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Three kinds of hybrid algorithms and their numerical realizations for a finite family of quasi-asymptotically pseudocontractive mappings

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Abstract The purpose of this article is to propose three new hybrid projection methods for a finite family of quasi-asymptotically pseudocontractive mappings. The strong convergence of the algorithms is proved in real Hilbert spaces. Some numerical experiments are also included to compare and explain the effectiveness of the proposed methods.

Keywords Quasi-asymptotically pseudocontractive mappings · Hybrid algorithms · Strong convergence · Hilbert spaces

Mathematics Subject Classification (2010) 47H05 · 47H09 · 47H10

1 Introduction

Suppose that H is a real Hilbert space. We use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to denote the inner product and the norm, respectively. Suppose that C is a closed convex nonempty

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subset of H. We use $F(T)$ to denote the fixed point set of a mapping $T : C \to C$, i.e., $F(T) = \{x \in C : x = Tx\}$. A mapping $T : C \to C$ is called a nonexpansive mappings if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x,\ y \in C$.

A mapping $T : C \to C$ is called a quasi-nonexpansive mapping if $F(T) \neq \emptyset$ such that

$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in C,\ p \in F(T)$.

A mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$.

A mapping $T : C \to C$ is called quasi-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|T^n x - p\| \leq k_n \|x - p\|,$$

for all $x \in C,\ p \in F(T)$ and all $n \geq 1$.

A mapping $T : C \to C$ is called asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\langle T^n x - T^n y, x - y\rangle \leq k_n \|x - y\|^2,$$

for all $x, y \in C$ and all $n \geq 1$.

A mapping $T : C \to C$ is called quasi-asymptotically pseudocontractive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\langle T^n x - p, x - p\rangle \leq k_n \|x - p\|^2,$$

for all $x \in C,\ p \in F(T)$ and all $n \geq 1$.

A mapping $T : C \to C$ is called κ-strictly asymptotically pseudocontractive if there exist some $\kappa \in [0, 1)$ and some real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\|^2 \leq k_n^2 \|x - y\|^2 + \kappa \|(I - T^n)x - (I - T^n)y\|^2,$$

for all $x, y \in C$ and all $n \geq 1$.

A mapping $T : C \to C$ is called quasi-κ-strictly asymptotically pseudocontractive if $F(T) \neq \emptyset$, and there exist some $\kappa \in [0, 1)$ and some real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|T^n x - p\|^2 \leq k_n^2 \|x - p\|^2 + \kappa \|(I - T^n)x\|^2,$$

for all $x \in C,\ p \in F(T)$ and all $n \geq 1$.

A mapping $T : C \to C$ is called uniformly L-Lipschitzian if there exists some $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$.

Remark 1.1 [15] We note that every κ-strictly asymptotically pseudocontractive mapping is uniformly L-Lipschitzian with the Lipschitz constant $L = \frac{M+\sqrt{\kappa}}{1-\sqrt{\kappa}}$, where $M = sup_n\{k_n\}$. In particular, every asymptotically nonexpansive mapping is uniformly L-Lipschitzian with $L = sup\{k_n : n \geq 1\}$.

Remark 1.2 [15] It is clear that every asymptotically nonexpansive mapping is 0-strictly asymptotically pseudocontractive, while every asymptotically pseudocontractive mapping with sequence $\{k_n\}$ is 1-strictly asymptotically pseudocontractive with sequence $\{2k_n - 1\}$.

Remark 1.3 [15] It is also clear that every asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$ is quasi-asymptotically pseudocontractive, but the converse may be not true in general.

Remark 1.4 [15] The class of asymptotically pseudocontractive mappings is a generalization of the class of pseudocontractive mappings, and the former contains properly the class of asymptotically nonexpansive mappings as a subclass.

We give an asymptotically pseudocontractive mapping in infinite dimensional Hilbert spaces.

Example 1.1 [25] Let B denote the unit ball in the Hilbert space l^2 and let F be defined as follows:

$$F : (x_1,\ x_2,\ x_3,\ \cdots) \to (0,\ x_1^2,\ A_2x_2,\ A_3x_3,\ \cdots)$$

where A_i is a sequence of numbers such that $0 < A_i < 1$ and $\Pi_{i=2}^{\infty} A_i = \dfrac{1}{2}$. Then, F is lipschitzian and $\|Fx - Fy\| \leq 2\|x - y\|,\ x,\ y \in B$; and moreover, $\|F^i x - F^i y\| \leq 2\Pi_{j=2}^{i} A_j \|x - y\|$ for $i = 2,\ 3,\ \cdots$. Thus,

$$\lim_{i\to\infty} k_i = \lim_{i\to\infty} 2\Pi_{j=2}^{i} A_j = 1.$$

Then, F is an asymptotically nonexpansive mapping, thereby F is also an asymptotically pseudocontractive mapping.

Construction of fixed points of nonlinear mappings is an important subject in the theory of nonlinear mappings and finds application in a number of applied areas. Recently, a great deal of literature on iteration algorithms for approximating fixed points of nonexpansive mappings has been published since these algorithms have a variety of applications in inverse problem, image recovery, and signal processing (see [1–10]). Mann's iteration process [1] is often used to approximate a fixed point of the operators, but it has only weak convergence in general (see [3] for an example). However, strong convergence is often much more desirable than weak convergence in many problems that arise in infinite dimensional spaces (see [7] and references therein). Consequently, in order to obtain strong convergence, one has to modify the normal Mann's iteration algorithm, and the so-called hybrid projection iteration method is such a modification.

Recently, the hybrid projection algorithm was developed rapidly for finding the nearest fixed point of certain quasi-nonexpansive mappings; see, for instance, Bauschke and Combettes [7] and the references therein.

By virtue of the hybrid projection methods, Nakajo and Takahashi [11] established some strong convergence results for nonexpansive mappings and nonexpansive semi-groups in a real Hilbert space; Marino and Xu [12] proved a strong convergence theorem for strict pseudo-contractions in a real Hilbert space; Zhou [13] extended Marino and Xu's strong convergence theorem to the more general class of Lipschitz pseudocontractive mappings; Zhou [14] generalized and extended the main results of [13] to the class of asymptotically pseudocontractive mappings; Zhou and Su [15] further extended the main results in [14] to a family of uniformly L-Lipschitz continuous and quasi-asymptotically pseudocontractive mappings.

Very recently, some authors constructed some hybrid projection methods for finding common fixed points of a finite family of nonexpansive mappings, quasi-nonexpansive mappings, quasi-asymptotically nonexpansive mappings, or quasi-asymptotically pseudocontractive mappings and gave some corresponding numerical experiments (see [16–23] and the references therein). Anh and Chung [21] proposed a parallel hybrid algorithm for a finite family of nonexpansive mappings in a Hilbert space H as follows:

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ z_k = P_C x_k, \\ y_k^i = \alpha_k z_k + (1-\alpha_k) T_i z_k, \ i = 1, \ 2, \ \cdots, \ N, \\ i_k = argmax\{\|y_k^i - x_k\| : i = 1, \ 2, \ \cdots, \ N\}, \\ C_k = \{v \in H : \|v - y_k^{i_k}\| \le \|v - x_k\|\}, \\ Q_k = \{u \in H : \langle x_0 - x_k, x_k - u\rangle \ge 0\}, \\ x_{k+1} = P_{C_k \cap Q_k} x_0 \end{cases} \tag{1.1}$$

and proved that $\{x_n\}$ produced by (1.1) converges to $P_{\bigcap_{i=1}^N F(T_i)} x_0$.

Dong et al. [18] proposed a cyclic algorithm for a finite family of nonexpansive mappings in a Hilbert space H as follows:

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ y_k^1 = \alpha_k x_k + (1-\alpha_k)T_1 x_k, \\ y_k^{i+1} = \alpha_k y_k^i + (1-\alpha_k)T_{i+1} y_k^i,\ i = 1, 2, \cdots, N-1, \\ i_k = argmax\{\|y_k^i - x_k\| : i = 1,\ 2,\ \cdots,\ N\}, \\ C_k = \{u \in C : \|u - y_k^{i_k}\| \le \|u - x_k\|\}, \\ Q_k = \{v \in C : \langle x_0 - x_k, x_k - v\rangle \ge 0\}, \\ x_{k+1} = P_{C_k \cap Q_k} x_0 \end{cases} \tag{1.2}$$

and proved that $\{x_n\}$ produced by (1.2) converges to $P_{\bigcap_{i=1}^N F(T_i)} x_0$.

We observe that the construction of the half-spaces C_n in [15] is complicated, and hence the computation of the metric projections $P_{C_n} x_0$ is difficult.

Our concern now is the following: Can one design some simple and new hybrid projection algorithms for finding a common fixed point for a finite family of quasi-asymptotically pseudocontractive mappings?

Motivated by the above work, the purpose of this paper is to propose three kinds of new hybrid projection algorithms for constructing a common fixed point of a finite family of quasi-asymptotically pseudocontractive mappings in a real Hilbert space. By using some new analysis techniques, we prove the strong convergence of the proposed algorithms. We also give some numerical experiments to compare and describe the effectiveness of the proposed algorithms. The results of this paper improve and extend the related ones obtained by some authors.

2 Preliminaries

Lemma 2.1 [12]*Suppose that C is a closed convex nonempty subset of H. Suppose that P_C is the projection from H onto C (that is, for any $x \in H$, $P_C x$ is the only point in C such that $\|x - P_C x\| = inf\{\|x - z\| : z \in C\}$). Then, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if*

$$\langle x - z,\ y - z\rangle \le 0$$

for all $y \in C$.

Lemma 2.2 [15] *Let C be a nonempty, bounded, and closed convex subset of H. Let $T : C \to C$ be a uniformly L-Lipschitzian and quasi-asymptotically pseudocontractive mapping. Then, F(T) is a closed convex subset of C.*

Lemma 2.3 [11, 12] *Suppose that C is a closed convex nonempty subset of H. Let P_C be the projection from H onto C. Then,*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \le \|x - y\|^2,\ \forall x \in H,\ \forall y \in C.$$

3 Main results

Theorem 3.1 *Suppose that C is a bounded closed convex nonempty subset of H. Suppose that $\{T_i\}_{i=1}^N : C \to C$ is a finite family of uniformly L_i-Lipschitzian and quasi-asymptotically pseudocontractive mappings such that $F = \bigcap_{i=1}^N F(T_i)$ is not empty. Assume the control sequences $\{\alpha_{ni}\}_{i=1}^N$ are chosen so that $\{\alpha_{ni}\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{1+L})$, where $L = max\{L_i : 1 \le i \le N\}$. Let a sequence $\{x_n\}$ be generated by the following manner:*

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ C_0^{(i)} = C, \ i = 1, \ 2, \ \cdots, \ N, \\ y_n^{(0)} = x_n, \ \forall n \ge 0, \\ y_n^{(1)} = (1-\alpha_{n1})x_n + \alpha_{n1}T_1^n x_n, \\ y_n^{(i)} = (1-\alpha_{ni})y_n^{(i-1)} + \alpha_{ni}T_i^n y_n^{(i-1)}, \ i = 2, \ \cdots, \ N, \\ C_{n+1}^{(1)} = \{w \in C_n^{(1)} : \alpha_{n1}[1-(1+L)\alpha_{n1}]\|x_n - T_1^n x_n\|^2 \\ \qquad\qquad \le \langle x_n - w, (I - T_1^n)y_n^{(1)}\rangle + (k_n - 1)(diamC)^2\}, \\ C_{n+1}^{(i)} = \{w \in C_n^{(i)} : \alpha_{ni}[1-(1+L)\alpha_{ni}]\|y_n^{(i-1)} - T_i^n y_n^{(i-1)}\|^2 \\ \qquad\qquad \le \langle y_n^{(i-1)} - w, (I - T_i^n)y_n^{(i)}\rangle + (k_n - 1)(diamC)^2\}, \ i = 2, \ \cdots, \ N, \\ C_{n+1} = \bigcap_{i=1}^N C_{n+1}^{(i)}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \ge 0, \end{cases} \tag{3.1}$$

where $k_n = max\{k_{ni} : 1 \le i \le N\}$, while k_{ni} are asymptotic sequences for $\{T_i\}_{i=1}^N$. Then, the sequence $\{x_n\}$ generated by (3.1) strongly converges to $P_F x_0$.

Proof By Lemma 2.2 and our assumption that $F \ne \emptyset$, we know that $P_F x_0$ is well defined for every $x_0 \in C$. It follows from the constructions of C_n that C_n is closed and convex, $\forall n \ge 0$. We omit the details.

Step 1 Show that $F \subset C_n$ for all $n \ge 0$.

It suffices to show that $F \subset C_n^{(i)}$ for all $n \ge 0, \ i = 1, 2, \ \cdots, \ N$. Firstly, we show that $F \subset C_n^{(1)}$ for all $n \ge 0$. In fact, $F \subset C_0^{(1)} = C$ is obvious. Assume that $F \subset C_n^{(1)}$ for some $n \ge 0$. For any $q \in F \subset C_n^{(1)}$, one has

$$\begin{aligned} \|x_n - T_1^n x_n\|^2 &= \langle x_n - T_1^n x_n, x_n - T_1^n x_n\rangle \\ &= \frac{1}{\alpha_{n1}}\langle x_n - y_n^{(1)}, (I - T_1^n)x_n\rangle \\ &= \frac{1}{\alpha_{n1}}\langle x_n - y_n^{(1)}, (I - T_1^n)x_n - (I - T_1^n)y_n^{(1)}\rangle \\ &\quad + \frac{1}{\alpha_{n1}}\langle x_n - y_n^{(1)}, (I - T_1^n)y_n^{(1)}\rangle \\ &\le \frac{1+L}{\alpha_{n1}}\|x_n - y_n^{(1)}\|^2 + \frac{1}{\alpha_{n1}}\langle x_n - q, (I - T_1^n)y_n^{(1)}\rangle \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\alpha_{n1}}\langle q-y_n^{(1)},(I-T_1^n)y_n^{(1)}\rangle\\
=&\ (1+L)\alpha_{n1}\|x_n-T_1^nx_n\|^2+\frac{1}{\alpha_{n1}}\langle x_n-q,(I-T_1^n)y_n^{(1)}\rangle\\
&+\frac{1}{\alpha_{n1}}[\langle q-y_n^{(1)},T_1^nq-T_1^ny_n^{(1)}\rangle-\langle q-y_n^1,q-y_n^1\rangle]\\
\le&\ (1+L)\alpha_{n1}\|x_n-T_1^nx_n\|^2+\frac{1}{\alpha_{n1}}\langle x_n-q,(I-T_1^n)y_n^{(1)}\rangle\\
&+\frac{1}{\alpha_{n1}}(k_n-1)(diamC)^2.
\end{aligned}
$$

It follows that

$$
\alpha_{n1}[1-(1+L)\alpha_{n1}]\|x_n-T_1^nx_n\|^2\le\langle x_n-q,(I-T_1^n)y_n^{(1)}\rangle+(k_n-1)(diamC)^2,
$$

which shows $q\in C_{n+1}^{(1)}$ and hence $F\subset C_n^{(1)}$ for all $n\ge 0$.

Secondly, we show that $F\subset C_n^{(i)}$ for all $n\ge 0,\ i=2,\cdots,N$. In fact, $F\subset C_0^{(i)}=C\ (i=2,\cdots,N)$ is obvious. Assume that $F\subset C_n^{(i)}\ (i=2,\cdots,N)$ for some $n\ge 0$. For any $q\in F\subset C_n^{(i)}\ (i=2,\cdots,N)$, one has

$$
\begin{aligned}
\|y_n^{(i-1)}-T_i^ny_n^{(i-1)}\|^2&=\langle y_n^{(i-1)}-T_i^ny_n^{(i-1)},y_n^{(i-1)}-T_i^ny_n^{(i-1)}\rangle\\
&=\frac{1}{\alpha_{ni}}\langle y_n^{(i-1)}-y_n^{(i)},(I-T_i^n)y_n^{(i-1)}\rangle\\
&=\frac{1}{\alpha_{ni}}\langle y_n^{(i-1)}-y_n^{(i)},(I-T_i^n)y_n^{(i-1)}-(I-T_i^n)y_n^{(i)}\rangle\\
&\quad+\frac{1}{\alpha_{ni}}\langle y_n^{(i-1)}-y_n^{(i)},(I-T_i^n)y_n^{(i)}\rangle\\
&\le\frac{1+L}{\alpha_{ni}}\|y_n^{(i-1)}-y_n^{(i)}\|^2+\frac{1}{\alpha_{ni}}\langle y_n^{(i-1)}-q,(I-T_i^n)y_n^{(i)}\rangle\\
&\quad+\frac{1}{\alpha_{ni}}\langle q-y_n^{(i)},(I-T_i^n)y_n^{(i)}\rangle\\
&=(1+L)\alpha_{ni}\|y_n^{(i-1)}-T_i^ny_n^{(i-1)}\|^2\\
&\quad+\frac{1}{\alpha_{ni}}\langle y_n^{(i-1)}-q,(I-T_i^n)y_n^{(i)}\rangle\\
&\quad+\frac{1}{\alpha_{ni}}[\langle q-y_n^{(i)},T_i^nq-T_i^ny_n^{(i)}\rangle-\langle q-y_n^{(i)},q-y_n^{(i)}\rangle]\\
&\le(1+L)\alpha_{ni}\|y_n^{(i-1)}-T_i^ny_n^{(i-1)}\|^2\\
&\quad+\frac{1}{\alpha_{ni}}\langle y_n^{(i-1)}-q,(I-T_i^n)y_n^{(i)}\rangle\\
&\quad+\frac{1}{\alpha_{ni}}(k_n-1)(diamC)^2.
\end{aligned}
$$

It follows that

$$\alpha_{ni}[1-(1+L)\alpha_{ni}]\|y_n^{(i-1)}-T_i^n y_n^{(i-1)}\|^2 \le \langle y_n^{(i-1)}-q,(I-T_i^n)y_n^{(i)}\rangle \\ +(k_n-1)(diamC)^2,$$

which shows $q \in C_{n+1}^{(i)}$ and hence $F \subset C_n^{(i)}$ for all $n \ge 0$, $i = 2, \cdots, N$. So $F \subset \bigcap_{i=1}^N C_n^{(i)} = C_n$ for all $n \ge 0$.

Step 2 Show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

In view of (3.1), one has $x_n = P_{C_n}x_0$. Since $C_{n+1} \subset C_n$ and $x_{n+1} \in C_{n+1}$, one has

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|, \ \forall n \ge 1. \tag{3.2}$$

On the other hand, as $F \subset C_n$ by step 1, it follows that

$$\|x_n - x_0\| \le \|z - x_0\|, \ \forall z \in F, \ \forall n \ge 1. \tag{3.3}$$

Combining (3.2) and (3.3), one sees that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Step 3 Show that $x_n \to v$ as $n \to \infty$, $v \in C$.

For $m > n \ge 1$, one has $x_m = P_{C_m}x_0 \in C_m \subset C_n$, by Lemma 2.3, one has

$$\|x_m - x_n\|^2 \le \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \tag{3.4}$$

Taking $m, n \to \infty$ in (3.4), one gets $x_m - x_n \to 0$ as $m, n \to \infty$, which proves that $\{x_n\}$ is a Cauchy sequence in C. By completeness of H and closedness of C, one has $v \in C$ such that $x_n \to v$ as $n \to \infty$.

Step 4 Show that $v \in F = \bigcap_{i=1}^N F(T_i)$.

Firstly, we show that $v = T_1 v$. In fact, it follows from Step 3 that $x_{n+1} - x_n \to 0$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1}$, one has $x_{n+1} \in C_{n+1}^{(i)}$, $i = 1, 2, \cdots, N$. It follows that

$$\alpha_{n1}[1-(1+L)\alpha_{n1}]\|x_n - T_1^n x_n\|^2 \le \langle x_n - x_{n+1}, (I-T_1^n)y_n^{(1)}\rangle \\ +(k_n-1)(diamC)^2. \tag{3.5}$$

Noting that $\{\alpha_{n1}\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{1+L})$, $\{y_n^{(1)}\}$ and $\{T_1^n y_n^{(1)}\}$ are all bounded and $k_n \to 1(n \to \infty)$, from (3.5), we obtain that

$$\|x_n - T_1^n x_n\| \to 0(n \to \infty). \tag{3.6}$$

Since $x_n \to v$ as $n \to \infty$ and T_1 is uniformly Lipschitzian, we have

$$v = T_1 v. \tag{3.7}$$

Secondly, we show that $v = T_2 v$. In fact, $x_{n+1} \in C_{n+1}^{(2)}$, we have

$$\begin{aligned}
&\alpha_{n2}[1-(1+L)\alpha_{n2}]\|y_n^{(1)} - T_2^n y_n^{(1)}\|^2\\
&\le \langle y_n^{(1)} - x_{n+1}, (I-T_2^n)y_n^{(2)}\rangle + (k_n-1)(diamC)^2\\
&= (1-\alpha_{n1})\langle x_n - x_{n+1}, (I-T_2^n)y_n^{(2)}\rangle + \alpha_{n1}\langle T_1^n x_n - x_{n+1}, (I-T_2^n)y_n^{(2)}\rangle\\
&\quad + (k_n-1)(diamC)^2\\
&= \langle x_n - x_{n+1}, (I-T_2^n)y_n^{(2)}\rangle + \alpha_{n1}\langle T_1^n x_n - x_n, (I-T_2^n)y_n^{(2)}\rangle + (k_n-1)(diamC)^2.
\end{aligned}$$

Since $x_{n+1} - x_n \to 0$, $x_n - T_1^n x_n \to 0$, $k_n \to 1$ as $n \to \infty$, while $\{(I - T_2^n)y_n^{(2)}\}$ is bounded and $\{\alpha_{n2}\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{1+L})$, we have

$$y_n^{(1)} - T_2^n y_n^{(1)} \to 0 (n \to \infty). \tag{3.8}$$

From $y_n^{(1)} = (1-\alpha_{n1})x_n + \alpha_{n1}T_1^n x_n$, (3.6) and $x_n \to v(n \to \infty)$, we have

$$y_n^{(1)} \to v(n \to \infty). \tag{3.9}$$

From (3.8), (3.9) and uniform Lipschitz continuity of T_2, we have

$$v = T_2 v. \tag{3.10}$$

Similarly, we obtain that

$$v = T_i v,\ i = 3, \cdots, N. \tag{3.11}$$

Combining (3.7), (3.10) and (3.11), we have $v \in F = \bigcap_{i=1}^N F(T_i)$.

Step 5 Show that $v = P_F x_0$.

Noting that $v \in F \subset C_n$ and $x_n = P_{C_n} x_0$, by Lemma 2.1, one concludes that

$$\langle z - x_n,\ x_0 - x_n\rangle \le 0,\ \forall z \in F,$$

which follows that

$$\langle z - v,\ x_0 - v\rangle \le 0,\ \forall z \in F.$$

By Lemma 2.1, one concludes that $v = P_F x_0$. □

Theorem 3.2 *Suppose that C is a bounded closed convex nonempty subset of H. Suppose that $\{T_i\}_{i=1}^N : C \to C$ is a finite family of uniformly L_i-Lipschitzian and quasi-asymptotically pseudocontractive mappings such that $F = \bigcap_{i=1}^N F(T_i)$ is not empty. Assume the control sequence $\{\alpha_n\}$ is chosen so that $\{\alpha_n\} \subset [a,\ b]$ for some $a,\ b \in (0,\ \frac{1}{1+L})$, where $L = max\{L_i : 1 \le i \le N\}$. Let a sequence $\{x_n\}$ be generated by the following manner:*

$$\begin{cases}
x_0 \in C \ \ chosen\ arbitrarily,\\
C_1 = C,\\
y_n^i = (1-\alpha_n)x_n + \alpha_n T_i^n x_n,\ i = 1,\ 2,\ \cdots,\ N,\\
i_n = argmax\{\|y_n^i - x_n\| : i = 1,\ 2,\ \cdots,\ N\},\\
C_{n+1} = \{w \in C_n : \alpha_n[1-(1+L)\alpha_n]\|x_n - T_{i_n}^n x_n\|^2\\
\qquad\qquad \le \langle x_n - w, y_n^{i_n} - T_{i_n}^n y_n^{i_n}\rangle + (k_n-1)(diamC)^2\},\\
x_{n+1} = P_{C_{n+1}} x_0,\ n \ge 0,
\end{cases} \tag{3.12}$$

where $k_n = max\{k_{n,i} : 1 \le i \le N\}$*, while* $k_{n,i}$ *are asymptotic sequences for* $\{T_i\}_{i=1}^N$*. Then, the sequence* $\{x_n\}$ *generated by* (3.12) *strongly converges to* $P_F x_0$*.*

Proof By Lemma 2.2 and our assumption that $F \neq \emptyset$, we know that $P_F x_0$ is well defined for every $x_0 \in C$. It follows from the constructions of C_n that C_n is closed and convex, $\forall n \ge 1$. We omit the details.

Step 1 Show that $F \subset C_n$ for all $n \ge 1$.

In fact, $F \subset C_1 = C$ is obvious. Assume that $F \subset C_n$ for some $n \ge 1$. For any $q \in F \subset C_n$, from (3.12), using the uniform L_i-Lipschitz continuity of T_i and quasi-asymptotic pseudo-contractiveness of T_i, we obtain that

$$\begin{aligned}
\|x_n - T_{i_n}^n x_n\|^2 &= \langle x_n - T_{i_n}^n x_n, x_n - T_{i_n}^n x_n\rangle \\
&= \frac{1}{\alpha_n}\langle x_n - y_n^{i_n}, x_n - T_{i_n}^n x_n\rangle \\
&= \frac{1}{\alpha_n}\langle x_n - y_n^{i_n}, (I - T_{i_n}^n)x_n - (I - T_{i_n}^n)y_n^{i_n}\rangle \\
&\quad + \frac{1}{\alpha_n}\langle x_n - y_n^{i_n}, (I - T_{i_n}^n)y_n^{i_n}\rangle \\
&\le \frac{1}{\alpha_n}(\|x_n - y_n^{i_n}\|^2 + \|x_n - y_n^{i_n}\| \cdot \|T_{i_n}^n x_n - T_{i_n}^n y_n^{i_n}\|) \\
&\quad + \frac{1}{\alpha_n}\langle x_n - q, y_n^{i_n} - T_{i_n}^n y_n^{i_n}\rangle + \frac{1}{\alpha_n}\langle q - y_n^{i_n}, y_n^{i_n} - T_{i_n}^n y_n^{i_n}\rangle \\
&\le \frac{1+L}{\alpha_n}\|x_n - y_n^{i_n}\|^2 + \frac{1}{\alpha_n}\langle x_n - q, y_n^{i_n} - T_{i_n}^n y_n^{i_n}\rangle \\
&\quad + \frac{1}{\alpha_n}\langle q - y_n^{i_n}, y_n^{i_n} - q + T_{i_n}^n q - T_{i_n}^n y_n^{i_n}\rangle \\
&= (1+L)\alpha_n\|x_n - T_{i_n}^n x_n\|^2 + \frac{1}{\alpha_n}\langle x_n - q, y_n^{i_n} - T_{i_n}^n y_n^{i_n}\rangle \\
&\quad + \frac{1}{\alpha_n}[\langle q - y_n^{i_n}, T_{i_n}^n q - T_{i_n}^n y_n^{i_n}\rangle - \langle q - y_n^{i_n}, q - y_n^{i_n}\rangle] \\
&= (1+L)\alpha_n\|x_n - T_{i_n}^n x_n\|^2 + \frac{1}{\alpha_n}\langle x_n - q, y_n^{i_n} - T_{i_n}^n y_n^{i_n}\rangle \\
&\quad + \frac{1}{\alpha_n}(k_n - 1)(diamC)^2,
\end{aligned}$$

which implies that

$$\alpha_n[1 - (1+L)\alpha_n]\|x_n - T_{i_n}^n x_n\|^2 \le \langle x_n - q, y_n^{i_n} - T_{i_n}^n y_n^{i_n}\rangle + (k_n - 1)(diamC)^2.$$

So $q \in C_{n+1}$ and hence $F \subset C_n$ for all $n \ge 1$.

Step 2 Show that $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

The proof is similar to the proof of Step 2 in Theorem 3.1 and omitted here.

Step 3 Show that $x_n \to v$ as $n \to \infty$, $v \in C$.

The proof is similar to the proof of Step 3 in Theorem 3.1 and omitted here.

Step 4 Show that $x_n - T_{i_n}^n x_n \to 0$ as $n \to \infty$.

It follows from Step 3 that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1}$, one has

$$\begin{aligned}\alpha_n[1-(1+L)\alpha_n]\|x_n - T_{i_n}^n x_n\|^2 &\le \langle x_n - x_{n+1}, y_n^{i_n} - T_{i_n}^n y_n^{i_n}\rangle + (k_n-1)(diamC)^2\\ &\le \|x_n - x_{n+1}\| \cdot \|y_n^{i_n} - T_{i_n}^n y_n^{i_n}\|\\ &\quad + (k_n-1)(diamC)^2.\end{aligned}$$

Noting that $\{\alpha_n\} \subset [a, b]$ for $a, b \in (0, \frac{1}{1+L})$, $\{y_n^{i_n}\}$ and $\{T_{i_n}^n y_n^{i_n}\}$ are all bounded, we have $\|x_n - T_{i_n}^n x_n\| \to 0$ as $n \to \infty$.

Step 5 Show that $x_n - T_i^n x_n \to 0$ as $n \to \infty,\ i = 1, 2, \cdots, N$.

By the definition of $y_n^{i_n}$, we have

$$\alpha_n(x_n - T_{i_n}^n x_n) = x_n - y_n^{i_n}.$$

Since $x_n - T_{i_n}^n x_n \to 0$ as $n \to \infty$, we have $x_n - y_n^{i_n} \to 0$ as $n \to \infty$. By the definition of i_n, one has

$$\|x_n - y_n^i\| \to 0(n \to \infty),\ i = 1, 2, \cdots, N.$$

Using (3.12), one has

$$\|x_n - T_i^n x_n\| = \frac{1}{\alpha_n}\|x_n - y_n^i\| \to 0(n \to \infty),\ i = 1, 2, \cdots, N.$$

Step 6 Show that $x_n - T_i x_n \to 0$ as $n \to \infty,\ i = 1, 2, \cdots, N$.

Since T_i is uniformly L_i-Lipschitzian, $i = 1, 2, \cdots, N$, we have

$$\begin{aligned}\|x_{n+1} - T_i x_{n+1}\| &\le \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\|\\ &\quad + \|T_i^{n+1} x_n - T_i x_{n+1}\|\\ &\le \|x_{n+1} - T_i^{n+1} x_{n+1}\| + L_i\|x_{n+1} - x_n\| + L_i\|T_i^n x_n - x_{n+1}\|\\ &\le \|x_{n+1} - T_i^{n+1} x_{n+1}\| + L_i\|x_{n+1} - x_n\| + L_i\|T_i^n x_n - x_n\|\\ &\quad + L_i\|x_n - x_{n+1}\|\\ &= \|x_{n+1} - T_i^{n+1} x_{n+1}\| + 2L_i\|x_{n+1} - x_n\| + L_i\|T_i^n x_n - x_n\|.\end{aligned}$$

In view of Steps 3 and 5, we obtain that $x_{n+1} - T_i x_{n+1} \to 0$ as $n \to \infty,\ i = 1, 2, \cdots, N$.

Step 7 Show that $v = P_F x_0$.

Since $x_n \to v$ as $n \to \infty$, $x_n - T_i x_n \to 0$ as $n \to \infty$ and T_i is uniformly L_i-Lipschitzian, $i = 1, 2, \cdots, N$, one has $v = T_i v$, $i = 1, 2, \cdots, N$. So $v \in F$. Noting that $F \subset C_n$ and $x_n = P_{C_n} x_0$, by Lemma 2.1, one concludes that

$$\langle z - x_n, x_0 - x_n \rangle \leq 0, \ \forall z \in F.$$

It follows that

$$\langle z - v, x_0 - v \rangle \leq 0, \ \forall z \in F.$$

By Lemma 2.1, one concludes that $v = P_F x_0$. □

Next, we give another kind of iterative algorithm for a finite family of quasi-asymptotically pseudocontractive mappings.

We put $I = \{1, 2, \cdots, N\}$. For any positive integer n, we write $n = (h(n) - 1)N + i(n)$, where $h(n) \to \infty$ as $n \to \infty$, and $i(n) \in I$, for all $n \geq 1$.

Theorem 3.3 *Suppose that C is a bounded closed convex nonempty subset of H. Suppose that $\{T_i\}_{i=1}^N : C \to C$ is a finite family of uniformly L_i-Lipschitzian and quasi-asymptotically pseudocontractive mappings such that $F = \bigcap_{i=1}^N F(T_i)$ is not empty. Assume the control sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\{\alpha_n\}, \{\beta_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{1+L})$, where $L = max\{L_i : 1 \leq i \leq N\}$. Let a sequence $\{x_n\}$ be generated by the following manner:*

$$\begin{cases} x_0 \in C \ \ chosen \ arbitrarily, \\ C_1 = C, \\ y_n = (1-\alpha_n)x_n + \alpha_n T_{i(n)}^{h(n)} x_n, \\ z_n = (1-\beta_n)x_n + \beta_n T_{i(n)}^{h(n)} y_n, \\ C_{n+1} = \{w \in C_n : \alpha_n \beta_n [1 - (1+L)\alpha_n] \|x_n - T_{i(n)}^{h(n)} x_n\|^2 \\ \qquad\qquad \leq \langle x_n - w, x_n - z_n \rangle + 2(k_{h(n)} - 1)(diamC)^2\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 1, \end{cases} \tag{3.13}$$

where $k_{h(n)} = max\{k_{h(n),i(n)} : 1 \leq i(n) \leq N\}$, while $k_{h(n),i(n)}$ are asymptotic sequences for $\{T_i\}_{i=1}^N$. Then, the sequence $\{x_n\}$ generated by (3.13) strongly converges to $P_F x_0$.

Proof By Lemma 2.2 and our assumption that $F \neq \emptyset$, we know that $P_F x_0$ is well defined for every $x_0 \in C$. It follows from the constructions of C_n that C_n is closed and convex, $\forall n \geq 1$. We omit the details.

Step 1 Show that $F \subset C_n$ for all $n \geq 1$.

In fact, $F \subset C_1 = C$ is obvious. Assume that $F \subset C_n$ for some $n \geq 1$. For any $q \in F \subset C_n$, from (3.13), using the uniform L_i-Lipschitz continuity of T_i and quasi-asymptotic pseudo-contractiveness of T_i, we obtain that

$$\begin{aligned}
&\|x_n - T_{i(n)}^{h(n)} x_n\|^2 \\
&= \langle x_n - T_{i(n)}^{h(n)} x_n, x_n - T_{i(n)}^{h(n)} x_n\rangle \\
&= \frac{1}{\alpha_n}\langle x_n - y_n, x_n - T_{i(n)}^{h(n)} x_n\rangle \\
&= \frac{1}{\alpha_n}\langle x_n - y_n, (I - T_{i(n)}^{h(n)})x_n - (I - T_{i(n)}^{h(n)})y_n\rangle + \frac{1}{\alpha_n}\langle x_n - y_n, (I - T_{i(n)}^{h(n)})y_n\rangle \\
&\leq \frac{1+L}{\alpha_n}\|x_n - y_n\|^2 + \frac{1}{\alpha_n}\langle x_n - q, (I - T_{i(n)}^{h(n)})y_n\rangle + \frac{1}{\alpha_n}\langle q - y_n, (I - T_{i(n)}^{h(n)})y_n\rangle \\
&= \frac{1+L}{\alpha_n}\cdot \alpha_n^2\|x_n - T_{i(n)}^{h(n)} x_n\|^2 + \frac{1}{\alpha_n}\langle x_n - q, (I - T_{i(n)}^{h(n)})y_n\rangle \\
&\quad + \frac{1}{\alpha_n}\langle q - y_n, y_n - q + T_{i(n)}^{h(n)} q - T_{i(n)}^{h(n)} y_n\rangle \\
&\leq (1+L)\alpha_n\|x_n - T_{i(n)}^{h(n)} x_n\|^2 + \frac{1}{\alpha_n}\langle x_n - q, (I - T_{i(n)}^{h(n)})y_n\rangle + \frac{1}{\alpha_n}(k_{h(n)} - 1)(diamC)^2 \\
&= (1+L)\alpha_n\|x_n - T_{i(n)}^{h(n)} x_n\|^2 + \frac{1}{\alpha_n}\langle x_n - q, (1-\alpha_n)x_n + \alpha_n T_{i(n)}^{h(n)} x_n \\
&\quad - \frac{z_n - (1-\beta_n)x_n}{\beta_n}\rangle + \frac{1}{\alpha_n}(k_{h(n)} - 1)(diamC)^2 \\
&= (1+L)\alpha_n\|x_n - T_{i(n)}^{h(n)} x_n\|^2 + \frac{1}{\alpha_n}\langle x_n - T_{i(n)}^{h(n)} x_n + T_{i(n)}^{h(n)} x_n - q, \\
&\quad - \alpha_n(x_n - T_{i(n)}^{h(n)} x_n) + \frac{1}{\beta_n}(x_n - z_n)\rangle + \frac{1}{\alpha_n}(k_{h(n)} - 1)(diamC)^2 \\
&= (1+L)\alpha_n\|x_n - T_{i(n)}^{h(n)} x_n\|^2 - \|x_n - T_{i(n)}^{h(n)} x_n\|^2 + \frac{1}{\alpha_n\beta_n}\langle x_n - T_{i(n)}^{h(n)} x_n, x_n - z_n\rangle \\
&\quad - \langle T_{i(n)}^{h(n)} x_n - q, x_n - T_{i(n)}^{h(n)} x_n\rangle + \frac{1}{\alpha_n\beta_n}\langle T_{i(n)}^{h(n)} x_n - q, x_n - z_n\rangle \\
&\quad + \frac{1}{\alpha_n}(k_{h(n)} - 1)(diamC)^2 \\
&= [(1+L)\alpha_n - 1]\|x_n - T_{i(n)}^{h(n)} x_n\|^2 + \frac{1}{\alpha_n\beta_n}\langle x_n - q, x_n - z_n\rangle + \|x_n - T_{i(n)}^{h(n)} x_n\|^2 \\
&\quad + \langle q - x_n, x_n - T_{i(n)}^{h(n)} x_n\rangle + \frac{k_{h(n)} - 1}{\alpha_n}(diamC)^2 \\
&\leq (1+L)\alpha_n\|x_n - T_{i(n)}^{h(n)} x_n\|^2 + \frac{1}{\alpha_n\beta_n}\langle x_n - q, x_n - z_n\rangle \\
&\quad + (k_{h(n)} - 1)(diamC)^2 + \frac{k_{h(n)} - 1}{\alpha_n}(diamC)^2,
\end{aligned}$$

from which it turns out that

$$\alpha_n\beta_n[1-(1+L)\alpha_n]\|x_n - T_{i(n)}^{h(n)}x_n\|^2 \leq \langle x_n - q, x_n - z_n\rangle + 2(k_{h(n)} - 1)(diamC)^2.$$

So $q \in C_{n+1}$ and hence $F \subset C_n$ for all $n \geq 1$.

Step 2 Show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

The proof is similar to the proof of Step 2 in Theorem 3.1 and omitted here.

Step 3 Show that $x_n \to v$ as $n \to \infty$, $v \in C$.

The proof is similar to the proof of Step 3 in Theorem 3.1 and omitted here.

Step 4 Show that $x_n - T_{i(n)}^{h(n)}x_n \to 0$ as $n \to \infty$.

It follows from step 3 that $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1}$, one has

$$\begin{aligned}\alpha_n\beta_n[1-(1+L)\alpha_n]\|x_n-T_{i(n)}^{h(n)}x_n\|^2 &\leq \langle x_n-x_{n+1}, x_n-z_n\rangle+2(k_{h(n)}-1)(diamC)^2\\ &\leq \|x_n-x_{n+1}\|\cdot\|x_n-z_n\|\\ &\quad +2(k_{h(n)}-1)(diamC)^2.\end{aligned}$$

Noting that $\{\alpha_n\}$, $\{\beta_n\} \subset [a, b]$ for $a,\ b \in (0, \frac{1}{1+L})$, $\{x_n\}$ and $\{z_n\}$ are all bounded, $k_{h(n)} \to 1$ as $n \to \infty$, we have $x_n - T_{i(n)}^{h(n)}x_n \to 0$ as $n \to \infty$.

Step 5 Show that $x_n - T_{i(n)}x_n \to 0$ as $n \to \infty$.

Since $n = (h(n) - 1)N + i(n)$, we have

$$n - N = (h(n) - 1 - 1)N + i(n).$$

On the other hand, since $n - N = (h(n - N) - 1)N + i(n - N)$, we have $h(n) - 1 = h(n - N)$ and $i(n) = i(n - N)$. Noting that

$$\begin{aligned}\|x_n-T_{i(n)}x_n\| &\leq \|x_n - T_{i(n)}^{h(n)}x_n\| + \|T_{i(n)}^{h(n)}x_n - T_{i(n)}x_n\|\\ &\leq \|x_n - T_{i(n)}^{h(n)}x_n\| + L\|T_{i(n)}^{h(n)-1}x_n - x_n\|\\ &\leq \|x_n - T_{i(n)}^{h(n)}x_n\| + L\|T_{i(n)}^{h(n-N)}x_n - T_{i(n-N)}^{h(n-N)}x_{n-N}\|\\ &\quad + \|T_{i(n-N)}^{h(n-N)}x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|\\ &\leq \|x_n-T_{i(n)}^{h(n)}x_n\|+(1+L^2)\|x_{n-N}-x_n\|+\|T_{i(n-N)}^{h(n-N)}x_{n-N}-x_{n-N}\|,\end{aligned}$$

from which it turns out that $x_n - T_{i(n)}x_n \to 0$ as $n \to \infty$ in view of Steps 3 and 4.

Step 6 Show that $\forall j \in I,\ x_n - T_{i(n)+j}x_n \to 0$ as $n \to \infty$.

Observing that

$$\begin{aligned}\|x_n - T_{i(n)+j}x_n\| &\le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{i(n)+j}x_{n+j}\| \\ &\quad + \|T_{i(n)+j}x_{n+j} - T_{i(n)+j}x_n\| \\ &\le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{i(n+j)}x_{n+j}\| + L\|x_{n+j} - x_n\| \\ &= (1+L)\|x_{n+j} - x_n\| + \|x_{n+j} - T_{i(n+j)}x_{n+j}\|,\end{aligned}$$

by using Steps 3 and 5, we reach the desired conclusion.

Step 7 Show that $\forall l \in I,\ x_n - T_l x_n \to 0$ as $n \to \infty$.

Indeed, for arbitrary given $l \in I$, we can choose $j \in I$ such that $j = l - i(n)$ if $l > i(n)$ and $j = N + l - i(n)$ if $l \le i(n)$. Then, we have $l = i(n+j) = i(n) + j$, for all $n \ge 1$. In view of Step 6, we obtain $x_n - T_l x_n = x_n - T_{i(n+j)}x_n = x_n - T_{i(n)+j}x_n \to 0$ as $n \to \infty$.

Step 8 Show that $v = P_F x_0$.

Since $x_n \to v$ as $n \to \infty$, $x_n - T_l x_n \to 0$ as $n \to \infty$ and T_l is uniformly L_l-Lipschitzian, $l = 1, 2, \cdots, N$, one has $v = T_l v$, $l = 1, 2, \cdots, N$. So $v \in F$. Noting that $F \subset C_n$ and $x_n = P_{C_n}x_0$, by Lemma 2.1, one concludes that

$$\langle z - x_n,\ x_0 - x_n\rangle \le 0,\ \forall z \in F.$$

It follows that

$$\langle z - v,\ x_0 - v\rangle \le 0,\ \forall z \in F.$$

By Lemma 2.1, one concludes that $v = P_F x_0$. □

4 Rate of convergence and numerical experiments

In this section, we provide some numerical examples to show our algorithms are effective. We also compare the rate of convergence of the algorithms (3.1), (3.12), and (3.13). In order to compare two fixed point iteration schemes, Rhoades [26] introduced the following concept in 1976.

Definition 4.1 [26] If $\{x_n\}$, $\{z_n\}$ are two iteration schemes which converge to the same fixed point p, we shall say that $\{x_n\}$ is better than $\{z_n\}$ if $\|x_n - p\| \le \|z_n - p\|$ for all n.

Berinde [27] introduced the following definition, which is slightly different from definition 4.1.

Definition 4.2 [27] Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

(a) If $l = 0$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b.

(b) If $0 < l < \infty$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Phuengrattana and Suantai [28] presented numerical examples to compare the convergence speed of Mann, Ishikawa, Noor and SP-iterations by using the definition 4.1.

Motivated by the above results, we present numerical examples to compare the convergence speed of algorithms (3.1), (3.12), and (3.13) by using the definition 4.1, the definition 4.2, and other methods.

In our numerical experiments, we consider the case of $N = 2$. Taking the mapping $T_1,\ T_2 : C \to \mathbb{R}$ by

$$T_1 x = \frac{1}{9} x \cos x,\ \ T_2 x = \frac{1}{8} x \sin x,\ \ x \in C,$$

where $C = [0,\ 2\pi]$, $\mathbb{R}$ denotes the set of real numbers. One has $F(T_1) = F(T_2) = \{0\}$ and T_1 and T_2 are two uniformly Lipschitzian and quasi-asymptotically pseudocontractive mappings. It is easy to see that asymptotic sequences $k_{n,1} \equiv 1$ and $k_{n,2} \equiv 1$. Obviously, $F(T_1) \cap F(T_2) = \{0\}$. For such a family $\{T_i\}_{i=1}^{2}$, we have $L_1 = \frac{1+2\pi}{9}$ and $L_2 = \frac{1+2\pi}{8}$; therefore, $L = \frac{1+2\pi}{8}$. Iterating algorithms (3.1), (3.12), and (3.13) to 60 steps, respectively.

For algorithm (3.1), we take $\alpha_{n,1} = 0.1,\ \alpha_{n,2} = 0.1$ for all $n \geq 0$. For algorithm (3.12), we take $\alpha_n = 0.1$ for all $n \geq 0$. For algorithm (3.13), we take $\alpha_n = 0.1,\ \beta_n = 0.1$ for all $n \geq 0$. Choosing $x_0 \in [0,\ 2\pi]$ arbitrarily, then for 50 different initial values, one can see all the results are convergent in Figs. 1, 2, and 3.

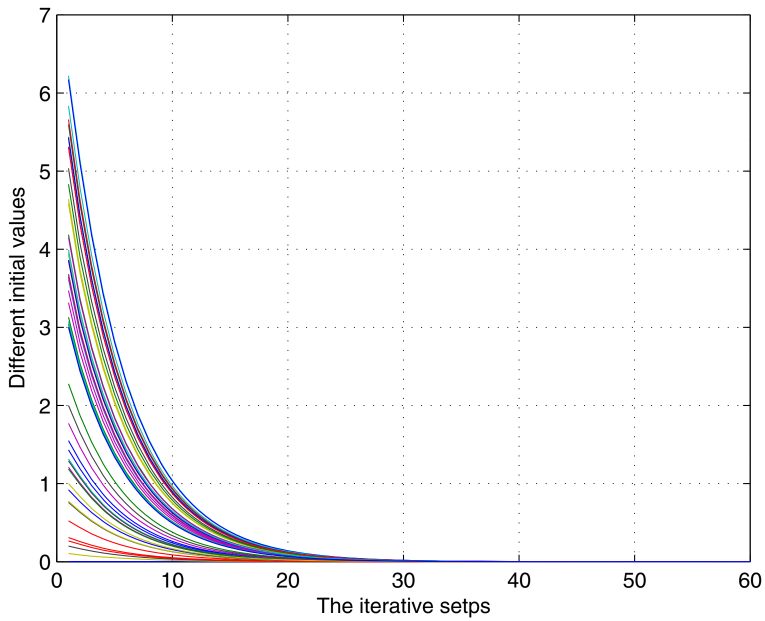

Fig. 1 The iterative curves of algorithm (3.1) under different initial value

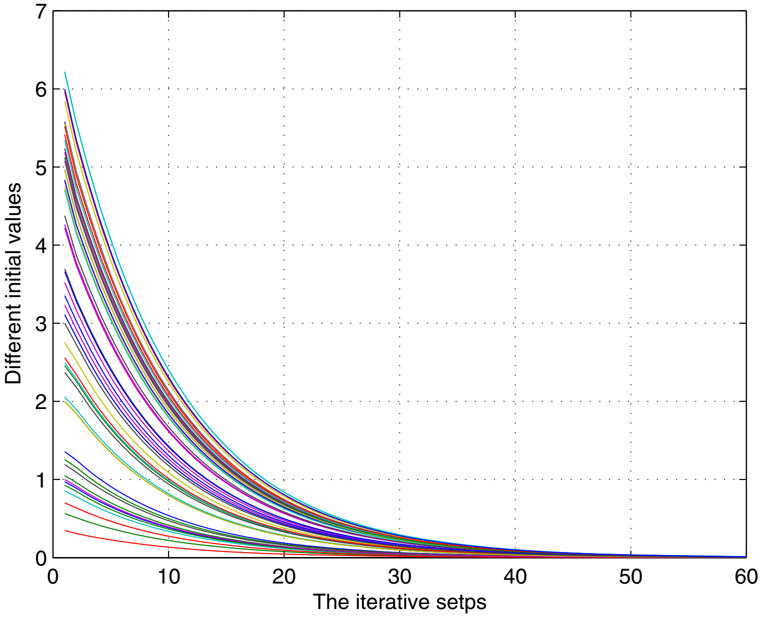

Fig. 2 The iterative curves of algorithm (3.12) under different initial value

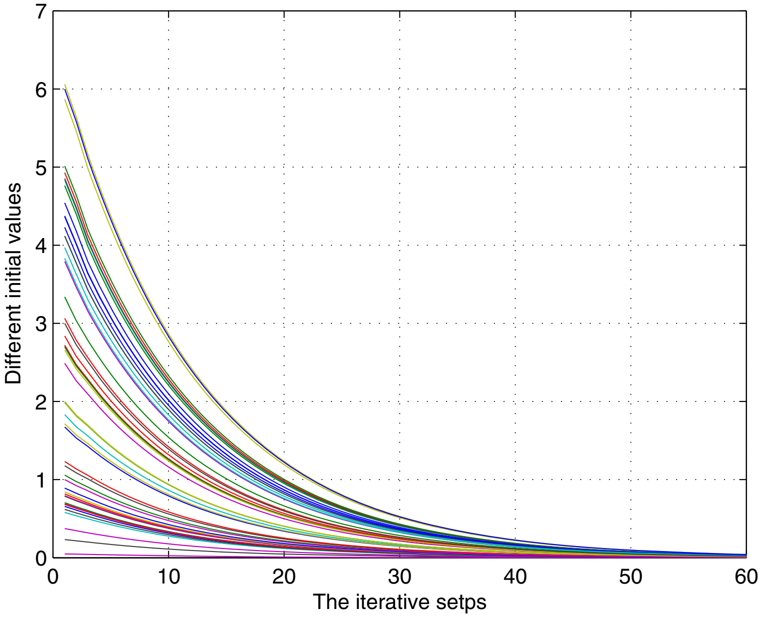

Fig. 3 The iterative curves of algorithm (3.13) under different initial value

Next, we compare algorithms (3.1), (3.12), and (3.13) for the above given examples. In the numerical results listed in the following tables, $Iter.$ and $Sec.$ denote the number of iterations and the running time in seconds, respectively. We chose $x_0 \in [0, 2\pi]$ as initial points. We took $\|x_n - 0\| \leq \varepsilon_1$ as the stopping criterion and $\varepsilon_1 = 10^{-6}$ in Table 1. We took $\|x_n - 0\| \leq \varepsilon_2$ as the stopping criterion and $\varepsilon_2 = 10^{-10}$ in Table 2. We took $\|x_n - 0\| \leq \varepsilon_3$ as the stopping criterion and $\varepsilon_3 = 10^{-14}$ in Table 3. The algorithms were coded in Matlab 2013 and run on a personal laptop. For different stopping criteria, the numerical results of the above algorithms (3.1), (3.12), and (3.13) with the initial points 1, 2, 3, and 6 are shown in Tables 1, 2, and 3, respectively. From Tables 1, 2, and 3, we observe that algorithm (3.1) is the best, from the points of view of number of iterations and running time.

In Tables 4 and 5, let $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence given by (3.1), $\{y_n\}_{n=0}^{\infty}$ be the iterative sequence given by (3.12) and $\{z_n\}_{n=0}^{\infty}$ be the iterative sequence given by (3.13). The comparison of the convergences of algorithms (3.1), (3.12), and (3.13) to the common fixed point $p = 0$ is given in Tables 4 and 5, with the initial point $x_0 = y_0 = z_0 = 3$.

From Table 4, we see that the algorithm (3.1) converges faster than the algorithms (3.12) and (3.13) by using the definition 4.1. From Table 5, we see that the algorithm (3.1) converges faster than the algorithm (3.12) and the algorithm (3.12) converges faster than the algorithm (3.13) by using the definition 4.2.

Let $\mathbb{R}^2$ be a two-dimensional Euclidean space with the usual inner product $\langle v^{(1)}, v^{(2)}\rangle = v_1^{(1)}v_1^{(2)} + v_2^{(1)}v_2^{(2)}$ for all $v^{(1)} = (v_1^{(1)}, v_2^{(1)})^T, v^{(2)} = (v_1^{(2)}, v_2^{(2)})^T \in \mathbb{R}^2$, and the norm $\|v\| = \sqrt{v_1^2 + v_2^2}(v = (v_1, v_2)^T \in \mathbb{R}^2)$. He et al. [24] defined a mapping:

$$T_1 : v = (v_1, v_2)^T \mapsto (\sin\frac{v_1 + v_2}{\sqrt{2}}, \cos\frac{v_1 + v_2}{\sqrt{2}})^T$$

and showed that T_1 is nonexpansive. So T_1 is a quasi-asymptotically pseudocontractive mapping. It is easily to observe that T_1 has a fixed point in the unit disk. Define a mapping $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$T_2(v) = P_K v$$

for all $v \in \mathbb{R}^2$, where $K = \{v \in \mathbb{R}^2 : \|v\| \leq 1\}$. It is well known that T_2 is nonexpansive. So T_2 is a quasi-asymptotically pseudocontractive mapping. And $F(T_2) = K$. Thus, we get $F(T_1) \cap F(T_2) = F(T_1) \neq \varnothing$.

Table 1 The numerical results of the three algorithms with $\varepsilon_1 = 10^{-6}$

x_0	Algorithm (3.1)		Algorithm (3.12)		Algorithm (3.13)	
	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
1	71	2.5189	133	17.567	165	10.7554
2	74	6.0655	139	16.022	174	19.747
3	76	8.6421	143	11.618	178	20.586
6	80	10.009	150	24.682	187	20.899

Table 2 The numerical results of the three algorithms with $\varepsilon_1 = 10^{-10}$

x_0	Algorithm (3.1)		Algorithm (3.12)		Algorithm (3.13)	
	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
1	117	11.123	220	31.779	275	37.463
2	120	4.6452	227	35.416	283	42.184
3	122	9.3935	231	44.532	287	38.304
6	126	11.187	237	39.732	296	45.05

Table 3 The numerical results of the three algorithms with $\varepsilon_1 = 10^{-14}$

x_0	Algorithm (3.1)		Algorithm (3.12)		Algorithm (3.13)	
	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
1	163	17.82	308	59.529	384	72.174
2	167	18.511	314	55.687	392	84.653
3	169	23.202	318	59.369	397	85.529
6	172	22.145	325	52.545	405	93.524

Table 4 Comparison of rate of convergence of the three algorithms by the def. 4.1

n	Algorithm (3.1) x_n	Algorithm (3.12) y_n	Algorithm (3.13) z_n
50	0.00013931	0.017295	0.047589
⋮	⋮	⋮	⋮
118	1.7819e-10	1.3504e-05	0.00015352
119	1.4595e-10	1.2155e-05	0.0001411
120	1.1955e-10	1.0941e-05	0.00012969

Table 5 Comparison of rate of convergence of the three algorithms by the def. 4.2

n	$\frac{\lvert x_n - 0\rvert}{\lvert y_n - 0\rvert}$	$\frac{\lvert x_n - 0\rvert}{\lvert z_n - 0\rvert}$	$\frac{\lvert y_n - 0\rvert}{\lvert z_n - 0\rvert}$
10	0.35047	0.29343	0.83725
30	0.053129	0.029307	0.55161
50	0.0080549	0.0029273	0.36342
100	0.000071979	9.2169e-06	0.12805
120	0.000010053	8.4817e-07	0.084362

Table 6 Comparison of Algorithm (3.1) with Algorithm (3.12) and (3.13)

x_0	Algorithm (3.1)		Algorithm (3.12)		Algorithm (3.13)	
	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
(0,0)	241	15.628	293	16.730	613	62.524
(2,7)	165	31.346	925	78.258	795	76.139
(-5,2)	535	46.857	1527	108.54	2308	360.45
(-3,-4)	514	62.732	1359	93.849	1037	125.21

Denote by $E(x) = \frac{\|x - T_1x\| + \|x - T_2x\|}{\|x\|}$ the relative rate of convergence of the algorithms since we do not know the exact value of the projection of x_0 onto common fixed points set of T_1 and T_2. We compare algorithms (3.1), (3.12), and (3.13). We took $\alpha_{n,1} = 0.1$, $\alpha_{n,2} = 0.1$ in Algorithm (3.1), $\alpha_n = 0.1$ in Algorithm (3.12) and $\alpha_n = 0.1$, $\beta_n = 0.1$ in Algorithm (3.13). We took $E(x) < \varepsilon$ as the stopping criterion and $\varepsilon = 10^{-4}$ unless specified otherwise. We chose different x_0 as initial point. The numerical results are shown in Table 6.

5 Conclusions

Three kinds of hybrid algorithms for a finite family of quasi-asymptotically pseudo-contractive mappings were proposed in this paper. Their strong convergences have been proven in Hilbert spaces. We also have given numerical examples to compare and explain the effectiveness of the introduced algorithms. The results given in this paper extend the well-known ones existing in the literature.

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