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# Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems

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**Abstract** In this paper, basing on the subgradient extragradient method and inertial method with line-search process, we introduce two new algorithms for finding a common element of the solution set of a variational inequality and the fixed point set of a quasi-nonexpansive mapping with a demiclosedness property. The weak convergence of the algorithms are established under standard assumptions imposed on cost operators. The proposed algorithms can be considered as an improvement of the previously known inertial extragradient method over each computational step. Finally, for supporting the convergence of the proposed algorithms, we also consider several preliminary numerical experiments on a test problem.

**Keywords** Subgradient extragradient method · Extragradient method · Inertial method · Variational inequality problem · Fixed point problem

**Mathematics Subject Classification (2010)** 47H09 · 47H10 · 47J20 · 47J25

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## 1 Introduction

Let $H$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $C$ be a nonempty closed convex subset in $H$. Let $A : H \to H$ be an operator. The variational inequality problem (VIP) for $A$ on $C$ is to find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \ \ \forall x \in C. \tag{1}$$

Let us denote $VI(C, A)$ by the solution set of VIP (1). Variational inequalities theory, which was introduced by Stampacchia [46], arise in various models for a large number of mathematical, physical, regional, social, engineering, and other problems. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. It has been shown that this theory provides a simple, natural, and unified framework for a general treatment of unrelated problems. In recent years, considerable interest has been shown in developing various extensions and generalizations of variational inequalities, both for their own sake and for their applications. Recently, much attention has been given to develop efficient and implementable numerical methods including projection method and its variant forms, see [12–15, 20, 32, 38–40, 50, 51, 54]. The basic idea consists of extending the projected gradient method for solving the problem of minimizing $f(x)$ subject to $x \in C$ given by

$$x_{n+1} = P_C(x_n - \alpha_n \bigtriangledown f(x_n)), \quad n \geq 0, \tag{2}$$

where $\{\alpha_n\}$ is a positive real sequence satisfying certain conditions and $P_C$ is the metric projection onto $C$. For convergence properties of this method for the case in which $f : H \to \mathbb{R}$ is convex and differentiable function, one may see [1]. An immediate extension of method (2) to VIP is the projected gradient method for optimization problems, substituting the operator $A$ for the gradient, so that we generate a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = P_C(x_n - \alpha_n A x_n), \quad n \geq 0.$$

However, the convergence of this method requires a slightly strong assumption that operators are strongly monotone or inverse strongly monotone, see, e.g., [55]. To avoid this strong assumption, Korpelevich [31] introduced the extragradient method for solving saddle point problems, and after that, this method was further extended to VIPs in both Euclidean spaces and Hilbert spaces. The convergence of the extragradient method only requires that the operator $A$ is monotone and $L$-Lipschitz continuous. More precisely, the extragradient method is of the form:

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ x_{n+1} = P_C(x_n - \tau A y_n), \end{cases} \tag{3}$$

where $\tau \in (0, \frac{1}{L})$ and $P_C$ is denoted by the metric projection from $H$ onto $C$. If the solution set $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by process (3) converges weakly to an element in $VI(C, A)$.

In fact, in the extragradient method, one needs to calculate two projections onto $C$ in each iteration. Note that the projection onto a closed convex set $C$ is related to a minimum distance problem. If $C$ is a general closed and convex set, this might require a prohibitive amount of computation time. To overcome this drawback, Y. Censor et al. in [13] modified this algorithm and called it the subgradient extragradient method. The purpose of this modification is to replace two projections onto $C$ by one projection onto $C$ and one onto a half-space. Let us note that the projection onto a half-space is easier to compute. The subgradient extragradient method is of the form:

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ T_n = \{x \in H | \langle x_n - \tau A x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \tau A y_n), \end{cases} \tag{4}$$

where $\tau \in (0, \frac{1}{L})$. The method subgradient extragradient for solving VIP (1) has received great attention by many authors (see, e.g., [32, 53] and the references therein).

On the other hand, related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. Let $T : H \to H$ be a mapping. A point $x^* \in H$ is called a fixed point of $T$ if $Tx = x$. The set of fixed points of $T$ is denoted by $Fix(T)$. The fixed point problem for $T$ is the problem

$$\text{Find} x^* \in H \text{ such that} T x^* = x^*. \tag{5}$$

For finding a common element of $Fix(T)$ and the solution set $VI(C, A)$ of VIP (1) in Hilbert space $H$, many iterative methods have been proposed, see, e.g., [13, 16, 17, 27, 42, 43, 53] and the references therein. The motivation for studying such a problem is in its possible application to mathematical models whose constraints can be expressed as fixed point problems and/or variational inequalities. This happens, in particular, in practical problems as signal processing, network resource allocation, and image recovery, see, for example [25, 26, 33, 34].

In 2003, Takahashi and Toyoda [48] introduced an iterative scheme to find a common point of solution set $VI(C, A)$ of VIP (1) and $Fix(T)$. Under the assumption that $A : H \to H$ is $\lambda$-inverse strongly monotone (where $\lambda$ is a positive constant) and $T : C \to C$ is nonexpansive such that $Fix(T) \cap VI(C, A) \neq \emptyset$, they proved that the sequence $\{x_n\}$ generated by their iterative scheme converges weakly to some point $z \in Fix(T) \cap VI(C, A)$. Their algorithm is of the form:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T P_C(x_n - \lambda_n A x_n). \tag{6}$$

Motivated by the ideal of Korpelevich's extragradient method of [31], under the assumption that $A : C \to H$ is monotone, $L$-Lipschitz continuous, and $T : C \to C$ is nonexpansive such that $Fix(T) \cap VI(C, A) \neq \emptyset$, Nadezhkina and Takahashi [43] proved that the sequence $\{x_n\}$, generated by their iterative process, converge weakly to some point $z \in Fix(T) \cap VI(C, A)$. Their algorithm is as follows:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T P_C(x_n - \lambda_n A y_n). \end{cases} \tag{7}$$

Recently, under the assumptions that $A : H \to H$ is monotone, Lipschitz continuous, $T : H \to H$ is nonexpansive such that $Fix(T) \cap VI(C, A) \neq \emptyset$, Censor et al.

[13] showed that the sequence generated by their algorithm weakly converge to the some point $z \in Fix(T) \cap VI(C, A)$. They introduced the following algorithm:

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ T_n = \{x \in H | \langle x_n - \tau A x_n - y_n, x - y_n \rangle \le 0\}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_{T_n}(x_n - \tau A y_n). \end{cases} \tag{8}$$

The concept of quasi-nonexpansive mapping was essentially introduced by Diaz and Metcalf [21]. It is well that every nonexpansive mapping with a nonempty set of fixed points is quasi-nonexpansive. Iterative approximation of fixed points of quasi-nonexpansive mappings has been studied extensively by various authors (see, for example, [19, 23, 24, 47, 49] and the references therein).

Now, let us mention an inertial type algorithm which is based upon a discrete version of a second-order dissipative dynamical system [3, 4] and can be regarded as procedure of acceleration of convergence properties (see, e.g., [2, 36, 37, 45]). In 2001, Alvarez and Attouch [2] applied the inertial technique to obtain an inertial proximal method for solving the problem of finding zero of a maximal monotone operator. It works as follows: given $x_{n-1}, x_n \in H$ and two parameters $\theta_n \in [0, 1)$, $\lambda_n > 0$, find $x_{n+1} \in H$ such that

$$0 \in \lambda_n A(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1}), \tag{9}$$

which can be written equivalently to the following

$$x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})), \tag{10}$$

where $J^A_{\lambda_n}$ is the resolvent of $A$ with parameter $\lambda_n$ and the inertia is induced by the term $\theta_n(x_n - x_{n-1})$.

Recently, a lot of researchers constructed fast iterative algorithms by using inertial extrapolation, including inertial forward-backward splitting methods [5, 35, 44], inertial Douglas-Rachford splitting method [10], inertial ADMM [11, 18], inertial forward-backward-forward method [6, 9], inertial proximal-extragradient method [7], inertial Tseng method [8], inertial contraction method [22], and inertial Mann method [52].

In this paper, motivated and inspired by the works in literature, and by the ongoing research in these directions, we propose two new inertial subgradient extragradient methods which combine the inertial subgradient extragradient method [53] with Mann method [41] for solving variational inequality problem and fixed point problem for quasi-nonexpansive mapping.

The remainder of this paper is organized as follows: In Section 2, we recall some definitions and preliminary results for further use. Section 3 deals with analyzing the convergence of the proposed algorithms. Finally, in Section 4, we perform several numerical examples to support the convergence of our algorithms.

## 2 Preliminaries

Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to $x$ is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to $x$ is written as $x_n \to x$ as $n \to \infty$. For each $x, y \in H$ and $\alpha \in \mathbb{R}$, we have

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2; \tag{11}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{12}$$

For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\| \ \forall y \in C$. $P_C$ is called the metric projection of $H$ onto $C$. It is known that $P_C$ is nonexpansive.

**Lemma 2.1** ([28]) *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Given $x \in H$ and $z \in C$. Then $z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0 \ \ \forall y \in C$.*

**Lemma 2.2** ([28]) *Let $C$ be a closed and convex subset in a real Hilbert space $H$, $x \in H$. Then*

(i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \ \forall y \in C$;
(ii) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2 \ \forall y \in C$.

For properties of the metric projection, the interested reader could be referred to Section 3 in [28].

**Definition 2.1** Let $T : H \to H$ be an operator. Then

– $T$ is called $L$-Lipschitz continuous with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in H. \tag{13}$$

– $T$ is called monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x, y \in H. \tag{14}$$

– A mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in H.$$

– A mapping $T : H \to H$ with $Fix(T) \neq \emptyset$ is said to be quasi-nonexpansive if

$$\|Tx - p\| \leq \|x - p\| \quad \forall x \in H, p \in Fix(T).$$

We give an example of a quasi-nonexpansive mapping which is not nonexpansive.

*Example 1* ([19]) Let $C := \{x \in l_\infty : \|x\|_\infty \leq 1\}$. Define $T : C \to C$ by $Tx := (0, x_1^2, x_2^2, ...)$ for $x = (x_1^2, x_2^2, ...)$ in $C$. Then it is clear that $T$ is continuous and maps $C$ into $C$. Moreover, $Tx^* = x^*$ if and only if $x^* = 0$. Furthermore,

$$\begin{aligned} \|Tx - x^*\| = \|Tx\|_\infty = \|(0, x_1^2, x_2^2, ...)\|_\infty &\leq \|(0, x_1, x_2, ...)\|_\infty \\ &= \|x\|_\infty = \|x - x^*\|_\infty \end{aligned}$$

for all $x \in C$. Therefore, $T$ is quasi-nonexpansive. However, $T$ is not nonexpansive, for $x = \left(\frac{3}{4}, \frac{3}{4}, ...\right)$ and $y = \left(\frac{1}{2}, \frac{1}{2}, ...\right)$, it is clear that $x$ and $y$ belong to $C$. Furthermore, $\|x - y\|_\infty = \|\left(\frac{1}{4}, \frac{1}{4}, ...\right)\|_\infty = \frac{1}{4}$ and $\|Tx - Ty\|_\infty = \|\left(0, \frac{5}{16}, \frac{5}{16}, ...\right)\|_\infty = \frac{5}{16} > \frac{1}{4} = \|x - y\|_\infty$.

**Definition 2.2** ([28]) Assume that $T : H \to H$ is a nonlinear operator with $Fix(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in $H$, the following implication holds:

$$x_n \rightharpoonup x \text{ and} (I - T)x_n \to 0 \Longrightarrow x \in Fix(T).$$

The following example shows that there exists a quasi-nonexpansive mapping $T$ but $I - T$ is not demiclosed at zero.

*Example 2* Let $H$ be the line real and $C = [0, \frac{3}{2}]$. Define $T$ on $C$ by

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 1], \\ x \cos 2\pi x & \text{if } x \in (1, \frac{3}{2}]. \end{cases}$$

Then $T$ is a quasi-nonexpansive mapping but $I - T$ is not demiclosed at zero. Indeed, it is easy to see that $Fix(T) = \{0\}$. For any $x \in [0, 1]$, we have

$$|Tx - 0| = |\frac{x}{2} - 0| \leq |x - 0|,$$

and for any $x \in (1, \frac{3}{2}]$, we have

$$|Tx - 0| = |x \cos 2\pi x - 0| = |x \cos 2\pi x| \leq |x| = |x - 0|.$$

Thus, $T$ is quasi-nonexpansive. By taking $\{x_n\} \subset (1, \frac{3}{2}]$ and $x_n \to 1$ as $n \to \infty$, we have

$$|(I - T)x_n| = |x_n - x_n \cos 2\pi x_n| = |x_n||1 - \cos 2\pi x_n| \to 0 \text{ as} n \to \infty.$$

But $1 \notin Fix(T)$, so $I - T$ is not demiclosed at zero.

**Lemma 2.3** ([2]) *Let $\{\varphi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0, +\infty)$ such that*

$$\varphi_{n+1} \leq \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \delta_n \ \forall n \geq 1, \sum_{n=1}^{+\infty} \delta_n < +\infty,$$

*and there exists a real number $\alpha$ with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following hold:*

(i) $\sum_{n=1}^{+\infty} [\varphi_n - \varphi_{n-1}]_+ < +\infty$, *where* $[t]_+ := \max\{t, 0\}$;
(ii) *there exists* $\varphi^* \in [0, +\infty)$ *such that* $\lim_{n \to +\infty} \varphi_n = \varphi^*$.

**Lemma 2.4** (Opial 1967) *Let $C$ be a nonempty set of $H$ and $\{x_n\}$ be a sequence in $H$ such that the following two conditions hold:*

(i) *for every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
(ii) *every sequential weak cluster point of $\{x_n\}$ is in $C$.*

*Then $\{x_n\}$ converges weakly to a point in $C$.*

**Lemma 2.5** ([32]) *Let $A : H \to H$ be a monotone and $L$-Lipschitz continuous mapping on C. Let $S = P_C(I - \mu A)$, where $\mu > 0$. If $\{x_n\}$ is a sequence in $H$ satisfying $x_n \rightharpoonup q$ and $x_n - Sx_n \to 0$ then $q \in VI(C, A) = Fix(S)$.*

## 3 Main results

In this section, we modify inertial subgradient extragradient algorithm for solving variational inequality problem and fixed point problem of a quasi-nonexpansive mapping $T$ in real Hilbert spaces. Under mild assumptions, the sequences generated by the proposed methods converge weakly to an element of $Fix(T) \cap VI(C, A)$. Throughout this section, we assume that $A : H \to H$ is monotone and Lipschitz continuous on $H$ with the constant $L$. However, the information of $L$ is not necessary to be known. Let $T : H \to H$ be a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero. In addition, we assume that the solution set $Fix(T) \cap VI(C, A) \neq \emptyset$.

First, we propose Mann-type subgradient extragradient-like algorithm. The algorithm is of the form:

**Lemma 3.1** *If $w_n = z_n = x_{n+1}$ then $w_n \in Fix(T) \cap VI(C, A)$.*

*Proof* Fix $p \in Fix(T) \cap VI(C, A)$. By Lemma 3.3, we have

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - (1-\mu)\|y_n - w_n\|^2.$$

This implies that $\|y_n - w_n\| = 0$ that is, $w_n = y_n$. Therefore, $w_n \in VI(C, A)$. On the other hand, if $w_n = z_n = x_{n+1}$, the by (16), we obtain $w_n = (1-\beta_n)w_n + \beta_n T w_n$; thus, $Tw_n = w_n$ that is $w_n \in Fix(T)$. So, we have $w_n \in Fix(T) \cap VI(C, A)$. □

The following lemmas are quite helpful to analyze the convergence of algorithms.

**Lemma 3.2** *The Armijo-like search rule (15) is well defined and*

$$\min\{\gamma, \frac{\mu l}{L}\} < \tau_n \le \gamma.$$

*Proof* Since $A$ is $L$-Lipschitz continuous on $H$, we have

$$\|A(w_n) - A(P_C(w_n - \gamma l^m A w_n))\| \le L\|w_n - P_C(x_n - \gamma l^m A w_n)\|;$$

**Algorithm 1**

**Initialization:** Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary

**Iterative Steps:** Calculate $x_{n+1}$ as follows:

**Step 1.** Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute

$$y_n = P_C(w_n - \tau_n A w_n),$$

where $\tau_n$ is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\|. \tag{15}$$

**Step 2.** Compute

$$z_n = P_{T_n}(w_n - \tau_n A y_n),$$

where $T_n := \{x \in H | \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \leq 0\}$.
**Step 3.** Compute

$$x_{n+1} = (1-\beta_n)w_n + \beta_n T z_n. \tag{16}$$

If $w_n = z_n = x_{n+1}$ then $w_n \in Fix(T) \cap VI(C, A)$.
Set $n := n+1$ and go to **Step 1**.

this is equivalent to

$$\frac{\mu}{L}\|A(w_n) - A(P_C(w_n - \gamma l^m A w_n))\| \leq \mu\|w_n - P_C(w_n - \gamma l^m A w_n)\|.$$

This implies that (15) holds for all $\gamma l^m \leq \frac{\mu}{L}$,so $\tau_n$ is well defined.

Obviously, $\tau_n \leq \gamma$. If $\tau_n = \gamma$, then this lemma is proved; otherwise, if $\tau_n < \gamma$ by the search rule (15), we know that $\frac{\lambda_n}{l}$ must violate inequality (15), i.e.,

$$\|A(w_n) - A(P_C(w_n - \frac{\tau_n}{l} A w_n))\| > \frac{\mu}{\frac{\tau_n}{l}}\|w_n - P_C(w_n - \frac{\tau_n}{l} A w_n)\|;$$

combining this with $A$ is $L$-Lipschitz continuous on $H$, we obtain

$$\tau_n > \frac{\mu l}{L}.$$

This completes the proof. □

**Lemma 3.3** *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then*

$$\|x_{n+1}-p\|^2 \leq \|w_n-p\|^2-(1-\mu)\|y_n-w_n\|^2-(1-\mu)\|x_{n+1}-y_n\|^2-2\tau_n\langle Ap, y_n-p\rangle, \tag{17}$$

*for all $p \in VI(C, A)$.*

*Proof* Since $p \in VI(C, A) \subset C \subset T_n$, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|P_{T_n}(w_n - \tau_n Ay_n) - P_{T_n}p\|^2 \leq \langle x_{n+1} - p, w_n - \tau_n Ay_n - p\rangle \\ &= \frac{1}{2}\|x_{n+1} - p\|^2 + \frac{1}{2}\|w_n - \tau_n Ay_n - p\|^2 - \frac{1}{2}\|x_{n+1} - w_n + \tau_n Ay_n\|^2 \\ &= \frac{1}{2}\|x_{n+1} - p\|^2 + \frac{1}{2}\|w_n - p\|^2 + \frac{1}{2}\tau_n^2\|Ay_n\|^2 - \langle w_n - p, \tau_n Ay_n\rangle \\ &\quad -\frac{1}{2}\|x_{n+1} - w_n\|^2 - \frac{1}{2}\tau_n^2\|Ay_n\|^2 - \langle x_{n+1} - w_n, \tau_n Ay_n\rangle \\ &= \frac{1}{2}\|x_{n+1} - p\|^2 + \frac{1}{2}\|w_n - p\|^2 - \frac{1}{2}\|x_{n+1} - w_n\|^2 - \langle x_{n+1} - p, \tau_n Ay_n\rangle. \quad (18)\end{aligned}$$

This implies that

$$\|x_{n+1} - p\|^2 \leq \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2 - 2\langle x_{n+1} - p, \tau_n Ay_n\rangle. \quad (19)$$

Since $A$ is monotone, we have $2\tau_n\langle Ay_n - Ap, y_n - p\rangle \geq 0$. Thus, adding this item to the right side of (19), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2 - 2\langle x_{n+1} - p, \tau_n Ay_n\rangle \\ &\quad +2\tau_n\langle Ay_n - Ap, y_n - p\rangle \\ &= \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2 + 2\langle y_n - x_{n+1}, \tau_n Ay_n\rangle - 2\tau_n\langle Ap, y_n - p\rangle \\ &= \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2 + 2\tau_n\langle y_n - x_{n+1}, Ay_n - Aw_n\rangle \\ &\quad +2\tau_n\langle Aw_n, y_n - x_{n+1}\rangle - 2\tau_n\langle Ap, y_n - p\rangle. \quad (20)\end{aligned}$$

We estimate $2\tau_n\langle y_n - x_{n+1}, Ay_n - Aw_n\rangle$. It follows that

$$\begin{aligned}2\tau_n\langle y_n - x_{n+1}, Ay_n - Aw_n\rangle &\leq 2\tau_n\|Ay_n - Aw_n\|\|y_n - x_{n+1}\| \\ &\leq 2\mu\|y_n - w_n\|\|y_n - x_{n+1}\| \\ &\leq \mu\|y_n - w_n\|^2 + \mu\|y_n - x_{n+1}\|^2. \quad (21)\end{aligned}$$

As $y_n = P_{T_n}(w_n - \tau_n Aw_n)$ and $x_{n+1} \in T_n$, we have

$$\langle w_n - \tau_n Aw_n - y_n, x_{n+1} - y_n\rangle \leq 0.$$

This implies that

$$\begin{aligned}2\tau_n\langle Aw_n, y_n - x_{n+1}\rangle &\leq 2\langle y_n - w_n, x_{n+1} - y_n\rangle \\ &= \|x_{n+1} - w_n\|^2 - \|y_n - w_n\|^2 - \|x_{n+1} - y_n\|^2. \quad (22)\end{aligned}$$

Using (21) and (22), we deduce in (20) that

$$\|x_{n+1}-p\|^2 \leq \|w_n-p\|^2-(1-\mu)\|y_n-w_n\|^2-(1-\mu)\|x_{n+1}-y_n\|^2-2\tau_n\langle Ap, y_n-p\rangle.$$

□

**Theorem 3.1** *Assume that the sequence $\{\alpha_n\}$ is non-decreasing such that $0 \leq \alpha_n \leq \alpha \leq \frac{1}{4}$ and $\{\beta_n\}$ is a sequence of real numbers such that $0 < \beta \leq \beta_n \leq \frac{1}{2}$. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to an element of $Fix(T) \cap VI(C, A)$.*

*Proof* Let $p \in Fix(T) \cap VI(C, A)$. By Lemma 3.3, we have

$$\|z_n - p\| \le \|w_n - p\|. \tag{23}$$

Using (12) and (23), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-\beta_n)w_n + \beta_n T z_n - p\|^2 \\ &= \|(1-\beta_n)(w_n - p) + \beta_n(T z_n - p)\|^2 \\ &= (1-\beta_n)\|w_n - p\|^2 + \beta_n\|T z_n - p\|^2 - (1-\beta_n)\beta_n\|T z_n - w_n\|^2 \\ &\le (1-\beta_n)\|w_n - p\|^2 + \beta_n\|z_n - p\|^2 - (1-\beta_n)\beta_n\|T z_n - w_n\|^2 \quad (24) \\ &\le (1-\beta_n)\|w_n - p\|^2 + \beta_n\|w_n - p\|^2 - (1-\beta_n)\beta_n\|T z_n - w_n\|^2 \\ &= \|w_n - p\|^2 - (1-\beta_n)\beta_n\|T z_n - w_n\|^2. \quad (25) \end{aligned}$$

On the other hand, we also have

$$T z_n - w_n = \frac{1}{\beta_n}(x_{n+1} - w_n). \tag{26}$$

Combining (21) and (22), we get

$$\|x_{n+1} - p\|^2 \le \|w_n - p\|^2 - \frac{1-\beta_n}{\beta_n}\|x_{n+1} - w_n\|^2. \tag{27}$$

From $\beta_n \le \frac{1}{2}$, we obtain $\frac{1-\beta_n}{\beta_n} \ge 1$. This implies that

$$\|x_{n+1} - p\|^2 \le \|w_n - p\|^2 - \|x_{n+1} - w_n\|^2. \tag{28}$$

By the definition of $w_n$, we have

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\|^2 \\ &= \|(1+\alpha_n)(x_n - p) - \alpha_n(x_{n-1} - p)\|^2 \\ &= (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2. \quad (29) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \alpha_n(x_n - x_{n-1})\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\ &\ge \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\ &\ge (1-\alpha_n)\|x_{n+1} - x_n\|^2 + \left(\alpha_n^2 - \alpha_n\right)\|x_n - x_{n-1}\|^2. \quad (30) \end{aligned}$$

Combining (28) with (20) and (30), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2\\ &\quad -(1-\alpha_n)\|x_{n+1}-x_n\|^2-\left(\alpha_n^2-\alpha_n\right)\|x_n-x_{n-1}\|^2\\ &= (1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2\\ &\quad +[\alpha_n(1+\alpha_n)-\left(\alpha_n^2-\alpha_n\right)]\|x_n-x_{n-1}\|^2\\ &= (1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2\\ &\quad +2\alpha_n\|x_n-x_{n-1}\|^2 \qquad (31)\\ &\le (1+\alpha_{n+1})\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2\\ &\quad +2\alpha_n\|x_n-x_{n-1}\|^2. \qquad (32)\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &- \alpha_{n+1}\|x_n-p\|^2+2\alpha_{n+1}\|x_{n+1}-x_n\|^2\\ &\le \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2\\ &\quad +2\alpha_{n+1}\|x_{n+1}-x_n\|^2-(1-\alpha_n)\|x_{n+1}-x_n\|^2.\end{aligned}$$

Put $\Gamma_n := \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2$. We get

$$\Gamma_{n+1}-\Gamma_n \le -(1-\alpha_n-2\alpha_{n+1})\|x_{n+1}-x_n\|^2. \qquad (33)$$

It follows from $\alpha_n \le \dfrac{1}{4}$ that $1-\alpha_n-2\alpha_{n+1} \ge \dfrac{1}{4}$. Therefore, we obtain

$$\Gamma_{n+1}-\Gamma_n \le -\delta\|x_{n+1}-x_n\|^2 \le 0, \qquad (34)$$

where $\delta = \dfrac{1}{4}$. This implies that the sequence $\{\Gamma_n\}$ is nonincreasing. On the other hand, we have

$$\begin{aligned}\Gamma_n &= \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+2\alpha_n\|x_n-x_{n-1}\|^2\\ &\ge \|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_n-p\|^2 &\le \alpha_n\|x_{n-1}-p\|^2+\Gamma_n\\ &\le \alpha\|x_{n-1}-p\|^2+\Gamma_1\\ &\le \ldots \le \alpha^n\|x_0-p\|^2+\Gamma_1(\alpha^{n-1}+\ldots+1)\\ &\le \alpha^n\|x_0-p\|^2+\frac{\Gamma_1}{1-\alpha}. \qquad (35)\end{aligned}$$

We also have

$$\begin{aligned}\Gamma_{n+1} &= \|x_{n+1}-p\|^2-\alpha_{n+1}\|x_n-p\|^2+2\alpha_{n+1}\|x_{n+1}-x_n\|^2\\ &\ge -\alpha_{n+1}\|x_n-p\|^2. \qquad (36)\end{aligned}$$

From (35) and (36), we obtain

$$-\Gamma_{n+1} \le \alpha_{n+1}\|x_n-p\|^2 \le \alpha\|x_n-p\|^2 \le \alpha^{n+1}\|x_0-p\|^2+\frac{\alpha\Gamma_1}{1-\alpha}.$$

It follows from (34) that

$$\delta \sum_{n=1}^{k} \|x_{n+1}-x_n\|^2 \le \Gamma_1 - \Gamma_{k+1} \le \alpha^{k+1}\|x_0-p\|^2 + \frac{\Gamma_1}{1-\alpha} \le \|x_0-p\|^2 + \frac{\Gamma_1}{1-\alpha}.$$

This implies

$$\sum_{n=1}^{\infty} \|x_{n+1}-x_n\|^2 < +\infty. \tag{37}$$

Therefore, we obtain

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0. \tag{38}$$

We have

$$\begin{aligned}\|x_{n+1}-w_n\| &= \|x_{n+1}-x_n-\alpha_n(x_n-x_{n-1})\| \\ &\le \|x_{n+1}-x_n\| + \alpha_n\|x_n-x_{n-1}\| \\ &\le \|x_{n+1}-x_n\| + \alpha\|x_n-x_{n-1}\|. \end{aligned} \tag{39}$$

From (38) and (39), we obtain

$$\lim_{n\to\infty} \|x_{n+1}-w_n\| = 0. \tag{40}$$

Since (31), we get

$$\|x_{n+1}-p\|^2 \le (1+\alpha_n)\|x_n-p\|^2 - \alpha_n\|x_{n-1}-p\|^2 + 2\alpha\|x_n-x_{n-1}\|^2. \tag{41}$$

By (37), (41), and Lemma 2.3, we have

$$\lim_{n\to\infty} \|x_n-p\|^2 = l, \tag{42}$$

and by (20), we obtain

$$\lim_{n\to\infty} \|w_n-p\|^2 = l. \tag{43}$$

We also have

$$0 \le \|x_n-w_n\| \le \alpha\|x_n-x_{n-1}\| \to 0. \tag{44}$$

It follows from (24) that

$$\|x_{n+1}-p\|^2 \le (1-\beta_n)\|w_n-p\|^2 + \beta_n\|z_n-p\|^2. \tag{45}$$

This implies that

$$\|z_n-p\|^2 \ge \frac{\|x_{n+1}-p\|^2 - \|w_n-p\|^2}{\beta_n} + \|w_n-p\|^2. \tag{46}$$

Since $\{\beta_n\}$ is bounded, it implies from (42), (43), and (46) that

$$\lim_{n\to\infty} \|z_n-p\|^2 \ge \lim_{n\to\infty} \|w_n-p\|^2 = l. \tag{47}$$

By (23), we get

$$\lim_{n\to\infty} \|z_n-p\|^2 \le \lim_{n\to\infty} \|w_n-p\|^2 = l. \tag{48}$$

Combining (47) and (48), we obtain

$$\lim_{n\to\infty} \|z_n - p\|^2 = l. \tag{49}$$

It implies from (17) that

$$(1-\mu)\|y_n - w_n\|^2 \le \|w_n - p\|^2 - \|z_n - p\|^2$$

and

$$(1-\mu)\|z_n - y_n\|^2 \le \|w_n - p\|^2 - \|z_n - p\|^2.$$

This implies that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0 \tag{50}$$

and

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{51}$$

Combining (50) and (51), we have

$$\lim \|z_n - w_n\| = 0 \tag{52}$$

Since $\beta_n \ge \beta$, it follows from (22) and (40) that

$$\lim_{n\to\infty} \|Tz_n - w_n\| = 0. \tag{53}$$

Combining (52) and (53), we get

$$\|Tz_n - z_n\| \le \|Tz_n - w_n\| + \|z_n - w_n\| \to 0. \tag{54}$$

Now, we show that the sequence $\{x_n\}$ converges weakly to an element of $Fix(T) \cap VI(C, A)$. Since $\{x_n\}$ is bounded sequence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in H$ such that $x_{n_k} \rightharpoonup q$. By (44), we get $w_{n_k} \rightharpoonup q$ and by (52) $z_{n_k} \rightharpoonup q$. It follows from (54) and the demiclosedness of $I - T$ that $q \in Fix(T)$.

Note that by Lemma 3.2, we have $\tau_n > \dfrac{\mu l}{L}$ for all $n$. Therefore, it implies from $\lim_{n\to\infty} \|w_n - y_n\| = 0$ and Lemma 2.5 that $q \in VI(C, A)$.

Therefore, we proved that:

(i) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in Fix(T) \cap VI(C, A)$;
(ii) If $x_{n_k} \rightharpoonup q$ then $q \in Fix(T) \cap VI(C, A)$. By Lemma 2.4, we get $\{x_n\}$ converges weakly to an element of $Fix(T) \cap VI(C, A)$.

□

Second, we introduce Mann-type inertial subgradient extragradient algorithm. The algorithm is of the form:

**Theorem 3.2** *Assume that the sequence $\{\alpha_n\}$ is non-decreasing such that $0 \le \alpha_n \le \alpha < \frac{1}{3}$ and the sequence $\{\beta_n\}$ is is non-decreasing such that $0 < \beta \le \beta_n \le \frac{3}{4}$. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to an element of $Fix(T) \cap VI(C, A)$.*

**Algorithm 2**

**Initialization:** Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary

**Iterative Steps:** Calculate $x_{n+1}$ as follows:

**Step 1.** Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute

$$y_n = P_C(w_n - \tau_n A w_n),$$

where $\tau_n$ is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau \|Aw_n - Ay_n\| \le \mu \|w_n - y_n\|. \tag{55}$$

**Step 2.** Compute

$$z_n = P_{T_n}(w_n - \tau_n A y_n),$$

where $T_n := \{x \in H | \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \le 0\}$.
**Step 3.** Compute

$$x_{n+1} = (1-\beta_n)x_n + \beta_n T z_n. \tag{56}$$

If $w_n = z_n = x_n = x_{n+1}$ then $x_n \in Fix(T) \cap VI(C, A)$.
Set $n := n+1$ and go to **Step 1**.

*Proof* Let $p \in Fix(T) \cap VI(C, A)$, by Lemma 3.3, we have

$$\|z_n - p\| \le \|w_n - p\|. \tag{57}$$

Using (12) and (57), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\beta_n)x_n + \beta_n T z_n - p\|^2 \\
&= \|(1-\beta_n)(x_n - p) + \beta_n(T z_n - p)\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T z_n - p\|^2 - (1-\beta_n)\beta_n\|T z_n - x_n\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 - (1-\beta_n)\beta_n\|T z_n - x_n\|^2 \quad (58) \\
&\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|w_n - p\|^2 - (1-\beta_n)\beta_n\|T z_n - x_n\|^2. \quad (59)
\end{aligned}$$

On the other hand, we also have

$$T z_n - x_n = \frac{1}{\beta_n}(x_{n+1} - x_n). \tag{60}$$

Combining (59) and (60), we get

$$\|x_{n+1} - p\|^2 \le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|w_n - p\|^2 - \frac{1-\beta_n}{\beta_n}\|x_{n+1} - x_n\|^2. \tag{61}$$

By the definition of $w_n$, we have

$$\begin{aligned}\|w_n - p\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\|^2 \\ &= \|(1+\alpha_n)(x_n - p) - \alpha_n(x_{n-1} - p)\|^2 \\ &= (1+\alpha_n)\|x_n - p\|^2 - \alpha_n\|x_{n-1} - p\|^2 + \alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2. \quad (62)\end{aligned}$$

Combining (61) with (62), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n(1+\alpha_n)\|x_n - p\|^2 - \beta_n\alpha_n\|x_{n-1} - p\|^2 \quad (63)\\ &\quad +\beta_n\alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 - \frac{1-\beta_n}{\beta_n}\|x_{n+1} - x_n\|^2 \\ &= (1+\beta_n\alpha_n)\|x_n - p\|^2 - \beta_n\alpha_n\|x_{n-1} - p\|^2 + \beta_n\alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ &\quad -\frac{1-\beta_n}{\beta_n}\|x_{n+1} - x_n\|^2. \quad (64)\end{aligned}$$

By the sequences $\{\beta_n\}$, $\{\alpha_n\}$ are non-decreasing we have the sequence $\{\beta_n\alpha_n\}$ is non-decreasing. Therefore, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le (1+\beta_{n+1}\alpha_{n+1})\|x_n - p\|^2 - \beta_n\alpha_n\|x_{n-1} - p\|^2 \\ &\quad +\beta_n\alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 - \frac{1-\beta_n}{\beta_n}\|x_{n+1} - x_n\|^2. \quad (65)\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 &- \beta_{n+1}\alpha_{n+1}\|x_n - p\|^2 + \beta_{n+1}\alpha_{n+1}(1+\alpha_{n+1})\|x_{n+1} - x_n\|^2 \\ &\le \|x_n - p\|^2 - \beta_n\alpha_n\|x_{n-1} - p\|^2 + \beta_n\alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 \\ &\quad +\beta_{n+1}\alpha_{n+1}(1+\alpha_{n+1})\|x_{n+1} - x_n\|^2 - \frac{1-\beta_n}{\beta_n}\|x_{n+1} - x_n\|^2. \quad (66)\end{aligned}$$

Put $\Gamma_n := \|x_n - p\|^2 - \beta_n\alpha_n\|x_{n-1} - p\|^2 + \beta_n\alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2$. It implies from (66) that

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n &\le -\frac{1-\beta_n}{\beta_n}\|x_{n+1} - x_n\|^2 + \beta_{n+1}\alpha_{n+1}(1+\alpha_{n+1})\|x_{n+1} - x_n\|^2 \\ &= -\left(\frac{1-\beta_n}{\beta_n} - \beta_{n+1}\alpha_{n+1}(1+\alpha_{n+1})\right)\|x_{n+1} - x_n\|^2. \quad (67)\end{aligned}$$

It follows from $\beta_n \le \frac{3}{4}$ that

$$\begin{aligned}\frac{1-\beta_n}{\beta_n} - \beta_{n+1}\alpha_{n+1}(1+\alpha_{n+1}) &= \frac{1}{\beta_n} - 1 - \beta_{n+1}\alpha_{n+1}(1+\alpha_{n+1}) \\ &\ge \frac{4}{3} - 1 - 3\frac{\alpha}{4} - 3\frac{\alpha^2}{4} \\ &= \frac{4 - 9\alpha - 9\alpha^2}{12}. \quad (68)\end{aligned}$$

Combining (67) and (68), we get

$$\Gamma_{n+1} - \Gamma_n \le -\delta\|x_{n+1} - x_n\|^2, \quad (69)$$

where $\delta = \dfrac{4-9\alpha-9\alpha^2}{12}$; since $\alpha < \dfrac{1}{3}$, we obtain $\delta > 0$.

The remains of proof are similar to that of Theorem 3.1; we leave the proof for the reader to verify. □

*Remark 3.1* It should be emphasized that, in this paper, we apply inertia technique to the subgradient extragradient method for solving variational inequality problems and fixed point problems. Furthermore, our algorithms also do not require Lipschitz constant as the input parameter. Therefore, the proposed algorithms are different from the algorithm studied in [13].

## 4 Numerical examples

In this section, we consider some examples in support of the convergence of Algorithms 1 and 2. We use two well-known algorithms as the extragradient method of Nadezhkina and Takahashi (NTEGM) [43, Theorem 3.1] and the modified subgradient extragradient method of Censor et al. (MSEGM) [13, Algorithm 6.1] to compare with our algorithms. In order to show the computational advantage of the proposed algorithms with others, we illustrate the behavior of the sequence $D_n = ||x_n - x^*||^2$, $n = 0, 1, 2, \ldots$ when the execution time in second elapses (elapsed time) where $x^*$ is the solution of the problem and $\{x_n\}$ is the sequence generated by each algorithm. The starting points in the first two examples are $x_0 = x_1 = (1, 1, \ldots, 1) \in \Re^m$ and those ones in the last example are $x_0(t) = x_1(t) = t+0.5\cos t$ or $x_0(t) = x_1(t) = t^2$. All the programs are written in Matlab 7.0 and performed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU at 2.50 GHz, RAM 2.00 GB. We take $\alpha_n = 1/4$ for the proposed algorithms, $\beta_n = 1/2$ for all the algorithms. We also choose the (possibly) best stepsize $\lambda_n = 0.99/L$ for two algorithms MSEGM and NTEGM. The following are the examples in details.

*Example 3* We first provide an example of Lipschitz continuous and monotone mapping $A$ and quasi-nonexpansive mapping $T$ with $Fix(T) \cap VI(C, A) \neq \emptyset$. Let $C = [-2, 5]$ and $H = \mathbb{R}$ with standard topology. Let $A : H \to H$ be given by

$$Ax := x + \sin x \tag{70}$$

and $T : H \to H$ be given by

$$Tx := \frac{x}{2}\sin x. \tag{71}$$

Now, first we show that $A$ is Lipschitz continuous and monotone with $L = 2$. Indeed, for all $x, y \in H$ we have

$$\|Ax - Ay\| = \|x + \sin x - y - \sin y\| \le \|x - y\| + \|\sin x - \sin y\| \le 2\|x - y\|.$$

This implies that $A$ is Lipschitz continuous. Next, we show that $A$ is monotone. Take arbitrarily $x, y \in H$, we have

$$\langle Ax-Ay, x-y\rangle = (x+\sin x-y-\sin y)(x-y) = (x-y)^2+(\sin x-\sin y)(x-y) \ge 0,$$

where the last inequality comes since $\|\sin x - \sin y\| \leq \|x - y\|$ for all $x, y \in H$. Second, we show that the only fixed point of $T$ is 0, since if $x \neq 0$ and $Tx = x$ then

$$x = \frac{x}{2}\sin x \quad \text{or} \quad \sin x = 2$$

which is impossible. Therefore, $Fix(T) = \{0\}$. $T$ is quasi-nonexpansive since

$$\|Tx - 0\| = \|\frac{x}{2}\sin x\| \leq \|\frac{x}{2}\| < \|x\| = \|x - 0\| \text{ for all} x \in H.$$

However, $T$ is not a nonexpansive mapping. Indeed, take $x = 2\pi$ and $y = \frac{3\pi}{2}$, then

$$\|Tx - Ty\| = \|\frac{2\pi}{2}\sin 2\pi - \frac{3\pi}{4}\sin\frac{3\pi}{2}\| = \frac{3\pi}{4} > \|2\pi - \frac{3\pi}{2}\| = \frac{\pi}{2}.$$

Furthermore, $T$ is continuous. Therefore, $I - T$ is demiclosed at 0 and it is easy to see that $Fix(T) \cap VI(C, A) = \{0\}$. The numerical results for this example are shown in Fig. 1.

*Example 4* Let $H = \mathbb{R}^n$ with standard topology and $T : H \to H$ be given by

$$Tx := -\frac{1}{2}x \qquad \text{or} \qquad Tx = P_{\mathbb{R}^n_+}(x). \tag{72}$$

Consider an operator $A : \Re^m \to \Re^m (m = 20, 50, 100, 150)$ in the form $A(x) = Mx + q$ [29, 30], where

$$M = NN^T + S + D,$$

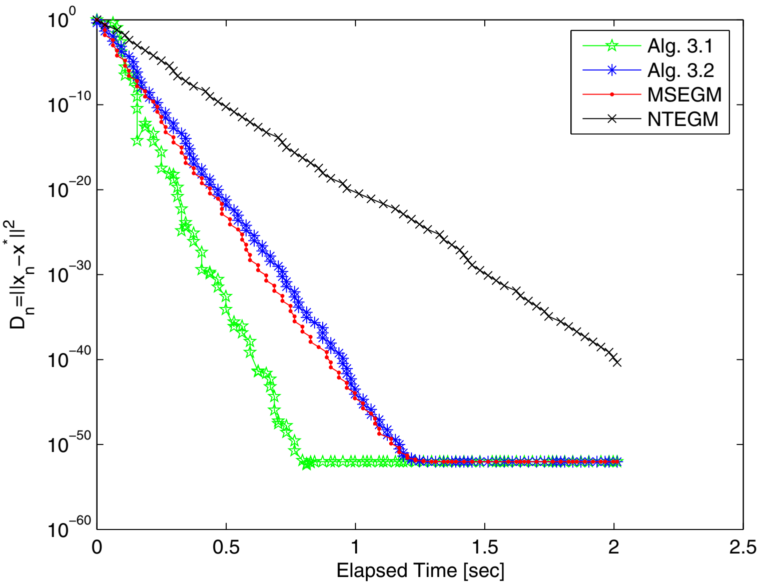


**Fig. 1** Experiment for Example 1

$N$ is a $m \times m$ matrix, $S$ is a $m \times m$ skew-symmetric matrix, $D$ is a $m \times m$ diagonal matrix, whose diagonal entries are nonnegative (so $M$ is positive definite), and $q$ is a vector in $\Re^m$. The feasible set is

$$C = \left\{x = (x_1, \ldots, x_m) \in \Re^m : -2 \leq x_i \leq 5, i = 1, \ldots, m\right\}.$$

It is clear that $A$ is monotone and Lipschitz continuous with the Lipschitz constant $L = ||M||$.

For experiments, all the entries of $N$, $S$ are generated randomly and uniformly in $[-2, 2]$, the diagonal entries of $D$ are in $(0, 2)$, and $q$ is equal to the zero vector. It is easy to see that the solution of the problem in this case is $x^* = 0$. Figures 2, 3, 4, and 5 show the numerical results for the case $Tx = -\frac{1}{2}x$, while Figs. 6–7 are for $Tx = P_{\mathbb{R}^n_+}(x)$.

*Example 5* Suppose that $H = L^2([0, 1])$ with the inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt \ \ \forall x, y \in H \tag{73}$$

and the induced norm

$$\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}} \ \ \forall x \in H. \tag{74}$$

Consider an operator $A : H \to H$ given by

$$(Ax)(t) = \max\{0, x(t)\}, t \in [0, 1] \text{ for all} x \in H.$$

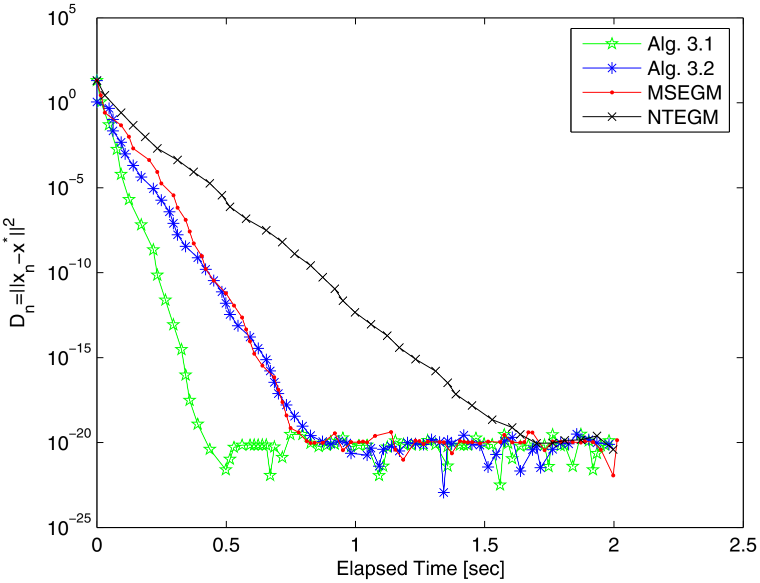


**Fig. 2** Experiment for Example 2 in $\Re^{20}$ (with $Tx = -\frac{1}{2}x$)

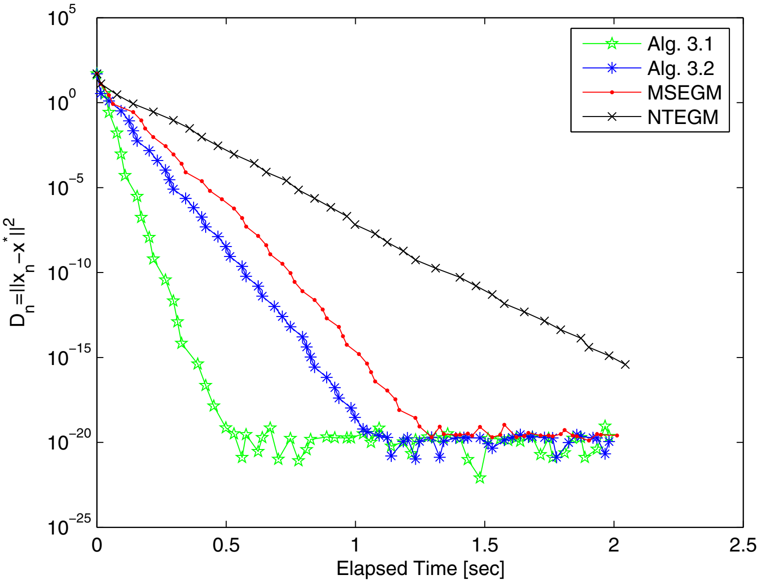


**Fig. 3** Experiment for Example 2 in $\Re^{50}$ (with $Tx = -\frac{1}{2}x$)

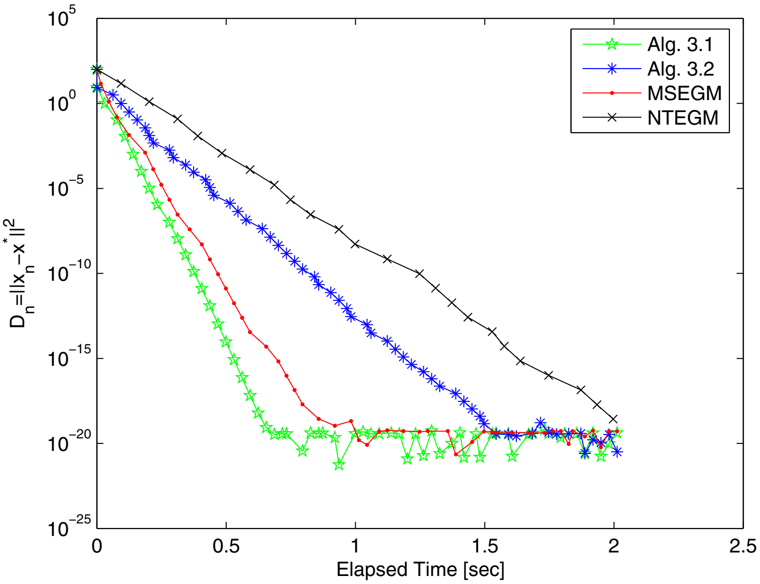


**Fig. 4** Experiment for Example 2 in $\Re^{100}$ (with $Tx = -\frac{1}{2}x$)

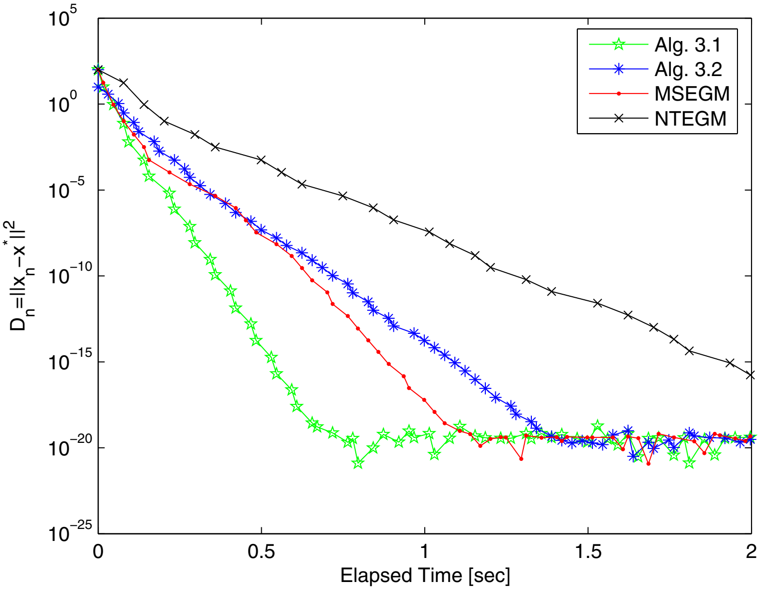


**Fig. 5** Experiment for Example 2 in $\Re^{150}$ (with $Tx = -\frac{1}{2}x$)

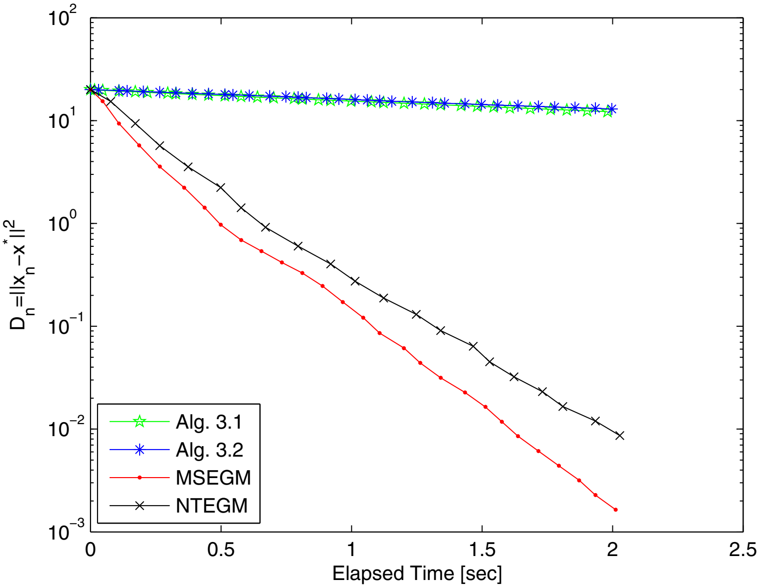


**Fig. 6** Experiment for Example 2 in $\Re^{20}$ (with $Tx = P_{\mathbb{R}^n_+}(x)$)

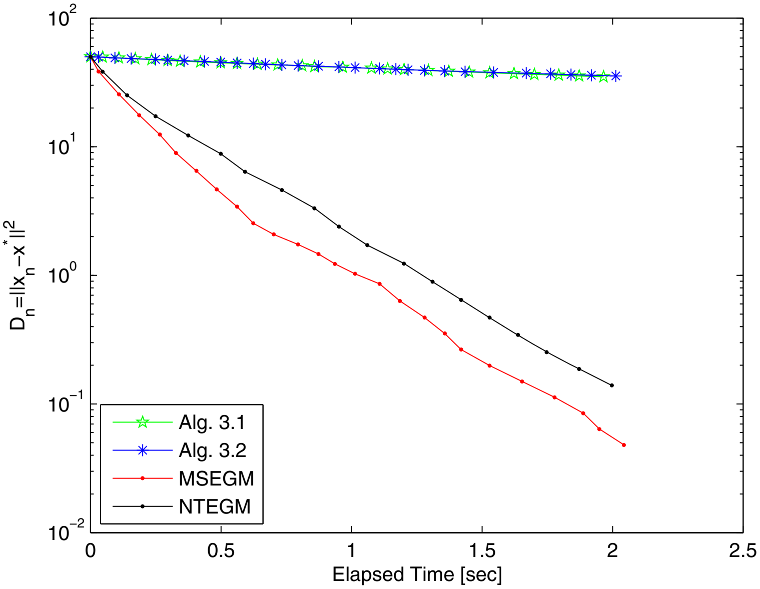


**Fig. 7** Experiment for Example 2 in $\Re^{50}$ (with $Tx = P_{\mathbb{R}^n_+}(x)$)

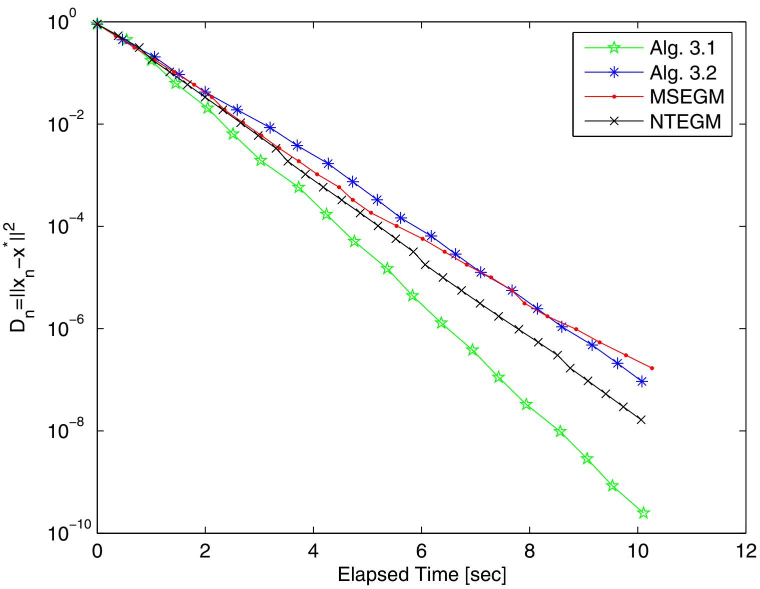


**Fig. 8** Experiment for Example 3 with the starting point $x_0(t) = t + 0.5\cos t$

It is easy to verify that $A$ is 1-Lipschitz continuous and monotone on $H$. Let $C := \{x \in H : \|x\| \le 1\}$ be the unit ball. The set of solution to the variational inequality (1) is given by $VI(C, A) = \{0\} \ne \emptyset$. It is known that

$$P_C(x) = \begin{cases} \frac{x}{\|x\|_{L^2}}, & \text{if } \|x\|_{L^2} > 1, \\ x, & \text{if } \|x\|_{L^2} \le 1. \end{cases}$$

Let $T : L^2([0, 1]) \to L^2([0, 1])$ is defined by

$$(Tx)(t) = \int_0^1 tx(s)ds, \quad t \in [0, 1]. \tag{75}$$

We see that $Fix(T) \ne \emptyset$ because $0 \in Fix(T)$. Moreover, $T$ is nonexpansive (so, quasi-nonexpansive). Indeed, we have

$$\begin{aligned} |Tx(t) - Ty(t)|^2 = |\textstyle\int_0^1 t(x(s) - y(s))ds|^2 &\le (\textstyle\int_0^1 t|x(s) - y(s)|ds)^2 \\ \le \quad & \textstyle\int_0^1 |x(s) - y(s)|^2 ds = \|x - y\|^2. \end{aligned}$$

Therefore,

$$\|Tx - Ty\|^2 = \textstyle\int_0^1 |Tx(t) - Ty(t)|^2 dt \le \|x - y\|^2.$$

The solution of the problem is $x^*(t) = 0$. Also, note here that although the result of this paper is the weak convergence of the algorithms, we wish to do a numerical example in the infinite dimensional space $L^2([0, 1])$. All the integrals in (73)–(75) and others are computed by the trapezoidal formula with the stepsize $\tau = 0.001$. Figures 8 and 9 show the behaviors of $D_n = ||x_n - x_0||^2$ generated by all the

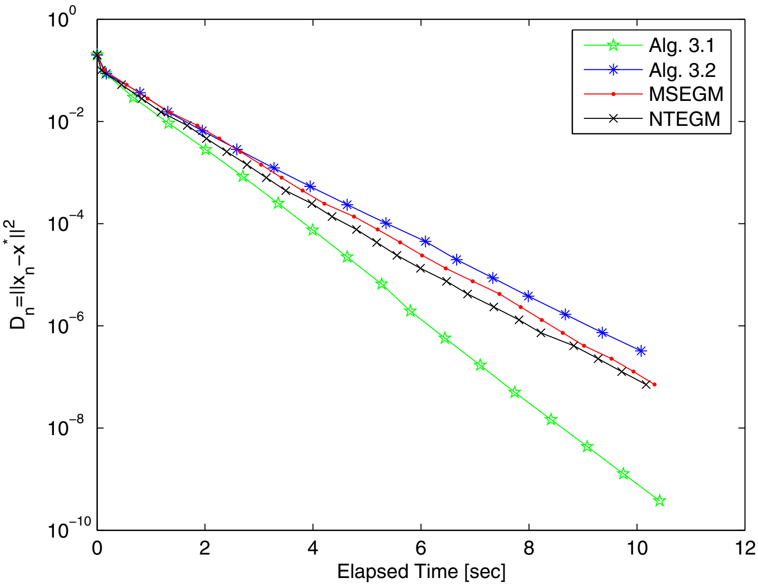


**Fig. 9** Experiment for Example 3 with the starting point $x_0(t) = t^2$

algorithms with two starting points $x_0(t) = t + 0.5\cos t$ and $x_0(t) = t^2$, respectively. The reported results have conformed that the proposed algorithms also have the competitive advantages over existing algorithms.

## 5 Conclusions

The paper has presented two iterative methods for the problem of finding a common element of the solution set of a variational inequality and of the set of fixed point of a quasi-nonexpansive mapping with a demiclosedness property. Since every nonexpansive mapping with a fixed point is quasi-nonexpansive and satisfies a demiclosedness property, it follows that our two methods improve and extend some known results existing in the literatures.

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