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Modified Kantorovich operators with better approximation properties

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Abstract In the present paper, we study a new kind of Bernstein-Kantorovich-type operators. Here, we discuss a uniform convergence estimate for this modified form. Also, some direct estimates, which involve the asymptotic-type results, are established. Some numerical examples which show the relevance of the results are considered.

Keywords Approximation by polynomials · Kantorovich polynomials · Voronovskaja-type theorem

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1 Introduction

The Bernstein operators were introduced by S.N. Berstein [3] in 1912 in order to prove Weierstrass's fundamental theorem [18]. For $f \in C[0,1]$, the Bernstein operators of degree n with respect to f are defined by

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x)\ f\left(\frac{k}{n}\right), \quad x \in [0,1], \tag{1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \ldots, n$, and $p_{n,k}(x) = 0$, if $k < 0$ or $k > n$. It is known that

$$p_{n,k}(x) = (1-x)\ p_{n-1,k}(x) + x\ p_{n-1,k-1}(x), \quad 0 < k < n. \tag{2}$$

For a long time, the order of approximation of Bernstein operators was intensively studied. Using the first modulus of continuity, T. Popoviciu gave a solution of this problem in [14] and [15]. An asymptotic error term of the Bernstein operators was first given by Voronovskaja [17]. Later, this result was extended by the authors of [6–8, 10, 11, 16].

In a recent paper, H. K. Arab, M. Dehghan, and M. R. Eslahchi [2] have introduced modified Bernstein operators to improve the degree of approximation as follows:

$$B_n^{M,1}(f,x) = \sum_{k=0}^{n} p_{n,k}^{M,1}(x)\ f\left(\frac{k}{n}\right), \quad x \in [0,1], \tag{3}$$

$$p_{n,k}^{M,1}(x) = a(x,n)\ p_{n-1,k}(x) + a(1-x,n)\ p_{n-1,k-1}(x) \tag{4}$$

and

$$a(x,n) = a_1(n)\ x + a_0(n),\ n = 0, 1, \ldots, \tag{5}$$

where $a_0(n)$ and $a_1(n)$ are two unknown sequences which are determined in an appropriate way. For $a_1(n) = -1$, $a_0(n) = 1$, obviously, (4) reduces to (2).

2 Kantorovich operators of order I

The Kantorovich variant of sequences of linear positive operators is a method to approximate an integrable function on [0, 1]. These operators were introduced by Kantorovich [12], and from the approximation point of view, Kantorovich operators attracted attention and have been extensively studied. Özarslan and Duman [13] considered a modified Kantorovich operators and showed that the order of approximation to a function by these operators is at least as good as that of ones classically used.

Dhamija and Deo [4] introduced a King-type modification of Kantorovich operators and proved that the error estimation of these operators is better than the classical operators. Inequalities for the Kantorovich-type operators in terms of moduli of continuity were studied in [1]. In this paper, the results of H. K. Arab et al. are extended to the classical Kantorovich operators as follows

$$K_n^{M,1}(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{M,1}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{6}$$

Lemma 2.1 *The moments of the operator $K_n^{M,1}$ are given by*

i) $K_n^{M,1}(e_0;x) = 2a_0(n) + a_1(n)$;

ii) $K_n^{M,1}(e_1;x) = \frac{1}{2(n+1)}\{2x(2a_0(n)+a_1(n))n - 4a_1(n)x + 4a_0(n) - 4xa_0(n) + 3a_1(n)\}$;

iii) $K_n^{M,1}(e_2;x) = \frac{1}{3(n+1)^2}\Big\{3x^2(a_1(n)+2a_0(n))n^2 - 3x(5a_1(n)x + 6xa_0(n) - 6a_0(n) - 4a_1(n))n + 12a_1(n)x^2 - 18xa_0(n) - 18a_1(n)x + 8a_0(n) + 7a_1(n) + 12x^2a_0(n)\Big\}$.

Lemma 2.2 *The central moments of the operators $K_n^{M,1}$ are given by*

i) $K_n^{M,1}(t-x;x) = \frac{1-2x}{2(n+1)}(3a_1(n)+4a_0(n))$;

ii) $K_n^{M,1}((t-x)^2;x) = -\frac{1}{3(n+1)^2}\{3x(x-1)(2a_0(n)+a_1(n))n - 8a_0(n) - 7a_1(n) + 27a_1(n)x - 27a_1(n)x^2 + 30xa_0(n) - 30x^2a_0(n)\}$;

iii) $K_n^{M,1}((t-x)^4;x) = \frac{1}{5(n+1)^4}\Big\{15x^2(1-x)^2(2a_0(n)+a_1(n))n^2 + 5x(1-x)(1-2x)^2(21a_1+26a_0)n - 285a_1(n)x + 32a_0(n) + 31a_1(n) + 930a_1(n)x^2 - 290xa_0(n) - 1290x^3a_1(n) + 645x^4a_1(n) - 1300x^3a_0(n) + 650x^4a_0(n) + 940x^2a_0(n)\Big\}$.

In order to study the uniform convergence, we consider that sequences $a_i(n)$, $i = 0, 1$ verify the condition

$$2a_0(n) + a_1(n) = 1. \tag{7}$$

We will consider the following two cases for unknown sequences $a_0(n)$ and $a_1(n)$:
Case 1. Let

$$a_0(n) \geq 0,\, a_0(n) + a_1(n) \geq 0. \tag{8}$$

Using condition (7) we obtain $0 \leq a_0(n) \leq 1$ and $-1 \leq a_1(n) \leq 1$, namely the sequences are bounded. In this case, the operator (6) is positive.

Case 2. Let

$$a_0(n) < 0 \text{ or } a_1(n) + a_0(n) < 0. \tag{9}$$

If $a_0(n) < 0$, then $a_1(n) + a_0(n) > 1$ and if $a_1(n) + a_0(n) < 0$, then $a_0(n) > 1$. In this case, the operator (6) is not positive.

Theorem 2.1 *Let $a_1(n)$, $a_0(n)$ be two sequence which verify the conditions* (7) *and* (8). *If $f \in C[0, 1]$, then*

$$\lim_{n\to\infty} K_n^{M,1}(f; x) = f(x),$$

uniformly on $[0, 1]$.

Proof Since the sequences $a_1(n)$, $a_0(n)$ are bounded using the well-known Korovkin Theorem and Lemma 2.1, it follows the uniform convergence of the operators $K_n^{M,1}$. □

The above result can be extended for Case 2. In order to prove this result, we recall the extended form of the Korovkin Theorem:

Theorem 2.2 [2, Theorem 10] *Let $0 < h \in C[a, b]$ be a function and suppose that $(L_n)_{n\geq 1}$ is a sequence of positive linear operators such that $\lim_{n\to\infty} L_n(e_i) = he_i$, $i = 0, 1, 2$, uniformly on [a,b]. Then for given function $f \in C[a, b]$ we have $\lim_{n\to\infty} L_n(f) = hf$ uniformly on $[a, b]$.*

Theorem 2.3 *Let $f \in C[0, 1]$. Then, for all bounded sequences $a_1(n)$ and $a_0(n)$ that satisfy the conditions* (7) *and* (9), *we have*

$$\lim_{n\to\infty} K_n^{M,1}(f; x) = f(x),$$

uniformly on $[0, 1]$.

Proof Denote

$$K_{n,1}^{M,1}(f; x) = (n + 1) \sum_{k=0}^{n} \left[-a_1(n)xp_{n-1,k}(x) - a_1(n)p_{n-1,k-1}(x)\right] \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt;$$

$$K_{n,2}^{M,1}(f; x) = (n + 1) \sum_{k=0}^{n} \Big[a_0(n)p_{n-1,k}(x) + (-a_1(n)x + a_0(n))\, p_{n-1,k-1}(x)\Big] \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt.$$

The operators $K_n^{M,1}$ can be written as follows:

$$K_n^{M,1}(f; x) = K_{n,2}^{M,1}(f; x) - K_{n,1}^{M,1}(f; x).$$

The moments of the operators $K_{n,i}^{M,1}$, i=1,2 are calculated below:

$$K_{n,1}^{M,1}(e_0;x) = -a_1(n)(x+1);\ K_{n,1}^{M,1}(e_1;x)$$
$$= -\frac{1}{2}a_1(n)\left[2x(x+1)\frac{n}{n+1} + \frac{(2x+3)(1-x)}{n+1}\right];$$
$$K_{n,1}^{M,1}(e_2;x) = -\frac{1}{3}a_1(n)\left[3x^2(x+1)\frac{n^2}{(n+1)^2} + 3x(3x+4)(1-x)\frac{n}{(n+1)^2}\right.$$
$$\left. + \frac{6x^3-11x+7}{(n+1)^2}\right];$$
$$K_{n,2}^{M,1}(e_0;x) = -a_1(n)x + 2a_0(n);$$
$$K_{n,2}^{M,1}(e_1;x) = \left[2x(-a_1(n)x+2a_0(n))\frac{n}{n+1}\right.$$
$$\left. + \frac{2a_1(n)x^2 - 4a_0(n)x - 3a_1(n)x + 4a_0(n)}{n+1}\right];$$
$$K_{n,2}^{M,1}(e_2;x) = \frac{1}{3}\left[3x^2(-a_1(n)x + 2a_0(n)\frac{n^2}{(n+1)^2} - 3x(-3a_1(n)x^2 + 6a_0(n)x\right.$$
$$+ 4a_1(n)x - 6a_0(n))\frac{n}{(n+1)^2}$$
$$\left.\frac{-6a_1(n)x^3 + 12a_0(n)x^2 + 12a_1(n)x^2 - 18a_0(n)x - 7a_1(n)x + 8a_0(n)}{(n+1)^2}\right].$$

From Case 2, it follows that $(a_0(n) < 0, a_1(n) > 0)$ or $(a_0(n) > 0, a_1(n) < 0)$; therefore, the extended form of Korovkin Theorem can be applied to $K_{n,1}^{M,1}$ and $K_{n,2}^{M,1}$.

Using the above relations and Theorem 2.2, we obtain

$$\lim_{n\to\infty} K_{n,1}^{M,1}(f;x) = -l_1(1+x)f(x),$$
$$\lim_{n\to\infty} K_{n,2}^{M,1}(f;x) = (1-l_1(1+x))f(x),$$

where $l_1 = \lim_{n\to\infty} a_1(n)$. Therefore, $\lim_{n\to\infty} K_n^{M,1}(f;x) = f(x)$. □

Theorem 2.4 *Let $a_i(n)$ be a convergent sequence that satisfies the conditions (7) and (8) and $l_i = \lim_{n\to\infty} a_i(n)$, $i = 0, 1$. If $f'' \in C[0,1]$, then*

$$\lim_{n\to\infty} n\left(K_n^{M,1}(f;x) - f(x)\right) = \frac{1}{2}(1-2x)(3l_1+4l_0)f'(x) + \frac{1}{2}x(1-x)(2l_0+l_1)f''(x),$$

uniformly on $[0,1]$.

Proof Applying the Kantorovich operators $K_n^{M,1}$ to the Taylor's formula, we obtain

$$K_n^{M,1}(f;x) - f(x) = K_n^{M,1}(t-x;x)f'(x) + \frac{1}{2}K_n^{M,1}\left((t-x)^2;x\right)f''(x)$$
$$+ K_n^{M,1}\left(\theta(t,x)(t-x)^2;x\right),$$

where $\theta \in C[0,1]$ and $\lim_{t\to x}\theta(t,x)=0$.

Using the Cauchy-Schwarz inequality for the positive operators $K_n^{M,1}$, we get

$$K_n^{M,1}\left(\theta(t,x)(t-x)^2;x\right) \le \sqrt{K_n^{M,1}\left(\theta^2(t,x);x\right)}\sqrt{K_n^{M,1}\left((t-x)^4;x\right)}.$$

Since $\theta^2(x,x)=0$ and $\theta^2(\cdot,x)\in C[0,1]$, by Theorem 2.1, we obtain

$$\lim_{n\to\infty} K_n^{M,1}\left(\theta^2(t,x);x\right)=0$$

uniformly with respect to $x\in[0,1]$. Therefore, from Lemma 2.2, we obtain

$$\lim_{n\to\infty} nK_n^{M,1}\left(\theta(t,x)(t-x)^2;x\right)=0.$$

Using the results from Lemma 2.2, the proof of this theorem is completed. □

Now, we extend the results from Theorem 2.4 when the operator $K_n^{M,1}$ is nonpositive, i.e., the sequences $a_0(n)$ and $a_1(n)$ satisfy (7) and (9).

Theorem 2.5 *Let $a_i(n)$, $i=0,1$ be bounded convergent sequences which satisfy* (7) *and* (9) *and $l_i=\lim_{n\to\infty} a_i(n)$, $i=0,1$. If $f\in C[0,1]$ and f'' exists at a certain point $x\in[0,1]$, then we have*

$$\lim_{n\to\infty} n[K_n^{M,1}(f;x)-f(x)] = \frac{1}{2}(1-2x)(3l_1+4l_0)f'(x) + \frac{1}{2}x(1-x)(2l_0+l_1)f''(x). \tag{10}$$

Moreover, the relation (10) *holds uniformly on* $[0,1]$ *if* $f''\in C[0,1]$.

Proof Again, as in the proof of Theorem 2.4, applying the Kantorovich variant $K_n^{M,1}$ to the Taylor's formula is sufficient to show that

$$\lim_{n\to\infty} nK_n^{M,1}(\theta(t,x)(t-x)^2;x)=0. \tag{11}$$

Here, we cannot use Cauchy-Schwarz inequality because $K_n^{M,1}$ is not a positive linear operator in this case. From (4) and (6), we may represent $K_n^{M,1}$ as

$$K_n^{M,1}(f;x)=(n+1)\sum_{k=0}^{n-1} p_{n-1,k}(x)\left\{a(x,n)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt + a(1-x,n)\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} f(t)dt\right\}. \tag{12}$$

Let $\varepsilon>0$ be given. There exists a $\delta>0$ such that if $|t-x|<\delta$ then $|\theta(t,x)|<\varepsilon$. We denote

$$A = \left\{k : \left|\frac{k}{n}-x\right| < \delta, k=0,1,2,\cdots,n\right\} \text{ and}$$
$$B = \left\{k : \left|\frac{k}{n}-x\right| \ge \delta, k=0,1,2,\cdots,n\right\}.$$

The boundedness of the sequences $a_i(n)$, $i = 0, 1$ implies that there is a constant $C > 0$ such that $|a(x,n)| < C$. From (12), it follows

$$\left|K_n^{M,1}(\theta(t,x)(t-x)^2;x)\right| \leq C(n+1)\sum_{k=0}^{n-1} p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |\theta(t,x)|(t-x)^2 dt$$
$$+C(n+1)\sum_{k=0}^{n-1} p_{n-1,k}(x)\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} |\theta(t,x)|(t-x)^2 dt. \tag{13}$$

Let $k \in A$. Hence, $|\theta(t,x)| < \varepsilon$. Therefore, we get

$$\left|K_n^{M,1}(\theta(t,x)(t-x)^2;x)\right| \leq \varepsilon C(n+1)\sum_{k=0}^{n-1} p_{n-1,k}(x)$$
$$\times\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^2 dt + \int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} (t-x)^2 dt\right)$$
$$= \frac{2\varepsilon C}{3(n+1)^2}\left\{3x(1-x)n + 15x^2 - 15x + 4\right\}. \tag{14}$$

Let $k \in B$. We denote $M = \sup_{0\leq t\leq 1} |\theta(t,x)|(t-x)^2$. Then $|\theta(t,x)|(t-x)^2 \leq \frac{M}{\delta^4}\left(\frac{k}{n} - x\right)^4$. From (13), we get the following upper bound

$$\left|K_n^{M,1}(\theta(t,x)(t-x)^2;x)\right| \leq \frac{2MC}{\delta^4}\sum_{k=0}^{n-1} p_{n-1,k}(x)\left(\frac{k}{n} - x\right)^4$$
$$= \frac{2MC}{\delta^4 n^4}\left\{3x^2(1-x)^2n^2 + x(1-x)(26x^2 - 16x + 1)n + (2x-1)(12x^2 - 12x + 1)x\right\}. \tag{15}$$

The relations (14) and (15) lead to (11) and the proof of Theorem 2.5 is completed. □

Remark 2.1 We point out that the proofs of Voronovskaja-type estimates [2, Theorem 8 and Theorem 13] are not correct because the relevant upper bounds (15) for modified Bernstein operator when using only the second central moment (instead of the fourth central moment) are not enough to get the proofs. Nonetheless, Theorem 8 and Theorem 13 are true.

Our next result is a direct estimate for the operators $K_n^{M,1}$:

Theorem 2.6 *If $f(x)$ is bounded for $x \in [0,1]$, $a_0(n)$, $a_1(n)$ satisfy (7) and $a_1(n)$ is a bounded sequence, then*

$$\|K_n^{M,1} f - f\| \leq \frac{3}{2}(3|a_1(n)| + 1)\,\omega\left(f; \frac{1}{\sqrt{n}}\right), \text{ for } n \geq 5, \tag{16}$$

where $\|\cdot\|$ *is the uniform norm on the interval* $[0, 1]$ *and* $\omega(f, \delta)$ *is the first-order modulus of continuity.*

Proof From Lemma 2.1 (i), condition (7), and representation (12), we obtain

$$\begin{aligned}\left|K_n^{M,1}(f;x)-f(x)\right| &\le |a_1(n)x+a_0(n)|(n+1)\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\\ &\quad |f(t)-f(x)|dt\\ &\quad +|a_1(n)(1-x)+a_0(n)|(n+1)\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}}\\ &\quad |f(t)-f(x)|dt\\ &\le |a_1(n)x+a_0(n)|(n+1)\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\omega(f;|t-x|)dt\\ &\quad +|a_1(n)(1-x)+a_0(n)|(n+1)\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}}\\ &\quad \omega(f;|t-x|)dt.\end{aligned}$$

It is known that

$$\omega(f;|t-x|)\le\left(1+\sqrt{n}|t-x|\right)\omega\left(f;\frac{1}{\sqrt{n}}\right). \tag{17}$$

Therefore,

$$\begin{aligned}\left|K_n^{M,1}(f;x)-f(x)\right| \le\ & |a_1(n)x+a_0(n)|\,\omega\left(f;\frac{1}{\sqrt{n}}\right)\\ &\times\left[1+(n+1)\sqrt{n}\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|t-x|dt\right]\\ &+|a_1(n)(1-x)+a_0(n)|\,\omega\left(f;\frac{1}{\sqrt{n}}\right)\\ &\times\left[1+(n+1)\sqrt{n}\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}}|t-x|dt\right].\end{aligned}$$

From Hölder's inequality it follows:

$$\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|t-x|dt\le\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2dt\right)^{\frac{1}{2}}\frac{1}{\sqrt{n+1}}. \tag{18}$$

Consequently, the relation (18) and the Cauchy-Schwarz inequality lead to the following relation:

$$\begin{aligned}\left|K_n^{M,1}(f;x)-f(x)\right| &\leq |a_1(n)x+a_0(n)|\,\omega\left(f;\frac{1}{\sqrt{n}}\right)\\ &\quad\times\left[1+(n+1)\left(\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2dt\right)^{\frac{1}{2}}\right]\\ &\quad+|a_1(n)(1-x)+a_0(n)|\,\omega\left(f;\frac{1}{\sqrt{n}}\right)\\ &\quad\times\left[1+(n+1)\left(\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}}(t-x)^2dt\right)^{\frac{1}{2}}\right]. \qquad (19)\end{aligned}$$

It is clear from (7) that

$$|a_1(n)x+a_0(n)| \leq |a_1(n)|+|a_0(n)| = |a_1(n)|+\left|\frac{1-a_1(n)}{2}\right| \leq \frac{3}{2}|a_1(n)|+\frac{1}{2}. \qquad (20)$$

The same upper bound (20) holds for $|a_1(n)(1-x)+a_0(n)|$. Hence, from (19) and (20), we obtain

$$\begin{aligned}\left|K_n^{M,1}(f;x)-f(x)\right| &\leq \frac{3|a_1(n)|+1}{2}\omega\left(f;\frac{1}{\sqrt{n}}\right)\\ &\quad\times\left[2+\sqrt{2}(n+1)\left(\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+2}{n+1}}(t-x)^2dt\right)^{\frac{1}{2}}\right]. \qquad (21)\end{aligned}$$

Obviously,

$$\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+2}{n+1}}(t-x)^2dt = \frac{2(5-n)}{(n+1)^3}\left[x^2-x+\frac{4}{3}\cdot\frac{1}{5-n}\right] := g(x). \qquad (22)$$

Simple calculations show that

$$\max_{x\in[0,1]} g(x) = g\left(\frac{1}{2}\right) = \frac{3n+1}{6(n+1)^3}, \text{ for } n \geq 5. \qquad (23)$$

Now, the relations (21), (22), and (23) imply

$$\left|K_n^{M,1}(f;x)-f(x)\right| \leq \frac{3}{2}\,(3|a_1(n)|+1)\,\omega\left(f;\frac{1}{\sqrt{n}}\right). \qquad (24)$$

The proof of Theorem 2.6 is completed. □

Remark 2.2 We point out that in [2, Theorem 9] it was supposed that the modified Bernstein operator $B_n^{M,1}$ is positive, although the proof holds true also for the cases when the operator is nonpositive. Of course we need that the sequence $a_1(n)$ is bounded in the last case.

Corollary 2.1 *If $f(x)$ is continuous on $[0,1]$, then $\lim_{n\to\infty}\omega\left(f;\frac{1}{\sqrt{n}}\right)=0$, and thus, another proof of the Theorems 2.1 and 2.3 is given.*

Corollary 2.2 *We say that f satisfies the Lipschitz condition of order r with the constant k on $[0,1]$, i.e.,*

$$|f(x_1)-f(x_2)|\le k|x_1-x_2|^r,\ r>0 \text{ for } x_1,x_2\in[0,1]$$

if and only if $\omega(f;\delta)\le k\delta^r$. Therefore, if f satisfies the Lipschitz condition of order r with constant k, then

$$\|K_n^{M,1}f-f\|\le\frac{3}{2}\left(3|a_1(n)|+1\right)kn^{-\frac{r}{2}}. \tag{25}$$

Corollary 2.3 *If $a_1(n)=-1$, the estimate (16) reduces to the original Kantorovich operator.*

Let us consider our modified Kantorovich operator $K_n^{M,1}$ in the case when e_0, e_1 are reproduced, i.e,

$$2a_0(n)+a_1(n)=1,\ \ 4a_0(n)+3a_1(n)=0.$$

Hence, $a_1(n)=-2$ and $a_0(n)=\frac{3}{2}$. From (9), it follows that $K_n^{M,1}$ is a nonpositive operator.

Let $g,g''\in C[0,1]$. We apply operator $K_n^{M,1}$ to the Taylor's formula and obtain

$$K_n^{M,1}(g;x)-g(x)-\frac{1}{2}K_n^{M,1}\left((t-x)^2;x\right)g''(x)=\frac{1}{2}K_n^{M,1}\left(\theta(t,x)(t-x)^2;x\right), \tag{26}$$

where $\theta(t,x)=g''(\xi_x)-g''(x)$ and ξ_x is a point between t and x. Therefore,

$$|\theta(t,x)|\le\omega\left(g'',|t-x|\right)\le\left(1+\sqrt{n}|t-x|\right)\omega\left(g'',\frac{1}{\sqrt{n}}\right).$$

It is clear that in our case, $|a_1(n)x+a_0(n)|\le\frac{3}{2}$ for $x\in[0,1]$. Consequently, the relation (12) implies that using the last two inequalities, we have

$$\left|K_n^{M,1}\left(\theta(t,x)(t-x)^2;x\right)\right|\le\frac{3}{2}(n+1)\omega\left(g'',\frac{1}{\sqrt{n}}\right)\left[\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+2}{n+1}}(t-x)^2dt\right.$$
$$\left.+\sqrt{n}\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+2}{n+1}}|t-x|(t-x)^2dt\right]. \tag{27}$$

From the same calculations made in (14), it follows

$$(n+1)\sum_{k=0}^{n-1}p_{n-1,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+2}{n+1}}(t-x)^2dt\le4\left[\frac{x(1-x)}{n-1}+\frac{4}{n^2}\right]+\frac{C'}{n}\le C_1\cdot\frac{1}{n} \tag{28}$$

with $C_1>0$ an absolute constant independent of n, x.

Concerning the second sum in (27), we observe that for $t \in \left[\frac{k}{n+1}, \frac{k+2}{n+1}\right]$ *we have*

$$\begin{aligned} |t-x|(t-x)^2 &= |t-x|^3 = \left|t - \frac{k}{n-1} + \frac{k}{n-1} - x\right|^3 \\ &\le C_2 \left(\left|t - \frac{k}{n-1}\right|^3 + \left|\frac{k}{n-1} - x\right|^3 \right) \\ &\le C_3 \left(\frac{1}{(n+1)^3} + \left|\frac{k}{n-1} - x\right|^3 \right). \end{aligned}$$

Consequently,

$$\begin{aligned} &(n+1)\sqrt{n} \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+2}{n+1}} |t-x|(t-x)^2 dt \\ &\le C_3 \left(\frac{\sqrt{n}}{n^3} + \sqrt{n} \sum_{k=0}^{n-1} p_{n-1,k}(x) \left|\frac{k}{n-1} - x\right|^3 \right) \\ &\le C_3 \left[\frac{1}{n^{5/2}} + \sqrt{n} \left(\sum_{k=0}^{n-1} p_{n-1,k}(x) \left(\frac{k}{n-1} - x\right)^2 \right)^{\frac{1}{2}} \right. \\ &\quad \left. \times \left(\sum_{k=0}^{n-1} p_{n-1,k}(x) \left(\frac{k}{n-1} - x\right)^4 \right)^{\frac{1}{2}} \right] \\ &\le C_4 \left[\frac{1}{n^{5/2}} + \sqrt{n} \sqrt{\frac{x(1-x)}{n-1}} \cdot \frac{1}{n} \right] \le C_5 \cdot \frac{1}{n}. \end{aligned} \tag{29}$$

Finally, the estimates (27), (28), *and* (29) *imply*

$$\left| K_n^{M,1}\left(\theta(t,x)(t-x)^2; x\right) \right| \le C_6 \frac{1}{n} \omega\left(f'', \frac{1}{\sqrt{n}}\right). \tag{30}$$

Hence, we gave the proof of the following quantitative form of Voronovskaja-type theorem for the operator $K_n^{M,1}$ *with* $a_1(n) = -2$, $a_0(n) = \frac{3}{2}$*:*

Theorem 2.7 *For* $g \in C^2[0,1]$, $x \in [0,1]$ *fixed and* $K_n^{M,1}$ *defined above, we have*

$$\left| K_n^{M,1}(g;x) - g(x) - \frac{1}{2} K_n^{M,1}\left((t-x)^2; x\right) g''(x) \right| \le C \frac{1}{n} \omega\left(g'', \frac{1}{\sqrt{n}}\right), \tag{31}$$

where $C > 0$ *is a constant independent of* n, x.

Corollary 2.4 *For* $g \in C^2[0,1]$, $x \in [0,1]$ *fixed, we have*

$$\lim_{n\to\infty} n\left[K_n^{M,1}(g;x) - g(x)\right] = \frac{x(1-x)}{2} g''(x).$$

Proof The proof follows immediately from Theorem 2.7 and Lemma 2.2, ii), i.e,

$$K_n^{M,1}\left((t-x)^2; x\right) = O\left(\frac{1}{n}\right). \tag{32}$$

□

Corollary 2.5 *For* $f \in C^2[0,1]$, *the following holds*

$$\|K_n^{M,1} g - g\|_{C[0,1]} \le \frac{C}{n}\|g''\|.$$

Proof The proof is a simple consequence from (32) and (31) and the property $\omega(h,\delta) \le 2\|h\|_{C[0,1]}$. □

Remark 2.3 Theorem 2.7 gives another proof of the Voronovskaja-type estimate in Theorem 2.5 for the special case $a_1(n) = -2$, $a_0(n) = \frac{3}{2}$.

Our next goal is to extend the direct estimate in Theorem 2.6 in terms of the second-order moduli $\omega_2\left(f; \frac{1}{\sqrt{n}}\right)$.

Theorem 2.8 *For* $f \in C[0,1]$, $a_1(n) = -2$, $a_0(n) = \frac{3}{2}$ *the following holds true*

$$\|K_n^{M,1} f - f\|_{C[0,1]} \le C\omega_2\left(f; \frac{1}{\sqrt{n}}\right). \tag{33}$$

Proof First, we observe that $K_n^{M,1} : C[0,1] \to C[0,1]$ is the bounded operator, namely from (12) it follows

$$\|K_n^{M,1} f\|_{C[0,1]} \le 2\left(|a_0(n)| + |a_1(n)|\right)\|f\|_{C[0,1]}. \tag{34}$$

It is known that the second-order modulus $\omega_2(f,t)$ is equivalent to the following K-functional

$$K_2(f,t^2) := \inf_{f\in C^2[0,1]}\left\{\|f-g\|_{C[0,1]} + t^2\|g''\|_{C[0,1]}\right\}.$$

More precisely from [9, Corrollary 2.7], we have

$$K_2(f,t^2) \le \frac{7}{2}\omega_2(f,t),\ t \ge 0,\ f \in C[0,1]. \tag{35}$$

Therefore,

$$\|K_n^{M,1}f - f\|_{C[0,1]} \le \|K_n^{M,1}(f-g) - (f-g)\|_{C[0,1]} + \|K_n^{M,1}g - g\|_{C[0,1]}$$
$$\le C_1\|f-g\|_{C[0,1]} + C_2\frac{1}{n}\|g''\|_{C[0,1]} \le C_3\left\{\|f-g\| + \frac{1}{n}\|g''\|\right\}.$$

We take the infimum over all $g \in C^2[0,1]$, and using (35), we complete the proof of Theorem 2.8. □

Remark 2.4 Obviously, Theorem 2.8 is better than Theorem 2.6 because now we have estimate in ω_2 instead of ω_1. Also, we observe that for e_0, e_1, e_2 in place of $g(x)$ in Theorem 2.7 in both sides of the inequality we have 0.

Remark 2.5 Voronovskaja's Theorem says that the optimal rate of approximation for the class $C[0,1]$ is exactly $O\left(\frac{1}{n}\right)$ independent of how smooth is the approximated function f (see [5, Theorem 5.1]). The same optimal rate of approximation (saturation order) $\frac{1}{n}$ is valid for Kantorovich operator (see [5, Theorem 6.3, p. 317]). In this paper for our modified Kantorovich operator, when $a_1(n) = -2$, $a_0(n) = \frac{3}{2}$, we have from Corollary 2.4 order of approximation again 1/n, but in next Section, we study a modification of the Kantorovich operator with order of approximation better than 1/n.

In the next section for the second modification of Kantorovich operator $K_n^{M,2}$, we will obtain even order of approximation $O\left(\frac{1}{n^2}\right)$. The price to be paid is that in these cases, we lose the positivity.

3 Kantorovich operators of order II

In this section, we will extend the previous results considering the modified Kantorovich-type operators as follows:

$$K_n^{M,2}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{M,2}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \tag{36}$$

where

$$p_{n,k}^{M,2}(x) = b(x,n)p_{n-2,k}(x) + d(x,n)p_{n-2,k-1}(x) + b(1-x,n)p_{n-2,k-2}(x) \tag{37}$$

and

$$b(x,n) = b_2(n)x^2 + b_1(n)x + b_0(n), \;\; d(x,n) = d_0(n)x(1-x),$$

where $b_i(n)$, $i = 0, 1, 2$ and $d_0(n)$ are two unknown sequences which are determined in the appropriate way. For $b_2(n) = b_0(n) = 1$, $b_1(n) = -2$, $d_0(n) = 2$, obviously (37) reduces to (2).

Lemma 3.1 *By simple computation, we have*

i) $K_n^{M,2}(e_0;x) = (2b_2(n) - d_0(n))x^2 - (2b_2(n) - d_0(n))x + b_2(n) + 2b_0(n) + b_1(n)$;

ii) $K_n^{M,2}(e_1;x) = \frac{1}{2(n+1)}\Big\{\Big[2(2b_2(n) - d_0(n))x^3 - 2(2b_2(n) - d_0(n))x^2 +2(b_2(n) + 2b_0(n) + b_1(n))x\Big]n -4(2b_2(n)-d_0(n))x^3+7(2b_2(n)-d_0(n))x^2+(-14b_2(n)+3d_0(n)-8b_0(n)-8b_1(n))x +5b_2(n) + 5b_1(n) + 6b_0(n)\Big\}$;

iii) $K_n^{M,2}(e_2;x) = \frac{1}{3(n+1)^2}\Big\{\Big[3(2b_2(n) - d_0(n))x^4 - 3(2b_2(n) - d_0(n))x^3 +3(b_1(n) + 2b_0(n) + b_2(n))x^2\Big]n^2 + \big[-15(2b_2(n) - d_0(n))x^4 + 27(2b_2(n) - d_0(n))x^3 + 3\,(4d_0(n) - 17b_2(n) -10b_0(n) - 9b_1(n))\,x^2 + 6(3b_2(n) + 4b_0(n) + 3b_1(n))x\big]n + 18(2b_2(n) - d_0(n))x^4 - 42(2b_2(n) - d_0(n))x^3 + (42b_1(n) + 110b_2(n) - 31d_0(n) + 36b_0(n))x^2 + (7d_0(n) - 74b_2(n) -48b_0(n) - 54b_1(n))x + 19b_2(n) + 20b_0(n) + 19b_1(n)\Big\}$.

Lemma 3.2 *The central moments of the operators $K_n^{M,2}$ are given by*

i) $K_n^{M,2}(t-x;x) = \frac{1}{2(n+1)}\Big\{-6(2b_2(n) - d_0(n))x^3 + 9(2b_2(n) - d_0)x^2 +(-16b_2(n) - 10b_1(n) + 3d_0(n) - 12b_0(n))x + 6b_0(n) + 5b_2(n) + 5b_1(n)\Big\}$;

ii) $K_n^{M,2}((t-x)^2;x) = \frac{1}{3(n+1)^2}\Big\{\Big[-3(2b_2(n)-d_0(n))x^4 + 6(2b_2(n)-d_0(n))x^3 +(-3b_1(n)-9b_2(n)+3d_0(n)-6b_0(n))x^2+(3b_1(n) + 6b_0(n)+3b_2(n))x\Big]n + 33(2b_2(n) - d_0(n))x^4 - 66(2b_2(n) - d_0(n))x^3 + (66b_0(n) + 69b_1(n) + 155b_2(n) - 40d_0(n))x^2 +(7d_0(n) - 66b_0(n) - 89b_2(n)-69b_1(n))x+19b_2(n) + 20b_0(n) + 19b_1(n)\Big\}$.

In order to study the uniform convergence, we set $K_n^{M,2}(e_0;x) = 1$, and this yields:

$$2b_2(n) - d_0(n) = 0,\ \ b_2(n) + 2b_0(n) + b_1(n) = 1.$$

Using the above relations, we obtain

$$\begin{aligned} K_n^{M,2}(e_1;x) &= x + \frac{(2x-1)(4b_0(n)-5)}{2(n+1)};\\ K_n^{M,2}(e_2;x) &= x^2 + \frac{1}{3(n+1)^2}\Big\{(24b_0(n)x^2 - 12b_0(n)x - 33x^2 + 18x)n\\ &\quad -60b_0(n)x^2 - 6b_1(n)x^2 + 72b_0(n)x + 6b_1(n)x + 45x^2\\ &\quad -18b_0(n) - 60x + 19\Big\}. \end{aligned}$$

In order to have $\lim_{n\to\infty} K_n^{M,2}(e_i;x)=x^i$, $i=0,1,2$, we consider the sequences $b_0(n)$ and $b_1(n)$ to verify the conditions

$$\lim_{n\to\infty}\frac{b_0(n)}{n}=0 \text{ and } \lim_{n\to\infty}\frac{b_1(n)}{n^2}=0.$$

We propose our analysis for the case $b_0(n)=\frac{5}{4}$ and $b_1(n)=-\frac{n}{2}$; therefore, $b_2(n)=\frac{n-3}{2}$ and $d_0(n)=n-3$.

With the above choices, the operator (36) becomes

$$\tilde{K}_n^{M,2}(f;x)=(n+1)\sum_{k=0}^{n}\tilde{p}_{n,k}^{M,2}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \tag{38}$$

where

$$\begin{aligned}\tilde{p}_{n,k}^{M,2}(x) &= \left(\frac{n-3}{2}x^2-\frac{n}{2}x+\frac{5}{4}\right)p_{n-2,k}(x)+(n-3)x(1-x)p_{n-2,k-1}(x)\\ &\quad+\left(\frac{n-3}{2}x^2+\frac{6-n}{2}x-\frac{1}{4}\right)p_{n-2,k-2}(x).\end{aligned}$$

Note that other choices for sequences $b_0(n)$ and $b_1(n)$ lead to some operators with order of approximation either one or two. In the following, we are concerning to study the uniform convergence of the operator (38).

Lemma 3.3 *The moments of the operators* (38) *are given by*

i) $\tilde{K}_n^{M,2}(e_0;x)=1$;
ii) $\tilde{K}_n^{M,2}(e_1;x)=x$;
iii) $\tilde{K}_n^{M,2}(e_2;x)=x^2+\frac{10x(1-x)}{(n+1)^2}-\frac{7}{6(n+1)^2}$.

Lemma 3.4 *The central moments of the operators* (38) *are given by*

i) $\tilde{K}_n^{M,2}\left((t-x)^2;x\right)=\frac{10x(1-x)}{(n+1)^2}-\frac{7}{6(n+1)^2}$,
ii) $\tilde{K}_n^{M,2}\left((t-x)^3;x\right)=\frac{1-2x}{4(n+1)^3}\left\{-14x(1-x)n-154x^2+154x-15\right\}$,
iii) $\tilde{K}_n^{M,2}\left((t-x)^4;x\right)=-3x^2(1-x)^2\frac{n^2}{(n+1)^4}+\mathcal{O}\left(\frac{1}{n^3}\right)$,
iv) $\tilde{K}_n^{M,2}\left((t-x)^5;x\right)=45x^2(1-x)^2\frac{n^2}{(n+1)^5}+\mathcal{O}\left(\frac{1}{n^4}\right)$,
v) $\tilde{K}_n^{M,2}\left((t-x)^6;x\right)=-30x^3(1-x)^3\frac{n^3}{(n+1)^6}+\mathcal{O}\left(\frac{1}{n^4}\right)$.

Theorem 3.1 *If $f \in C^6[0,1]$ and $x \in [0,1]$, then for sufficiently large n, we have*

$$\tilde{K}_n^{M,2}(f;x) - f(x) = \mathcal{O}\left(\frac{1}{n^2}\right).$$

Proof Applying the Kantorovich operators $\tilde{K}_n^{M,2}$ to the Taylor's formula, we obtain

$$\tilde{K}_n^{M,2}(f;x) - f(x) = \sum_{k=1}^{6} f^{(k)}(x)\tilde{K}_n^{M,2}\left((t-x)^k;x\right) + \tilde{K}_n^{M,2}\left(\theta(t,x)(t-x)^6;x\right),$$

where $\lim_{t\to x}\theta(t,x)$.

It is easy to check

$$\begin{aligned}\tilde{K}_n^{M,2}(f;x) = &(n+1)\left(\frac{n-3}{2}x^2 - \frac{n}{2}x + \frac{5}{4}\right)\sum_{k=0}^{n-2} p_{n-2,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt\\ &+(n+1)(n-3)x(1-x)\sum_{k=0}^{n-2} p_{n-2,k}(x)\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}} f(t)dt\\ &+(n+1)\left(\frac{n-3}{2}x^2 + \frac{6-n}{2}x - \frac{1}{4}\right)\sum_{k=0}^{n-2} p_{n-2,k}(x)\int_{\frac{k+2}{n+1}}^{\frac{k+3}{n+1}} f(t)dt.\end{aligned} \tag{39}$$

Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that if $|t-x| < \delta$ then $|\theta(t,x)| < \varepsilon$. We denote

$$\begin{aligned}A &= \left\{k : \left|\frac{k}{n} - x\right| < \delta, k = 0,1,2,\cdots,n\right\} \text{ and}\\ B &= \left\{k : \left|\frac{k}{n} - x\right| \geq \delta, k = 0,1,2,\cdots,n\right\}.\end{aligned}$$

Let $k \in A$. Since $|\theta(t,x)| < \varepsilon$, from (39), we obtain

$$\begin{aligned}&\left|\tilde{K}_n^{M,2}\left(\theta(t,x)(t-x)^6;x\right)\right| \leq \frac{\varepsilon}{8}(n+1)(n+10)\sum_{k=0}^{n-2} p_{n-2,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^6 dt\\ &+\frac{\varepsilon}{4}(n+1)(n-3)\sum_{k=0}^{n-2} p_{n-2,k}(x)\int_{\frac{k+1}{n+1}}^{\frac{k+2}{n+1}}(t-x)^6 dt + \frac{\varepsilon}{8}(n+1)(n+10)\\ &\times\sum_{k=0}^{n-2} p_{n-2,k}(x)\int_{\frac{k+2}{n+1}}^{\frac{k+3}{n+1}}(t-x)^6 dt\\ &= \frac{\varepsilon}{2}(n+1)(n+4)\left\{\frac{105x^3(1-x)^3 n^3}{7(n+1)^7} + \mathcal{O}\left(\frac{1}{n^5}\right)\right\} = \mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned} \tag{40}$$

Let $k \in B$. We denote $M = \sup_{0\le t\le 1} |\theta(t,x)|(t-x)^6$. Then, $|\theta(t,x)|\,(t-x)^6 \le \frac{M}{\delta^6}\left(\frac{k}{n}-x\right)^6$. We get the following upper bound

$$\left|\tilde{K}_n^{M,2}\left(\theta(t,x)(t-x)^6; x\right)\right| \le \frac{(n+4)M}{2\delta^6}\sum_{k=0}^{n-2} p_{n-2,k}(x)\left(\frac{k}{n}-x\right)^6$$
$$= \frac{(n+4)M}{2\delta^6}\left\{\frac{15x^3(1-x)^3}{n^3} + \mathcal{O}\left(\frac{1}{n^2}\right)\right\} = \mathcal{O}\left(\frac{1}{n^2}\right). \tag{41}$$

From (40) and (41), the proof of theorem is completed. □

4 Kantorovich operators of order III

Using the approach as in the previous section, we construct a third-order approximation formula. Let us consider the following modification of Kantorovich operators:

$$K_n^{M,3}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{M,3}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \tag{42}$$

where

$$p_{n,k}^{M,3}(x) = \tilde{b}(x,n)p_{n-4,k}(x) + \tilde{d}(x,n)p_{n-4,k-1}(x) + \tilde{e}(x,n)p_{n-4,k-2}(x)$$
$$+\tilde{d}(1-x,n)p_{n-4,k-3}(x) + \tilde{b}(1-x,n)p_{n-4,k-4}(x)$$

and

$$\tilde{b}(x,n) = \tilde{b}_4(n)x^4 + \tilde{b}_3(n)x^3 + \tilde{b}_2(n)x^2 + \tilde{b}_1(n)x + \tilde{b}_0(n),$$
$$\tilde{d}(x,n) = \tilde{d}_4(n)x^4 + \tilde{d}_3(n)x^3 + \tilde{d}_2(n)x^2 + \tilde{d}_1(n)x + \tilde{d}_0(n),$$
$$\tilde{e}(x,n) = \tilde{e}_0(n)(x(1-x))^2.$$

We note that $b_i(n)$, $d_i(n)$, $i = 0, 1, \ldots, 4$, and $e_0(n)$ are some unknown sequences which are determined in an appropriate way. Let $\tilde{K}_n^{M,3}$ be the operator (42) with the following sequences:

$$\tilde{b}_0(n) = \frac{137}{72},\ \tilde{b}_1(n) = -\frac{69}{8} - \frac{17}{24}n,\ \tilde{b}_2(n) = \frac{n^2}{8},\ \tilde{b}_3(n) = \frac{115}{6} + \frac{5}{2}n - \frac{1}{4}n^2,$$
$$\tilde{b}_4(n) = -\frac{101}{8} - \frac{43}{24}n + \frac{1}{8}n^2,$$
$$\tilde{d}_0(n) = -\frac{43}{36},\ \tilde{d}_1(n) = \frac{45}{4} + \frac{5}{4}n,\ \tilde{d}_2(n) = \frac{115}{4} - \frac{1}{2}n^2 + \frac{15}{4}n,$$
$$\tilde{d}_3(n) = -\frac{533}{6} + n^2 - \frac{73}{6}n,$$
$$\tilde{d}_4(n) = \frac{101}{2} + \frac{43}{6}n - \frac{1}{2}n^2,\ \tilde{e}_0(n) = -\frac{303}{4} - \frac{43}{4}n + \frac{3}{4}n^2.$$

Lemma 4.1 *The central moments of the operators* $\tilde{K}_n^{M,3}$ *are given by*

i) $\tilde{K}_n^{M,3}(t-x;x)=\tilde{K}_n^{M,3}((t-x)^2;x)=\tilde{K}_n^{M,3}((t-x)^3;x)=0;$

ii) $\tilde{K}_n^{M,3}((t-x)^4;x)=x(1-x)(135x^2-135x+23)\dfrac{n}{(n+1)^4}+\mathcal{O}\left(\dfrac{1}{n^4}\right);$

iii) $\tilde{K}_n^{M,3}((t-x)^5;x)=-\dfrac{55}{2(n+1)^5}(2x-1)x^2(1-x)^2n^2+\mathcal{O}\left(\dfrac{1}{n^4}\right);$

v) $\tilde{K}_n^{M,3}((t-x)^6;x)=\dfrac{15x^3(1-x)^3n^3}{(n+1)^6}+\mathcal{O}\left(\dfrac{1}{n^4}\right);$

iv) $\tilde{K}_n^{M,3}((t-x)^7;x)=\tilde{K}_n^{M,3}((t-x)^8;x)=\mathcal{O}\left(\dfrac{1}{n^4}\right);$

v) $\tilde{K}_n^{M,3}((t-x)^9;x)=\tilde{K}_n^{M,3}((t-x)^{10};x)=\mathcal{O}\left(\dfrac{1}{n^5}\right).$

The asymptotic order of approximation of $\tilde{K}_n^{M,3}$ to f when n goes to infinity is given in the following result:

Theorem 4.1 *If* $f\in C^{10}[0,1]$ *and* $x\in[0,1]$, *then for sufficiently large n, we have*

$$\tilde{K}_n^{M,3}(f;x)-f(x)=\mathcal{O}\left(\frac{1}{n^3}\right).$$

5 Numerical results

In this section, we will analyse the theoretical results presented in the previous sections by numerical examples.

Example 5.1 The convergence of the new modifications of the Kantorovich operators is illustrated in Fig. 1, where $f(x)=\sin(2\pi x)$, $n=10$, $a_0=\dfrac{n}{2n+1}$ and $a_1=\dfrac{1}{2n+1}$. Note that the approximation by the modified Kantorovich operators $K_n^{M,i}$, $i=1,2,3$ is better than using the classical Kantorovich operator K_n. Let $E_n(f;x)=|f(x)-K_n(f;x)|$ and $E_n^{M,i}(f;x)=\left|f(x)-\tilde{K}_n^{M,i}(f;x)\right|$, $i=1,2,3$ be the error function of the Kantorovich operators, respectively, the modified Kantorovich operators. Also, the error of approximation is illustrated in Fig. 2.

Example 5.2 Let us consider the following function $f(x)=\left|x-\frac{1}{2}\right|\cos(2\pi x)$. It can be observed that $f\in C[0,1]$, but it is not differentiable at the point $x=0.5$. For $n=10$, $a_0=\dfrac{n}{2n+1}$ and $a_1=\dfrac{1}{2n+1}$, the convergence of the modified Kantorovich operators to function $f(x)$ is illustrated in Fig. 3. Also, the error functions E_n and $E_n^{M,i}$, $i=1,2,3$ are given in Fig. 4.

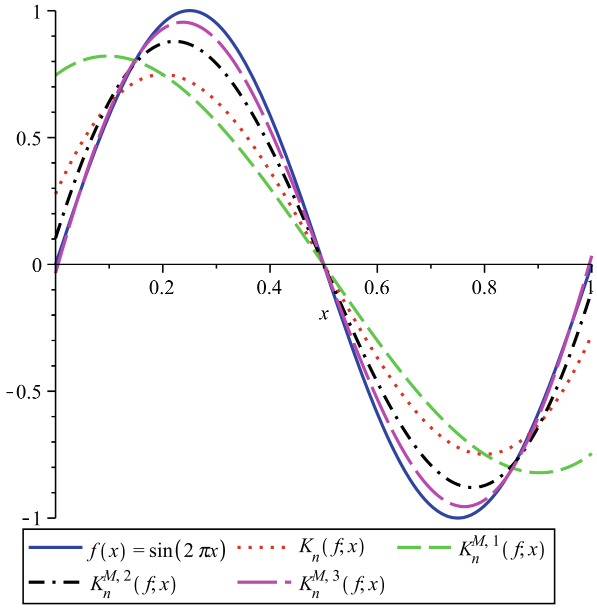

Fig. 1 Approximation process

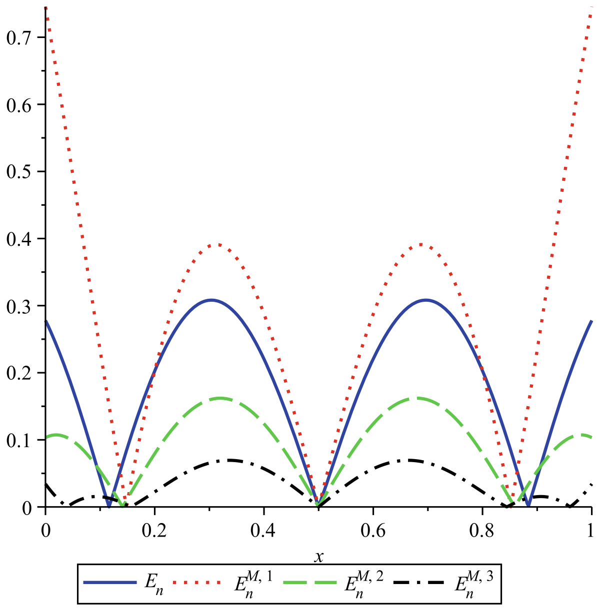

Fig. 2 Error of approximation

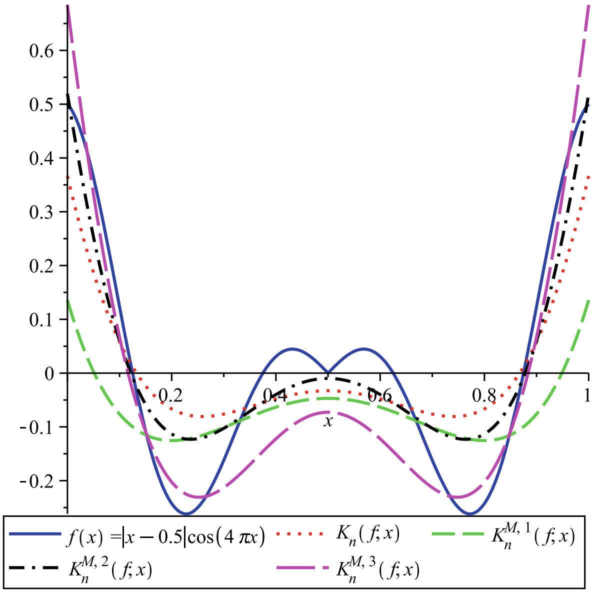

Fig. 3 Approximation process

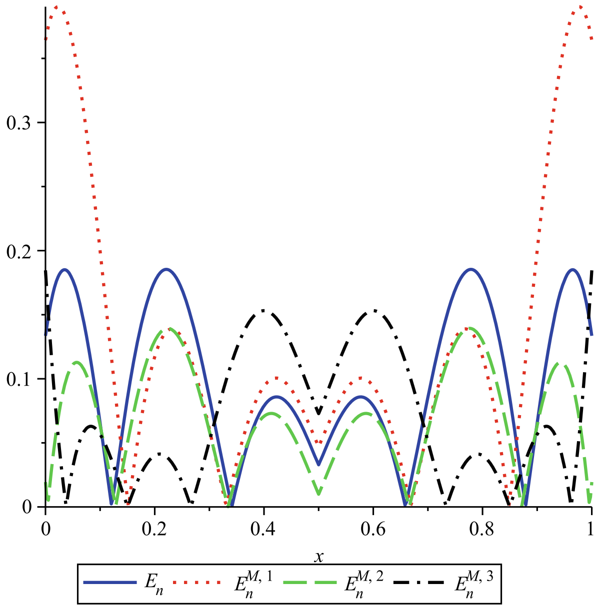

Fig. 4 Error of approximation

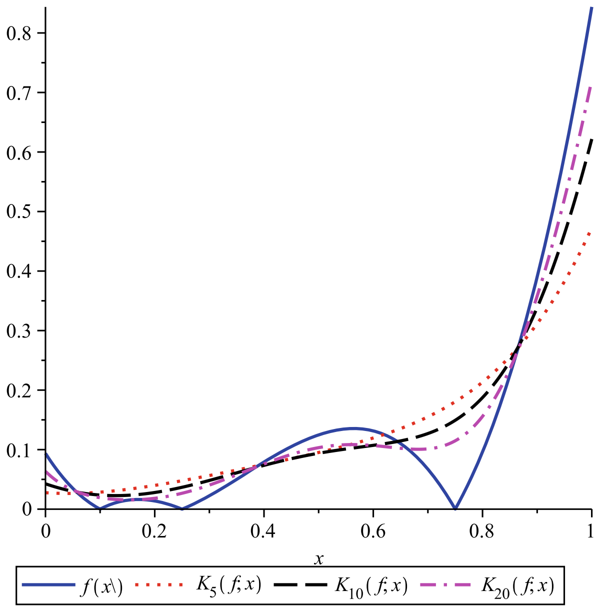

Fig. 5 Approximation process of K_n

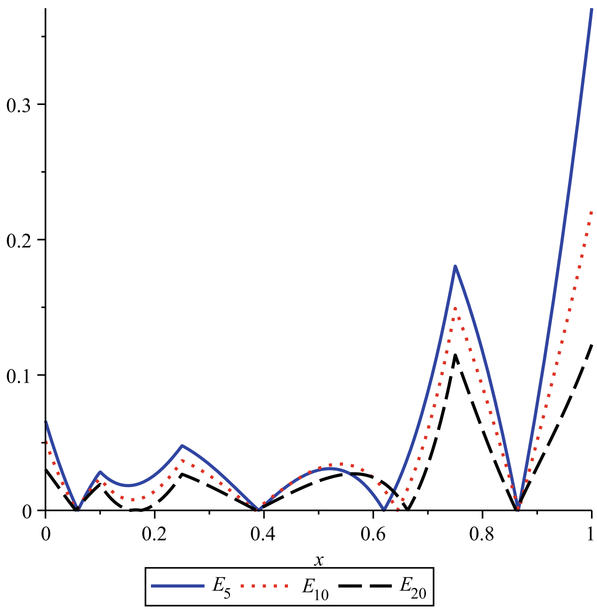

Fig. 6 Error of approximation E_n

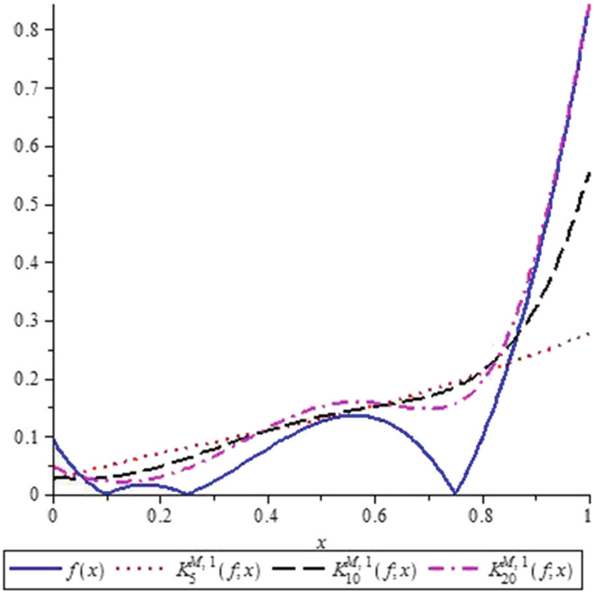

Fig. 7 Approximation process of $\tilde{K}_n^{M,1}$

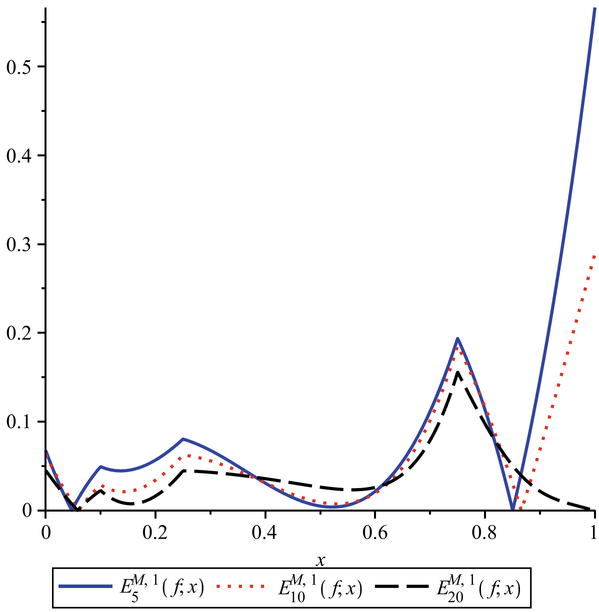

Fig. 8 Error of approximation $E_n^{M,1}$

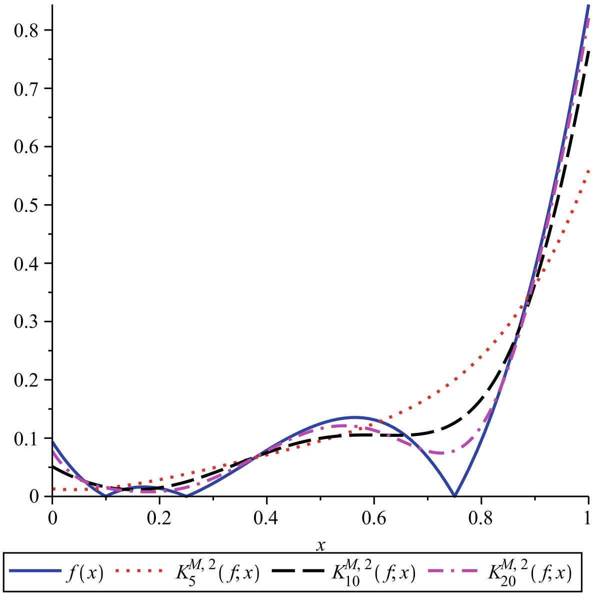

Fig. 9 Approximation process of $\tilde{K}_n^{M,2}$

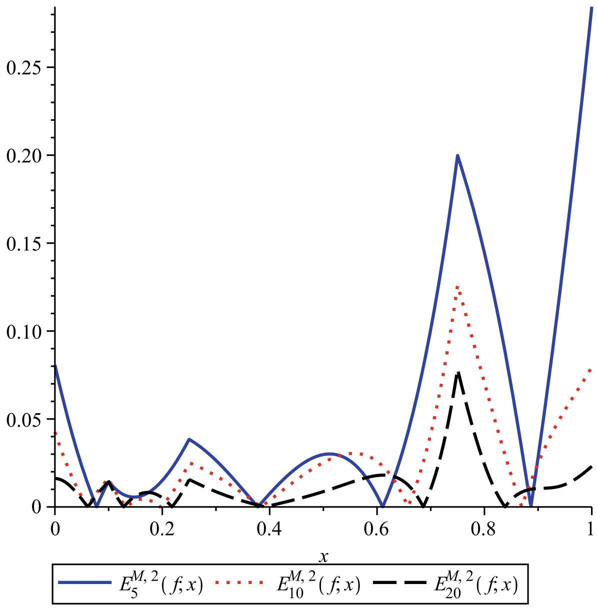

Fig. 10 Error of approximation $E_n^{M,2}$

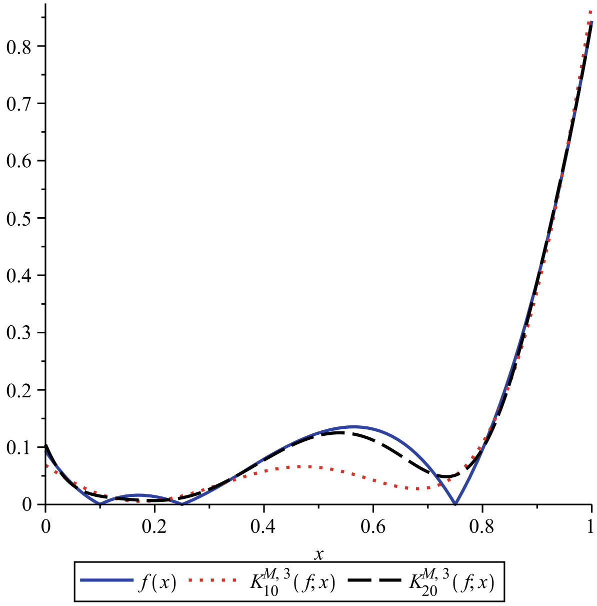

Fig. 11 Approximation process $\tilde{K}_n^{M,3}$

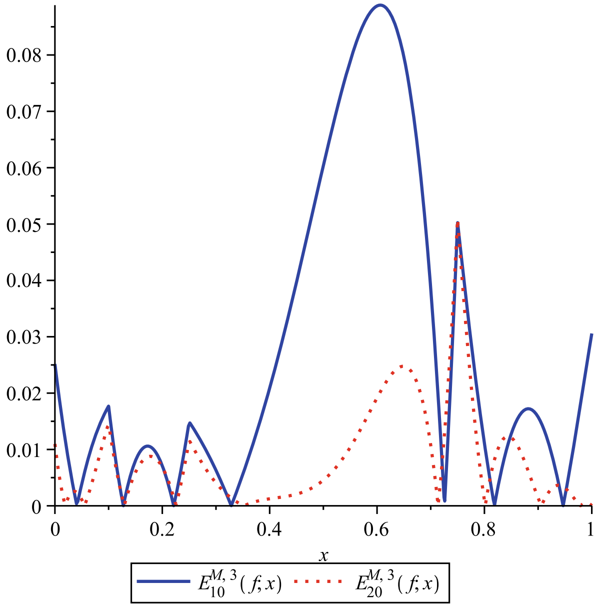

Fig. 12 Error of approximation $E_n^{M,3}$

Example 5.3 Let us consider the following function $f(x) = \left|\left(x - \frac{1}{4}\right)\left(5x - \frac{1}{2}\right)\left(x - \frac{3}{4}\right)\right|$. The behaviors of the approximations $K_n(f;x)$, $\tilde{K}_n^{M,i}(f;x)$, $i = 1, 2, 3$ and their error functions $E_n(f;x)$, $E_n^{M,i}(f;x)$ for $n = 5, 10, 20$, $a_0 = \frac{n}{2n+1}$, $a_1 = \frac{1}{2n+1}$ are illustrated in the Figs. 5, 6, 7, 8, 9, 10, 11, and 12.

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