get

$$2\langle x_{n_k} - t_{n_k}, x_{n_k} - z_{n_k}\rangle = \|x_{n_k} - z_{n_k}\|^2 + \|x_{n_k} - t_{n_k}\|^2 - \|z_{n_k} - t_{n_k}\|^2.$$

Therefore,

$$\|x_{n_k} - t_{n_k}\|^2 - \|z_{n_k} - t_{n_k}\|^2 < (\sigma - 1)\|x_{n_k} - z_{n_k}\|^2 - 2\langle A\bar{y}_k - Ax_{n_k}, x_{n_k} - z_{n_k}\rangle \ \forall k \in \mathbb{N}.$$

Since A is uniformly continuous on bounded subsets of C and (28), if $a > 0$ then the right hand side of the last inequality converges to $(\sigma - 1)a < 0$ as $k \to \infty$. From the last inequality, we have

$$\limsup_{k\to\infty} \left(\|x_{n_k} - t_{n_k}\|^2 - \|z_{n_k} - t_{n_k}\|^2\right) \le (\sigma - 1)a < 0.$$

For $\epsilon = -(\sigma - 1)a/2 > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x_{n_k} - t_{n_k}\|^2 - \|z_{n_k} - t_{n_k}\|^2 \le (\sigma - 1)a + \epsilon = (\sigma - 1)a/2 < 0 \quad \forall k \in \mathbb{N}, k \ge N,$$

leading to

$$\|x_{n_k} - t_{n_k}\| < \|z_{n_k} - t_{n_k}\| \quad \forall k \in \mathbb{N}, k \ge N,$$

which is a contradiction to the definition of $z_{n_k} = P_C(x_{n_k} - Ax_{n_k})$. Hence, $a = 0$.

From the Algorithm 3.3, we have

$$\|y_{n_k} - x_{n_k}\| = \eta_{n_k}\|r(x_{n_k})\| \to 0, \ \ k \to \infty. \tag{29}$$

Furthermore, we get

$$\|x_{n_k+1} - w_{n_k}\| = \alpha_{n_k}\|x_1 - w_{n_k}\| \to 0, \ k \to \infty. \tag{30}$$

Since $\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0$, we obtain

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0, \ k \to \infty.$$

Since $x_{n_k} \rightharpoonup p \in \mathrm{SOL}$ and $\|x_{n_k+1} - x_{n_k}\| \to 0$ as $k \to \infty$, we get $x_{n_k+1} \rightharpoonup p \in \mathrm{SOL}$. So

$$\begin{aligned} \limsup_{n\to\infty} b_n &\le 2 \lim_{k\to\infty} \langle x_1 - z, x_{n_k+1} - z\rangle \\ &= 2\langle x_1 - z, p - z\rangle \le 0, \end{aligned} \tag{31}$$

where the last inequality follows by the fact that $z = P_{\text{SOL}}x_1$ and $p \in \text{SOL}$. Using Lemma 2.6 and (31) in (24), we obtain $\lim_{n\to\infty} \|x_n - z\| = 0$. Thus, we have that $x_n \to z$ as $n \to \infty$.

Case 2: Assume that there is no $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}_{n=n_0}^{\infty}$ is monotonically decreasing. The technique of proof used here is adapted from [33, 34]. Set $\Gamma_n = \|x_n - z\|^2$ for all $n \geq 1$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\},$$

i.e. $\tau(n)$ is the largest number k in $\{1, \ldots, n\}$ such that Γ_k increases at $k = \tau(n)$; note that, in view of Case 2, this $\tau(n)$ is well-defined for all sufficiently large n. Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

From (24), we get

$$\begin{aligned}
\|x_{\tau(n)+1} - z\|^2 &\leq (1 - \alpha_{\tau(n)})\|x_{\tau(n)} - z\|^2 + 2\alpha_{\tau(n)}\langle x_1 - z, x_{\tau(n)+1} - z\rangle \\
&\quad -\left(\frac{(1 - \alpha_{\tau(n)})\sigma\eta_{\tau(n)}}{M}\|x_{\tau(n)} - z_{\tau(n)}\|^2\right)^2 \\
&\leq \|x_{\tau(n)} - z\|^2 - \left(\frac{(1 - \alpha_{\tau(n)})\sigma\eta_{\tau(n)}}{M}\|x_{\tau(n)} - z_{\tau(n)}\|^2\right)^2 \\
&\quad +\alpha_{\tau(n)}M_3,
\end{aligned}$$

for some $M_3 > 0$. Therefore,

$$\begin{aligned}
\left(\frac{(1 - \alpha_{\tau(n)})\sigma\eta_{\tau(n)}}{M}\|x_{\tau(n)} - z_{\tau(n)}\|^2\right)^2 &\leq \|x_{\tau(n)} - z\|^2 - \|x_{\tau(n)+1} - z\|^2 + \alpha_{\tau(n)}M_3 \\
&\leq \alpha_{\tau(n)}M_3 \to 0, \quad n \to \infty.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \eta_{\tau(n)}\|x_{\tau(n)} - z_{\tau(n)}\|^2 = 0.$$

Just as in Case 1, we can show that

$$\lim_{n\to\infty} \|x_{\tau(n)} - z_{\tau(n)}\|^2 = 0.$$

Similarly, we can obtain from (25) that

$$\lim_{n\to\infty} \|w_{\tau(n)} - x_{\tau(n)}\| = 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$, which converges weakly to some $p \in \text{SOL}$. Similarly, as in Case 1 above, we can show that

$$\|y_{\tau(n)} - x_{\tau(n)}\| = \eta_{\tau(n)}\|r(x_{\tau(n)})\| \to 0, \quad n \to \infty. \tag{32}$$

and

$$\|x_{\tau(n)+1} - w_{\tau(n)}\| = \alpha_{\tau(n)}\|x_1 - w_{\tau(n)}\| \to 0, \; n \to \infty. \tag{33}$$

Since $\lim_{n\to\infty} \|w_{\tau(n)} - x_{\tau(n)}\| = 0$, we obtain

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \leq \|x_{\tau(n)+1} - w_{\tau(n)}\| + \|w_{\tau(n)} - x_{\tau(n)}\| \to 0, \; n \to \infty.$$

Since $x_{\tau(n)} \rightharpoonup p \in SOL$ and $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0$ as $n \to \infty$, we get $x_{\tau(n)+1} \rightharpoonup p \in SOL$. So $\limsup_{n\to\infty} \langle x_1 - z, x_{\tau(n)+1} - z\rangle \leq 0$. Following (24), we obtain

$$\|x_{\tau(n)+1} - z\|^2 \leq (1-\alpha_{\tau(n)})\|x_{\tau(n)} - z\|^2 + 2\alpha_{\tau(n)}\langle x_1 - z, x_{\tau(n)+1} - z\rangle. \quad (34)$$

By Lemma 2.6 and using Assumptions 3.2, we have from (34) that $\lim_{n\to\infty} \|x_{\tau(n)} - z\| = 0$ which, in turn, implies $\lim_{n\to\infty} \|x_{\tau(n)+1} - z\| = 0$. Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ (observe that $\tau(n) \leq n$ for $n \geq n_0$ and consider the three cases: $\tau(n) = n$, $\tau(n) = n - 1$ and $\tau(n) < n - 1$. For the first and second cases, it is obvious that $\Gamma_n \leq \Gamma_{\tau(n)+1}$, for $n \geq n_0$. For the third case $\tau(n) \leq n - 2$, we have from the definition of $\tau(n)$ and for any integer $n \geq n_0$ that $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n - 1$. Thus, $\Gamma_{\tau(n)+1} \geq \Gamma_{\tau(n)+2} \geq \cdots \geq \Gamma_{n-1} \geq \Gamma_n$). As a consequence, we obtain for all sufficiently large n that $0 \leq \Gamma_n \leq \Gamma_{\tau(n)+1}$. Hence $\lim_{n\to\infty} \Gamma_n = 0$. Therefore, $\{x_n\}$ converges strongly to z.

□

Remark 4.5 (a) We emphasize here that in this paper, the iterative method presented for solving variational inequality involving continuous pseudomonotone mapping, which is weaker than Lipschitz continuous, monotone mapping assumed in [11, 12]. Also, our result complements the weak convergence results obtained in [32, 57].

(b) All our results in this paper are obtained without any extra condition either on the sequence of iterates or on the operator involved unlike the results of Ceng and Yao [10] and Zeng and Yao [58].

5 Applications to optimal control and numerical experiments

In this section, we provide computational experiments insulating our newly proposed method considered in Section 3 for solving variational inequality arising in optimal control problem. Let $0 < T \in \mathbb{R}$, we denote by $L_2([0, T], \mathbb{R}^m)$ the Hilbert space of square integrable, measurable vector function $u : [0, T] \to \mathbb{R}^m$ with inner product

$$\langle u, v\rangle = \int_0^T \langle u(t), v(t)\rangle dt,$$

and norm

$$\|u\|_2 = \sqrt{\langle u, u\rangle} < \infty.$$

We consider the following optimal control problem:

$$u^*(t) = \text{argmin}\{f(u) : u \in U\} \quad (35)$$

on the interval $[0, T]$, assuming that such a control exists. Here, U is the set of admissible controls, which has the form of an m-dimensional box and consists of piecewise continuous function:

$$U = \left\{u(t) \in L_2([0, T], \mathbb{R}^m) : u_i(t) \in [u_i^-, u_i^+], i = 1, 2, \ldots, m\right\}.$$

Specially, the control can be bang-bang (piecewise constant function).

The terminal objective has the form

$$f(u) = \phi(x(T)),$$

where ϕ is a convex and differentiable function, defined on the attainability set.

Suppose that the trajectory $x(t) \in L_2([0, T]$ satisfies constrains in the form of a system of linear differential equation:

$$\dot{x}(t) = D(t)x(t) + B(t)u(t), \quad x(0) = x_0, \quad t \in [0, T],$$

where $D(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ are given continuous matrices for every $t \in [0, T]$. By the Pontryagin maximum principle, there exists a function $p^* \in L_2([0, T]$ such that the triple (x^*, p^*, u^*) solves for a.e. $t \in [0, T]$ the system

$$\begin{cases} \dot{x}^*(t) = D(t)x^*(t) + B(t)u^*(t) \\ x^*(0) = x_0, \end{cases} \tag{36}$$

$$\begin{cases} \dot{p}^*(t) = -D(t)^\top p^*(t) \\ p^*(T) = \nabla g(x(T)), \end{cases} \tag{37}$$

$$0 \in B(t)^\top p^*(t) + N_U(u^*(t)), \tag{38}$$

where $N_U(u)$ is the normal cone to U at u defined by

$$N_U(u) := \begin{cases} \emptyset & \text{if } u \notin U \\ \{\ell \in H : \langle \ell, v - u\rangle \le 0, \ \forall v \in U\} & \text{if } u \in U. \end{cases}$$

Denoting $Gu(t) := B(t)^\top p(t)$, it is known that Gu is the gradient of the objective cost function f [26, 43]. We can write (38) as a variational inequality

$$\langle Gu^*, v - u^*\rangle \ge 0, \ \forall v \in U. \tag{39}$$

An extragradient method for solving (39) has been recently investigated [26]. However, only weak convergence was obtained. Our newly algorithm guarantees the strong convergence.

We now present some numerical examples to confirm the theoritical finding in Section 4. To make the algorithm implementable, we discretize the continuous functions. We choose a natural number N and define the *mesh size* $h := T/N$. We identity any discretized control $u^N := (u_0, u_1, \ldots, u_{N-1})$ with its piece-wise constant extension:

$$u^N(t) = u_i \text{ for } t \in [t_i, t_{i+1}), \ i = 0, 1, \ldots, N.$$

Moreover, we identity any discretized state $x^N := (x_0, x_1, \ldots, x_N)$ with its piecewise linear interpolation

$$x^N(t) = x_i + \frac{t - t_i}{h}(x_{i+1} - x_i), \text{ for } t \in [t_i, t_{i+1}), \ i = 0, 1, \ldots, N-1.$$

Similarly for the co-state variable $p^N := (p_0, p_1, \ldots, p_N)$.

There are several discretization techniques in the theory of numerical ODE, such as Euler and Runge-Kutta methods [8]. In this paper, we consider the Euler one. That is, at each iteration, the system of ODEs (36) and (37) is solved by the Euler method

$$\begin{cases} x_{i+1}^N = x_i^N + h\left[A(t_i)x_i^N + B(t_i)u_i^N\right] \\ x(0) = x_0, \end{cases} \tag{40}$$

$$\begin{cases} p_i^N = p_{i+1}^N + hA(t_i)^T p_{i+1}^N \\ p(N) = \nabla g(x_N). \end{cases} \tag{41}$$

It is well known that the Euler discretization has the error estimate $O(h)$ [1, 6, 13]. This means that the difference between the discretized solution $u^N(t)$ and the original solution $u^*(t)$ is proportional to the mesh size h, i.e., there exists a constant $C > 0$ such that

$$\|u^N - u^*\| \leq Ch.$$

All codes are implement in Matlab 2010*b* and we perform all computation on a Windows Desktop with an Itel(R) Core(TM) i7-2600CPU at 3.4 GHz and 8.00 GB of memory.

The following example is taken from [44, Example 7].

Example 5.1 (Control of a harmonic oscillator)

$$\begin{array}{ll} \text{minimize} & x_2(3\pi) \\ \text{subject to} & \dot{x_1}(t) = x_2(t), \\ & \dot{x_2}(t) = -x_1(t) + u(t), \ \forall t \in [0, 3\pi], \\ & x(0) = 0, \\ & u(t) \in [-1, 1]. \end{array} \tag{42}$$

The exact optimal control in this problem is known:

$$u^*(t) = \begin{cases} 1 & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2) \\ -1 & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

We choose the following parameters for Algorithm 3.3:

$$N = 100, \ \sigma = 0.5, \ \gamma = 0.5, \ \alpha_n = 10^{-4}/(n+1), \ \epsilon = 10^{-4}.$$

The initial control $u_0(t)$ is chosen randomly in $[-1, 1]$, and the stopping condition is $\|u_{n+1} - u_n\| \leq \epsilon$. The approximate solution is obtained after 92 iteration in 0.27856 seconds of CPU time. In Fig. 1, we display the approximate optimal control and the corresponding trajectories.

We now consider examples in which the terminal function is not linear. The following is the Rocket Car example in [1, 45].

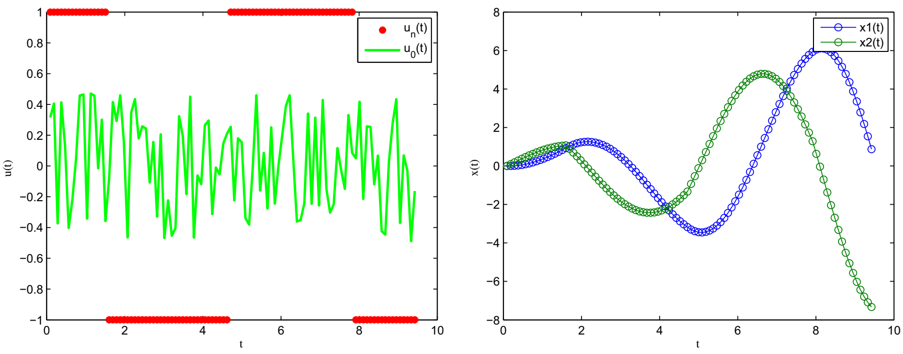

Fig. 1 Random initial control (green) and optimal control (red) on the left and optimal trajectories on the right for Example 5.1 computed by Algorithm 3.3

Example 5.2 (Rocket car)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\left((x_1(5))^2 + (x_2(5))^2\right) \\ \text{subject to} & \dot{x_1}(t) = x_2(t), \\ & \dot{x_2}(t) = u(t), \ \forall t \in [0, 5], \\ & x_1(0) = 6, \ x_2(0) = 1, \\ & u(t) \in [-1, 1]. \end{array} \tag{43}$$

The exact optimal control is

$$u^* = \begin{cases} 1 & \text{if } t \in (\tau, 5] \\ -1 & \text{if } t \in (0, \tau], \end{cases}$$

where $\tau = 3.5174292$. Parameters are chosen as in Example 5.1. The approximate solution is obtained after 199 iteration in 0.60509 s of CPU time (Fig. 2).

The following example is taken from [7].

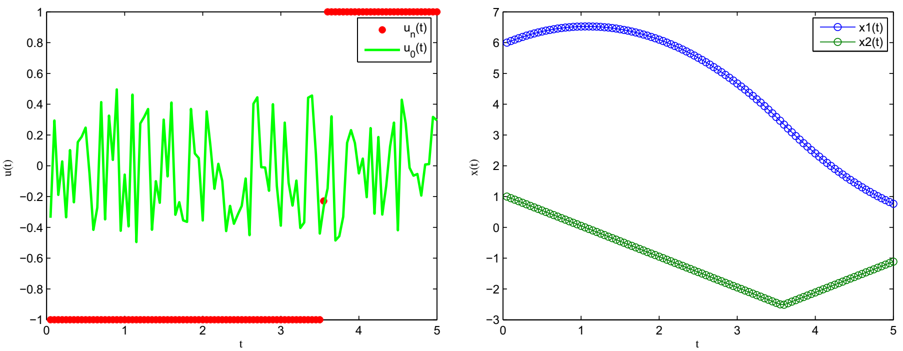

Fig. 2 Random initial control (green) and optimal control (red) on the left and optimal trajectories on the right for Example 5.2 computed by Algorithm 3.3

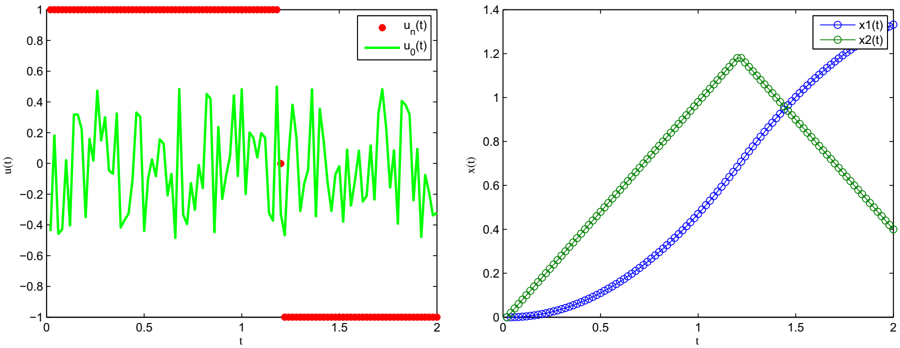

Fig. 3 Random initial control (green) and optimal control (red) on the left and optimal trajectories on the right for Example 5.3 computed by Algorithm 3.3

Example 5.3 (See [7, Example 6.3])

$$\begin{array}{ll} \text{minimize} & -x_1(2) + (x_2(2))^2 \\ \text{subject to} & \dot{x_1}(t) = x_2(t), \\ & \dot{x_2}(t) = u(t), \ \forall t \in [0, 2], \\ & x_1(0) = 0, \ x_2(0) = 0, \\ & u(t) \in [-1, 1]. \end{array} \tag{44}$$

The exact optimal control is

$$u^* = \begin{cases} 1 & \text{if } t \in [0, 6/5) \\ -1 & \text{if } t \in (6/5, 2], \end{cases}$$

Parameters are chosen as in Example 5.1. The approximate solution is obtained after 288 iteration in 0.87635 seconds of CPU time.

The following figure displays the error estimate $\|u_{n+1} - u_n\|$ for three examples considered above with $u_0(t) = 1$ (Figs. 3 and 4).

To conclude this section, let us consider an academic example in which the mapping A is pseudo monotone, but not monotone [24, 27].

Example 5.4 Let $H = \ell_2$, the real Hilbert space whose elements are the square summable sequences of real scalars, i.e.,

$$H = \left\{ x = (x_1, x_2, \ldots, x_k, \ldots) \middle| \sum_{k=1}^{\infty} x_k < +\infty \right\}.$$

The inner product and the norm on H are given by

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k \quad \text{and} \quad \|x\| = \sqrt{\langle x, x \rangle},$$

where $x = (x_1, x_2, \ldots, x_k, \ldots)$, $y = (y_1, y_2, \ldots, y_k, \ldots)$.

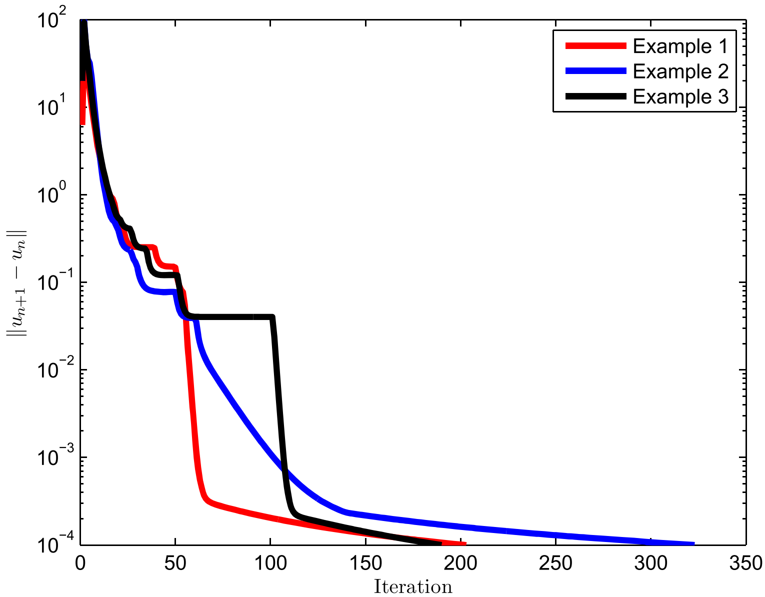

Fig. 4 Error estimate $\|u_{n+1} - u_n\|$

Let $\alpha, \beta \in \mathbb{R}$ be such that $\beta > \alpha > \beta/2 > 0$ and

$$C = \{x \in H : \|x\| \leq \alpha\} \quad \text{and} \quad Ax := (\beta - \|x\|)x.$$

It is easy to verify that the solution set $SOL = \{0\}$. Now let $x, y \in C$ be such that $\langle Ax, y - x\rangle \geq 0$, i.e.,

$$(\beta - \|x\|)\langle x, y - x\rangle \geq 0.$$

Since $\beta > \alpha > \beta/2 > 0$, the last inequality implies $\langle x, y - x\rangle \geq 0$. Hence,

$$\begin{aligned}\langle Ay, y - x\rangle &= (\beta - \|y\|)\langle y, y - x\rangle \\ &\geq (\beta - \|y\|)\langle y, y - x\rangle - (\beta - \|y\|)\langle x, y - x\rangle \\ &= (\beta - \|y\|)\|y - x\|^2 \geq 0.\end{aligned}$$

This means that A is pseudo monotone on C. To see that it is not monotone on C, let us consider

$$x = (\beta/2, 0, \ldots, 0, \ldots), \;\; y = (\alpha, 0, \ldots, 0, \ldots) \in C,$$

then we have

$$\langle Ax - Ay, x - y\rangle = \left(\frac{\beta}{2} - \alpha\right)^3 < 0.$$

In the following figure, we display the behavior of Algorithm 3.3 applying to Example 5.4, where H is $\mathbb{R}^m$ for some m. The starting points are chosen randomly in $\mathbb{B}_\alpha$. We choose $\alpha = 2$, $\beta = 3$ and $\sigma = \gamma = 0.5$ when $m = 5$, $\sigma = 0.1$, $\gamma = 0.6$ when $m = 10$ and $\sigma = 0.5$, $\gamma = 0.4$ when $m = 20$ (Fig. 5).

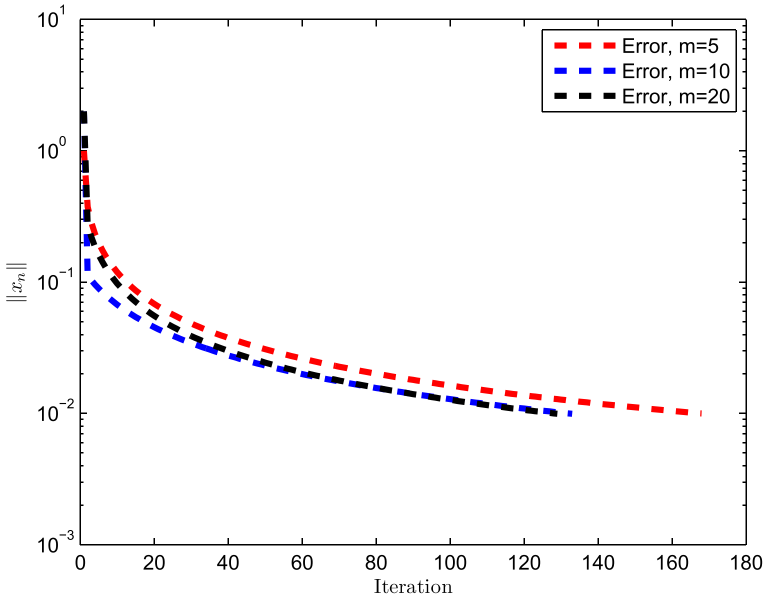

Fig. 5 Error estimate $\|x_n - x^*\|$

6 Final remarks

In this paper, we propose a variant of Korpelevich's method for solving variational inequality problems involving uniformly continuous pseudomonotone operators in real Hilbert spaces and obtain strong convergence under reasonable assumptions on the problem data. Applications to optimal control are considered and numerical experiments are presented confirming the theoretical results finding. Part of our future research concentrates on extending our results to Banach spaces.

Acknowledgments The authors would like to thank Nguyen Thanh Qui and two anonymous referees for their useful comments and suggestions on the first version the paper.

References

1. Alt, W., Baier, R., Gerdts, M., Lempio, F.: Error bounds for Euler approximation of linear-quadratic control problems with bang-bang solutions. Numer. Algebra Control Optim. **2**, 547–570 (2012)
2. Aubin, J.-P., Ekeland, I.: Applied Nonlinear Analysis. Wiley, New York (1984)
3. Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities; Applications to Free Boundary Problems. Wiley, New York (1984)
4. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces, CMS Books in Mathematics. Springer, New York (2011)
5. Bello Cruz, J.Y., Iusem, A.N.: An explicit algorithm for monotone variational inequalities. Optimization **61**, 855–871 (2012)
6. Bonnans, J.F., Festa, A.: Error estimates for the Euler discretization of an optimal control problem with first-order state constraints. SIAM J. Numer. Anal. **55**, 445–471 (2017)

7. Bressan, B., Piccoli, B.: Introduction to the mathematical theory of control, AIMS series on applied mathematics (2007)
8. Butcher, J.C.: Numerical Methods for Ordinary Differential Equations. Wiley, New York (2003)
9. Ceng, L.-C., Teboulle, M., Yao, J.-C.: Weak convergence of an iterative method for pseudomonotone variational inequalities and fixed-point problems. J. Optim. Theory Appl. **146**, 19–31 (2010)
10. Ceng, L.-C., Yao, J.-C.: An extragradient-like approximation method for variational inequality problems and fixed point problems. Appl. Math. Comput. **190**, 205–215 (2007)
11. Censor, Y., Gibali, A., Reich, S.: Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim. Methods Softw. **26**, 827–845 (2011)
12. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
13. Dontchev, A.L., Hager, W.W.: Lipschitzian stability in nonlinear control and optimization. SIAM J. Control Optim. **31**, 569–603 (1993)
14. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. II. Springer Series in Operations Research, Springer, New York (2003)
15. Glowinski, R., Lions, J.-L., Trémolières, R.: Numerical Analysis of Variational Inequalities. North-Holland, Amsterdam (1981)
16. Halpern, B.: Fixed points of nonexpanding maps. Proc. Amer. Math. Soc. **73**, 957–961 (1967)
17. He, Y.R.: A new double projection algorithm for variational inequalities. J. Comput. Appl. Math. **185**, 166–173 (2006)
18. Iusem, A.N.: An iterative algorithm for the variational inequality problem. Comput. Appl. Math. **13**, 103–114 (1994)
19. Iusem, A.N., Nasri, M.: Korpelevich's method for variational inequality problems in Banach spaces. J. Global Optim. **50**, 59–76 (2011)
20. Iusem, A.N., Svaiter, B.F.: A variant of Korpelevich's method for variational inequalities with a new search strategy. Optimization **42**, 309–321 (1997)
21. Kanzow, C., Shehu, Y.: Strong convergence of a double projection-type method for monotone variational inequalities in Hilbert spaces, J. Fixed Point Theory Appl. **20**. Article 51 (2018)
22. Karamardian, S., Schaible, S.: Seven kinds of monotone maps. J. Optim. Theory Appl. **66**, 37–46 (1990)
23. Khanh, P.D.: A modified extragradient method for infinite-dimensional variational inequalities. Acta. Math. Vietnam **41**, 251–263 (2016)
24. Khanh, P.D., Vuong, P.T.: Modified projection method for strongly pseudomonotone variational inequalities. J. Global Optim **58**, 341–350 (2014)
25. Khobotov, E.N.: Modification of the extragradient method for solving variational inequalities and certain optimization problems. USSR Comput. Math. Math. Phys. **27**, 120–127 (1989)
26. Khoroshilova, E.V.: Extragradient-type method for optimal control problem with linear constraints and convex objective function. Optim. Lett. **7**, 1193–1214 (2013)
27. Kim, D.S., Vuong, P.T., Khanh, P.D.: Qualitative properties of strongly pseudomonotone variational inequalities. Optim. Lett. **10**, 1669–1679 (2016)
28. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)
29. Konnov, I.V.: Combined Relaxation Methods for Variational Inequalities. Springer, Berlin (2001)
30. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ékon. Mat. Metody **12**, 747–756 (1976). (In Russian)
31. Kraikaew, R., Saejung, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **163**, 399–412 (2014)
32. Maingé, P.-E., Gobinddass, M.L.: Convergence of one-step projected gradient methods for variational inequalities. J. Optim. Theory Appl. **171**, 146–168 (2016)
33. Maingé, P.-E.: The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces. Comput. Math. Appl. **59**, 74–79 (2010)
34. Maingé, P.-E.: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**, 899–912 (2008)
35. Malitsky, Y.V., Semenov, V.V.: A hybrid method without extrapolation step for solving variational inequality problems. J. Global Optim. **61**, 193–202 (2015)
36. Malitsky, Y.: Projected reflected gradient methods for monotone variational inequalities. SIAM J. Optim. **25**, 502–520 (2015)

37. Marcotte, P.: Applications of Khobotov's algorithm to variational and network equlibrium problems. INFOR Inf. Syst. Oper. Res. **29**, 258–270 (1991)
38. Mashreghi, J., Nasri, M.: Forcing strong convergence of Korpelevich's method in Banach spaces with its applications in game theory. Nonlinear Anal. **72**, 2086–2099 (2010)
39. Nadezhkina, N., Takahashi, W.: Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **128**, 191–201 (2006)
40. Nadezhkina, N., Takahashi, W.: Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings. SIAM J. Optim. **16**, 1230–1241 (2006)
41. Nadezhkina, N., Takahashi, W.: Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings. SIAM J. Optim. **16**, 1230–1241 (2006)
42. Nagurney, A.: Network Economics: a Variational Inequality Approach. Kluwer Academic Publishers, Dordrecht (1999)
43. Nikol'skii, M.S.: Convergence of the gradient projection method in optimal control problems. Comp. Math. Model. **18**, 148–156 (2007)
44. Pietrus, A., Scarinci, T., Veliov, V.M.: High order discrete approximations to Mayer's problems for linear systems. SIAM J. Control Optim. **56**, 102–119 (2018)
45. Preininger, J., Vuong, P.T.: On the convergence of the gradient projection method for convex optimal control problems with bang-bang solutions. Comput. Optim. Appl. **70**, 221–238 (2018)
46. Solodov, M.V., Svaiter, B.F.: A new projection method for variational inequality problems. SIAM J. Control Optim. **37**, 765–776 (1999)
47. Shehu, Y., Iyiola, O.S.: Strong convergence result for monotone variational inequalities. Numer. Algorithms **76**, 259–282 (2017)
48. Shehu, Y., Iyiola, O.S.: Iterative Algorithms for solving Fixed Point Problems and Variational Inequalities with Uniformly Continuous Monotone Operators, Numer. Algorithms. https://doi.org/10.1007/s11075-017-0449-z
49. Shehu, Y., Iyiola, O.S.: On a modified extragradient method for variational inequality problem with application to industrial electricity production, J. Ind. Manag. Optim. https://doi.org/10.3934/jimo.2018045
50. Sun, D.F.: A class of iterative methods for solving nonlinear projection equations. J. Optim. Theory Appl. **91**, 123–140 (1996)
51. Sun, D.: A projection and contraction method for the nonlinear complementarity problem and its extensions. Math. Numer. Sin. **16**, 183–194 (1994)
52. Vuong, P.T.: On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities. J. Optim. Theory Appl. **176**, 399–409 (2018)
53. Wang, Y.J., Xiu, N.H., Wang, C.Y.: Unified framework for extragradient-type methods for pseudomonotone variational inequalities. J. Optim. Theory Appl. **111**, 641–656 (2001)
54. Xiu, N.H., Wang, C.Y., Zhang, J.Z.: Convergence properties of projection and contraction methods for variational inequality problems. Appl. Math. Optim. **43**, 147–168 (2001)
55. Xu, H.-K.: Iterative algorithms for nonlinear operators. J. London. Math. Soc. **66**, 240–256 (2002)
56. Yao, Y., Postolache, M.: Iterative methods for pseudomonotone variational inequalities and fixed-point problems. J. Optim. Theory Appl. **155**, 273–287 (2012)
57. Ye, M., He, Y.: A double projection method for solving variational inequalities without monotonicity. Comput. Optim. Appl. **60**, 141–150 (2015)
58. Zeng, L.-C., Yao, J.-C.: Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems. Taiwanese J. Math. **10**, 1293–1303 (2006)