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On the zero point problem of monotone operators in Hadamard spaces

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Abstract In this paper, by using products of finitely many resolvents of monotone operators, we propose an iterative algorithm for finding a common zero of a finite family of monotone operators and a common fixed point of an infinitely countable family of nonexpansive mappings in Hadamard spaces. We derive the strong convergence of the proposed algorithm under appropriate conditions. A common fixed point of an infinitely countable family of quasi-nonexpansive mappings and a common zero of a finite family of monotone operators are also approximated in reflexive Hadamard spaces. In addition, we define a norm on $X^\diamond := spanX^*$ and give an application of this norm, where X is an Hadamard space with dual space X^*. A numerical example to solve a nonconvex optimization problem will be exhibited in an Hadamard space to support our main results.

Keywords Proximal point algorithm · $CAT(0)$ space · Monotone operator

Mathematics Subject Classification (2010) 47H09 · 47H10

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1 Introduction

A valuable tool in the study of gradient and subdifferential mappings and other mappings that appear in many problems, such as optimization, equilibrium or in variational inequality problems, is the concept of monotonicity. In the case of Hilbert spaces, the problem of finding zeros of monotone operators has been investigated by many authors (see, for example, [6–8, 20, 30, 37, 38, 40, 41, 46, 48]). Bearing in mind the fact that zeros of a maximal monotone operator are fixed points of its resolvent, which is a nonexpansive mapping, Rockafellar defined the proximal point algorithm for monotone operators by means of the following iterative scheme:

$$0 \in A(x_{n+1}) + \lambda_n (x_{n+1} - x_n), \quad n = 0, 1, 2, ...,$$

where $\{\lambda_n\}$ is a sequence of real positive numbers and x_0 is an initial point. Rockafellar showed the weak convergence of the sequence generated by the proximal point algorithm to a zero of the maximal monotone operator in Hilbert spaces [48]. Güler's counterexample [26] showed that the sequence generated by the proximal point algorithm does not necessarily converge strongly even if the maximal monotone operator is the subdifferential of a convex, proper and lower semicontinuous function. In this connection, see also [4]. In recent years, some algorithms defined to solve nonlinear equations, variational inequalities and minimization problems, which involve monotone operators, have been extended from the Hilbert space framework to the more general setting of Riemannian manifolds, especially Hadamard manifolds and the Hilbert unit ball (see, for example, [13, 14, 23, 24, 33, 35, 39]). This popularization is due to the fact that several non-convex problems may be viewed as a convex problem under such perspective. Studying of the proximal point method in Hadamard spaces was started by Bačák [2, Theorem 6.3.1], but only in terms of Yosida-Moreau regularization for convex optimization (see also [3]). The proximal point method in Hadamard spaces has been modified to obtain strong convergence by Cholamjiak [11] using the Halpern procedure (see also [51]). Very recently, Khatibzadeh and Ranjbar [31] generalized monotone operators and their resolvents to Hadamard spaces by using the duality theory introduced in [29].

In this paper, by using products of finitely many resolvents of monotone operators (see also [44, 45]), we propose an iterative algorithm for finding a common zero of a finite family of monotone operators and a common fixed point of an infinitely countable family of nonexpansive mappings in Hadamard spaces. We derive the strong convergence of the proposed algorithm under appropriate conditions. A common fixed point of an infinitely countable family of quasi-nonexpansive mappings and a common zero of a finite family of monotone operators is also approximated in reflexive Hadamard spaces. In addition, we define a norm on $X^\diamond := span X^*$ and give an application of this norm, where X is an Hadamard space with dual space X^*. Two numerical examples to support our main results will be exhibited. The results presented in this paper generalize and improve related results in the literature.

2 Preliminaries and lemmas

First, we collect the preliminaries on Hadamard spaces required for our proof. For the details, we refer to [2]. Let (X, d) be a metric space. A geodesic path joining x to y in X is a mapping c from a closed interval $[0, l] \subseteq \mathbb{R}$ to X such that c(0) = x, $c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$. In particular, the mapping c is an isometry and $d(x, y) = l$. The image of c is called geodesic segment joining x and y which when unique is denoted by [x, y]. We denote the unique point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ and $d(y, z) = (1 - \alpha)d(x, y)$, by $(1 - \alpha)x \oplus y$, where $0 \leq \alpha \leq 1$. The metric space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic segment joining x and y for each $x, y \in X$. A geodesic space (X, d) is a $CAT(0)$ space if it satisfies the (CN) inequality:

$$d^2((1 - \alpha)x \oplus \alpha y, z) \leq (1 - \alpha)d^2(x, z) + \alpha d^2(y, z) - \alpha(1 - \alpha)d^2(x, y), \quad (1)$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$. In particular, if x, y, z are points in X and $\alpha \in [0, 1]$, then we have

$$d((1 - \alpha)x \oplus \alpha y, z) \leq (1 - \alpha)d(x, z) + \alpha d(y, z). \quad (2)$$

A subset C of a $CAT(0)$ space is convex if $[x, y] \subseteq C$ for all $x, y \in C$. Complete $CAT(0)$ spaces are often called Hadamard spaces.

Let $\{\alpha_i\}_{i=1}^n$ be any finite subset of (0, 1) with $\Sigma_{i=1}^n \alpha_i = 1$ and $\{x_i\}_{i=1}^n \subseteq X$. By induction, the convex combination "$\bigoplus_{i=1}^n \alpha_i x_i$" is defined as follows [34, 45]:

$$\bigoplus_{i=1}^{n} \alpha_i x_i := (1 - \alpha_n)\left(\frac{\alpha_1}{1 - \alpha_n}x_1 \oplus \frac{\alpha_2}{1 - \alpha_n}x_2 \oplus \ldots \oplus \frac{\alpha_{n-1}}{1 - \alpha_n}x_{n-1}\right) \oplus \alpha_n x_n. \quad (3)$$

It is remarkable that Dhompongsa et al. [17] defined an infinite sum '$\bigoplus$' as follows:

Let $\{\alpha_i\}_{i=1}^\infty$ be a sequence in (0, 1) such that $\Sigma_{i=1}^\infty \alpha_i = 1$, $\{x_n\}$ be a bounded sequence in an Hadamard space X and u be an arbitrary element of X. Suppose that $\acute{\alpha}_n = \Sigma_{i=n+1}^\infty \alpha_i$ and $\Sigma_{i=n}^\infty \acute{\alpha}_i \to 0$ as $n \to \infty$. Set $s_n = \alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \ldots \oplus \alpha_n x_n \oplus \acute{\alpha}_n u$. Thus, by (3),

$$s_n = (\Sigma_{i=1}^n \alpha_i)w_n \oplus \acute{\alpha}_n u, \quad (4)$$

where $w_1 = x_1$ and for each $n \geq 2$,

$$w_n = \frac{\alpha_1}{\Sigma_{i=1}^n \alpha_i}x_1 \oplus \ldots \oplus \frac{\alpha_n}{\Sigma_{i=1}^n \alpha_i}x_n.$$

The sequence $\{s_n\}$ is a Cauchy sequence [17]. Therefore, $\{s_n\}$ converges to some point $x \in X$ and it is denoted by $\oplus_{i=1}^\infty \alpha_i x_i$. It follows from $d(s_n, w_n) = \acute{\alpha}_n d(u, w_n)$ that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} w_n$. Thus, the definition of $\oplus_{i=1}^\infty \alpha_i x_i$ is independent of the choice of u.

Let C be a nonempty subset of an Hadamard space X and $T : C \to C$ be a mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. Recall that a mapping T is called

- Nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$.
- Quasi nonexpansive, if $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$, for all $x \in C$ and $p \in F(T)$.

Chaoha et al. [10] showed that the fixed point set of a quasi nonexpansive mapping T is closed and convex.

Lemma 2.1 [17] *Let C be a nonempty closed convex subset of an Hadamard space X and $\{T_n\}_{n=1}^{\infty}$ be a family of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Define $T : C \to C$ by $Tx = \oplus_{n=1}^{\infty}\alpha_n T_n x$ for all $x \in C$, where $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$ with $\Sigma_{n=1}^{\infty}\alpha_n = 1$ and $\Sigma_{i=n}^{\infty}\acute{\alpha}_i \to 0$ as $n \to \infty$. Then T is a nonexpansive mapping and $F(T) = \cap_{n=1}^{\infty} F(T_n)$.*

Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius $r(x_n)$ of $\{x_n\}$ is given by:

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point [18]. A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.2 [19] *Let C be a closed and convex subset of an Hadamard space X and $T : C \to C$ be a nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n\to\infty} x_n = x$. Then $x = Tx$.*

Definition 2.3 [43] (Asymptotic fixed point). Let C be a nonempty closed convex subset of an Hadamard space X. A point $x \in C$ is said to be an asymptotic fixed point of $T : C \to C$ if there exists a sequence $\{x_n\}$ in C such that $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and $d(x_n, Tx_n) \to 0$. We denote the asymptotic fixed point set of T by $\widehat{F}(T)$.

Berg and Nikolaev in [5] introduced the concept of quasilinearization in a metric space X. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \longrightarrow \mathbb{R}$ defined by:

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\big[d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\big], \tag{5}$$

for all $a, b, c, d \in X$. It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d),$$

for all $a, b, c, d \in X$. It is known [5] that a geodesically connected metric space is a $CAT(0)$ space if and only if it satisfies the Cauchy-Schwarz inequality.

Kakavandi and Amini [29] have introduced the concept of dual space of a complete CAT(0) space X, based on a work of Berg and Nikolaev [5], as follows.

Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \quad (a, b, x \in X, t \in \mathbb{R}),$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on $\mathbb{R} \times X \times X$. Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$, for all $t \in \mathbb{R}$ and $a, b \in X$, where $L(\varphi) = \sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)}; x, y \in X, x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \quad (a, b, c, d \in X, t, s \in \mathbb{R}).$$

For an Hadamard space (X, d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, R), L)$. By [29, Lemma 2.1], $D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x, y \in X$. Thus, 'D' induces an equivalence relation on $\mathbb{R} \times X \times X$ where the equivalence class of (t, a, b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd}; t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle, \quad \forall x, y \in X\}.$$

The set $X^* := \{[t\overrightarrow{ab}]; (t, a, b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([tab], [scd]) := D((t, a, b), (s, c, d))$, which is called the dual space of (X, d). It is clear that $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a, b \in X$. Fix $o \in X$, we write $\mathbf{0} = [\overrightarrow{oo}]$ as the zero of the dual space. Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \quad (x^* = [t\overrightarrow{ab}] \in X^*, x, y \in X).$$

We also use the following notation in the subsequent work,

$$\left\langle \sum_{i=1}^{N} \alpha_i x_i^*, . \right\rangle := \sum_{i=1}^{N} \alpha_i \langle x_i^*, . \rangle, \quad (\forall x_i^* \in X^*, \alpha_i \in \mathbb{R}).$$

Definition 2.4 [29] A sequence $\{x_n\}$ in the Hadamard space X w-converges (weakly converges) to $x \in X$ if $\lim_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0$, for all $y \in X$.

It is obvious that convergence in the metric implies w-convergence, and it is easy to check that w-convergence implies Δ-convergence [29], but the converses are not valid in general. It is well known that every bounded sequence in X has a Δ-convergent subsequence (see [32]), but this is not always true for the w-convergence as was noted in [28]. Therefore, it is sensible to consider the following concept of reflexivity.

Definition 2.5 [9] An Hadamard space X is reflexive if closed balls are weakly sequentially compact.

Examples of reflexive Hadamard spaces are Hilbert spaces, or more generally, Hadamard spaces that satisfy property (S) (see [28] for definition).

Let X be an Hadamard space with dual X^* and let $A : X \rightrightarrows X^*$ be a multivalued operator with domain $D(A) := \{x \in X, Ax \neq \emptyset\}$, range $R(A) := \bigcup_{x\in X} Ax$, $A^{-1}(x^*) = \{x \in X, x^* \in Ax\}$ and graph $gra(A) := \{(x, x^*) \in X \times X^*, x \in D(A), x^* \in Ax\}$.

Definition 2.6 [31] Let X be an Hadamard space with dual X^*. The multivalued operator $A : X \rightrightarrows X^*$ is said to be monotone if the inequality $\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0$ holds for every $(x, x^*), (y, y^*) \in gra(A)$.

A monotone operator $A : X \rightrightarrows X^*$ is maximal if there exists no monotone operator $B : X \rightrightarrows X^*$ such that $gra(B)$ properly contains $gra(A)$ (that is, for any $(y, y^*) \in X \times X^*$, the inequality $\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0$ for all $(x, x^*) \in gra(A)$ implies that $y^* \in Ay$).

Definition 2.7 [31] Let X be an Hadamard space with dual X^*, $\lambda > 0$ and let $A : X \rightrightarrows X^*$ be a multivalued operator. The resolvent of A of order λ, is the multivalued mapping $J_\lambda^A : X \rightrightarrows X$, defined by $J_\lambda^A(x) := \{z \in X, [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}$. Indeed

$$J_\lambda^A = (\overrightarrow{oI} + \lambda A)^{-1} \circ \overrightarrow{oI},$$

where o is an arbitrary member of X and $\overrightarrow{oI}(x) := [\overrightarrow{ox}]$. It is obvious that this definition is independent of the choice of o.

Theorem 2.8 [31] *Let X be a $CAT(0)$ space with dual X^* and let $A : X \rightrightarrows X^*$ be a multivalued mapping. Then*

- (i) *For any $\lambda > 0$, $R(J_\lambda^A) \subset D(A)$, $F(J_\lambda^A) = A^{-1}(\boldsymbol{0})$,*
- (ii) *If A is monotone, then J_λ^A is a single-valued on its domain and*

$$d^2(J_\lambda^A x, J_\lambda^A y) \leq \langle \overrightarrow{J_\lambda^A x J_\lambda^A y}, \overrightarrow{xy}\rangle, \quad \forall x, y \in D(J_\lambda^A).$$

In particular J_λ^A is a nonexpansive mapping.

- (iii) *If A is monotone and $0 < \lambda \leq \mu$, then $d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu-\lambda}{\mu+\lambda} d^2(x, J_\mu^A x)$, which implies that $d(x, J_\lambda^A x) \leq 2d(x, J_\mu^A x)$.*

Remark 1 It is well known that if T is a nonexpansive mapping on a subset C of a $CAT(0)$ space X, then $F(T)$ is closed and convex. Thus, if A is a monotone operator on a $CAT(0)$ space X, then, by parts (i) and (ii) of Theorem 2.8, $A^{-1}(\boldsymbol{0})$ is closed and convex. Also by using part (ii) of this theorem for all $u \in F(J_\lambda^A)$ and $x \in D(J_\lambda^A)$, we have

$$d^2(J_\lambda^A x, x) \leq d^2(u, x) - d^2(u, J_\lambda^A x). \tag{6}$$

We say that $A : X \rightrightarrows X^*$ satisfies the range condition if, for every $\lambda > 0$, $D(J_\lambda^A) = X$. It is known that if A is a maximal monotone operator on a Hilbert space H, then $R(I + \lambda A) = H$ for all $\lambda > 0$. Thus, every maximal monotone operator A on a Hilbert space satisfies the range condition. Also as it has been shown in [35] if A is a maximal monotone operator on an Hadamard manifold, then A satisfies the range condition. Some examples of monotone operators in $CAT(0)$ spaces which satisfying the range condition are presented in [31].

The following theorem states that any convex closed subset of an Hadamard space is Chebyshev and summarizes the basic properties of the projection.

Theorem 2.9 [2] *Let X be an Hadamard space and $C \subset X$ be a closed convex set. Then:*

(i) *For every $x \in X$, there exists a unique point $P_C(x) \in C$ such that*

$$d(x, P_C(x)) = d(x, C).$$

(ii) *If $x \in X$ and $y \in C$, then*

$$d^2(x, P_C x) + d^2(P_C x, y) \leq d^2(x, y),$$

(iii) *The mapping P_C is a nonexpansive mapping from X onto C.*

Lemma 2.10 [16] *Let C be a nonempty closed convex subset of a $CAT(0)$ space X, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if $\langle \overrightarrow{xu}, \overrightarrow{uy} \rangle \geq 0$ for all $y \in C$.*

Lemma 2.11 [52] *Let X be an Hadamard space. Then for all $u, x, y \in X$, the following inequality holds:*

$$d^2(x, y) \leq d^2(y, u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Lemma 2.12 [52] *Let X be a $CAT(0)$ space. For any $u, v \in X$ and $t \in [0, 1]$, let $u_t = tu \oplus (1-t)v$.Then, for all $x, y \in X$,*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$,
(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$, *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (a_1, a_2, a_3, ...) \in l^\infty$. Sometimes, we denote by $\mu_n(a_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a Banach limit if $\mu(1, 1, ...) = \|\mu\| = 1$, and $\mu_n(a_{n+1}) = \mu_n(a_n)$.

Lemma 2.13 [50] *Let* $(a_1, a_2, a_3, ...) \in l^\infty$ *be such that* $\mu_n(a_n) \leq 0$ *for all Banach limits* μ *and* $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$. *Then* $\limsup_{n\to\infty} a_n \leq 0$.

Lemma 2.14 [25, 49] *Let C be a closed convex subset of an Hadamard space* (X, d) *and let* $T : C \to C$ *be a nonexpansive mapping. Let* $u \in C$ *be fixed. Then* $F(T) \neq \emptyset$ *if and only if* $\{x_t\}$ *given by the* $x_t = tu \oplus (1-t)Tx_t$ *for all* $t \in (0, 1)$ *remains bounded as* $t \to 0$. *In this case, the following hold:*

(1) $\{x_t\}$ *converges to the unique fixed point* z_0 *of* T *which is nearest* u;
(2) $d^2(u, z_0) \leq \mu_n d^2(u, x_n)$ *for all Banach limits* μ *and bounded sequences* $\{x_n\}$ *with* $d(x_n, Tx_n) \to 0$.

Similar results of Lemma 2.14 have been obtained in a uniformly smooth Banach space by Reich [42] (see also [47]).

The $AKTT$-condition was introduced in [1] as follows:

Definition 2.15 Let C be a nonempty subset of metric space X, and let $\{T_n\}_{n=1}^\infty$ be a sequence of mappings from C into C such that $\cap_{n=1}^\infty F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^\infty$ is said to satisfy the $AKTT$-condition if for each bounded subset K of C,

$$\sum_{n=1}^{\infty} sup\{d(T_{n+1}z, T_n z) : z \in K\} < \infty.$$

We also need the following lemma [1].

Lemma 2.16 *Let C be a nonempty subset of metric space X, and let* $\{T_n\}_{n=1}^\infty$ *be a sequence of mappings from C into C which satisfies the $AKTT$-condition. Then, for each* $x \in C$, $\{T_n x\}_{n=1}^\infty$ *converges strongly to a point in C. Furthermore, define the mapping* $T : C \to C$ *by:*

$$Tx := \lim_{n\to\infty} T_n x, \qquad \forall x \in C.$$

Then, for each bounded subset K of C,

$$\lim_{n\to\infty} sup\{d(T_n z, Tz) : z \in K\} = 0.$$

In the sequel, we will write that $(\{T_n\}_{n=1}^\infty, T)$ satisfies the $AKTT$-condition if $\{T_n\}_{n=1}^\infty$ satisfies the $AKTT$-condition and T is defined by Lemma 2.16 with $F(T) = \cap_{n=1}^\infty F(T_n)$.

In [27], the authors constructed a sequence of nonexpansive mappings satisfying the AKTT condition by choosing an appropriate control sequence under certain conditions. Applying the same argument as in [27], we give a sequence of quasi-nonexpansive mappings satisfying the AKTT condition as follows:

Example 2.17 Let C be a nonempty closed convex subset of an Hadamard space X, $\{T_n\}_{n=1}^\infty$ be a family of quasi-nonexpansive mappings on C with $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and $\{\gamma_{n,k} : k \leq n\} \subset (0, 1)$ be a sequence satisfying:

(i) $\sum_{k=1}^{n} \gamma_{n,k} = 1$, $\forall n \in \mathbb{N}$,
(ii) $\lambda_k = \lim_{n\to\infty} \gamma_{n,k} > 0$, $\forall k \in \mathbb{N}$, and $\lim_{n\to\infty} \sum_{k=n}^{\infty} \acute{\lambda_k} = 0$,
(iii) $\sum_{n=1}^{\infty} \sum_{k=1}^{n+1} |\bar{\gamma}_{n+1,k} - \bar{\gamma}_{n,k}| < \infty$, where $\gamma_{n,n+1} = 0$ and

$$\bar{\gamma}_{n,k} = \frac{\gamma_{n,k}}{\gamma_{n,1} + ... + \gamma_{n,k}}, \quad k = 1, ..., n+1.$$

For each $n \in \mathbb{N}$, define the mapping $S_n : C \to C$ by $S_n x = \bigoplus_{k=1}^{n} \gamma_{n,k} T_k x$. Then, $\{S_n\}$ is a family of quasi-nonexpansive mappings satisfying the $AKTT$-condition and $\bigcap_{n=1}^{\infty} F(S_n) = \bigcap_{n=1}^{\infty} F(T_n)$. Moreover, the mapping $S : C \to C$ defined by $Sx = \lim_{n\to\infty} S_n x$ is also quasi-nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$.

The following lemmas concerning properties of real sequences.

Lemma 2.18 [53] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the inequality:*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n \delta_n, \qquad \forall n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$, *or* $\sum_{n=0}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.19 [36] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3 Main results

This section is divided into three subsections. In the first subsection, we propose a new iterative algorithm for finding a common zero of finitely many monotone mappings and a common fixed point of an infinitely countable family of nonexpansive mappings in Hadamard spaces. In the second subsection, a common zero of a finite family of monotone operators and a common fixed point of an infinitely countable family of quasi-nonexpansive mappings is approximated in reflexive Hadamard spaces. Finally, in the third subsection, we define a norm on $X^\diamond$ by the formula:

$$\|x^\diamond\|_\diamond := \sup\left\{\frac{\left|\langle x^\diamond, \overrightarrow{ab}\rangle - \langle x^\diamond, \overrightarrow{cd}\rangle\right|}{d(a,b) + d(c,d)}, \quad (a, b, c, d \in X, a \neq b \text{ or } c \neq d)\right\},$$

in addition an application of this norm will be exhibited.

3.1 Algorithm in Hadamard spaces

A problem of interest in optimization theory is to find zeros of mapping $A : X \rightrightarrows X^*$. The concept of monotonicity is a valuable tool in studying important operators, such as the gradient or subdifferential of a convex function, which appear in many problems in optimization, variational inequality problems or differential equations. In this subsection, we propose an iterative algorithm for finding a common zero of a finite family of monotone operators and a common fixed point of an infinitely countable family of nonexpansive mappings in Hadamard spaces (see also [21, 22]). Before this, we state the following lemma.

Lemma 3.1 *Let C be a nonempty closed convex subset of an Hadamard space X and $A_i : X \rightrightarrows X^*$, $i = 1, 2, ..., N$, be N monotone operators such that satisfy the range condition and $D(A_N) \subset C$. Let $T : C \longrightarrow C$ be a nonexpansive mapping and $\lambda_i > 0$ $(i = 1, 2, ..., N)$. If $F(T) \cap \bigcap_{i=1}^N F(J_{\lambda_i}^{A_i}) \neq \emptyset$, then*

$$F(T \circ J_{\lambda_N}^{A_N} \circ ... \circ J_{\lambda_1}^{A_1}) = F(T) \cap \bigcap_{i=1}^{N} F(J_{\lambda_i}^{A_i}).$$

Proof It follows from the range condition and Theorem 2.8 that the mapping $T \circ J_{\lambda_N}^{A_N} \circ ... \circ J_{\lambda_1}^{A_1}$ is well defined. Set $S^i = J_{\lambda_i}^{A_i} \circ ... \circ J_{\lambda_1}^{A_1}$ $(1 \leq i \leq N)$ and $S^0 = I$, where I is the identity operator. It is clear that $F(T) \cap \bigcap_{i=1}^N F(J_{\lambda_i}^{A_i}) \subseteq F(T \circ J_{\lambda_N}^{A_N} \circ ... \circ J_{\lambda_1}^{A_1})$. To prove the reverse inclusion, let $x \in F(T \circ J_{\lambda_N}^{A_N} \circ ... \circ J_{\lambda_1}^{A_1})$ and suppose that $y \in F(T) \cap \bigcap_{i=1}^N F(J_{\lambda_i}^{A_i})$. Then, By Theorem 2.8, for any $i = 1, 2, ..., N$, we obtain

$$\begin{aligned} d(S^{i-1}x, y) &\leq d(x, y) = d(T(S^N x), T(S^N y)) \\ &\leq d(S^N x, S^N y) \leq d(S^i x, S^i y) \\ &= d(S^i x, y). \end{aligned} \tag{7}$$

By using (6), for any $i = 1, 2, ..., N$, we have

$$d^2(S^i x, y) \leq d^2(S^{i-1}x, y) - d^2(S^i x, S^{i-1}x).$$

This together with (7) implies that

$$d^2(S^{i-1}x, y) \leq d^2(S^{i-1}x, y) - d^2(S^i x, S^{i-1}x).$$

Therefore, for any $i = 1, 2, ..., N$, we have $S^i x = S^{i-1}x$. It follows that $x \in \bigcap_{i=1}^N F(J_{\lambda_i}^{A_i})$. On the other hand, we derive from $x = TS^N x = Tx$ that $x \in F(T)$ and hence, $x \in F(T) \cap \bigcap_{i=1}^N F(J_{\lambda_i}^{A_i})$. □

Proposition 3.2 *Let C be a nonempty closed convex subset of an Hadamard space X and $A_i : X \rightrightarrows X^*$, $i = 1, 2, ..., N$, be N monotone operators such that satisfy the range condition and $D(A_N) \subset C$. Let $T : C \to C$ be a nonexpansive mapping such* *that* $\Omega = F(T) \cap \bigcap_{i=1}^{N} A_i^{-1}(\boldsymbol{0}) \neq \emptyset$. *Assume that* f *is a k-contraction on* C *into itself with* $k \in [0, \frac{1}{2})$. *For* $x_1 \in X$, *let* $\{x_n\}$ *be a sequence defined by:*

$$\begin{cases} z_n = J_{\lambda_n^N}^{A_N} \circ \ldots \circ J_{\lambda_n^1}^{A_1} x_n, \\ y_n = \beta_n x_n \oplus (1-\beta_n) T z_n, \\ x_{n+1} = \alpha_n f(y_n) \oplus (1-\alpha_n) y_n, \end{cases} \tag{8}$$

where $\{\beta_n\}, \{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n^i\} \subset (0,\infty)$ *satisfy the following conditions:*

(i) $\liminf_{n\to\infty} \lambda_n^i > 0$ *for all* $1 \le i \le N$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ *converges strongly to* $u \in \Omega$ *which solves the variational inequality:*

$$\langle \overrightarrow{yf(y)}, \overrightarrow{xy} \rangle \ge 0, \qquad \forall x \in \Omega.$$

Proof It follows from Remark 1 that Ω is closed and convex. Since $P_\Omega f$ is a contraction mapping, by the Banach contraction principle, there exists a unique $u \in \Omega$ such that $u = P_\Omega f(u)$. First, we show that the sequence $\{x_n\}$ is bounded. By using Theorem 2.8, we have

$$d(z_n, u) = d(J_{\lambda_n^N}^{A_N} \circ \ldots \circ J_{\lambda_n^1}^{A_1} x_n, u) \le d(J_{\lambda_n^{N-1}}^{A_{N-1}} \circ \ldots \circ J_{\lambda_n^1}^{A_1} x_n, u) \le \cdots \le d(x_n, u), \tag{9}$$

Using (2) and (9), we obtain

$$\begin{aligned} d(y_n, u) &\le \beta_n d(x_n, u) + (1-\beta_n) d(Tz_n, u) \\ &\le \beta_n d(x_n, u) + (1-\beta_n) d(z_n, u) \\ &\le \beta_n d(x_n, u) + (1-\beta_n) d(x_n, u) \\ &= d(x_n, u). \end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned} d(x_{n+1}, u) &\le \alpha_n d(f(y_n), u) + (1-\alpha_n) d(y_n, u) \\ &\le \alpha_n \big[d(f(y_n), f(u)) + d(f(u), u) \big] + (1-\alpha_n) d(y_n, u) \\ &\le \alpha_n k d(y_n, u) + \alpha_n d(f(u), u) + (1-\alpha_n) d(y_n, u) \\ &\le \alpha_n k d(x_n, u) + \alpha_n d(f(u), u) + (1-\alpha_n) d(x_n, u) \\ &= (1-\alpha_n(1-k)) d(x_n, u) + \alpha_n (1-k) \frac{d(f(u), u))}{1-k} \\ &\le max\{d(x_n, u), \frac{d(f(u), u))}{1-k}\} \\ &\vdots \\ &\le max\{d(x_1, u), \frac{d(f(u), u))}{1-k}\}. \end{aligned}$$

Thus, $\{x_n\}$ is bounded. Consequently, we conclude that $\{y_n\}$, $\{f(y_n)\}$ and $\{z_n\}$ are also bounded. Now, we consider two cases:

Case 1 Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{d(x_n, u)\}_{n=n_0}^{\infty}$ is nonincreasing, in this situation, $\{d(x_n, u)\}$ is convergent. This shows that $\lim_{n\to\infty} d(x_{n+1}, u) - d(x_n, u) = 0$. Denote by S_n^i the composition $J_{\lambda_n^i}^{A_i} \circ \ldots \circ J_{\lambda_n^1}^{A_1}$ for any $i = 1, 2, ..., N$ and $n \in \mathbb{N}$. Therefore $z_n = S_n^N x_n$. We also assume that $S_n^0 = I$. Using Theorem 2.8, we get $d(S_n^i x_n, u) - d(x_n, u) \le d(x_n, u) - d(x_n, u)$, for any $i = 1, 2, ..., N$, and hence,

$$\limsup_{n\to\infty} \left[d(S_n^i x_n, u) - d(x_n, u)\right] \le 0. \tag{11}$$

On the other hand, from (2), for any $i = 1, 2, ..., N$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x_{n+1}, u) &\le \alpha_n d(f(y_n), u) + (1-\alpha_n) d(y_n, u) \\ &\le \alpha_n d(f(y_n), u) + (1-\alpha_n)\left[\beta_n d(x_n, u) + (1-\beta_n) d(Tz_n, u)\right] \\ &\le \alpha_n d(f(y_n), u) + (1-\alpha_n)\left[\beta_n d(x_n, u) + (1-\beta_n) d(z_n, u)\right] \\ &\le \alpha_n d(f(y_n), u) + (1-\alpha_n)\left[\beta_n d(x_n, u) + (1-\beta_n) d(S_n^i x_n, u)\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_{n+1}, u) - d(x_n, u) \le \alpha_n\left[d(f(y_n), u) - \beta_n d(x_n, u) - (1-\beta_n) d(S_n^i x_n, u)\right] \\ +(1-\beta_n)\left[d(S_n^i x_n, u) - d(x_n, u)\right]. \end{aligned}$$

From the boundedness of the sequences $\{f(y_n)\}$, $\{x_n\}$, $\{S_n^i x_n\}$, and the conditions (i) and (ii), we get

$$\liminf_{n\to\infty} \left[d(S_n^i x_n, u) - d(x_n, u)\right] \ge 0. \tag{12}$$

Using inequalities (11) and (12), for any $i = 1, 2, ..., N$, we have

$$\lim_{n\to\infty} \left[d(S_n^i x_n, u) - d(x_n, u)\right] = 0. \tag{13}$$

Applying (6), we get

$$\begin{aligned} d^2(J_{\lambda_n^i}^{A_i}(S_n^{i-1} x_n), S_n^{i-1} x_n) &\le d^2(u, S_n^{i-1} x_n) - d^2(u, S_n^i x_n) \\ &\le d^2(u, x_n) - d^2(u, S_n^i x_n). \end{aligned}$$

This together with (13) implies that $\lim_{n\to\infty} d(S_n^i x_n, S_n^{i-1} x_n) = 0$, and hence for any $i = 1, 2, ..., N$, we obtain

$$d(x_n, S_n^i x_n) \le d(x_n, S_n^1 x_n) + ... + d(S_n^{i-1} x_n, S_n^i x_n) \to 0. \tag{14}$$

Since $\liminf_{n\to\infty} \lambda_n^i > 0$, there exists $\lambda_0 \in \mathbb{R}$ such that $\lambda_n^i \ge \lambda_0 > 0$ for all $n \in \mathbb{N}$ and $1 \le i \le N$. By using Theorem 2.8, for all $1 \le i \le N$, we have

$$\begin{aligned} d(J_{\lambda_0}^{A_i}(S_n^{i-1} x_n), S_n^i x_n)) &\le d(J_{\lambda_0}^{A_i}(S_n^{i-1} x_n), S_n^{i-1} x_n) + d(S_n^{i-1} x_n, S_n^i x_n) \\ &\le 2d(J_{\lambda_n^i}^{A_i}(S_n^{i-1} x_n), S_n^{i-1} x_n) + d(S_n^{i-1} x_n, S_n^i x_n) \\ &= 3d(S_n^i x_n, S_n^{i-1} x_n) \to 0, \quad as \quad n \to \infty. \end{aligned}$$

So for all $1 \le i \le N$, we have

$$\begin{aligned} d(J_{\lambda_0}^{A_i}x_n, x_n) &\le d(J_{\lambda_0}^{A_i}x_n, J_{\lambda_0}^{A_i}(S_n^{i-1}x_n)) + d(J_{\lambda_0}^{A_i}(S_n^{i-1}x_n), S_n^i x_n) + d(S_n^i x_n, x_n) \\ &\le d(x_n, S_n^{i-1}x_n) + d(J_{\lambda_0}^{A_i}(S_n^{i-1}x_n), S_n^i x_n) + d(S_n^i x_n, x_n) \to 0, \quad (15) \end{aligned}$$

as $n \to \infty$. Now, we show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By using (1) and (9), we get

$$\begin{aligned} d^2(x_{n+1}, u) &\le \alpha_n d^2(f(y_n), u) + (1-\alpha_n)d^2(y_n, u) \\ &\le \alpha_n d^2(f(y_n), u) + (1-\alpha_n)\big[\beta_n d^2(x_n, u) + (1-\beta_n)d^2(Tz_n, u) \\ &\quad -\beta_n(1-\beta_n)d^2(x_n, Tz_n)\big] \\ &\le \alpha_n d^2(f(y_n), u) + (1-\alpha_n)\big[\beta_n d^2(x_n, u) + (1-\beta_n)d^2(Tz_n, u) \\ &\quad -\beta_n(1-\beta_n)d^2(x_n, Tz_n)\big] \\ &\le \alpha_n d^2(f(y_n), u) + (1-\alpha_n)\big[\beta_n d^2(x_n, u) + (1-\beta_n)d^2(x_n, u) \\ &\quad -\beta_n(1-\beta_n)d^2(x_n, Tz_n)\big] \\ &= \alpha_n\big[d^2(f(y_n), u) - d^2(x_n, u) + \beta_n(1-\beta_n)d^2(x_n, Tz_n)\big] \\ &\quad +d^2(x_n, u) - \beta_n(1-\beta_n)d^2(x_n, Tz_n). \end{aligned}$$

Hence,

$$\begin{aligned} d^2(x_n, Tz_n) \le \frac{\alpha_n}{\beta_n(1-\beta_n)}\big[d^2(f(y_n), u) - d^2(x_n, u) + \beta_n(1-\beta_n)d^2(x_n, Tz_n)\big] \\ +\frac{1}{\beta_n(1-\beta_n)}\big[d^2(x_n, u) - d^2(x_{n+1}, u)\big]. \end{aligned}$$

Using the above inequality and the conditions (i) and (ii), we get

$$\lim_{n\to\infty} d(x_n, Tz_n) = 0. \quad (16)$$

Applying (14) and (16), we obtain

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, Tz_n) + d(Tz_n, Tx_n) \quad (17) \\ &\le d(x_n, Tz_n) + d(z_n, x_n) \to 0, \end{aligned}$$

as $n \to \infty$. Let $g = T \circ J_{\lambda_0}^{A_N} \circ ... \circ J_{\lambda_0}^{A_1}$. Set $S^i := J_{\lambda_0}^{A_i} \circ ... \circ J_{\lambda_0}^{A_1}$ $(1 \le i \le N)$ and $S^0 = I$. We show that for any $i = 1, 2, ..., N$,

$$\lim_{n\to\infty} d(S^i x_n, S^{i-1}x_n) = 0. \quad (18)$$

For this purpose, we use the principle of strong induction. From (15), it is obvious that (18) holds for $i = 1$. Also for $i = 2$, we have

$$\begin{aligned} d(S^2 x_n, S^1 x_n) &= d(J_{\lambda_0}^{A_2} \circ J_{\lambda_0}^{A_1}x_n, J_{\lambda_0}^{A_1}x_n) \\ &\le d(J_{\lambda_0}^{A_2} \circ J_{\lambda_0}^{A_1}x_n, J_{\lambda_0}^{A_2}x_n) + d(J_{\lambda_0}^{A_2}x_n, x_n) + d(x_n, J_{\lambda_0}^{A_1}x_n) \\ &\le 2d(x_n, J_{\lambda_0}^{A_1}x_n) + d(x_n, J_{\lambda_0}^{A_2}x_n) \to 0, \quad as \quad n \to \infty. \end{aligned}$$

Now, suppose that (18) holds for $i = 1, 2, ..., N-1$. Then, we have

$$\begin{aligned} d(S^N x_n, S^{N-1}x_n) &\leq d\big(J_{\lambda_0}^{A_N}(S^{N-1}x_n), J_{\lambda_0}^{A_N}(S^{N-2}x_n)\big) \\ &\quad + d\big(J_{\lambda_0}^{A_N}(S^{N-2}x_n), S^{N-2}x_n\big) + d\big(S^{N-2}x_n, S^{N-1}x_n\big) \\ &\leq 2d\big(S^{N-2}x_n, S^{N-1}x_n\big) + d\big(J_{\lambda_0}^{A_N}(S^{N-2}x_n), S^{N-2}x_n\big) \\ &\leq 2d\big(S^{N-2}x_n, S^{N-1}x_n\big) + d\big(J_{\lambda_0}^{A_N}(S^{N-2}x_n), J_{\lambda_0}^{A_N}x_n\big) \\ &\quad + d\big(J_{\lambda_0}^{A_N}x_n, x_n\big) + d(x_n, S^{N-2}x_n) \\ &\leq 2d\big(S^{N-2}x_n, S^{N-1}x_n\big) + d\big(J_{\lambda_0}^{A_N}x_n, x_n\big) + 2d(x_n, S^{N-2}x_n) \\ &\leq 2d\big(S^{N-2}x_n, S^{N-1}x_n\big) + d\big(J_{\lambda_0}^{A_N}x_n, x_n\big) \\ &\quad + 2\Big[d(x_n, S^1x_n) + d(S^1x_n, S^2x_n) + ... + d(S^{N-3}x_n, S^{N-2}x_n)\Big] \to 0, \end{aligned}$$

as $n \to \infty$. Hence,

$$d(x_n, S^N x_n) \leq d(x_n, S^1x_n) + d(S^1x_n, S^2x_n) + ... + d(S^{N-1}x_n, S^N x_n) \to 0,$$

as $n \to \infty$. Using this and (17), we get

$$\begin{aligned} d(x_n, g(x_n)) &\leq d(x_n, Tx_n) + d(Tx_n, TS^N x_n) \\ &\leq d(x_n, Tx_n) + d(x_n, S^N x_n) \to 0, \quad as\ n \to \infty. \end{aligned}$$

Also from (16), we have

$$d(x_n, y_n) \leq \beta_n d(x_n, x_n) + (1-\beta_n)d(x_n, Tz_n) \to 0, \tag{19}$$

as $n \to \infty$, and hence,

$$d(x_n, x_{n+1}) \leq \alpha_n d(x_n, f(y_n)) + (1-\alpha_n)d(x_n, y_n) \to 0.$$

Applying Lemma 3.1, we get that $F(g) = \Omega$. For each $t \in (0,1)$, let $x_t = tf(u) \oplus (1-t)g(x_t)$. From Lemma 2.14, we know that $x_t \to u = P_\Omega f(u)$, as $t \to 0$. Since $d(x_n, g(x_n)) \to 0$, again from Lemma 2.14, we obtain

$$\mu_n\big[d^2(u, f(u)) - d^2(f(u), x_n)\big] \leq 0, \tag{20}$$

for all Banach limits μ. Since $d(x_n, x_{n+1}) \to 0$, we get

$$\begin{aligned} &\limsup_{n\to\infty}\big[d^2(u, f(u)) - d^2(f(u), x_{n+1}) - (d^2(u, f(u)) - d^2(f(u), x_n))\big] \\ &= \limsup_{n\to\infty}\big[d(f(u), x_n) - d(f(u), x_{n+1})\big]\big[d(f(u), x_n) + d(f(u), x_{n+1})\big] \\ &\leq \limsup_{n\to\infty} d(x_{n+1}, x_n)\big[d(f(u), x_n) + d(f(u), x_{n+1})\big] = 0. \end{aligned} \tag{21}$$

Using inequalities (20), (21) and Lemma 2.13, we get

$$\limsup_{n\to\infty}\big[d^2(u, f(u)) - d^2(f(u), x_n)\big] \leq 0. \tag{22}$$

For each $n \in \mathbb{N}$, set $w_n = \alpha_n u \oplus (1-\alpha_n)y_n$. It follows from Lemmas 2.11 and 2.12 that

$$\begin{aligned}
d^2(x_{n+1},u) &\le d^2(w_n,u) + 2\langle \overrightarrow{x_{n+1}w_n}, \overrightarrow{x_{n+1}u}\rangle \\
&= (1-\alpha_n)^2 d^2(y_n,u) + 2\langle \overrightarrow{x_{n+1}w_n}, \overrightarrow{x_{n+1}u}\rangle \\
&\le (1-\alpha_n)^2 d^2(y_n,u) + 2\big[\alpha_n\langle \overrightarrow{f(y_n)w_n}, \overrightarrow{x_{n+1}u}\rangle + (1-\alpha_n)\langle \overrightarrow{y_n w_n}, \overrightarrow{x_{n+1}u}\rangle\big] \\
&\le (1-\alpha_n)^2 d^2(x_n,u) + 2\big[\alpha_n^2\langle \overrightarrow{f(y_n)u}, \overrightarrow{x_{n+1}u}\rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(y_n)y_n}, \overrightarrow{x_{n+1}u}\rangle \\
&\quad +\alpha_n(1-\alpha_n)\langle \overrightarrow{y_n u}, \overrightarrow{x_{n+1}u}\rangle + (1-\alpha_n)^2\langle \overrightarrow{y_n y_n}, \overrightarrow{x_{n+1}u}\rangle\big] \\
&= (1-\alpha_n)^2 d^2(x_n,u) + 2\big[\alpha_n^2\langle \overrightarrow{f(y_n)u}, \overrightarrow{x_{n+1}u}\rangle + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(y_n)u}, \overrightarrow{x_{n+1}u}\rangle\big] \\
&= (1-\alpha_n)^2 d^2(x_n,u) + 2\alpha_n\langle \overrightarrow{f(y_n)u}, \overrightarrow{x_{n+1}u}\rangle \\
&= (1-\alpha_n)^2 d^2(x_n,u) + 2\alpha_n\langle \overrightarrow{f(y_n)f(u)}, \overrightarrow{x_{n+1}u}\rangle + 2\alpha_n\langle \overrightarrow{f(u)u}, \overrightarrow{x_{n+1}u}\rangle \\
&\le (1-\alpha_n)^2 d^2(x_n,u) + 2\alpha_n k d(x_n,u)d(x_{n+1},u) + 2\alpha_n\langle \overrightarrow{f(u)u}, \overrightarrow{x_{n+1}u}\rangle \\
&\le (1-\alpha_n)^2 d^2(x_n,u) + \alpha_n k\big[d^2(x_n,u) + d^2(x_{n+1},u)\big] \\
&\quad +\alpha_n\big[d^2(f(u),u) + d^2(x_{n+1},u) - d^2(x_{n+1},f(u))\big].
\end{aligned}$$

This implies that

$$\begin{aligned}
d^2(x_{n+1},u) &\le \Big(1-\frac{\alpha_n(1-2k)}{1-\alpha_n(k+1)}\Big)d^2(x_n,u) + \frac{\alpha_n^2}{1-\alpha_n(k+1)}d^2(x_n,u) \\
&\quad +\frac{\alpha_n}{1-\alpha_n(k+1)}\big[d^2(f(u),u) - d^2(x_{n+1},f(u))\big] \\
&\le \Big(1-\frac{\alpha_n(1-2k)}{1-\alpha_n(k+1)}\Big)d^2(x_n,u) + \frac{\alpha_n(1-2k)}{1-\alpha_n(k+1)} \times \Big[\frac{\alpha_n L}{1-2k} \\
&\quad +\frac{d^2(f(u),u) - d^2(x_{n+1},f(u))}{1-2k}\Big] \\
&= (1-\gamma_n)d^2(x_n,u) + \gamma_n\delta_n,
\end{aligned}$$

where

$$\gamma_n = \frac{\alpha_n(1-2k)}{1-\alpha_n(k+1)}, \qquad L = \sup\{d^2(x_n,u), n \in \mathbb{N}\},$$

and

$$\delta_n = \frac{\alpha_n L}{1-2k} + \frac{d^2(f(u),u) - d^2(x_{n+1},f(u))}{1-2k}.$$

It follows from the condition (ii) and (22) that $\{\gamma_n\} \subset (0,1)$, $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Utilizing Lemma 2.18, we deduce that the sequence $\{x_n\}$ converges strongly to $u = P_\Omega f(u)$.

Case 2 There exists a subsequence $\{n_j\}$ of $\{n\}$ such that $d(u, x_{n_j}) < d(u, x_{n_j+1})$, for all $j \in \mathbb{N}$. Then by Lemma 2.19, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$d(u, x_{m_k}) \le d(u, x_{m_k+1}) \quad \textit{and} \quad d(u, x_k) \le d(u, x_{m_k+1}),$$

for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned} 0 &\leq \liminf_{k\to\infty} \left[d(u, x_{m_k+1}) - d(u, x_{m_k})\right] \\ &\leq \limsup_{k\to\infty} \left[d(u, x_{m_k+1}) - d(u, x_{m_k})\right] \\ &\leq \limsup_{k\to\infty} \left[\alpha_{m_k} d(f(y_{m_k}), u) + (1-\alpha_{m_k}) d(y_{m_k}, u) - d(u, x_{m_k})\right] \\ &\leq \limsup_{k\to\infty} \left[\alpha_{m_k} d(f(y_{m_k}), u) + (1-\alpha_{m_k}) d(x_{m_k}, u) - d(u, x_{m_k})\right] \\ &= \limsup_{k\to\infty} \left[\alpha_{m_k}\left(d(f(y_{m_k}), u) - d(u, x_{m_k})\right)\right] = 0. \end{aligned}$$

This shows that

$$\lim_{k\to\infty} \left[d(u, x_{m_k+1}) - d(u, x_{m_k})\right] = 0. \tag{23}$$

By a similar argument to that used in Case 1, we get

$$\limsup_{k\to\infty} \left[d^2(u, f(u)) - d^2(f(u), x_{m_k})\right] \leq 0, \tag{24}$$

and

$$d^2(x_{m_k+1}, u) \leq (1-\gamma_{m_k}) d^2(x_{m_k}, u) + \gamma_{m_k}\delta_{m_k}.$$

Hence,

$$\gamma_{m_k} d^2(x_{m_k}, u) \leq d^2(x_{m_k}, u) - d^2(x_{m_k+1}, u) + \gamma_{m_k}\delta_{m_k} \leq \gamma_{m_k}\delta_{m_k}. \tag{25}$$

On the other hand,

$$d^2(f(u), x_{m_k}) \leq d^2(f(u), x_{m_k+1}) + d^2(x_{m_k}, x_{m_k+1}) + 2d(f(u), x_{m_k+1}) d(x_{m_k}, x_{m_k+1}).$$

So, we have

$$\begin{aligned} d^2(u, f(u)) - d^2(f(u), x_{m_k+1}) &\leq d^2(u, f(u)) - d^2(f(u), x_{m_k}) + d^2(x_{m_k+1}, x_{m_k}) \\ &\quad + 2d(x_{m_k+1}, x_{m_k}) d(f(u), x_{m_k+1}). \end{aligned} \tag{26}$$

By using inequalities (24), (26), the condition (ii) and $d(x_{m_k+1}, x_{m_k}) \to 0$, we get $\limsup_{k\to\infty} \delta_{m_k} \leq 0$. From this, (25) and $\gamma_{m_k} > 0$, we obtain $\lim_{k\to\infty} d(x_{m_k}, u) = 0$. This together with (23), implies $\lim_{k\to\infty} d(u, x_{m_k+1}) = 0$. Therefore from $d(u, x_k) \leq d(u, x_{m_k+1})$, we get $x_k \to u$. This completes the proof. □

We are now ready for the main result of this subsection.

Theorem 3.3 *Let C be a nonempty closed convex subset of an Hadamard space X and $A_i : X \rightrightarrows X^*$, $i = 1, 2, ..., N$, be N monotone operators such that satisfy the range condition and $D(A_N) \subset C$. Let $\{T_n\}_{n=1}^{\infty}$ be a family of nonexpansive mappings on C such that $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \cap \bigcap_{i=1}^{N} A_i^{-1}(\mathbf{0}) \neq \emptyset$. Assume that f is a k-contraction on C into itself with $k \in [0, \frac{1}{2})$. For $x_1 \in X$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} z_n = J_{\lambda_n^N}^{A_N} \circ ... \circ J_{\lambda_n^1}^{A_1} x_n, \\ y_n = \beta_n x_n \oplus (1-\beta_n) \oplus_{m=1}^{\infty} \eta_m T_m z_n, \\ x_{n+1} = \alpha_n f(y_n) \oplus (1-\alpha_n) y_n, \end{cases} \tag{27}$$

where $\{\beta_n\}, \{\eta_n\}, \{\alpha_n\} \subset (0, 1)$ *and* $\{\lambda_n^i\} \subset (0, \infty)$ *satisfy the following conditions:*

(i) $\liminf_{n\to\infty} \lambda_n^i > 0$ *for all* $1 \le i \le N$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \eta_n = 1$ *and* $\sum_{j=n}^{\infty} \acute{\eta_j} \to 0$ *as* $n \to \infty$.

Then $\{x_n\}$ *converges strongly to* $u \in \Omega$ *which solves the variational inequality:*

$$\langle \overrightarrow{yf(y)}, \overrightarrow{xy} \rangle \ge 0, \qquad \forall x \in \Omega.$$

Proof The proof follows from Lemma 2.1 and Proposition 3.2. □

3.2 Algorithm in reflexive Hadamard spaces

In this subsection, we propose the iterative algorithm (28) for finding a common zero of a finite family of monotone operators and a common fixed point of an infinitely countable family of quasi-nonexpansive mappings in reflexive Hadamard spaces.

Theorem 3.4 *Let* C *be a nonempty closed convex subset of a reflexive Hadamard space* X *and* $A_i : X \rightrightarrows X^*$, $i = 1, 2, ..., N$, *be* N *monotone operators such that satisfy the range condition and* $D(A_N) \subset C$. *Let* $\{T_n\}_{n=1}^{\infty}$ *be an infinitely countable family of quasi-nonexpansive mappings from* C *into* C *such that* $F(T_n) = \widehat{F}(T_n)$ *for all* $n \ge 1$. *Suppose in addition that* $T : C \to C$ *is a quasi-nonexpansive mapping such that* $(\{T_n\}_{n=1}^{\infty}, T)$ *satisfies the* $AKTT$*-condition and* $F(T) = \widehat{F}(T)$. *Let* $\Omega = \cap_{n=1}^{\infty} F(T_n) \bigcap \cap_{i=1}^{N} A_i^{-1}(\boldsymbol{0}) \ne \emptyset$. *Assume that* f *is a k-contraction of* C *into itself with constant* $k \in [0, 1)$. *For* $x_1 \in X$, *let* $\{x_n\}$ *be a sequence defined by:*

$$\begin{cases} z_n = J_{\lambda_n^N}^{A_N} \circ ... \circ J_{\lambda_n^1}^{A_1} x_n, \\ y_n = \beta_n x_n \oplus (1-\beta_n) T_n z_n, \\ x_{n+1} = \alpha_n f(y_n) \oplus (1-\alpha_n) y_n, \end{cases} \tag{28}$$

where $\{\beta_n\}, \{\alpha_n\} \subset (0, 1)$ *and* $\{\lambda_n^i\} \subset (0, \infty)$ *satisfy the following conditions:*

(i) $\liminf_{n\to\infty} \lambda_n^i > 0$ *for all* $1 \le i \le N$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then, $\{x_n\}$ *converges strongly to* $u \in \Omega$ *which solves the variational inequality:*

$$\langle \overrightarrow{yf(y)}, \overrightarrow{xy} \rangle \ge 0, \qquad \forall x \in \Omega.$$

Proof As in the proof of Proposition 3.2, we can obtain that $\{x_n\}$ is bounded. Now, we consider two cases:

Case 1 Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{d(x_n, u)\}_{n=n_0}^{\infty}$ is nonincreasing, where $u = P_\Omega f(u)$. By the same argument as in the proof of Proposition 3.2, we see that

$$\lim_{n\to\infty} d(z_n, T_n z_n) = 0, \qquad \lim_{n\to\infty} d(x_n, J_{\lambda_0}^{A_i} x_n) = 0. \tag{29}$$

Since $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT condition, we conclude that

$$\begin{aligned} d(z_n, Tz_n) &\leq d(z_n, T_n z_n) + d(T_n z_n, Tz_n) \\ &\leq d(z_n, T_n z_n) + sup\{d(T_n z, Tz) : z \in K\}, \end{aligned}$$

where $\{z_n\} \subseteq K$ and K is bounded subset of C. By using Lemma 2.16 and (29), we get

$$\lim_{n\to\infty} d(z_n, Tz_n) = 0. \tag{30}$$

Since $\{x_n\}$ is bounded and X is a reflexive Hadamard space, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ w-converges to $\bar{x}$. Since $\{x_{n_k}\}$ w-converges to $\bar{x}$ and $d(x_{n_k}, z_{n_k}) \to 0$, by using the Cauchy-Schwarz inequality, for all $y \in X$, we have

$$\begin{aligned} \lim_{k\to\infty} |\langle \overrightarrow{z_{n_k}\bar{x}}, \overrightarrow{\bar{x}y}\rangle| &= \lim_{k\to\infty} |\langle \overrightarrow{z_{n_k}x_{n_k}}, \overrightarrow{\bar{x}y}\rangle + \langle \overrightarrow{x_{n_k}\bar{x}}, \overrightarrow{\bar{x}y}\rangle| \\ &\leq \lim_{k\to\infty} d(x_{n_k}, z_{n_k}) d(\bar{x}, y) + \lim_{k\to\infty} |\langle \overrightarrow{x_{n_k}\bar{x}}, \overrightarrow{\bar{x}y}\rangle| \\ &= 0. \end{aligned}$$

So, $\{z_{n_k}\}$ w-converges to $\bar{x}$, and hence by (30), $\bar{x} \in \widehat{F}(T) = F(T)$. Also from Lemma 2.2, Theorem 2.8 and (29), we get $\bar{x} \in A_i^{-1}(\mathbf{0})$ for any $i = 1, 2, ..., N$ and so, $\bar{x} \in \Omega$.

Next, we show that $\limsup_{n\to\infty} \langle \overrightarrow{f(u)u}, \overrightarrow{x_n u}\rangle \leq 0$. We can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(u)u}, \overrightarrow{x_n u}\rangle = \lim_{j\to\infty} \langle \overrightarrow{f(u)u}, \overrightarrow{x_{n_j} u}\rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ which w-converges to $p \in \Omega$. So, by using Lemma 2.10, we get

$$\lim_{j\to\infty} \langle \overrightarrow{f(u)u}, \overrightarrow{x_{n_j} u}\rangle = \lim_{k\to\infty} \langle \overrightarrow{f(u)u}, \overrightarrow{x_{n_{j_k}} u}\rangle = \langle \overrightarrow{f(u)u}, \overrightarrow{pu}\rangle \leq 0. \tag{31}$$

For each $n \in \mathbb{N}$, set $w_n = \alpha_n u \oplus (1 - \alpha_n) y_n$. Similar to the proof of Theorem 3.2, we get

$$d^2(x_{n+1}, u) \leq (1 - \alpha_n)^2 d^2(x_n, u) + \alpha_n k\big[d^2(x_n, u) + d^2(x_{n+1}, u)\big] + 2\alpha_n \langle \overrightarrow{f(u)u}, \overrightarrow{x_{n+1}u}\rangle.$$

This implies that

$$\begin{aligned} d^2(x_{n+1}, u) &\leq (1 - \frac{2\alpha_n(1-k)}{1-\alpha_n k}) d^2(x_n, u) + \frac{\alpha_n^2}{1-\alpha_n k} d^2(x_n, u) \\ &\quad + \frac{2\alpha_n}{1-\alpha_n k} \langle \overrightarrow{f(u)u}, \overrightarrow{x_{n+1}u}\rangle \\ &\leq (1 - \frac{2\alpha_n(1-k)}{1-\alpha_n k}) d^2(x_n, u) + \frac{2\alpha_n(1-k)}{1-\alpha_n k} \times \Big[\frac{\alpha_n L}{2(1-k)} \\ &\quad + \frac{1}{1-k} \langle \overrightarrow{f(u)u}, \overrightarrow{x_{n+1}u}\rangle\Big] \\ &= (1 - \gamma_n) d^2(x_n, u) + \gamma_n \delta_n, \end{aligned}$$

where

$$\gamma_n = \frac{2\alpha_n(1-k)}{1-\alpha_n k}, \qquad L = \sup\{d^2(x_n, u), n \in \mathbb{N}\},$$

and

$$\delta_n = \frac{\alpha_n L}{2(1-k)} + \frac{\langle \overrightarrow{f(u)u}, \overrightarrow{x_{n+1}u} \rangle}{1-k}.$$

It is easy to see that $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Utilizing Lemma 2.18, we deduce that the sequence $\{x_n\}$ converges strongly to $u = P_\Omega f(u)$.

Case 2 There exists a subsequence $\{n_j\}$ of $\{n\}$ such that $d(u, x_{n_j}) < d(u, x_{n_j+1})$, for all $j \in \mathbb{N}$. The remaining assertion goes through in a similar way to the one in the corresponding part of Theorem 3.2. □

3.3 Norm on $X^\diamond$

Now, we define a norm on $X^\diamond$ and give an application of this norm.

Proposition 3.5 *Let X be an Hadamard space with daul X^* and $X^\diamond = Span\ X^*$. Then*

$$\|x^\diamond\|_\diamond := \sup\left\{\frac{\left|\langle x^\diamond, \overrightarrow{ab}\rangle - \langle x^\diamond, \overrightarrow{cd}\rangle\right|}{d(a,b)+d(c,d)}, \quad (a,b,c,d \in X, a \neq b \text{ or } c \neq d)\right\}.$$

is a norm on $X^\diamond$. In particular, $\|[t\overrightarrow{xy}]\|_\diamond = |t|d(x,y)$.

Proof The only thing that needs verification is to show that $\|x^\diamond\|_\diamond < \infty$, for all $x^\diamond \in X^\diamond$. Let $x^* = [t\overrightarrow{xy}]$. By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|x^*\|_\diamond &= \sup\left\{\frac{\left|t\langle \overrightarrow{xy}, \overrightarrow{ab}\rangle - t\langle \overrightarrow{xy}, \overrightarrow{cd}\rangle\right|}{d(a,b)+d(c,d)}, \quad (a,b,c,d \in X, a \neq b \text{ or } c \neq d)\right\} \qquad (32)\\
&\leq \sup\left\{\frac{\left|t\langle \overrightarrow{xy}, \overrightarrow{ab}\rangle\right| + \left|t\langle \overrightarrow{xy}, \overrightarrow{cd}\rangle\right|}{d(a,b)+d(c,d)}, \quad (a,b,c,d \in X, a \neq b \text{ or } c \neq d)\right\}\\
&\leq \sup\left\{\frac{|t|d(x,y)d(a,b) + |t|d(x,y)d(c,d)}{d(a,b)+d(c,d)}, \quad (a,b,c,d \in X, a \neq b \text{ or } c \neq d)\right\}\\
&= |t|d(x,y).
\end{aligned}$$

On the other hand, substituting $a = x$, $b = y$ and $c = d$ into (32), we get

$$\frac{\left|t\langle \overrightarrow{xy}, \overrightarrow{xy}\rangle - t\langle \overrightarrow{xy}, \overrightarrow{cc}\rangle\right|}{d(x,y)} = |t|d(x,y).$$

Therefore, $\|x^*\|_\diamond = |t|d(x,y)$. Since $x^\diamond$ is a finite linear combinations of elements of X^*, we conclude that $\|x^\diamond\|_\diamond < \infty$. □

Proposition 3.6 *Let $\{x_n\}$ be a bounded sequence in an Hadamard space X with dual X^* and $\{x_n^\diamond\}$ be a sequence in $X^\diamond$. If $\{x_n\}$ is w-convergent to x and $x_n^\diamond \to x^\diamond$, then for all $z \in X$, $\langle x_n^\diamond, \overrightarrow{x_n z}\rangle \to \langle x^\diamond, \overrightarrow{xz}\rangle$.*

Proof Using the definition of $\|.\|_\diamond$ on $X^\diamond$, we have

$$\begin{aligned}|\langle x_n^\diamond, \overrightarrow{x_n z}\rangle - \langle x^\diamond, \overrightarrow{xz}\rangle| &\leq |\langle x_n^\diamond, \overrightarrow{x_n z}\rangle - \langle x^\diamond, \overrightarrow{x_n z}\rangle| + |\langle x^\diamond, \overrightarrow{x_n z}\rangle - \langle x^\diamond, \overrightarrow{xz}\rangle| \\ &= |\langle x_n^\diamond - x^\diamond, \overrightarrow{x_n z}\rangle - \langle x_n^\diamond - x^\diamond, \overrightarrow{zz}\rangle| + |\langle x^\diamond, \overrightarrow{x_n z}\rangle - \langle x^\diamond, \overrightarrow{xz}\rangle| \\ &\leq \|x_n^\diamond - x^\diamond\|_\diamond d(x_n, z) + |\langle x^\diamond, \overrightarrow{x_n z}\rangle - \langle x^\diamond, \overrightarrow{xz}\rangle| \to 0,\end{aligned}$$

as $n \to \infty$. □

4 Application

One of the major problems in optimization is to find $x \in X$ such that $f(x) = \min_{y\in X} f(y)$. In this section, we apply our results to solving this convex problem.

Definition 4.1 [29] Let X be an Hadamard space with dual X^* and let $f : X \to (-\infty, +\infty]$ be a proper function with efficient domain $D(f) := \{x : f(x) < +\infty\}$. Then, the subdifferential of f is the multivalued mapping $\partial f : X \rightrightarrows X^*$ defined by:

$$\partial f(x) = \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz}\rangle \quad (z \in X)\},$$

when $x \in D(f)$ and $\partial f(x) = \emptyset$, otherwise.

Theorem 4.2 [29] *Let $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function on an Hadamard space X with dual X^*. Then*

(i) *f attains its minimum at $x \in X$ if and only if $\mathbf{0} \in \partial f(x)$,*
(ii) *$\partial f : X \rightrightarrows X^*$ is a monotone operator,*
(iii) *for any $y \in X$ and $\alpha > 0$, there exists a unique point $x \in X$ such that $[\alpha\overrightarrow{xy}] \in \partial f(x)$.*

Part (iii) of Theorem 4.2 shows that the subdifferential of a convex, proper and lower semicontinuous function satisfies the range condition.

Lemma 4.3 [31] *Let $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function on an Hadamard space X with dual X^*. Then*

$$J_\lambda^{\partial f}(x) = argmin_{y\in X}\big[f(y) + \frac{1}{2\lambda} d^2(y, x)\big],$$

for all $\lambda > 0$ and $x \in X$.

Using Theorems 3.4, and 4.2, we have the following corollary.

Corollary 4.4 *Let C be a nonempty closed convex subset of a reflexive Hadamard space X and $f_i : C \to (-\infty, +\infty]$, $i = 1, 2, ..., N$, be N proper convex and lower semicontinuous functions. Let $\{T_n\}_{n=1}^\infty$ be an infinitely countable family of quasi-nonexpansive mappings on C such that $F(T_n) = \widehat{F}(T_n)$. Suppose in addition that $T : C \to C$ is a quasi-nonexpansive mapping such* *that* $(\{T_n\}_{n=1}^{\infty}, T)$ *satisfies the* $AKTT$*-condition and* $F(T) = \widehat{F}(T)$. *Let* $\Omega = \cap_{n=1}^{\infty} F(T_n) \bigcap \cap_{i=1}^{N} argmin_{y \in C}\ f_i(y) \neq \emptyset$. *Assume that* f *be a contraction of* C *into itself with constant* $k \in [0, 1)$. *For* $x_1 \in X$, *let* $\{x_n\}$ *be a sequence defined by:*

$$\begin{cases} z_n = J_{\lambda_n^N}^{\partial f_N} \circ ... \circ J_{\lambda_n^1}^{\partial f_1}(x_n), \\ y_n = \beta_n z_n \oplus (1-\beta_n) T_n z_n, \\ x_{n+1} = \alpha_n f(y_n) \oplus (1-\alpha_n) y_n, \end{cases} \tag{33}$$

where $\{\beta_n\}, \{\alpha_n\} \subset (0, 1)$ *and* $\{\lambda_n^i\} \subset (0, \infty)$ *satisfy the following conditions:*

(i) $\liminf_{n\to\infty} \lambda_n^i > 0$ *for all* $1 \leq i \leq N$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ *converges strongly to* $u \in \Omega$ *which solves the variational inequality:*

$$\langle \overrightarrow{yf(y)}, \overrightarrow{xy} \rangle \geq 0, \qquad \forall x \in \Omega.$$

5 Numerical examples

In this section, we will present two numerical examples in the three-dimensional space of real number and in an Hadamard space X to show that our algorithms are efficient.

Example 5.1 Let $X = \mathbb{R}^3$ with the Euclidean norm and $C = \{x = (x_1, x_2, x_3), \quad -1 \leq x_1, x_2, x_3 \leq 1\}$. For each $x \in C$ and $n = 1, 2, ...,$ we define the mappings T_n and f on C as follows: $T_n x = \dfrac{x}{n}$, $f(x) = (0.1, -0.2, 0.4)$. For each $x \in C$, define $f_1, f_2 : C \to (-\infty, +\infty]$ by:

$$f_1(x) = \frac{1}{2}\|A_1 x - b_1\|^2 \quad and \quad f_2(x) = \frac{1}{2}\|A_2 x - b_2\|^2,$$

where

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b_1 = \begin{bmatrix} 0 \\ 4 \\ -5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}.$$

Using the proximity operator [12], algorithm (28) becomes

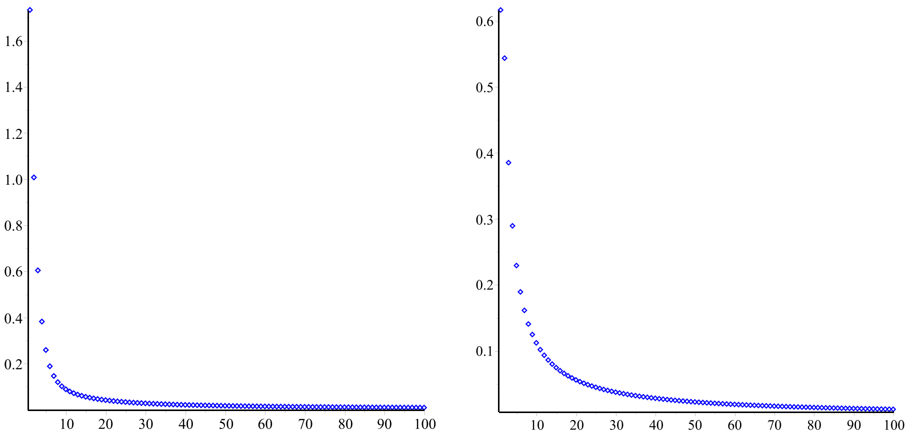
$$\begin{cases} w_n = (I + A_1^t A_1)^{-1}(x_n + A_1^t b_1), \\ z_n = (I + A_2^t A_2)^{-1}(w_n + A_2^t b_2), \\ y_n = \beta_n z_n + (1-\beta_n) T_n z_n, \\ x_{n+1} = \alpha_n f(y_n) + (1-\alpha_n) y_n. \end{cases} \tag{34}$$

Table 1 Numerical results of Example 5.1

n	x_n	$\|x_n\|_2$
1	(1.00000000, – 1.00000000, 1.00000000)	1.73205080
2	(0.67500000, – 0.47500000, 0.57500000)	1.00591997
3	(0.40104166, – 0.27395833, 0.35729166)	0.60294686
4	(0.24370659, – 0.16879340, 0.23927951)	0.38097087
5	(0.15615632, – 0.11009367, 0.17285120)	0.25764889
⋮	⋮	⋮
98	(0.00344929, – 0.00273763, 0.00661979)	0.00795072
99	(0.00341395, – 0.00270980, 0.00655248)	0.00786976
100	(0.00337934, – 0.00268254, 0.00648651)	0.00779043

We choose $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{2}$. It can be observed that all the assumptions of Theorem 3.4 are satisfied and $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \cap \bigcap_{i=1}^{2} argmin_{x\in C} f_i(x) = \{0\}$. Using the algorithm (34) with the initial point $x_1 = (1, -1, 1)$, we have the numerical results in Table 1 and Fig. 1.

Now, we shall illustrate another numerical experiment performed by our method. The problem we are setting up is to solve a nonconvex optimization problem by reducing the problem into convex optimization and apply our algorithm to solve such problem in an Hadamard space. For more details, we refer to [15].



(a) (b)

Fig. 1 **a** Plotting of $\|x_n\|_2$ in Table 1. **b** Plotting of $d_H(x_n, 0)$ in Table 2

Example 5.2 Let $f_1 : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by:

$$f_1(x_1, x_2) = 100\Big((x_2 + 1) - (x_1 + 1)^2\Big)^2 + x_1^2.$$

The function f_1 is not convex in the classical sense. We define a metric on $\mathbb{R}^2$ by:

$$d_H(x, y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, $(\mathbb{R}^2, d_H)$ is an Hadamard space with the geodesics:

$$\gamma_{x,y}(t) = \Big((1-t)x_1 + ty_1, \big((1-t)x_1 + ty_1\big)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\Big).$$

Let $f_2 : \mathbb{R}^2 \to \mathbb{R}$, $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be mappings defined by $f_2(x_1, x_2) = 100x_1^2$, $T(x_1, x_2) = (-x_1, x_2)$ and $f(x_1, x_2) = (1, 1)$. Then, T is a nonexpansive mapping, f_1 and f_2 are convex in $(\mathbb{R}^2, d_H)$ (see [15]). Therefore, algorithm (8) takes the following form:

$$\begin{cases} w_n = argmin_{y\in X}\big[f_1(y) + \frac{1}{2\lambda_n^1} d^2(y, x_n)\big], \\ z_n = argmin_{y\in X}\big[f_2(y) + \frac{1}{2\lambda_n^2} d^2(y, w_n)\big], \\ y_n = \beta_n x_n \oplus (1 - \beta_n)T z_n, \\ x_{n+1} = \alpha_n f(y_n) \oplus (1 - \alpha_n) y_n, \end{cases} \tag{35}$$

We choose $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{10}$ and $\lambda_n^1 = \lambda_n^2 = n$, for all $n \in \mathbb{N}$. It can be observed that all the assumptions of Theorem 3.2 are satisfied and $\Omega = F(T) \cap \bigcap_{i=1}^2 argmin_{x\in X} f_i(x) = \{0\}$. Using the algorithm (35) with the initial point $x_1 = (0.6, 0.5)$, we have the numerical results in Table 2 and Fig. 1.

Table 2 Numerical results of Example 5.2

n	x_n	$d_H(x_n, 0)$
1	(0.60000000, 0.50000000)	0.61611687
2	(0.52971819, 0.40063700)	0.54314815
3	(0.36851985, 0.24640539)	0.38475825
4	(0.27757877, 0.15772162)	0.28906382
5	(0.22217595, 0.10444313)	0.22890187
⋮	⋮	⋮
98	(0.01133786, 0.00024499)	0.01133846
99	(0.01122334, 0.00024006)	0.01122392
100	(0.01111110, 0.00023527)	0.01111167

References

1. Aoyama, K., Kamimura, Y., Takahashi, W., Toyoda, M.: Approximation of common fixed point of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 2350–2360 (2007)
2. Bačák, M.: Convex Analysis and Optimization in Hadamard Spaces, Volume 22 of De Gruyter Series in Nonlinear Analysis and Applications. De Gruyter, Berlin (2014)
3. Bačák, M., Reich, S.: The asymptotic behavior of a class of nonlinear semigroups in Hadamard spaces. J. Fixed Point Theory Appl. **16**, 189–202 (2014)
4. Bauschke, H.H., Matoušková, E., Reich, S.: Projection and proximal point methods: Convergence results and counterexamples. Nonlinear Anal. **56**, 715–738 (2004)
5. Berg, I.D., Nikolaev, I.G.: Quasilinearization and curvature of Alexandrov spaces. Geom. Dedi- cata **133**, 195–218 (2008)
6. Bruck, R.E.: A strongly convergent iterative solution of $0 \in U(x)$ for a maximal monotone operator in Hilbert spaces. J. Math. Anal. Appl. **48**, 114–126 (1974)
7. Bruck, R.E., Reich, S.: Nonexpansive projections and resolvents of accretive operators in Banach spaces. Houston J. Math. **3**, 459–470 (1977)
8. Bruck, R.E., Reich, S.: A general convergence principle in nonlinear functional analysis. Nonlinear Anal. **4**, 939–950 (1980)
9. Chaipunya, P., Kumam, P.: On the proximal point method in Hadamard spaces. Optimization **66**, 1647–1665 (2017)
10. Chaoha, P., Phon-on, A.: A note on fixed point sets in $CAT(0)$ spaces. J. Math. Anal. Appl. **320**, 983–987 (2006)
11. Cholamjiak, P.: The modified proximal point algorithm in $CAT(0)$ spaces. J. Optim. Lett. **9**, 1401–1410 (2014)
12. Combettes, P.L., Pesquet, J.C.: Proximal splitting methods in signal processing. In: Bauschke, H.H., Burachik, R., Combettes, P.L., Elser, V., Luke, D.R., Wolkowicz, H. (eds.) Fixed-Point Algorithms for Inverse Problems in Science and Engineering, pp. 185–212. Springer, New York (2011)
13. Da Cruz Neto, J.X., Ferreira, O.P., Lucambio Pérez, L.R.: Monotone point-to-set vector fields. Balkan J. Geom. Appl. **5**, 69–79 (2000)
14. Da Cruz Neto, J.X., Ferreira, O.P., Lucambio Pérez, L.R.: Contributions to the study of monotone vector fields. Acta Math. Hungar. **94**, 307–320 (2002)
15. Da Cruz Neto, J.X., Ferreira, O.P., Lucambio Prez, L.R., Németh, S.Z.: Convex- and monotone-transformable mathematical programming problems and a proximal-like point method. J. Global Optim. **35**(1), 53–69 (2006)
16. Dehghan, H., Rooin, J.: A characterization of metric projection in $CAT(0)$ spaces. In: International Conference on Functional Equation, Geometric Functions and Applications (ICFGA 2012) 11-12 th May 2012, pp. 41–43. Payame University, Tabriz (2012)
17. Dhompongsa, S., Kaewkhao, A., Panyanak, B.: On Kirk's strong convergence theorem for multivalued nonexpansive mappings on $CAT(0)$ spaces. Nonlinear Anal. **75**, 459–468 (2012)
18. Dhompongsa, S., Kirk, W.A., Sims, B.: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762–772 (2006)
19. Dhompongsa, S., Panyanak, B.: On Δ-convergence theorems in $CAT(0)$ spaces. Comput. Math. Appl. **56**, 2572–2579 (2008)
20. Eskandani, G.Z., Raeisi, M.: A new algorithm for finding fixed points of Bregman quasi-nonexpansive mappings and zeros of maximal monotone operators by using products of resolvents. Result Math. **71**, 1307–1326 (2017)
21. Eskandani, G.Z., Azarmi, S., Eslamian, M.: Composition of resolvents and quasi-nonexpansive multivalued mappings in Hadamared spaces. Bult. Iran. Math. Soc. **6**, 1939–1955 (2017)
22. Eskandani, G.Z., Azarmi, S., Raeisi, M.: Products of resolvents and multivalued hybrid mappings in $CAT(0)$ spaces. Acta Math. Sci. **38**, 791–804 (2018)
23. Ferreira, O.P., Lucambio Pérez, L.R., Németh, S.Z.: Singularities of monotone vector fields and an extragradient-type algorithm. J. Global Optim. **31**, 133–151 (2005)
24. Ferreira, O.P., Oliveira, P.R.: Proximal point algorithm on Riemannian manifolds. Optimization **51**, 257–270 (2002)
25. Goebel, K., Reich, S.: Uniform convexity, hyperbolic geometry, and nonexpansive mappings. Volume 83 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker Inc., New York (1984)

26. Güler, O.: On the convergence of the proximal point algorithm for convex minimization. SIAM J. Control Optim. **29**, 403–419 (1991)
27. Huang, S., Kimura, Y.: A projection method for approximating fixed points of quasi-nonexpansive mappings in Hadamard spaces. Fixed Point Theory Appl. **36**, 1–13 (2016)
28. Kakavandi, B.A.: Weak topologies in complete $CAT(0)$ metric spaces. Proc. Amer. Math. Soc. **141**, 1029–1039 (2013)
29. Kakavandi, B.A., Amini, M.: Duality and subdifferential for convex functions on complete $CAT(0)$ metric spaces. Nonlinear Anal. **73**, 3450–3455 (2010)
30. Kamimura, S., Takahashi, W.: Approximating solutions of maximal monotone operators in Hilbert spaces. J. Approx. Theory **13**, 226–240 (2000)
31. Khatibzadeh, H., Ranjbar, S.: Monotone operators and the proximal point algorithm in complete $CAT(0)$ metric spaces. J. Aust. Math. Soc. **103**, 70–90 (2017)
32. Kirk, W.A., Panyanak, B.: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**, 3689–3696 (2008)
33. Kopecká, E., Reich, S.: Hyperbolic monotonicity in the Hilbert ball. Fixed Point Theory Appl. Art. ID 78104, 15 pages (2006)
34. Kopecká, E., Reich, S.: Asymptotic behavior of resolvents of coaccretive operators in the Hilbert ball. Nonlinear Anal. **70**, 3187–3194 (2009)
35. Li, C., López, G., Martín-Márquez, V.: Monotone vector fields and the proximal point algorithm on Hadamard manifolds. J. Lond. Math. Soc. **79**, 663–683 (2009)
36. Maingé, P.E.: Strong convergence of projected pubgradient methods for nonsmooth and nonstrictly convex minimization. Set-valued Anal. **16**, 899–912 (2008)
37. Martinet, B.: Régularisation dinéquations variationelles par approximations successives. Revue Fr. Inform. Rech. Oper. **4**, 154–159 (1970)
38. Nevanlinna, O., Reich, S.: Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces. Isr J. Math. **32**, 44–58 (1979)
39. Németh, S.Z.: Monotone vector fields. Publ. Math. Debrecen. **54**, 437–449 (1999)
40. Raeisi, M., Eskandani, G.Z., Eslamian, M.: A general algorithm for multiple-sets split feasibility problem involving resolvents and Bregman mappings. Optimization **68**, 309–327 (2018)
41. Reich, S.: Constructive techniques for accretive and monotone operators. In: Applied Nonlinear Analysis, pp. 335–345. Academic Press, New York (1979)
42. Reich, S.: Strong convergence theorems for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl. **75**, 287–292 (1980)
43. Reich, S.: A weak convergence theorem for the alternating method with Bregman distances. In: Theory and Applications of Nonlinear Operators, pp. 313–318. Marcel Dekker, New York (1996)
44. Reich, S., Salinas, Z.: Infinite products of discontinuous operators in Banach and metric spaces. Linear Nonlinear Anal. **1**, 169–200 (2015)
45. Reich, S., Salinas, Z.: Weak convergence of infinite products of operators in Hadamard spaces. Rend. Circ. Mat. Palermo **65**, 55–71 (2016)
46. Reich, S., Shafrir, I.: Nonexpansive iterations in hyperbolic spaces. Nonlinear Anal. **15**, 537–558 (1990)
47. Reich, S., Shemen, L.: A note on Halpern's algorithm in the Hilbert ball. J. Nonlinear Convex Anal. **14**, 853–862 (2013)
48. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**, 877–898 (1976)
49. Saejung, S.: Halpern's iteration in $CAT(0)$ spaces. Fixed Point Theory Appl. **2010**, 471781 (2010)
50. Shioji, N., Takahashi, W.: Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces. Proc. Amer. Math. Soc. **125**, 3641–3645 (1997)
51. Suparatulatorn, R., Cholamjiak, P., Suantai, S.: On solving the minimization problem and the fixed-point problem for nonexpansive mapping in $CAT(0)$ spaces. Optim. Meth. Soft., 182–192 (2017)
52. Wangkeeree, R., Preechasilp, P.: Viscosity approximation methods for nonexpansive mappings in $CAT(0)$ spaces. J. Inequal. Appl. **2013**, 93 (2013)
53. Xu, H.K.: Another control condition in an iterative method for nonexpansive mappings. Bull. Austral. Math. Soc. **65**, 109–113 (2002)