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ORIGINAL PAPER

A high-order numerical method for solving the 2D fourth-order reaction-diffusion equation

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Abstract In the present work, orthogonal spline collocation (OSC) method with convergence order $O(\tau^{3-\alpha}+h^{r+1})$ is proposed for the two-dimensional (2D) fourth-order fractional reaction-diffusion equation, where τ, h, r, and α are the time-step size, space size, polynomial degree of space, and the order of the time-fractional derivative ($0<\alpha<1$), respectively. The method is based on applying a high-order finite difference method (FDM) to approximate the time Caputo fractional derivative and employing OSC method to approximate the spatial fourth-order derivative. Using the argument developed recently by Lv and Xu (SIAM J. Sci. Comput. **38**, A2699–A2724, 2016) and mathematical induction method, the optimal error estimates of proposed fully discrete OSC method are proved in detail. Then, the theoretical analysis is validated by a number of numerical experiments. To the best of our knowledge, this is the first proof on the error estimates of high-order numerical method with convergence order $O(\tau^{3-\alpha}+h^{r+1})$ for the 2D fourth-order fractional equation.

Keywords Fourth-order fractional equation · Orthogonal spline collocation · Finite difference method · Error estimate

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1 Introduction

In this paper, we will consider numerical methods for solving following 2D fourth-order fractional reaction-diffusion equation:

$$\partial_t^\alpha u + \Delta^2 u - \kappa \Delta u = f(x, y, t), \quad (x, y) \in \Omega,\ t \in (0, T], \tag{1.1}$$

subject to

$$u(x, y, 0) = u_0(x, y), \ \ (x, y) \in \Omega, \tag{1.2}$$

$$u(x, y, t) = \Delta u(x, y, t) = 0, \quad (x, y) \in \partial\Omega,\ t \in (0, T], \tag{1.3}$$

where $\kappa \geq 0$ is a constant, $\Omega = [0, L] \times [0, L]$, Δ is the Laplace operator, f and u_0 are given functions assumed to be sufficiently regular and $\partial_t^\alpha u$ is Caputo fractional derivative defined by

$$\partial_t^\alpha u(x, y, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, y, s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1. \tag{1.4}$$

To solve (1.1)–(1.3) numerically, one needs to approximate the time fractional order derivative (1.4). One main approximation formula is L1 scheme, in which the truncation error is $O(\tau^{2-\alpha})$. We refer the reader to [1, 2] and the references there in. Recently, Gao and Sun et al. [3] proposed a formula to approximate Caputo fractional derivative with convergence order $O(\tau^{3-\alpha})$, which is called L1-2 formula and applied this formula for solving a time fractional diffusion equation. But there are no error estimates in [3]. Based on the idea of [3], Alikhanov [4] constructed a new formula (called L2-1_σ formula) to approximate the Caputo fractional derivative with the approximation order $O(\tau^{3-\alpha})$. Li et al. [5–7] derived a $(3-\alpha)$-th order numerical scheme to Caputo derivatives and applied it to the Caputo type advection-diffusion equations. However, the error estimates and stability analysis in [5] are provided only for $\alpha \in (0, \alpha_0)$ with some positive $\alpha_0 \in (0, 1)$. In very recent, Lv and Xu [8] proposed a higher order numerical method which is slightly different the method in Gao and Sun et al. [3] for solving time fractional diffusion equation and proved that the scheme has the convergence order $O(\tau^{3-\alpha})$ for all $\alpha \in (0, 1)$. Li and Yan et al. [9–11] use a new numerical method to approximate the Hadamard finite-part integral with the approximation order $O(\tau^{3-\alpha})$ and applied it to time fractional diffusion equation. Dehghan et al. [12–14] proposed some interesting and efficient numerical method for fractional reaction-diffusion. In very recent, Dehghan and Abbaszadeh [15, 16] constructed the element-free Galerkin (EFG) meshless method and moving Kriging collocation meshless technique for solving 2D time fractional partial differential equations with the approximation order $O(\tau^{3-\alpha})$ in time, and proved the unconditional stability and obtained an error bound for the EFG method using the energy method.

However, it is especially worth mentioning that the work above we list all obtain high-order ($3-\alpha$-order) temporal accuracy, which are merely concentrated on fractional partial differential equations with second-order space derivative. For some applications, a fourth-order space derivative term is needed to describe special phenomena such as wave propagation in beams and modeling formation of grooves on a flat surface and the propagation of intense laser beams in a bulk medium with Kerr nonlinearity [17, 18]. Recently, Ji and Sun et al. [19], Hu and Zhang [20, 21], Guo and Li et al. [22], and Vong and Wang [23] proposed the compact FDM for 1D fourth-order fractional sub-diffusion or diffusion-wave equations with the temporal accuracy of order less than or equal to two. More recently, Zhang and Pu [24] proposed the compact FDM for 1D fourth-order fractional sub-diffusion equations with the temporal accuracy of order equal to two. Wei and He [25] presented a fully discrete local discontinuous Galerkin method for 1D time-fractional fourth-order equation with the convergence order $O(\tau^{2-\alpha}+\tau^{-\alpha}h^{r+1}+\tau^{\frac{-\alpha}{2}}h^{r+\frac{1}{2}}+h^{r+1})$, where τ, h and r were the time step, space step and degree of approximate solution, respectively. Liu and Fang et al. [26] solved the 2D fourth-order fractional reaction-diffusion equation (1.1)-(1.3) by applying L1 method in time and mixed finite element method (FEM) in space with the convergence order $O(\tau^{2-\alpha}+\tau^{-\alpha}h^{r+1}+h^{r+1})$. However, the undesirable factors $\tau^{-\alpha}$ in the space error term grow with decreasing time step, which is not the optimal. Subsequently, Liu and Du et al. [27, 28] solved a nonlinear 2D time-dependent fourth-order fractional reaction-diffusion equation by same idea. Siddiqi and Arshed [29] dealt with time-fractional fourth-order PDE by using L1 scheme for time direction and a quintic B-spline collocation technique for space approximation.

Be that as it may, we find that there are few reports on the high-order numerical methods for the 2D fourth-order fractional reaction-diffusion equation with convergence order $O(\tau^{3-\alpha}+h^{r+1})$. In this paper, we will consider the high-order numerical method for solving the 2D fourth-order fractional reaction-diffusion (1.1)–(1.3). The time discretization is based on high-order FDM in [3, 8, 30] and the space discretization is based on the OSC method. The optimal error estimates with convergence order $O(\tau^{3-\alpha}+h^{r+1})$ are proved in detail. In particular, we prove the error estimate for the first step solution so that we attain the desired $3-\alpha$ order global error estimate in time. We further improve the error estimates in [26] and removing the factor $\tau^{-\alpha}$ in the space error term, which grows with decreasing time step. The reason for considering OSC method is that this method is much superior to B-splines (see [31]) in terms of stability, efficiency, and conditioning of the resulting matrix. Compared to FEM, the calculation of the coefficients of the mass and stiffness matrices determining the approximate solution is very fast since no integrals need to be evaluated or approximated [31]. In comparison with FDM, OSC methods show continuous approximations to the solution and its spatial derivatives at all points of the domain of the problem and allow for arbitrarily high-order accuracy in the spatial approximation [31]. Moreover, OSC scheme always lead to the almost block diagonal linear systems, which can be implemented efficiently using the software package COLROW [32]. For elliptic problems, OSC is based on replacing the exact solution by its piecewise polynomial approximation and on satisfying the partial differential equation at the Gauss points (see, e.g., [33] and references therein).

The rest of the paper is organized as follows. In Section 2, we introduce some notations and preliminary lemmas. In Section 3, the fully discrete scheme based on high-order FDM in time direction and OSC method in space is constructed by introducing the auxiliary variable $v = \Delta u$. In Section 4, we derive the full discrete error estimates. Some numerical examples are given in Section 5 and some conclusions are drawn in Section 6.

Throughout this paper, C will denote a generic positive constant which is independent of the time step and space step, but possibly with different values at different places.

2 Preliminaries

In this section, we first introduce some notations. For a positive integer N_x and N_y, a uniform partition of $\overline{I} = [0, 1]$ is defined as follows:

$$\delta_x : 0 = x_0 < x_1 < ... < x_{N_x} = 1, \qquad \delta_y : 0 = y_0 < y_1 < ... < y_{N_y} = 1.$$

Let $\delta = \delta_x \times \delta_y$ of Ω be quasi-uniform [34], $h_k^x = x_k - x_{k-1}$, $h_l^y = y_l - y_{l-1}$ and $h = \max(\max\limits_{1\le k\le N_x} h_k^x, \max\limits_{1\le l\le N_y} h_l^y)$.

Set $\mathscr{M}_r(\delta)$ be the space of piecewise polynomials of degree at most r in the x- and y- directions, with $r \geq 3$, defined by

$$\mathscr{M}_r(\delta) = \mathscr{M}(r, \delta_x) \otimes \mathscr{M}(r, \delta_y),$$

where $\mathscr{M}(r, \delta_x) = \left\{ v | v \in C^1(\bar{I}), v|_{\overline{I}_k^x} \in P_r, k = 1, 2, ..., N_x, v(0) = v(1) = 0 \right\}$, and P_r denotes the set of polynomials of degree at most r. With $\mathscr{M}(r, \delta_y)$ defined similarly.

Define Gauss collocation points set in Ω: $\Lambda_r = \{\xi | \xi = (\xi^x, \xi^y), \xi^x \in \Lambda_x, \xi^y \in \Lambda_y\}$, where $\Lambda_x = \{\xi_{i,k}^x\}_{i,k=1}^{N_x, r-1}$, $\xi_{i,k}^x = x_{i-1} + \lambda_k h_i^x$, and $\{\lambda_k\}_{k=1}^{r-1}$ be the nodes of the $(r-1)$-point Gauss quadrature rule on $\overline{I}$. With Λ_y defined similarly.

For any function U and V defined on Λ_r, the discrete inner product and norm are defined as follows

$$\langle U, V \rangle = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h_i^x h_j^y \sum_{k=1}^{r-1} \sum_{l=1}^{r-1} \omega_k \omega_l (UV)(\xi_{i,k}^x, \xi_{j,l}^y), \qquad \|V\|_{\mathscr{M}_r}^2 = \langle V, V \rangle. \quad (2.5)$$

Lemma 1 [31] *The norms $\|\cdot\|_{\mathscr{M}_r}$ and $\|\cdot\|$ is equivalent on $\mathscr{M}_r(\delta)$, where $\|\cdot\|$ is the usual L^2 norm.*

If X is a normed space with norm $\|\cdot\|_X$, then we denote $L^P(X)$ by

$$L^P(X) = \{v : v(\cdot, t) \in X, t \in [0, T]; \|v\|_{L^P(X)} < \infty\},$$

where

$$\|v\|_{L^P(X)} = \left(\int_0^T \|v\|_X^P dt \right)^{1/p}, \quad \|v\|_{L^\infty(X)} = \sup_{0\le t\le T} \|v\|_X.$$

In the error analysis, we shall use the intermediate projections as differentiable maps $\{\widehat{U}, \widehat{V}\}$: $[0, T] \to \mathscr{M}_r(\delta) \times \mathscr{M}_r(\delta)$ satisfying

$$\begin{cases} \langle \Delta \left(u - \widehat{U}\right), \chi_h \rangle = 0 \ \text{on} \ \Lambda_r \times [0, T], \ \ \forall \chi_h \in \mathscr{M}_r(\delta); \\ \langle \Delta \left(v - \widehat{V}\right), \psi_h \rangle = 0 \ \text{on} \ \Lambda_r \times [0, T], \ \ \forall \psi_h \in \mathscr{M}_r(\delta), \end{cases} \tag{2.6}$$

The next estimates are well known [34, 35].

Lemma 2 *If* $\frac{\partial^\ell u}{\partial t^\ell}, \frac{\partial^\ell v}{\partial t^\ell} \in L^p(H^{r+3-j})$, *for* $t \in [0, T]$, $\ell, j = 0, 1, 2$, $p = 2, \infty$, *then there exists a constant C, independent of h, such that*

$$\left\| \frac{\partial^\ell (u - \widehat{U})}{\partial t^\ell} \right\|_{H^j} \le Ch^{r+1-j} \left\| \frac{\partial^\ell u}{\partial t^\ell} \right\|_{H^{r+3-j}}, \quad \ell = 0, 1, 2, \quad j = 0, 1, 2,$$

$$\left\| \frac{\partial^\ell (v - \widehat{V})}{\partial t^\ell} \right\|_{H^j} \le Ch^{r+1-j} \left\| \frac{\partial^\ell v}{\partial t^\ell} \right\|_{H^{r+3-j}}, \quad \ell = 0, 1, 2, \quad j = 0, 1, 2.$$

Lemma 3 *If* $\frac{\partial^i u}{\partial t^i}, \frac{\partial^j v}{\partial t^j} \in L^p(H^{r+3})$, *for* $t \in [0, T]$, $i, j = 0, 1, 2$, *then there exists a constant C, independent of h, such that*

$$\left\| \frac{\partial^{\ell+i} (u - \widehat{U})}{\partial x^{\ell_1} \partial y^{\ell_2} \partial t^i} \right\|_{\mathscr{M}_r} \le Ch^{r+1-\ell} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{H^{r+3}},$$

$$\left\| \frac{\partial^{\ell+j} (v - \widehat{V})}{\partial x^{\ell_1} \partial y^{\ell_2} \partial t^j} \right\|_{\mathscr{M}_r} \le Ch^{r+1-\ell} \left\| \frac{\partial^j v}{\partial t^j} \right\|_{H^{r+3}},$$

where $0 \le \ell = \ell_1 + \ell_2 \le 4$.

3 The fully discrete OSC scheme

Let $v(x, y, t) = -\Delta u(x, y, t)$, then (1.1)–(1.3) can be split into the following coupled system:

$$\partial_t^\alpha u - \kappa \Delta u - \Delta v = f(x, y, t), \ \ (x, y, t) \in \Omega \times (0, T], \tag{3.7}$$

$$v(x, y, t) = -\Delta u(x, y, t), \ \ (x, y, t) \in \Omega \times (0, T], \tag{3.8}$$

with the initial condition

$$u(x, y, 0) = u_0(x, y), \ \ (x, y) \in \Omega, \tag{3.9}$$

and boundary conditions

$$u(x, y, t) = v(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T]. \tag{3.10}$$

Then, the corresponding continuous-time OSC approximation to (3.7)–(3.8) are: Find a pair of differentiable maps $\{u_h, v_h\}$: $(0, T] \to \mathscr{M}_r(\delta) \times \mathscr{M}_r(\delta)$, such that, for $\forall t \in (0, T]$, $1 \le k, l \le r-1$, $1 \le i \le N_x$, $1 \le j \le N_y$,

$$[\partial_t^\alpha u_h - \kappa \Delta u_h - \Delta v_h](\xi_{i,k}^x, \xi_{j,l}^y, t) = f(\xi_{i,k}^x, \xi_{j,l}^y, t), \ \ (\xi_{i,k}^x, \xi_{j,l}^y) \in \Lambda_r, \tag{3.11}$$

and

$$v_h(\xi^x_{i,k},\xi^y_{j,l},t) = -\Delta u_h(\xi^x_{i,k},\xi^y_{j,l},t),\quad (\xi^x_{i,k},\xi^y_{j,l}) \in \Lambda_r, \tag{3.12}$$

with the appropriate initial approximation to $u_h(\cdot,0) = u_0(\cdot,0)$.

To derive the time discretization for (3.11)–(3.12), we first introduce the new fractional numerical differential formula proposed in [3, 8, 30]. Let $0 < t_0 < t_1 < ... < t_N = T$ be a given partition of the time interval, then we have the time step $\tau = T/K$ and $t_k = k\tau$, $k = 0, 1, \cdots, K$. Defining

$$\begin{aligned}
a_j &= -\frac{3}{2}(2-\alpha)(j+1)^{1-\alpha} + \frac{1}{2}(2-\alpha)j^{1-\alpha} + (j+1)^{2-\alpha} - j^{2-\alpha},\\
b_j &= 2(2-\alpha)(j+1)^{1-\alpha} - 2(j+1)^{2-\alpha} + 2j^{2-\alpha},\\
c_j &= -(2-\alpha)((j+1)^{1-\alpha} + j^{1-\alpha})/2 + (j+1)^{2-\alpha} - j^{2-\alpha},
\end{aligned}$$

$$\alpha_0 = \Gamma(3-\alpha)\tau^\alpha,\quad \widetilde{\alpha}_0 = \Gamma(2-\alpha)\tau^\alpha,\quad \beta_0 = c_1 + 2 - \alpha/2. \tag{3.13}$$

For a smooth function u on $[0, T]$, we denote

$$\begin{aligned}
&u^n(\cdot) = u(\cdot, t_n),\quad 0 \le n \le K,\\
&L^\alpha_t u^n = \begin{cases} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(u^1 - u^0), & n = 1;\\ \frac{(2+\alpha)\tau^{-\alpha}}{\Gamma(3-\alpha)2^\alpha}(u^n - \sum\limits_{k=1}^{n} d^n_{n-k}u^{n-k}), & 2 \le n \le K, \end{cases}
\end{aligned} \tag{3.14}$$

where

$$\text{for}\quad n = 2,\quad d^n_{n-k} = \begin{cases} -(b_1 - 2)\beta_0^{-1}, & k = 1;\\ (-a_1 - \frac{\alpha}{2})\beta_0^{-1}, & k = 2, \end{cases}$$

$$\text{for}\quad n = 3,\quad d^n_{n-k} = \begin{cases} -(b_1 + c_2 - 2)\beta_0^{-1}, & k = 1;\\ (-a_1 - b_2 - \frac{\alpha}{2})\beta_0^{-1}, & k = 2;\\ -a_2\beta_0^{-1}, & k = 3, \end{cases}$$

$$\text{for}\quad n \ge 4,\quad d^n_{n-k} = \begin{cases} -(b_1 + c_2 - 2)\beta_0^{-1}, & k = 1;\\ (-a_1 - b_2 - c_3 - \frac{\alpha}{2})\beta_0^{-1}, & k = 2;\\ (-a_{k-1} - b_k - c_{k+1})\beta_0^{-1}, & k = 3, 4, \cdots, n-2;\\ -(a_{n-2} - b_{n-1})\beta_0^{-1}, & k = n-1;\\ -a_{n-1}\beta_0^{-1}, & k = n. \end{cases}$$

Using the Lemma 2.1 in Lv and Xu [8], we easily obtain

$$|R^1_\tau| = |\partial^\alpha_t u(t_1) - L^\alpha_t u^1| < c^\alpha_0 \max_{0\le t\le T}|\partial^2_t u(x,y,t)|\tau^{2-\alpha},\quad \forall (x,y) \in \Omega, \tag{3.15}$$

$$|R^n_\tau| = |\partial^\alpha_t u(t_n) - L^\alpha_t u^n| < c^\alpha_1 \max_{0\le t\le T}|\partial^3_t u(x,y,t)|\tau^{3-\alpha},\ 2 \le n \le K,\ \forall (x,y) \in \Omega, \tag{3.16}$$

where c^α_0 and c^α_1 depend only on α.

Lv and Xu [8] have proved the coefficients d^n_{n-2}, $n \ge 4$, may change sign from one coefficient to another. Therefore, the formula (3.14) is bad for stability and convergence analysis. Using the argument developed recently in [8], we now rearrange formula (3.14) by using a new technique.

Firstly, by a direct calculation, we have

$$\frac{(2+\alpha)\tau^{-\alpha}}{\Gamma(3-\alpha)2^{\alpha}} = \alpha_0^{-1}\beta_0.$$

Now, we introduce a parameter $p = \frac{1}{2}d_{n-1}^n$, then through a recombination of the terms in the formula (3.14), for $2 \le n \le K$, we have

$$\begin{aligned}
\alpha_0\beta_0^{-1}L_t^{\alpha}u^n &= u^n - \sum_{k=1}^{n} d_{n-k}^n u^{n-k} \qquad (3.17)\\
&= u^n - pu^{n-1} - p(u^{n-1} - pu^{n-2}) - (p^2 + d_{n-2}^n)u^{n-2} - d_{n-3}^n u^{n-3}\\
&\quad - \cdots - d_1^n u^1 - d_0^n u^0\\
&= u^n - pu^{n-1} - p(u^{n-1} - pu^{n-2}) - (p^2 + d_{n-2}^n)u^{n-2}\\
&\quad -(p^3 + d_{n-2}^n p + d_{n-3}^n)u^{n-3} - d_{n-4}^n u^{n-4} - \cdots - d_1^n u^1 - d_0^n u^0\\
&= u^n - pu^{n-1} - p(u^{n-1} - pu^{n-2}) - (p^2 + d_{n-2}^n)(u^{n-2} - pu^{n-3})\\
&\quad -(p^3 + d_{n-2}^n p + d_{n-3}^n)(u^{n-3} - pu^{n-4})\\
&\quad - \cdots\\
&\quad -(p^{n-2} + d_{n-2}^n p^{n-4} + \cdots + d_3^n p + d_2^n)(u^2 - pu^1)\\
&\quad -(p^{n-1} + d_{n-2}^n p^{n-3} + \cdots + d_2^n p + d_1^n)(u^1 - pu^0)\\
&\quad -(p^n + d_{n-2}^n p^{n-2} + \cdots + d_1^n p + d_0^n)u^0.
\end{aligned}$$

We further denote

$$\begin{aligned}
\overline{d}_{n-k}^n &= p^k + \sum_{i=2}^{k} p^{k-i} d_{n-i}^n, \quad 2 \le k \le n, \quad 2 \le n \le K,\\
\overline{u}^k &= u^k - pu^{k-1}, \quad 1 \le k \le K.
\end{aligned}$$

Then, (3.17) can be rewritten as

$$\alpha_0\beta_0^{-1}L_t^{\alpha}u^n = \overline{u}^n - p\overline{u}^{n-1} - \sum_{k=2}^{n-1}\overline{d}_{n-k}^n\overline{u}^{n-k} - \overline{d}_0^n u^0, \quad 2 \le n \le K. \qquad (3.18)$$

Now, the new coefficients $\overline{d}_{n-k}^n$ have some more beautiful properties than d_{n-k}^n.

Lemma 4 [8] *For* $0 < \alpha < 1$, *we have*

(1) $0 < p < 2/3, \quad \beta_0 > 0, \quad 0 < \alpha_0\beta_0^{-1} < \widetilde{\alpha}_0$;

(2) $\overline{d}_{n-k}^n > 0, \quad 2 \le k \le n, \quad 2 \le n \le K$;

(3) $p + \sum_{k=2}^{n-1}\overline{d}_{n-k}^n + \overline{d}_0^n \le 1, \quad 2 \le n \le K$;

(4) $\frac{1}{\overline{d}_0^n} < \frac{n^{\alpha}}{(2-\alpha)(1-\alpha)}, \quad 2 \le n \le K.$

Therefore, using (3.18), (3.14) is equivalent to

$$L_t^\alpha u^n = \begin{cases} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(u^1 - u^0), & n = 1; \\ \alpha_0^{-1}\beta_0(\overline{u}^n - p\overline{u}^{n-1} - \sum_{k=2}^{n-1} \overline{d}_{n-k}^n \overline{u}^{n-k} - \overline{d}_0^n u^0), & 2 \le n \le K, \end{cases} \tag{3.19}$$

The reformulations (3.14), (3.18), and (3.19) motivate the following time scheme:

$$\begin{cases} L_t^\alpha u^n - \kappa \Delta u^n - \Delta v^n = f^n, \\ v^n = -\Delta u^n. \end{cases} \tag{3.20}$$

Remark 1 The temporal semi-discretized problem (3.20) is unconditionally stable. Moreover, the following estimate holds

$$\begin{aligned} &\|u^n\| + \sqrt{\kappa\alpha_0\beta_0^{-1}}\|\nabla u^n\| + \sqrt{\alpha_0\beta_0^{-1}}\|\Delta u^n\| \\ &\le 4\sqrt{2}\left(\|u^0\| + \frac{T^\alpha \Gamma(2-\alpha)}{(2-\alpha)(1-\alpha)} \max_{0\le j\le n} \|f^j\|\right). \end{aligned} \tag{3.21}$$

The proof details of Remark 1 is provided in Appendix A.

Further, using (3.11)–(3.12) and (3.19), we can construct the following fully discrete OSC method: Find $\{U_h^n, V_h^n\} \in \mathscr{M}_r(\delta) \times \mathscr{M}_r(\delta)$, for $2 \le n \le K$, such that

$$\begin{cases} \overline{U}_h^n - \kappa\alpha_0\beta_0^{-1}\Delta U_h^n - \alpha_0\beta_0^{-1}\Delta V_h^n = p\overline{U}_h^{n-1} + \sum_{k=2}^{n-1} \overline{d}_{n-k}^n \overline{U}_h^{n-k} + \overline{d}_0^n U_h^0 + \alpha_0\beta_0^{-1} f_h^n, \\ V_h^n = -\Delta U_h^n, \quad 2 \le n \le K, \end{cases} \tag{3.22}$$

when $n = 1$

$$\begin{cases} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(U_h^1 - U_h^0) - \kappa\Delta U_h^1 - \Delta V_h^1 = f_h^1, \\ V_h^1 = -\Delta U_h^1. \end{cases} \tag{3.23}$$

Obviously, if $\{U_h^j\}_{j=0}^{n-1}$ are given for linear problem, the existence and uniqueness of the solution U_h^n to (3.22)–(3.23) can be guaranteed by the Lax-Milgram lemma.

We note from (3.15) that the first step solution U_h^1 is less than $(3-\alpha)$-order accuracy. In order to obtain a global required $(3-\alpha)$-order accurate scheme, U_h^1 will have to be computed with the same accuracy. We use the sub-stepping scheme similar to those in Lv and Xu [8] within the interval $(0, t_1)$. Let n_1 be some positive integer, we divide the interval $[0, t_1]$ by the equispaced nodes $0 < t_{(0),1} < t_{(1),1} < \ldots < t_{(n_1),1} = t_1$ with step size $\widetilde{\tau}$ such that $\widetilde{\tau}^{2-\alpha} \approx \tau^{3-\alpha}$. Then, we construct the first step fully-discrete OSC method: Find $\{U_h^{(m),1}, V_h^{(m),1}\} \in \mathscr{M}_r(\delta) \times \mathscr{M}_r(\delta)$, for $m = 1, 2, \cdots, n_1$, such that

$$\begin{cases} \frac{\widetilde{\tau}^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{m-1} \widetilde{b}_k (U_h^{(m-k),1} - U_h^{(m-k-1),1}) - \kappa\Delta U_h^{(m),1} - \Delta V_h^{(m),1} = f_h^{(m),1}, \\ V_h^{(m),1} = -\Delta U_h^{(m),1}, \end{cases} \tag{3.24}$$

with $\widetilde{b}_k = (k+1)^{1-\alpha} - k^{1-\alpha}, \ U_h^{(0),1} = \widehat{U}^0, \ U_h^1 = U_h^{(n_1),1}$.

4 Error estimates of the fully discrete OSC scheme

In this section, we state the error estimates for the fully discrete OSC scheme (3.22) and (3.24). Let $\eta = u - \widehat{U}$ and $\rho = v - \widehat{V}$. For our error analysis, we first split the error $E_u = u - U_h$ and $E_v = v - V_h$, respectively, as

$$\begin{cases} E_u^n = u(t_n) - U_h^n = (u(t_n) - \widehat{U}^n) - (U_h^n - \widehat{U}^n) = \eta^n - \xi^n, \ 1 \le n \le K; \\ E_v^n = v(t_n) - V_h^n = (v(t_n) - \widehat{V}^n) - (V_h^n - \widehat{V}^n) = \rho^n - \theta^n, \ 1 \le n \le K. \end{cases} \tag{4.25}$$

The next Theorem gives the error estimates of the fully discrete scheme.

Theorem 1 *Let u be the exact solution of (1.1). If U_h^1 is the approximate solution of the problem (3.24) with $U_h^{(0),1} = \widehat{U}^0$ and $U_h^1 = U_h^{(n_1),1}$. Suppose U_h^n, $n = 2, \cdots, K$ be the approximate solution of the problem (3.22) with $U_h^0 = \widehat{U}^0$. Assume $\partial_t^3 u \in L^\infty((0,T], H^{r+3})$, $r \ge 3$. Then for $1 \le n \le K$, we have*

$$\begin{aligned} &\|u(t_n) - U_h^n\|_{\mathscr{M}_r} + \sqrt{\frac{\kappa\alpha_0}{\beta_0}}\|\nabla u(t_n) - \nabla U_h^n\|_{\mathscr{M}_r} + \sqrt{\frac{\alpha_0}{\beta_0}}\|\Delta u(t_n) - \Delta U_h^n\|_{\mathscr{M}_r} \\ &\le C\Big((\|\partial_t u\|_{L^\infty(H^{r+3})} + \|\partial_t^\alpha u\|_{L^\infty(H^{r+3})} + \|\Delta u\|_{L^\infty(H^{r+3})} + \|u\|_{L^\infty(H^{r+3})})h^{r+1} \\ &\quad + (\|\partial_t^2 u\|_{L^\infty(L^1)} + \|\partial_t^3 u\|_{L^\infty(L^1)})\tau^{3-\alpha}\Big). \end{aligned} \tag{4.26}$$

Further, we have

$$\left\|u(t_n) - U_h^n\right\| \le C\left(h^{r+1} + \tau^{3-\alpha}\right). \tag{4.27}$$

Proof Since estimates of η^n and ρ^n are known from Lemma 2, it suffices to bound the term ξ^n, and then use the triangle inequality to complete the proof.

Firstly, let's prove the error estimate for the first step solution.

By taking the discrete inner product between (3.7) and (3.8) with $t = t_1$ by $\forall \chi_h, \psi_h \in \mathscr{M}_r(\delta)$, respectively and the resulting equations, we subtract from (3.24), respectively and using (2.6), (4.25), for $m = 1, 2, \cdots, n_1$, we obtain

$$\begin{aligned} &\left\langle \xi^{(m),1}, \chi_h \right\rangle - \kappa\widetilde{\tau}^\alpha\Gamma(2-\alpha)\left\langle \Delta\xi^{(m),1}, \chi_h \right\rangle - \widetilde{\tau}^\alpha\Gamma(2-\alpha)\langle \Delta\theta^{(m),1}, \chi_h\rangle \\ &= -\sum_{j=0}^{m-1} b_j^m \left\langle \xi^{(j),1}, \chi_h \right\rangle + \left\langle \sum_{j=0}^{m} b_j^m \eta^{(j),1}, \chi_h \right\rangle + \widetilde{\tau}^\alpha\Gamma(2-\alpha)\left\langle \widetilde{R}^m, \chi_h \right\rangle, \ \forall \chi_h \in \mathscr{M}_r(\delta), \end{aligned} \tag{4.28}$$

$$-\langle \Delta\xi^{(m),1}, \psi_h\rangle - \langle \theta^{(m),1}, \psi_h\rangle = -\langle \rho^{(m),1}, \psi_h\rangle, \ \ 1 \le m \le n_1, \ \ \forall \psi_h \in \mathscr{M}_r(\delta), \tag{4.29}$$

where

$$\begin{aligned} &\widetilde{R}^m = O(\widetilde{\tau}^{2-\alpha}) = O(\tau^{3-\alpha}), \\ &b_m^m = 1, b_0^m = (m-1)^{1-\alpha} - m^{1-\alpha}, b_j^m = (m-j+1)^{1-\alpha} - 2(m-j)^{1-\alpha} + (m-j-1)^{1-\alpha}. \end{aligned} \tag{4.30}$$

For $j = 0, 1, \cdots, m-1$, it is easy to see

$$b_j^m < 0, \quad 0 \le j \le m-1. \tag{4.31}$$

From [1, 36–38], we have the following beautiful results which is critical for establishing an estimate of the first step solution.

If $\phi^j \ge 0$, $j = 1, 2, \cdots, n_1$, $\gamma > 0$, and satisfy $\phi^m \le -\sum\limits_{j=1}^{m-1} b_j^m \phi^j + \gamma$, then

$$\phi^m \le C\widetilde{\tau}^{-\alpha}\gamma. \tag{4.32}$$

The result (4.32) was given in Lemma 3.5 of [37]. We now continue our main goal, to attain an estimate for the first step solution.

Choosing $\chi_h = \xi^{(m),1}$, $\psi_h = \widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\Delta\xi^{(m),1}$, and subtracting (4.28) from (4.29), we obtain

$$\begin{aligned}
&\left\|\xi^{(m),1}\right\|^2_{\mathscr{M}_r} - \kappa\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\left\langle\Delta\xi^{(m),1}, \xi^{(m),1}\right\rangle + \widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\|\Delta\xi^{(m),1}\|^2_{\mathscr{M}_r} \\
&= -\sum_{j=0}^{m-1} b_j^m \left\langle\xi^{(j),1}, \xi^{(m),1}\right\rangle + \left\langle\sum_{j=0}^{m} b_j^m \eta^{(j),1}, \xi^{(m),1}\right\rangle \\
&+\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\left\langle\widetilde{R}^m, \xi^{(m),1}\right\rangle + \widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\langle\rho^{(m),1}, \Delta\xi^{(m),1}\rangle, \quad 1 \le m \le n_1.
\end{aligned} \tag{4.33}$$

From Lemma 2.1 in Manickam and Moudgalya et al. [39], for $\xi^{(m),1} \in \mathscr{M}_r(\delta)$, we have

$$\|\nabla\xi^{(m),1}\|^2_{\mathscr{M}_r} \le -\langle\Delta\xi^{(m),1}, \xi^{(m),1}\rangle, \quad 1 \le m \le n_1. \tag{4.34}$$

Also, from (4.31), we have $-b_j^m > 0$, $j = 0, 1, \cdots, m-1$. So, using inequality (4.34) to the second term on the LHS of (4.33), and then applying the Schwartz inequality to the four terms on the RHS of the resulting expression, (4.33) can be rearranged as

$$\begin{aligned}
&\left\|\xi^{(m),1}\right\|^2_{\mathscr{M}_r} + \kappa\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\|\nabla\xi^{(m),1}\|^2_{\mathscr{M}_r} + \widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\|\Delta\xi^{(m),1}\|^2_{\mathscr{M}_r} \\
&\le \left(-\sum_{j=0}^{m-1} b_j^m \left\|\xi^{(j),1}\right\|_{\mathscr{M}_r} + \left\|\sum_{j=0}^{m} b_j^m \eta^{(j),1}\right\|_{\mathscr{M}_r} + \widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\left\|\widetilde{R}^m\right\|_{\mathscr{M}_r}\right)\left\|\xi^{(m),1}\right\|_{\mathscr{M}_r} \\
&+\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\left\|\rho^{(m),1}\right\|_{\mathscr{M}_r}\left\|\Delta\xi^{(m),1}\right\|_{\mathscr{M}_r}, \quad 1 \le m \le n_1.
\end{aligned} \tag{4.35}$$

Using Cauchy inequality to the terms on right-hand of (4.35), then using inequality $\frac{1}{3}(a+b+c)^2 \le a^2+b^2+c^2$ on the left-hand side, we have

$$\begin{aligned}
&\left\|\xi^{(m),1}\right\|_{\mathscr{M}_r} + \sqrt{\kappa\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)}\|\nabla\xi^{(m),1}\|_{\mathscr{M}_r} + \sqrt{\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)}\|\Delta\xi^{(m),1}\|_{\mathscr{M}_r} \\
&\le C\left(-\sum_{j=0}^{m-1} b_j^m \left\|\xi^{(j),1}\right\|_{\mathscr{M}_r} + \left\|\sum_{j=0}^{m} b_j^m \eta^{(j),1}\right\|_{\mathscr{M}_r}\right. \\
&\left.+\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\left\|\widetilde{R}^m\right\|_{\mathscr{M}_r} + \widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\left\|\rho^{(m),1}\right\|_{\mathscr{M}_r}\right), \quad 1\le m\le n_1.
\end{aligned} \tag{4.36}$$

Now, we only need to estimate the last three terms on the right hand side of (4.36). Since

$$\left\|\sum_{j=0}^{m} b_j^m \eta^{(j),1}\right\|_{\mathscr{M}_r} = \widetilde{\tau}\left\|\sum_{j=0}^{m-1} \tilde{b}_j \frac{\eta^{(m-j),1}-\eta^{(m-j-1),1}}{\widetilde{\tau}}\right\|_{\mathscr{M}_r}, \quad 1\le m\le n_1, \tag{4.37}$$

further, by Lemma 3, we have

$$\begin{aligned}
\left\|\frac{\eta^{(m-j),1}-\eta^{(m-j-1),1}}{\widetilde{\tau}}\right\|_{\mathscr{M}_r} &= \frac{1}{\widetilde{\tau}}\left\|\int_{t_{(m-j-1),1}}^{t_{(m-j),1}} \frac{\partial\eta}{\partial s}(\cdot,s)ds\right\|_{\mathscr{M}_r} \\
&\le \frac{1}{\widetilde{\tau}}\int_{t_{(m-j-1),1}}^{t_{(m-j),1}} \left\|\frac{\partial\eta}{\partial s}(\cdot,s)\right\|_{\mathscr{M}_r} ds \\
&\le Ch^{r+1}\|\partial_t u\|_{L^\infty(H^{r+3})}, \quad 1\le m\le n_1,
\end{aligned}$$

note that, $\sum\limits_{j=0}^{m-1}\tilde{b}_j = m^{1-\alpha}$, then we have

$$\left\|\sum_{j=0}^{m} b_j^m \eta^{(j),1}\right\|_{\mathscr{M}_r} \le C\widetilde{\tau}^{\alpha} t_{(m),1}^{1-\alpha} h^{r+1}\|\partial_t u\|_{L^\infty(H^{r+3})}, \quad 1\le m\le n_1. \tag{4.38}$$

Using (3.15) and (4.30), we have

$$\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\left\|\widetilde{R}^m\right\|_{\mathscr{M}_r} \le C\Gamma(2-\alpha)\widetilde{\tau}^{\alpha}\max_{0\le t\le t_1}|\partial_t^2 u(x,y,t)|\tau^{3-\alpha}, \quad 1\le m\le n_1. \tag{4.39}$$

By Lemma 3 with $j=0$, $\ell=0$, we obtain

$$\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)\left\|\rho^{(m),1}\right\|_{\mathscr{M}_r} \le C\Gamma(2-\alpha)\widetilde{\tau}^{\alpha}h^{r+1}\left\|v^{(m),1}\right\|_{H^{r+3}}, \quad 1\le m\le n_1. \tag{4.40}$$

Substituting (4.38)–(4.40) to (4.36), and using (4.32), for $1 \le m \le n_1$, we can attain

$$\begin{aligned}&\left\|\xi^{(m),1}\right\|_{\mathscr{M}_r}+\sqrt{\kappa\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)}\|\nabla\xi^{(m),1}\|_{\mathscr{M}_r}+\sqrt{\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)}\|\Delta\xi^{(m),1}\|_{\mathscr{M}_r} \qquad (4.41)\\ &\le C\left((t_{(m),1}^{1-\alpha}\|\partial_t u\|_{L^\infty(H^{r+3})}+\Gamma(2-\alpha)\|\Delta u\|_{L^\infty(H^{r+3})})h^{r+1}+\Gamma(2-\alpha)\|\partial_t^2 u\|_{L^\infty(L^1)}\tau^{3-\alpha}\right).\end{aligned}$$

Therefore, using the triangle inequality, (4.41), Lemma 2, and the equivalence of the norms on $\mathscr{M}_r(\delta)$ in Lemma 1, we have

$$\begin{aligned}\left\|u(t_{(m),1})-U_h^{(m),1}\right\|_{\mathscr{M}_r}&+\sqrt{\kappa\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)}\|\nabla(u(t_{(m),1})-U_h^{(m),1})\|_{\mathscr{M}_r} \qquad (4.42)\\ &+\sqrt{\widetilde{\tau}^{\alpha}\Gamma(2-\alpha)}\|\Delta(u(t_{(m),1})-U_h^{(m),1})\|_{\mathscr{M}_r}\\ &\le C\left((\|\partial_t u\|_{L^\infty(H^{r+3})}+\|u\|_{L^\infty(H^{r+3})}+\|\Delta u\|_{L^\infty(H^{r+3})})h^{r+1}\right.\\ &\left.+\|\partial_t^2 u\|_{L^\infty(L^1)}\tau^{3-\alpha}\right).\end{aligned}$$

So, we prove an estimate for this first step solution.

We now turn to derive the error estimate of the OSC scheme for the time steps $n \ge 2$.

First, we obtain from (3.7) and (3.8)

$$\begin{aligned}&\overline{u}(t_n)-\kappa\alpha_0\beta_0^{-1}\Delta u(t_n)-\alpha_0\beta_0^{-1}\Delta v(t_n) \qquad (4.43)\\ &=p\overline{u}(t_{n-1})+\sum_{k=2}^{n-1}\overline{d}_{n-k}^{n}\overline{u}(t_{n-k})+\overline{d}_0^{n}u(t_0)+\alpha_0\beta_0^{-1}R_\tau^n,\quad 2\le n\le K,\end{aligned}$$

$$v(t_n)=-\Delta u(t_n),\quad 2\le n\le K. \qquad (4.44)$$

Using (4.43)–(4.44), (3.22), (2.6), (4.25), and forming the discrete inner product between the resulting equations with $\chi_h \in \mathscr{M}_r(\delta)$ and $\psi_h \in \mathscr{M}_r(\delta)$, respectively, we obtain

$$\begin{aligned}&\left\langle\overline{\xi}^n,\chi_h\right\rangle-\kappa\alpha_0\beta_0^{-1}\langle\Delta\xi^n,\chi_h\rangle-\alpha_0\beta_0^{-1}\langle\Delta\theta^n,\chi_h\rangle=p\left\langle\overline{\xi}^{n-1},\chi_h\right\rangle \qquad (4.45)\\ &+\left\langle\sum_{k=2}^{n-1}\overline{d}_{n-k}^{n}\overline{\xi}^{n-k},\chi_h\right\rangle+\left\langle\overline{d}_0^n\xi^0,\chi_h\right\rangle+\alpha_0\beta_0^{-1}\left\langle L_t^{\alpha}\eta^n+R_\tau^n,\chi_h\right\rangle,\ 2\le n\le K,\ \chi_h\in\mathscr{M}_r(\delta),\end{aligned}$$

$$-\langle\Delta\xi^n,\psi_h\rangle-\langle\theta^n,\psi_h\rangle=-\langle\rho^n,\psi_h\rangle,\ \ 2\le n\le K,\ \ \psi_h\in\mathscr{M}_r(\delta). \qquad (4.46)$$

Now, choosing $\chi_h = 2\overline{\xi}^n$, $\psi_h = 2\alpha_0\beta_0^{-1}\Delta\overline{\xi}^n$ in (4.45) and (4.46), respectively, we subtract the resulting equations to obtain

$$\begin{aligned} &\left\langle\overline{\xi}^n, 2\overline{\xi}^n\right\rangle - \kappa\alpha_0\beta_0^{-1}\left\langle\Delta\xi^n, 2\overline{\xi}^n\right\rangle + \alpha_0\beta_0^{-1}\langle\Delta\xi^n, 2\Delta\overline{\xi}^n\rangle \\ = \ & p\left\langle\overline{\xi}^{n-1}, 2\overline{\xi}^n\right\rangle + \sum_{k=2}^{n-1}\overline{d}_{n-k}^n\left\langle\overline{\xi}^{n-k}, 2\overline{\xi}^n\right\rangle + \overline{d}_0^n\left\langle\xi^0, 2\overline{\xi}^n\right\rangle \\ &+\alpha_0\beta_0^{-1}\left\langle R_\tau^n, 2\overline{\xi}^n\right\rangle + \alpha_0\beta_0^{-1}\left\langle L_t^\alpha\eta^n, 2\overline{\xi}^n\right\rangle + \alpha_0\beta_0^{-1}\langle\rho^n, 2\Delta\overline{\xi}^n\rangle, \quad 2 \le n \le K. \end{aligned} \tag{4.47}$$

By using the definition of $\overline{\xi}^n$, we have

$$\langle\Delta\xi^n, 2\Delta\overline{\xi}^n\rangle = \langle\Delta\xi^n, \Delta\xi^n\rangle + \langle\Delta\overline{\xi}^n, \Delta\overline{\xi}^n\rangle - p^2\langle\Delta\xi^{n-1}, \Delta\xi^{n-1}\rangle, \quad 2 \le n \le K, \tag{4.48}$$

and

$$-\langle\Delta\xi^n, 2\overline{\xi}^n\rangle = -\langle\Delta\overline{\xi}^n, \overline{\xi}^n\rangle - \langle\Delta\xi^n, \xi^n\rangle + p^2\langle\Delta\xi^{n-1}, \xi^{n-1}\rangle, \quad 2 \le n \le K. \tag{4.49}$$

The proof details of (4.48) is provided in Appendix B.

From Lemma 2.1 in Manickam and Moudgalya et al. [39], for $\xi^n \in \mathscr{M}_r(\delta)$, we have

$$\|\nabla\xi^n\|_{\mathscr{M}_r}^2 \le -\langle\Delta\xi^n, \xi^n\rangle, \quad 1 \le n \le K. \tag{4.50}$$

Substituting (4.48)–(4.50) to (4.47), and using Lemma 4 , Schwartz inequality, noting that $\xi^0 = 0$, we have

$$\begin{aligned} &2\|\overline{\xi}^n\|_{\mathscr{M}_r}^2 + \kappa\alpha_0\beta_0^{-1}\|\nabla\overline{\xi}^n\|_{\mathscr{M}_r}^2 + \kappa\alpha_0\beta_0^{-1}\|\nabla\xi^n\|_{\mathscr{M}_r}^2 - p^2\kappa\alpha_0\beta_0^{-1}\|\nabla\xi^{n-1}\|_{\mathscr{M}_r}^2 \\ &+\alpha_0\beta_0^{-1}\|\Delta\xi^n\|_{\mathscr{M}_r}^2 + \alpha_0\beta_0^{-1}\|\Delta\overline{\xi}^n\|_{\mathscr{M}_r}^2 - p^2\alpha_0\beta_0^{-1}\|\Delta\xi^{n-1}\|_{\mathscr{M}_r}^2 \\ \le \ & p\|\overline{\xi}^{n-1}\|_{\mathscr{M}_r}^2 + \sum_{k=2}^{n-1}\overline{d}_{n-k}^n\|\overline{\xi}^{n-k}\|_{\mathscr{M}_r}^2 + 2\overline{d}_0^n\|(\overline{d}_0^n)^{-1}\alpha_0\beta_0^{-1}R_\tau^n\|_{\mathscr{M}_r}^2 + 2\|\alpha_0\beta_0^{-1}L_t^\alpha\eta^n\|_{\mathscr{M}_r}^2 \\ &+(p + \sum_{k=2}^{n-1}\overline{d}_{n-k}^n + \overline{d}_0^n)\|\overline{\xi}^n\|_{\mathscr{M}_r}^2 + \alpha_0\beta_0^{-1}\|\rho^n\|_{\mathscr{M}_r}^2 + \alpha_0\beta_0^{-1}\|\Delta\overline{\xi}^n\|_{\mathscr{M}_r}^2, \quad 2 \le n \le K. \end{aligned} \tag{4.51}$$

From (1) and (3) of Lemma 4, we attain

$$
\begin{aligned}
&\|\overline{\xi}^n\|^2_{\mathscr{M}_r} + \kappa\alpha_0\beta_0^{-1}\|\nabla\xi^n\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\Delta\xi^n\|^2_{\mathscr{M}_r} \qquad (4.52)\\
\le\; & p\|\overline{\xi}^{n-1}\|^2_{\mathscr{M}_r} + p^2\kappa\alpha_0\beta_0^{-1}\|\nabla\xi^{n-1}\|^2_{\mathscr{M}_r} + p^2\alpha_0\beta_0^{-1}\|\Delta\xi^{n-1}\|^2_{\mathscr{M}_r} + \sum_{k=2}^{n-1}\overline{d}^n_{n-k}\|\overline{\xi}^{n-k}\|^2_{\mathscr{M}_r}\\
&+2\overline{d}^n_0\|(\overline{d}^n_0)^{-1}\alpha_0\beta_0^{-1}R^n_\tau\|^2_{\mathscr{M}_r} + 2\|\alpha_0\beta_0^{-1}L^\alpha_t\eta^n\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\rho^n\|^2_{\mathscr{M}_r}\\
\le\; & p\left(\|\overline{\xi}^{n-1}\|^2_{\mathscr{M}_r} + \kappa\alpha_0\beta_0^{-1}\|\nabla\xi^{n-1}\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\Delta\xi^{n-1}\|^2_{\mathscr{M}_r}\right) + \sum_{k=2}^{n-1}\overline{d}^n_{n-k}\|\overline{\xi}^{n-k}\|^2_{\mathscr{M}_r}\\
&+2\overline{d}^n_0\|(\overline{d}^n_0)^{-1}\alpha_0\beta_0^{-1}R^n_\tau\|^2_{\mathscr{M}_r} + 2\|\alpha_0\beta_0^{-1}L^\alpha_t\eta^n\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\rho^n\|^2_{\mathscr{M}_r}\\
\le\; & p\left(\|\overline{\xi}^{n-1}\|^2_{\mathscr{M}_r} + \kappa\alpha_0\beta_0^{-1}\|\nabla\xi^{n-1}\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\Delta\xi^{n-1}\|^2_{\mathscr{M}_r}\right)\\
&+\sum_{k=2}^{n-1}\overline{d}^n_{n-k}\left(\|\overline{\xi}^{n-k}\|^2_{\mathscr{M}_r} + \kappa\alpha_0\beta_0^{-1}\|\nabla\xi^{n-k}\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\Delta\xi^{n-k}\|^2_{\mathscr{M}_r}\right)\\
&+2\overline{d}^n_0\|(\overline{d}^n_0)^{-1}\alpha_0\beta_0^{-1}R^n_\tau\|^2_{\mathscr{M}_r} + 2\|\alpha_0\beta_0^{-1}L^\alpha_t\eta^n\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\rho^n\|^2_{\mathscr{M}_r}.
\end{aligned}
$$

Now, we estimate the last three terms in (4.52). By using (3.16), (1) and (4) in Lemma 4, we have

$$
\|(\overline{d}^n_0)^{-1}\alpha_0\beta_0^{-1}R^n_\tau\|_{\mathscr{M}_r} \le \frac{n^\alpha\tau^\alpha(\Gamma(2-\alpha))}{(2-\alpha)(1-\alpha)}\|\partial^3_t u\|_{L^\infty(L^1)}\tau^{3-\alpha}. \qquad (4.53)
$$

According to (3.18), and (1), (2), (3) in Lemma 4, we have

$$
\begin{aligned}
\|\alpha_0\beta_0^{-1}L^\alpha_t\eta^n\|_{\mathscr{M}_r} &= \|\overline{\eta}^n - p\overline{\eta}^{n-1} - \sum_{k=2}^{n-1}\overline{d}^n_{n-k}\overline{\eta}^{n-k} - \overline{d}^n_0\eta^0\|_{\mathscr{M}_r} \qquad (4.54)\\
&\le \|\overline{\eta}^n\|_{\mathscr{M}_r} + p\|\overline{\eta}^{n-1}\|_{\mathscr{M}_r} + \sum_{k=2}^{n-1}\overline{d}^n_{n-k}\|\overline{\eta}^{n-k}\|_{\mathscr{M}_r} + \overline{d}^n_0\|\eta^0\|_{\mathscr{M}_r}\\
&\le (1+p+\sum_{k=2}^{n-1}\overline{d}^n_{n-k} + \overline{d}^n_0)\max_{0\le j\le n}\|\overline{\eta}^j\|_{\mathscr{M}_r}\\
&\le 2\max_{0\le j\le n}\|\overline{\eta}^j\|_{\mathscr{M}_r},\quad 2\le n\le K.
\end{aligned}
$$

Since $\overline{\eta}^j = \eta^j - p\eta^{j-1}$, then combining (4.55) and Lemma 3, we have

$$
\begin{aligned}
\|\alpha_0\beta_0^{-1}L^\alpha_t\eta^n\|_{\mathscr{M}_r} &\le 8/3\max_{0\le j\le n}\|\eta^j\|_{\mathscr{M}_r} \qquad (4.55)\\
&\le Ch^{r+1}\|u\|_{L^\infty(H^{r+3})},\quad 2\le n\le K.
\end{aligned}
$$

Also, using Lemma 3, (1) and (4) in Lemma 4, we can obtain

$$(\overline{d}_0^n)^{-1}\alpha_0\beta_0^{-1}\|\rho^n\|_{\mathscr{M}_r} \leq \frac{n^\alpha\tau^\alpha\Gamma(2-\alpha)}{(2-\alpha)(1-\alpha)}\|\rho^n\|_{\mathscr{M}_r} \leq Ch^{r+1}\|\Delta u\|_{L^\infty(H^{r+3})}, \quad 2\leq n\leq K. \tag{4.56}$$

Thus, combining the resulting (4.53), (4.55) and (4.56), (4.52) can be rearranged as

$$\begin{aligned}
&\|\overline{\xi}^n\|^2_{\mathscr{M}_r} + \kappa\alpha_0\beta_0^{-1}\|\nabla\xi^n\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\Delta\xi^n\|^2_{\mathscr{M}_r} \\
\leq\ & p\left(\|\overline{\xi}^{n-1}\|^2_{\mathscr{M}_r} + \kappa\alpha_0\beta_0^{-1}\|\nabla\xi^{n-1}\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\Delta\xi^{n-1}\|^2_{\mathscr{M}_r}\right) \\
&+\sum_{k=2}^{n-1}\overline{d}^n_{n-k}\left(\|\overline{\xi}^{n-k}\|^2_{\mathscr{M}_r} + \kappa\alpha_0\beta_0^{-1}\|\nabla\xi^{n-k}\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\Delta\xi^{n-k}\|^2_{\mathscr{M}_r}\right) \\
&+\overline{d}^n_0\left(C_1(\|u\|^2_{L^\infty(H^{r+3})} + \|\Delta u\|^2_{L^\infty(H^{r+3})})h^{2r+2} + C_2\|\partial_t^3 u\|^2_{L^\infty(L^1)}\tau^{6-2\alpha}\right).
\end{aligned} \tag{4.57}$$

According to the estimate (4.41) and (4.42) for the first step, we now turn to prove an estimate result for the time step $n\geq 2$. Next, we will prove the following estimates (4.58) by the mathematical induction.

$$\begin{aligned}
&\|\overline{\xi}^n\|_{\mathscr{M}_r} + \sqrt{\kappa\alpha_0\beta_0^{-1}}\|\nabla\xi^n\|_{\mathscr{M}_r} + \sqrt{\alpha_0\beta_0^{-1}}\|\Delta\xi^n\|_{\mathscr{M}_r} \\
\leq\ & C\Big((\|\partial_t u\|_{L^\infty(H^{r+3})} + \|u\|_{L^\infty(H^{r+3})} + \|\Delta u\|_{L^\infty(H^{r+3})})h^{r+1} \\
&+(\|\partial_t^2 u\|_{L^\infty(L^1)} + \|\partial_t^3 u\|_{L^\infty(L^1)})\tau^{3-\alpha}\Big), \quad 1\leq n\leq K.
\end{aligned} \tag{4.58}$$

By using (4.41), it is easy to check that (4.58) hold immediate for the case $n=1$.

Assuming the estimate (4.58) is true for $n=2,3,\cdots,i-1$, we want to prove that it also true for $n=i$. It can be done by (4.57)

$$\begin{aligned}
&\|\overline{\xi}^i\|^2_{\mathscr{M}_r} + \kappa\alpha_0\beta_0^{-1}\|\nabla\xi^i\|^2_{\mathscr{M}_r} + \alpha_0\beta_0^{-1}\|\Delta\xi^i\|^2_{\mathscr{M}_r} \\
\leq\ & C\left(p+\sum_{k=2}^{i-1}\overline{d}^i_{i-k}+\overline{d}^i_0\right)\Big((\|\partial_t u\|_{L^\infty(H^{r+3})} + \|u\|_{L^\infty(H^{r+3})} + \|\Delta u\|_{L^\infty(H^{r+3})})h^{r+1} \\
&+(\|\partial_t^2 u\|_{L^\infty(L^1)} + \|\partial_t^3 u\|_{L^\infty(L^1)})\tau^{3-\alpha}\Big)^2 \\
\leq\ & C\Big((\|\partial_t u\|_{L^\infty(H^{r+3})} + \|u\|_{L^\infty(H^{r+3})} + \|\Delta u\|_{L^\infty(H^{r+3})})h^{r+1} \\
&+(\|\partial_t^2 u\|_{L^\infty(L^1)} + \|\partial_t^3 u\|_{L^\infty(L^1)})\tau^{3-\alpha}\Big)^2.
\end{aligned} \tag{4.59}$$

Thus (4.58) is proven.

Further, by the triangle inequality and (4.58), we have

$$
\begin{aligned}
&\|\xi^n\|_{\mathscr{M}_r} + \sqrt{\kappa\alpha_0\beta_0^{-1}}\|\nabla\xi^n\|_{\mathscr{M}_r} + \sqrt{\alpha_0\beta_0^{-1}}\|\Delta\xi^n\|_{\mathscr{M}_r} \qquad (4.60)\\
\le\ & p\|\xi^{n-1}\|_{\mathscr{M}_r} + \|\overline{\xi}^n\|_{\mathscr{M}_r} + \sqrt{\kappa\alpha_0\beta_0^{-1}}\|\nabla\xi^n\|_{\mathscr{M}_r} + \sqrt{\alpha_0\beta_0^{-1}}\|\Delta\xi^n\|_{\mathscr{M}_r}\\
\le\ & p\left(\|\overline{\xi}^{n-1}\|_{\mathscr{M}_r} + p\|\xi^{n-2}\|_{\mathscr{M}_r}\right) + C\left((\|\partial_t u\|_{L^\infty(H^{r+3})} + \|u\|_{L^\infty(H^{r+3})}\right.\\
&\left. +\|\Delta u\|_{L^\infty(H^{r+3})})h^{r+1} + (\|\partial_t^2 u\|_{L^\infty(L^1)} + \|\partial_t^3 u\|_{L^\infty(L^1)})\tau^{3-\alpha}\right)\\
\le\ & p^2\left(\|\overline{\xi}^{n-2}\|_{\mathscr{M}_r} + p\|\xi^{n-3}\|_{\mathscr{M}_r}\right) + C(1+p)\left((\|\partial_t u\|_{L^\infty(H^{r+3})} + \|u\|_{L^\infty(H^{r+3})}\right.\\
&\left. +\|\Delta u\|_{L^\infty(H^{r+3})})h^{r+1} + (\|\partial_t^2 u\|_{L^\infty(L^1)} + \|\partial_t^3 u\|_{L^\infty(L^1)})\tau^{3-\alpha}\right)\\
\le\ & C(1+p+p^2+\cdots+p^n)\left((\|\partial_t u\|_{L^\infty(H^{r+3})} + \|u\|_{L^\infty(H^{r+3})} + \|\Delta u\|_{L^\infty(H^{r+3})})h^{r+1}\right.\\
&\left. +(\|\partial_t^2 u\|_{L^\infty(L^1)} + \|\partial_t^3 u\|_{L^\infty(L^1)})\tau^{3-\alpha}\right)\\
\le\ & C\left((\|\partial_t u\|_{L^\infty(H^{r+3})} + \|u\|_{L^\infty(H^{r+3})} + \|\Delta u\|_{L^\infty(H^{r+3})})h^{r+1}\right.\\
&\left. +(\|\partial_t^2 u\|_{L^\infty(L^1)} + \|\partial_t^3 u\|_{L^\infty(L^1)})\tau^{3-\alpha}\right).
\end{aligned}
$$

Finally, using the triangle inequality, (4.60), Lemma 3, and the equivalence of the norms on $\mathscr{M}_r(\delta)$ in Lemma 1 and combining (4.42), we complete the proof of (4.26). □

5 Numerical experiments

In this section, we carry out some numerical experiments to illustrate our theoretical statements by using the new developed numerical algorithms (3.22) and (3.24). In our implementations, we use the space of piecewise Hermite bi-cubics, $\mathscr{M}_3(\delta)$, with the standard value and scaled slope basis functions [40] on identical uniform partitions of [0, 1]. In all of the test problems, we consider uniform partitions in the x and y directions with $N_x = N_y = N$. The initial condition is approximated by the OSC elliptic projection of u_0, as specified in Theorem 1. The forcing function f is approximated by using interpolant projection in the collocation point. For our method, we give temporal and spatial errors in L^∞ and L^2 norms and the corresponding convergence rates determined by the formula

$$
Convergence\ Rate \approx \frac{log(e_m/e_{m+1})}{log(h_m/h_{m+1})},
$$

where $h_m = 1/N_m$ is the step size with $h = 1/N = 1/N_m$, and e_m is the error the corresponding N_m.

Example 1 Consider the following 2D fourth-order reaction-diffusion equation:

$$\begin{cases} \partial_t^{\alpha} u - \Delta u + \Delta^2 u = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \\ u(x, y, 0) = 0, \quad (x, y) \in \Omega, \\ u(x, y, t) = \Delta u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T], \end{cases} \tag{5.61}$$

where $\Omega = [0, 1] \times [0, 1]$, $T = 1$, and

$$f(x, y, t) = (\frac{\Gamma(5-\alpha)}{\Gamma(5-2\alpha)} t^{4-2\alpha} + 2\pi^2 t^{4-\alpha} + 4\pi^4 t^{4-\alpha}) \sin(\pi x) \sin(\pi y).$$

The exact solution to (5.61) is $u(x, y, t) = t^{4-\alpha} \sin(\pi x) \sin(\pi y)$. In this case, the exact solution $u \in C^3$ satisfy the condition of Theorem 1.

Firstly, we take the fixed space step $h = 1/200$ which is sufficiently small such that the error will be dominated by the time discretization of the method. Table 1 presents the computational errors and the temporal convergence orders in both the maximum L^∞ norm and L^2-norm for different α ($\alpha = 0.3, 0.5, 0.7, 0.9$). As predicted by the theoretical estimates, the proposed scheme yields a temporal approximation order close to $3 - \alpha$. Then, in order to check the convergence order in space, the time step τ and space step h are chosen such that $\tau^{3-\alpha} \approx h^4$ as in [41], and $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$; the maximum L^∞ and L^2 errors are shown in Table 2.

Table 1 Example 1: the computational errors and convergence orders in time with $h = 1/200$

α	τ	L^2 error	Rate	L^∞ error	Rate
$\alpha = 0.3$	1/10	6.4759e−06		1.2952e−05	
	1/20	9.5628e−07	2.7596	1.9126e−06	2.7596
	1/40	1.3943e−07	2.7779	2.7886e−07	2.7779
	1/80	2.0251e−08	2.7835	4.0501e−08	2.7835
	1/160	2.9170e−09	2.7954	5.8341e−09	2.7954
$\alpha = 0.5$	1/10	1.8307e−05		3.6614e−05	
	1/20	3.2263e−06	2.5158	6.4526e−06	2.5044
	1/40	5.6508e−07	2.5150	1.1302e−06	2.5133
	1/80	9.8859e−08	2.5134	1.9772e−07	2.5150
	1/160	1.7286e−08	2.5044	3.4571e−08	2.5158
$\alpha = 0.7$	1/10	3.9612e−05		7.9224e−05	
	1/20	8.0895e−06	2.2918	1.6179e−05	2.2918
	1/40	1.6449e−06	2.2981	3.2898e−06	2.2981
	1/80	3.3393e−07	2.3004	6.6785e−07	2.3004
	1/160	6.7734e−08	2.3016	1.3547e−07	2.3016
$\alpha = 0.9$	1/10	7.2527e−05		1.4505e−04	
	1/20	1.6971e−05	2.0954	3.3941e−05	2.0954
	1/40	3.9641e−06	2.0980	7.9282e−06	2.0980
	1/80	9.2523e−07	2.0991	1.8505e−06	2.0991
	1/160	2.1585e−07	2.0998	4.3170e−07	2.0998

Table 2 Example 1: the computational errors and convergence orders in space

α	$1/\tau$	$1/h$	L^2 error	Rate	L^∞ error	Rate
$\alpha = 0.1$	9	5	2.5858e−05		4.9185e−05	
	24	10	1.4847e−06	4.1224	2.9694e−06	4.0500
	62	20	8.9819e−08	4.0470	1.7964e−07	4.0470
	162	40	5.4915e−09	4.0317	1.0983e−08	4.0318
$\alpha = 0.3$	11	5	2.9517e−05		5.6145e−05	
	30	10	1.7390e−06	4.0852	3.4779e−06	4.0129
	85	20	1.0486e−07	4.0517	2.0972e−07	4.0517
	236	40	6.4643e−09	4.0198	1.2929e−08	4.0198
$\alpha = 0.5$	13	5	3.4286e−05		6.5216e−05	
	40	10	2.0099e−06	4.0924	4.0197e−06	4.0201
	121	20	1.2369e−07	4.0223	2.4737e−07	4.0223
	366	40	7.6907e−09	4.0075	1.5381e−08	4.0074
$\alpha = 0.7$	16	5	3.8551e−05		7.3329e−05	
	55	10	2.2527e−06	4.0970	4.5054e−06	4.0247
	183	20	1.3956e−07	4.0127	2.7913e−07	4.0126
	611	40	8.6932e−09	4.0049	1.7386e−08	4.0049
$\alpha = 0.9$	21	5	4.0632e−05		7.7286e−05	
	80	10	2.4038e−06	4.0792	4.8077e−06	4.0068
	301	20	1.4813e−07	4.0204	2.9625e−07	4.0205
	1126	40	9.2398e−09	4.0029	1.8480e−08	4.0028

Clearly, numerical solutions fit well with the exact solutions, and fourth-order convergence in space are observed for $r = 3$, which is in agreement with the theoretical analysis in Theorem 1.

In the following Example 2, we mainly compare the numerical results obtained by the present method with the compact finite difference method (CFDM) proposed by Vong and Wang [23], in which time was discretized by the L1 method and space was approximated by the fourth-order CFDM.

Example 2 Let $L = 1$, $T = 1$ and $\kappa = 0$. We compute the fourth-order fractional diffusion problem in [23]:

$$\begin{aligned}
&\partial_t^\alpha u + \frac{\partial^4 u}{\partial x^4} = \frac{\Gamma(3+\alpha)}{2} t^2 e^x x^2 (1-x)^2 \\
&+(x^4 + 14x^3 + 49x^2 + 32x - 12)t^{2+\alpha} e^x, \quad (x,t) \in (0,1) \times (0,1], \\
&u(x,0) = 0, \quad x \in [0,1], \\
&u(0,t) = u(1,t) = 0, \ \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(1,t)}{\partial x^2} = 0, \ t \in (0,T].
\end{aligned}$$

The exact solution is $u(x,t) = e^x x^2 (1-x)^2 t^{2+\alpha}$.

Table 3 Comparison of L^∞ errors in time for Example 2, $N = 500$

α	$1/\tau$	The present scheme	Rate	L1-CFDM in [23]	Rate in [23]
$\alpha = 0.5$	10	3.7141e−07		5.2354e−06	
	20	6.2895e−08	2.5620	1.9190e−06	1.4479
	40	1.0761e−08	2.5471	6.8823e−07	1.4794
	80	1.8529e−09	2.5380	2.5251e−07	1.4466
	160	3.1354e−10	2.5631	8.8027e−08	1.5203
$\alpha = 0.7$	10	1.8551e−06		1.4947e−05	
	20	3.7008e−07	2.3256	6.2160e−06	1.2658
	40	7.4391e−08	2.3146	2.5627e−06	1.2783
	80	1.4913e−08	2.3186	1.0473e−06	1.2910
	160	2.8959e−09	2.3645	4.2522e−07	1.3003
$\alpha = 0.9$	10	7.0351e−06		3.8832e−05	
	20	1.6328e−06	2.1072	1.8456e−05	1.0731
	40	3.7984e−07	2.1039	8.6927e−06	1.0862
	80	8.8377e−08	2.1036	4.0794e−06	1.0915
	160	2.0482e−08	2.1093	1.9105e−06	1.0944

We choose the same parameters N, τ and α as in [23], Table 3 displays the L^∞ errors and the convergence orders with $\alpha = 0.5, 0.7, 0.9$; the last two columns present the numerical results obtained in [23]. From Table 3, we find that the proposed method in this paper show better performance than that in [23] for this example, since we use higher-order approximation in time. Further, we give illustrations in Table 3 to show that $3 - \alpha$ order temporal accuracy is obtained even if u is not sufficiently smooth in time. That is to say, $3-\alpha$ order accuracy in temporal direction can be still achieved even if the smooth assumption of the solution with respect to the time variable in Theorem 1 is no longer satisfied. The regularity assumption of solution in Theorem 1 may be a sufficient condition, not a necessary and sufficient condition. It is one of most interesting problems relate to the fourth-order fractional diffusion model is how to establish an estimate with respect to the data regularity. In the future, we will further explore convergence analysis of the OSC method for fractional diffusion equations, where the exact solutions have limited regularity.

Table 4 Comparison of L^∞ errors in space for Example 2 with $\alpha = 0.5$

N	The present scheme	Rate	L1-CFDM in [23]	Rate in [23]
5	8.7108e−04		3.6349e−02	
10	3.8926e−05	4.4840	2.3726e−03	3.9374
20	2.3364e−06	4.0584	1.4936e−04	3.9895
40	1.4268e−07	4.0334	9.3776e−06	3.9935
80	8.8737e−09	4.0071	5.8628e−07	3.9995

Choosing the same parameters α and N as in [23], we also compare the L^∞ errors and the convergence orders in space with $\alpha = 0.5$ in the Table 4. The parameters τ is chosen as $\tau = 1/50000$ in [23] (see Table 2 in [23]), while we choose $\tau = 1/5000$. From Table 4, we can see that the numerical results of our method exhibit a slightly better accuracy than the published results in [23].

In the next example, the main purpose is to further compare the numerical results obtained by present method with the high-order method proposed in [9] with convergence order $O(\tau^{3-\alpha} + h^2)$, where time was discretized by the high-order FDM as given in this paper and space was approximated by the standard FEM.

Example 3 We consider the following second-order time fractional sub-diffusion problem by Li, Liang and Yan [9]:

$$\begin{cases} \partial_t^\alpha u - \Delta u = f(x, y, t), & (x, y, t) \in \Omega \times (0, T], \\ u(x, y, 0) = 0, & (x, y) \in \Omega, \\ u(x, y, t) = 0, & (x, y, t) \in \partial\Omega \times (0, T], \end{cases} \tag{5.62}$$

where $\Omega = [0, 1] \times [0, 1]$, $T = 1$, the exact solution is $u(x, y, t) = t^m \sin(2\pi x) \sin(2\pi y)$ for some $m > 0$ and

$$f(x, y, t) = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} t^{m-\alpha} \sin(2\pi x) \sin(2\pi y) + 8\pi^2 t^m \sin(2\pi x) \sin(2\pi y).$$

We first choose the same parameters as those in [9], except that the space step size $h = 1/2^9$ is used in this paper, since we have fourth-order approximation in space.

Table 5 Comparison of L^2 errors in time for Example 3, $h = 1/2^9$

α	τ	Our method	Rate	Method [9]	Rate [9]	Method [3]	Rate [3]
0.3	$1/2^3$	2.4818e−5		2.6121e−5		4.0872e−5	
	$1/2^4$	3.6483e−6	2.77	3.9008e−6	2.74	6.8168e−6	2.58
	$1/2^5$	5.3059e−7	2.78	5.6947e−7	2.77	1.1087e−6	2.62
	$1/2^6$	7.7074e−8	2.78	8.2783e−8	2.78	1.7751e−7	2.64
	$1/2^7$	1.1239e−8	2.78	1.2035e−8	2.78	2.8063e−8	2.66
0.5	$1/2^3$	8.5474e−5		9.1162e−5		1.1424e−4	
	$1/2^4$	1.5132e−5	2.50	1.6239e−5	2.49	2.1282e−5	2.42
	$1/2^5$	2.6538e−6	2.51	2.8516e−6	2.51	3.8813e−6	2.45
	$1/2^6$	4.6444e−7	2.51	4.9885e−7	2.52	6.9912e−7	2.47
	$1/2^7$	8.1343e−8	2.51	8.6983e−8	2.52	1.2458e−7	2.49
0.9	$1/2^3$	5.4612e−4		5.8723e−4		6.0022e−4	
	$1/2^4$	1.3062e−4	2.06	1.4046e−4	2.06	1.4406e−4	2.06
	$1/2^5$	3.0828e−5	2.08	3.3137e−5	2.08	3.4057e−5	2.08
	$1/2^6$	7.2313e−6	2.09	7.7557e−6	2.09	7.9814e−6	2.09
	$1/2^7$	1.6914e−6	2.10	1.7963e−6	2.11	1.8500e−6	2.11

While the space step size in [9] is chosen as $h = 1/2^6$. In the Table 5, the time step sizes are chosen as τ ($\tau = 2^{-k}$, $k = 3, 4, 5, 6, 7$), and $m = 3.5$ as in [9]. We compares the L^2 error in time with different α ($\alpha = 0.3, 0.5, 0.9$) in Table 5. We can see that the present method have similar temporal convergence accuracy as in [9] and [3] for the second-order fractional diffusion equation (5.62).

Table 6 displays the L^2 and L^∞ errors and convergence orders in space with $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$, and the time step τ and space step h are chosen such that $\tau^{3-\alpha} \approx h^4$ as in [41]. Clearly, numerical solutions fit well with the exact solutions, and fourth-order convergence rate are observed for $r = 3$, which is in agreement with the theoretical analysis.

In the following Example 4, we mainly test problem based on the Gaussian pulse (see [16]) to show the efficiency of the developed technique.

Table 6 Example 3: the computational errors and convergence orders in space

α	$1/\tau$	$1/h$	L^2 error	Rate	L^∞ error	Rate
$\alpha = 0.1$	7	2^2	2.4513e−03		4.9026e−03	
	18	2^3	1.3915e−04	4.1388	2.7831e−04	4.1388
	46	2^4	8.4904e−06	4.0347	1.6981e−05	4.0347
	119	2^5	5.2720e−07	4.0094	1.0544e−06	4.0094
	310	2^6	3.2879e−08	4.0031	6.5759e−08	4.0031
$\alpha = 0.3$	8	2^2	2.4793e−03		4.9585e−03	
	22	2^3	1.4109e−04	4.1352	2.8218e−04	4.1352
	61	2^4	8.6124e−06	4.0341	1.7225e−05	4.0340
	170	2^5	5.3482e−07	4.0093	1.0696e−06	4.0094
	474	2^6	3.3359e−08	4.0029	6.6718e−08	4.0246
$\alpha = 0.5$	9	2^2	2.5287e−03		5.0574e−03	
	28	2^3	1.4416e−04	4.1327	2.8832e−04	4.1327
	84	2^4	8.8149e−06	4.0316	1.7630e−05	4.0316
	256	2^5	5.4752e−07	4.0090	1.0950e−06	4.0090
	776	2^6	3.4165e−08	4.0023	6.8330e−08	4.0023
$\alpha = 0.7$	11	2^2	2.5898e−03		5.1796e−03	
	37	2^3	1.4849e−04	4.1244	2.9698e−04	4.1244
	124	2^4	9.0830e−06	4.0311	1.8166e−05	4.0311
	415	2^5	5.6453e−07	4.0080	1.1291e−06	4.0080
	1384	2^6	3.5242e−08	4.0017	7.0484e−08	4.0017
$\alpha = 0.9$	14	2^2	2.6653e−03		5.3306e−03	
	53	2^3	1.5349e−04	4.1181	3.0697e−04	4.1181
	197	2^4	9.4155e−06	4.0270	1.8831e−05	4.0269
	736	2^5	5.8578e−07	4.0066	1.1716e−06	4.0066
	2756	2^6	3.6546e−08	4.0026	7.3093e−08	4.0026

Table 7 The computational errors and convergence orders in time for Example 4 with $T = 1$ and $h = 1/2^9$

α	τ	L^2 error	Rate	L^∞ error	Rate	CPU time (s)
0.3	$1/2^3$	5.3941e−5		1.1165e−4		33.2376
	$1/2^4$	7.8334e−6	2.7837	1.6214e−5	2.7837	92.6708
	$1/2^5$	1.1334e−6	2.7890	2.3460e−6	2.7890	280.4970
	$1/2^6$	1.6438e−7	2.7856	3.4025e−7	2.7855	932.2557
0.7	$1/2^3$	8.3205e−4		1.7229e−3		33.3610
	$1/2^4$	1.7438e−4	2.2544	3.6104e−4	2.2546	92.6660
	$1/2^5$	3.5868e−5	2.2815	7.4259e−5	2.2815	280.7256
	$1/2^6$	7.3188e−6	2.2930	1.5152e−5	2.2931	932.3922

Example 4 Let $L = 1$ and $\kappa = 1$. We compute the fourth-order fractional reaction-diffusion problem:

$$\begin{aligned}
&\partial_t^\alpha u - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = f, \ \ (x,t) \in (0,1) \times (0,T], \\
&u(x,0) = 0, \quad x \in [0,1], \\
&u(0,t) = u(1,t) = 0, \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(1,t)}{\partial x^2} = 0, \ t \in (0,T].
\end{aligned}$$

Choosing the forcing function f so that the exact solution is $u(x,t) = e^{-\frac{(x-0.5)^2}{\beta}} \sin(\pi x) t^{3+\alpha}$.

Table 7 shows the temporal accuracy and convergence rates for Example 4 with $T = 1$, $\alpha = 0.3, 0.7$, $h = 1/2^9$, the time step sizes are chosen as τ ($\tau = 2^{-k}$, $k = 3,4,5,6$). Also, the temporal accuracy and convergence rates with $T = 5$, $\alpha = 0.5, 0.8$, $h = 1/2^9$ are presented in the Table 8. The last one column of Tables 7 and 8 present the CPU time.

Table 8 The computational errors and convergence orders in time for Example 4 with $T = 5$ and $h = 1/2^9$

α	τ	L^2 error	Rate	L^∞ error	Rate	CPU time (s)
0.5	$1/2^3$	3.2202e−2		6.6567e−2		33.2293
	$1/2^4$	5.7030e−3	2.4974	1.1789e−2	2.4974	92.7359
	$1/2^5$	1.0003e−3	2.5113	2.0678e−3	2.5113	280.9090
	$1/2^6$	1.7507e−4	2.5144	3.6190e−4	2.5144	931.5525
0.8	$1/2^3$	1.9372e−1		4.0031e−1		33.4181
	$1/2^4$	4.3810e−2	2.1446	9.0528e−2	2.1447	92.5233
	$1/2^5$	9.7022e−3	2.1749	2.0048e−2	2.1749	280.1219
	$1/2^6$	2.1284e−3	2.1885	4.3979e−3	2.1886	930.4146

Table 9 The computational errors and convergence orders in space for Example 4 with $T = 1$ and $\tau^{3-\alpha} \approx h^4$

α	$1/\tau$	$1/h$	L^2 error	Rate	L^∞ error	Rate	CPU time (s)
$\alpha = 0.2$	20	2^3	1.7749e−04		4.2363e−04		0.1089
	53	2^4	1.0215e−05	4.1190	2.4499e−05	4.1120	1.2129
	141	2^5	6.2590e−07	4.0286	1.5025e−06	4.0273	22.7992
	380	2^6	3.8930e−08	4.0070	9.3398e−08	4.0078	510.5582
$\alpha = 0.6$	32	2^3	1.7720e−04		4.6467e−04		0.2038
	102	2^4	1.0237e−05	4.1135	2.7032e−05	4.1035	3.9263
	323	2^5	6.2785e−07	4.0272	1.6620e−06	4.0237	112.6466
	1024	2^6	3.9058e−08	4.0067	1.0346e−07	4.0058	3631.0016

Table 10 The computational errors and convergence orders in space for Example 4 with $T = 5$ and $\tau^{3-\alpha} \approx h^4$

α	$1/\tau$	$1/h$	L^2 error	Rate	L^∞ error	Rate	CPU time (s)
$\alpha = 0.4$	25	2^3	4.2176e−02		1.0084e−01		0.1445
	71	2^4	2.4255e−03	4.1201	5.8433e−03	4.1091	2.0331
	207	2^5	1.4860e−04	4.0288	3.5852e−04	4.0267	47.5816
	601	2^6	9.2418e−06	4.0071	2.2302e−05	4.0068	1263.5514
$\alpha = 0.8$	44	2^3	8.0026e−02		1.9860e−01		0.3278
	155	2^4	4.6059e−03	4.1189	1.1542e−02	4.1049	8.6043
	545	2^5	2.8222e−04	4.0286	7.0948e−04	4.0240	317.5087
	1923	2^6	1.7553e−05	4.0070	4.4149e−05	4.0063	12733.1994

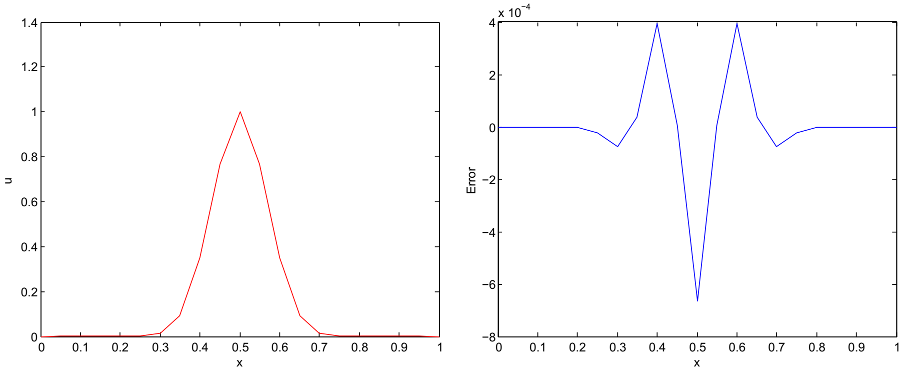

Fig. 1 Graphs of approximate solution and absolute error for Example 4 with $T = 1$, $\alpha = 0.1$, $\beta = 0.01$, $N = 20$, $K = 62$

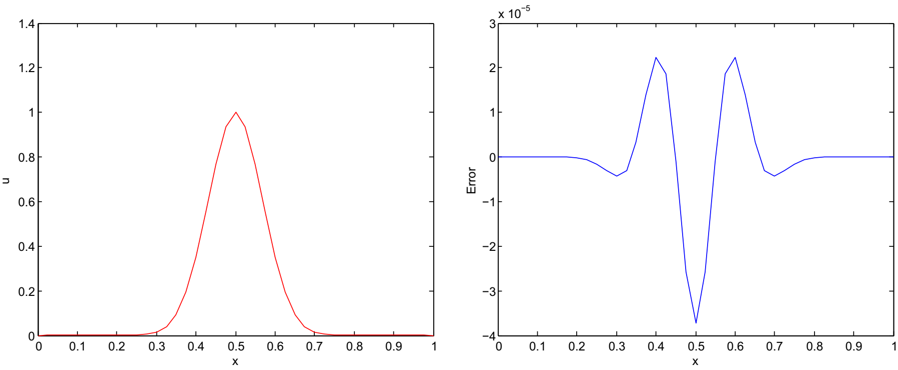

Fig. 2 Graphs of approximate solution and absolute error for Example 4 with $T = 1, \alpha = 0.5, \beta = 0.01$, $N = 40$, $K = 366$

Table 9 displays the L^2 and L^∞ errors and convergence orders in space with $T = 1, \alpha = 0.2, 0.6$, and the time step τ and space step h are chosen such that $\tau^{3-\alpha} \approx h^4$ as in [41], and Table 10 displays the numerical results with $T = 5, \alpha = 0.4, 0.8$. Figure 1 presents the graphs of approximate solution and absolute error for Example 4 with $T = 1, \alpha = 0.1, \beta = 0.01, N = 20, K = 62$. Figure 2 presents the graphs of approximate solution and absolute error for Example 4 with $T = 1, \alpha = 0.5$, $\beta = 0.01, N = 40, K = 366$.

From Tables 7, 8, 9, and 10 and Figs. 1 and 2, we can find the numerical results are in agreement with the theoretical analysis in this case.

6 Conclusion

In this paper, we propose a high-order finite difference/orthogonal spline collocation method for the two-dimensional fourth-order fractional reaction-diffusion equation (1.1). We give strict convergence analysis, and the convergence orders are $O(\tau^{3-\alpha} + h^{r+1})$. To the best knowledge of the authors, there are few works on numerical methods with convergence order $3 - \alpha$ for the fourth-order fractional diffusion equation. We present enough numerical experiments to verify the theoretical analysis, and the comparisons with other methods are also given, which exhibit better accuracy than some the existing numerical methods. In future work, we would extend the present methods with the alternating direct implicit technique to deal with high-order problems with high-order accuracy in both time and space.

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Appendix A

Proof Since the first step equation is easier to treat, we will only consider the case for the time steps $n \geq 2$. By using the reformulations (3.14), (3.18) and (3.19), (3.20) can be rewritten as

$$(\overline{u}^n,\chi)-\kappa\alpha_0\beta_0^{-1}(\Delta u^n,\chi)-\alpha_0\beta_0^{-1}(\Delta v^n,\chi) \tag{A.63}$$
$$=p(\overline{u}^{n-1},\chi)+\sum_{k=2}^{n-1}\overline{d}_{n-k}^n(\overline{u}^{n-k},\chi)+\overline{d}_0^n(u^0,\chi)+\alpha_0\beta_0^{-1}(f^n,\chi),\ \forall\chi$$
$$\in H_0^1(\Omega),\ 2\leq n\leq K,$$

$$(v^n,\psi)=-(\Delta u^n,\psi),\ \forall\psi\in H_0^1(\Omega),\quad 2\leq n\leq K, \tag{A.64}$$

where $(\cdot,\cdot)$ is the usual L^2-inner product, $\|\cdot\|$ is corresponding norm.

By choosing $\chi=2\overline{u}^n$, $\psi=2\alpha_0\beta_0^{-1}\Delta\overline{u}^n$ in (A.63) and (A.64), respectively, we subtract the resulting equations to obtain

$$(\overline{u}^n,2\overline{u}^n)-\kappa\alpha_0\beta_0^{-1}(\Delta u^n,2\overline{u}^n)+\alpha_0\beta_0^{-1}(\Delta u^n,2\Delta\overline{u}^n) \tag{A.65}$$
$$=p(\overline{u}^{n-1},2\overline{u}^n)+\sum_{k=2}^{n-1}\overline{d}_{n-k}^n(\overline{u}^{n-k},2\overline{u}^n)+\overline{d}_0^n(u^0,2\overline{u}^n)$$
$$+\alpha_0\beta_0^{-1}(f^n,2\overline{u}^n),\ \ 2\leq n\leq K.$$

In virtue of the identity

$$2(\Delta u^n,\Delta\overline{u}^n)=\|\Delta u^n\|^2+\|\Delta\overline{u}^n\|^2-p^2\|\Delta u^{n-1}\|^2,$$

and

$$2(\nabla u^n,\nabla\overline{u}^n)=\|\nabla\overline{u}^n\|^2+\|\nabla u^n\|^2-p^2\|\nabla u^{n-1}\|^2.$$

Substituting the above two equation to (A.65), using Lemma 4 and Schwartz inequality, we have

$$2\|\overline{u}^n\|^2+\kappa\alpha_0\beta_0^{-1}\|\nabla\overline{u}^n\|^2+\kappa\alpha_0\beta_0^{-1}\|\nabla u^n\|^2-p^2\kappa\alpha_0\beta_0^{-1}\|\nabla u^{n-1}\|^2$$
$$+\alpha_0\beta_0^{-1}\|\Delta u^n\|^2+\alpha_0\beta_0^{-1}\|\Delta\overline{u}^n\|^2-p^2\alpha_0\beta_0^{-1}\|\Delta u^{n-1}\|^2$$
$$\leq p\|\overline{u}^{n-1}\|^2+\sum_{k=2}^{n-1}\overline{d}_{n-k}^n\|\overline{u}^{n-k}\|^2+2\overline{d}_0^n\|u^0\|^2+2\overline{d}_0^n\|(\overline{d}_0^n)^{-1}\alpha_0\beta_0^{-1}f^n\|^2$$
$$+(p+\sum_{k=2}^{n-1}\overline{d}_{n-k}^n+\overline{d}_0^n)\|\overline{u}^n\|^2,\ \ 2\leq n\leq K.$$

From (1) and (3) of Lemma 4, we attain

$$\begin{aligned}
&\|\overline{u}^n\|^2+\kappa\alpha_0\beta_0^{-1}\|\nabla u^n\|^2+\alpha_0\beta_0^{-1}\|\Delta u^n\|^2 \\
\le\ & p\|\overline{u}^{n-1}\|^2+p^2\kappa\alpha_0\beta_0^{-1}\|\nabla u^{n-1}\|^2+p^2\alpha_0\beta_0^{-1}\|\Delta u^{n-1}\|^2 \\
&+\sum_{k=2}^{n-1}\overline{d}_{n-k}^n\|\overline{u}^{n-k}\|^2+2\overline{d}_0^n\|u^0\|^2+2\overline{d}_0^n\|(\overline{d}_0^n)^{-1}\alpha_0\beta_0^{-1}f^n\|^2 \\
\le\ & p\left(\|\overline{u}^{n-1}\|^2+\kappa\alpha_0\beta_0^{-1}\|\nabla u^{n-1}\|^2+\alpha_0\beta_0^{-1}\|\Delta u^{n-1}\|^2\right) \\
&+\sum_{k=2}^{n-1}\overline{d}_{n-k}^n\left(\|\overline{u}^{n-k}\|^2+\kappa\alpha_0\beta_0^{-1}\|\nabla u^{n-k}\|^2+\alpha_0\beta_0^{-1}\|\Delta u^{n-k}\|^2\right) \\
&+2\overline{d}_0^n\|u^0\|^2+2\overline{d}_0^n\|(\overline{d}_0^n)^{-1}\alpha_0\beta_0^{-1}f^n\|^2,\quad 2\le n\le K.
\end{aligned}\tag{A.66}$$

Next we will prove the following estimates (A.67) by the mathematical induction.

$$\begin{aligned}
&\|\overline{u}^n\|^2+\kappa\alpha_0\beta_0^{-1}\|\nabla u^n\|^2+\alpha_0\beta_0^{-1}\|\Delta u^n\|^2 \\
\le\ & 2(\|u^0\|^2+\|(\overline{d}_0^n)^{-1}\alpha_0\beta_0^{-1}f^n\|^2),\quad 2\le n\le K.
\end{aligned}\tag{A.67}$$

It is easy to check that (A.67) hold for the case $n=2$. Assuming the estimate (A.67) is true for $n=3,4,\cdots,i-1$, we want to prove that it also true for $n=i$, and deduce from (A.66)

$$\begin{aligned}
&\|\overline{u}^i\|^2+\kappa\alpha_0\beta_0^{-1}\|\nabla u^i\|^2+\alpha_0\beta_0^{-1}\|\Delta u^i\|^2 \\
\le\ & 2(p+\sum_{k=2}^{i-1}\overline{d}_{i-k}^i+\overline{d}_0^i)(\|u^0\|^2+\|(\overline{d}_0^i)^{-1}\alpha_0\beta_0^{-1}f^i\|^2) \\
\le\ & 2(\|u^0\|^2+\|(\overline{d}_0^i)^{-1}\alpha_0\beta_0^{-1}f^i\|^2).
\end{aligned}$$

Thus the estimate (A.67) is proven.

From Lemma 4(1) and (4), we have

$$\|(\overline{d}_0^i)^{-1}\alpha_0\beta_0^{-1}f^i\|\le\frac{i^\alpha\tau^\alpha\Gamma(2-\alpha)}{(2-\alpha)(1-\alpha)}\|f^i\|.\tag{A.68}$$

Through a recombination of the terms in (A.67) and (A.68), we obtain

$$\begin{aligned}
&\|\overline{u}^n\|^2+\kappa\alpha_0\beta_0^{-1}\|\nabla u^n\|^2+\alpha_0\beta_0^{-1}\|\Delta u^n\|^2 \\
\le\ & 2(\|u^0\|^2+\frac{T^{2\alpha}(\Gamma(2-\alpha))^2}{(2-\alpha)^2(1-\alpha)^2}\|f^n\|^2),\quad 2\le n\le K.
\end{aligned}\tag{A.69}$$

Finally, since $\overline{u}^n=u^n-pu^{n-1}$, we now turn to estimate $\|u^n\|^2$. Applying the triangle inequality and (A.69) yields

$$\begin{aligned}
\|u^n\|=\|\overline{u}^n+pu^{n-1}\|&\le\|\overline{u}^n\|+p\|u^{n-1}\| \\
&\le\sqrt{2}\left(\|u^0\|+\frac{T^\alpha\Gamma(2-\alpha)}{(2-\alpha)(1-\alpha)}\|f^n\|\right)+p\|u^{n-1}\|,
\end{aligned}\tag{A.70}$$

using arguments similar to (A.70), we have

$$\begin{aligned}\|u^n\| &\leq \sqrt{2}\left(\|u^0\| + \frac{T^\alpha\Gamma(2-\alpha)}{(2-\alpha)(1-\alpha)}\|f^n\|\right)\\ &\quad +p\left(\sqrt{2}(\|u^0\| + \frac{T^\alpha\Gamma(2-\alpha)}{(2-\alpha)(1-\alpha)}\|f^{n-1}\|) + p\|u^{n-2}\|\right)\\ &\leq \sqrt{2}\left(1+p+p^2+\cdots+p^n\right)\left(\|u^0\| + \frac{T^\alpha\Gamma(2-\alpha)}{(2-\alpha)(1-\alpha)}\max_{0\leq j\leq n}\|f^j\|\right)\\ &\leq 3\sqrt{2}\left(\|u^0\| + \frac{T^\alpha\Gamma(2-\alpha)}{(2-\alpha)(1-\alpha)}\max_{0\leq j\leq n}\|f^j\|\right).\end{aligned}$$

Combining the above estimate with (A.69) gives (3.21). The proof is completed. □

Appendix B

Proof Firstly, by using the definition $\Delta\overline{\xi}^n = \Delta\xi^n - p\Delta\xi^{n-1}$, we have

$$\begin{aligned}\langle\Delta\xi^n, 2\Delta\overline{\xi}^n\rangle &= \langle\Delta\xi^n, 2\Delta\xi^n - 2p\Delta\xi^{n-1}\rangle\\ &= 2\langle\Delta\xi^n, \Delta\xi^n\rangle - 2p\langle\Delta\xi^n, \Delta\xi^{n-1}\rangle,\end{aligned}$$

then,

$$2p\langle\Delta\xi^n, \Delta\xi^{n-1}\rangle = 2\langle\Delta\xi^n, \Delta\xi^n\rangle - \langle\Delta\xi^n, 2\Delta\overline{\xi}^n\rangle, \tag{B.71}$$

Also,

$$\begin{aligned}\langle\Delta\xi^n, 2\Delta\overline{\xi}^n\rangle &= \langle\Delta\overline{\xi}^n + p\Delta\xi^{n-1}, 2\Delta\overline{\xi}^n\rangle\\ &= 2\langle\Delta\overline{\xi}^n, \Delta\overline{\xi}^n\rangle + 2p\langle\Delta\xi^{n-1}, \Delta\xi^n - p\Delta\xi^{n-1}\rangle\\ &= 2\langle\Delta\overline{\xi}^n, \Delta\overline{\xi}^n\rangle + 2p\langle\Delta\xi^{n-1}, \Delta\xi^n\rangle - 2p^2\langle\Delta\xi^{n-1}, \Delta\xi^{n-1}\rangle.\end{aligned} \tag{B.72}$$

Substituting (B.71) into (B.72) to get (4.48), the proof is completed. □

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