Numer Algor (2018) 79:941–956
https://doi.org/10.1007/s11075-017-0468-9

ORIGINAL PAPER

An inertial subgradient-type method for solving single-valued variational inequalities and fixed point problems

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Received: 12 April 2017 / Accepted: 27 December 2017 / Published online: 10 February 2018

Abstract In this paper, we introduce an inertial subgradient-type algorithm to find the common element of fixed point set of a family of nonexpansive mappings and the solution set of the single-valued variational inequality problem. Under the assumption that the mapping is monotone and Lipschitz continuous, we show that the sequence generated by our algorithm converges strongly to some common element of the fixed set and the solution set. Moreover, preliminary numerical experiments are also reported.

Keywords Inertial subgradient-type method · Single-valued variational inequalities · Nonexpansive mapping · Lipschitz continuous mapping · Fixed point problems

1 Introduction

The well-known variational inequality problem is to find $x^* \in C$ satisfying

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall \ y \in C, \tag{1.1}$$

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where C is a nonempty closed convex subset in real Hilbert space $\mathbb{H}$, F is a single-valued mapping on $\mathbb{H}$, and $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the usual inner product and norm in $\mathbb{H}$, respectively. Let $VI(F, C)$ be the solution set of the problem (1.1), i.e., $VI(F, C) = \{x^* \in C : \langle F(x^*), y - x^*\rangle \geq 0, \ \forall \ y \in C\}$. Let $P_C(x)$ be the projection of x onto the nonempty closed convex set C, i.e, $P_C(x) = \arg\min_{y\in C} \|x - y\|$.

A mapping $F : \mathbb{H} \to \mathbb{H}$ is called monotone, if

$$\langle Fx - Fy, x - y\rangle \geq 0, \ \forall \ x, y \in \mathbb{H}. \tag{1.2}$$

A mapping $F : \mathbb{H} \to \mathbb{H}$ is called strongly monotone, if there exists a constant $\gamma > 0$ satisfying

$$\langle Fx - Fy, x - y\rangle \geq \gamma\|x - y\|^2, \ \forall \ x, y \in \mathbb{H}. \tag{1.3}$$

A mapping $F : \mathbb{H} \to \mathbb{H}$ is called $\kappa-$inverse strongly monotone, if there exists a constant $\kappa > 0$ satisfying

$$\langle Fx - Fy, x - y\rangle \geq \kappa\|F(x) - F(y)\|^2, \ \forall \ x, y \in \mathbb{H}; \tag{1.4}$$

A mapping $F : \mathbb{H} \to \mathbb{H}$ is called $L-$Lipschitz continuous on $\mathbb{H}$, if there exists a scalar $L > 0$ satisfying

$$\|Fx - Fy\| \leq L\|x - y\|, \ \forall \ x, y \in \mathbb{H}. \tag{1.5}$$

A mapping $T : C \to C$ is called nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall \ x, y \in C. \tag{1.6}$$

Let $\mathrm{Fix}(T)$ be the set of fixed point of T in C, *i.e.*, $\mathrm{Fix}(T) = \{x \in C : T(x) = x\}$.

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems. Due to this, the research of algorithms for solving variational inequality problem (1.1) has received extensive attention; see [10, 11, 13–19, 26, 28–30] and the references therein. In 1970, Sibony [25] proposed the following gradient projection algorithm:

$$x_{i+1} := P_C[x_i - \alpha F(x_i)]. \tag{1.7}$$

In [25], the mapping is required to be strongly monotone and Lipschitz continuous. In order to weak the assumption, Korpelevich [20] introduced an extragradient algorithm as follows:

$$\begin{cases} y_i := P_C[x_i - \alpha F(x_i)], \\ x_{i+1} := P_C[x_i - \alpha F(y_i)]. \end{cases} \tag{1.8}$$

In this method, the mapping is required to be monotone and Lipschitz continuous. We note that the projection onto a closed convex set C is related to a minimum distance problem. If C is a general closed convex set, this might be computationally expensive. To overcome the difficulty, Censor [11] proposed the following subgradient extragradient algorithm for solving single-valued variational inequality:

$$\begin{cases} y_i = P_C(x_i - \mu F(x_i)), \\ H_i = \{x \in \mathbb{H} : \langle (x_i - \mu F(x_i)) - y_i, x - y_i\rangle \leq 0\}, \\ x_{i+1} = P_{H_i}(x_i - \mu F(y_i)), \end{cases} \tag{1.9}$$

Under the assumption that the mapping F is monotone and Lipschitz continuous, they proved that the sequences $\{x_i\}$ and $\{y_i\}$ generated by (1.9) converge weakly to the same point $x^* \in VI(F, C)$ with $x^* = \lim_{i\to\infty} P_{VI(F,C)}(x_i)$.

The inertial methods have been studied extensively in the literature[1–3, 5, 7–9, 22]. Recently, Bot and Csetnek [4] proposed an inertial version of the Krasnosel'skiĭ-Mann algorithm for approximating the set of fixed points of a nonexpansive operator. Specifically, given x_{i-1} and x_i , the next point x_{i+1} is determined via

$$\begin{cases} w_i := x_i + \theta_i(x_i - x_{i-1}), \\ x_{i+1} := (1-\alpha_i)w_i + \alpha_i F(w_i), \ \forall i \geq 1, \end{cases} \tag{1.10}$$

where $(\theta_i)_{i\geq 1}$ is nondecreasing with $\theta_1 = 0$ and $0 \leq \theta_i \leq \theta < 1$ for every $i \geq 1$ and $\beta, \sigma, \delta > 0$ such that

$$\delta > \frac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}, \tag{1.11}$$

and

$$0 < \alpha \leq \alpha_i \leq \frac{\delta - \theta[\theta(1+\theta)+\theta\delta+\sigma]}{\delta[1+\theta(1+\theta)+\theta\delta+\sigma]}, \ \ \forall \ i \geq 1. \tag{1.12}$$

It was showed in [4] that the sequence $\{x_i\}$ generated by (1.10) converges weakly to some element in $\mathrm{Fix}(F)$.

Dong [12] proposed the following algorithm by combining inertial terms with the extragradient method:

$$\begin{cases} w_i := x_i + \theta_i(x_i - x_{i-1}), \\ y_i := P_C(w_i - \lambda F(w_i)), \\ x_{i+1} := (1-\alpha_i)w_i + \alpha_i P_C(w_i - \lambda F(y_i)), \end{cases} \tag{1.13}$$

where $\{\theta_i\}$ is nondecreasing with $\theta_1 = 0, 0 \leq \theta_i \leq \theta < 1$ for every $i \geq 1$ and $\tau, \sigma, \delta > 0$ satisfying

$$\delta > \frac{\theta[(1+\lambda L)^2\theta(1+\theta)+(1-\lambda^2L^2)\theta\sigma+\sigma(1+\lambda L)^2]}{1-\lambda^2L^2}, \tag{1.14}$$

and

$$0<\lambda\leq\alpha_i\leq\frac{\delta(1-\lambda^2L^2)-\theta[(1+\lambda L)^2\theta(1+\theta)+(1-\lambda^2L^2)\theta\sigma+\sigma(1+\lambda L)^2]}{\delta[(1+\lambda L)^2\theta(1+\theta)+(1-\lambda^2L^2)\theta\sigma+\sigma(1+\lambda L)^2]}. \tag{1.15}$$

Under the assumptions that the mapping F is monotone and Lipschitz continuous, they proved that the sequence $\{x_i\}$ generated by (1.13) converges weakly to some element of $VI(F, C)$.

Motivated by the research works mentioned above, we presented an inertial subgradient-type algorithm for solving the single-valued variational inequality problems and fixed point problems. In our method, the mapping F is assumed to be monotone and Lipschitz continuous, $T_i : C \to C$ is a nonexpansive mapping for every $i \in N$, and $\Psi := VI(F, C) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Under those assumptions above, we prove that the iterative sequence $\{x_i\}$ generated by our method converges strongly to $P_\Psi(x_0)$. We also compare the performance of our method with the algorithm iEgA in [12] through numerical experiments. Now let us compare our Algorithm 3.1 with the method (1.13). First, we incorporate the inertial effects into subgradient-type method rather than extragradient method in [12]. Secondly, the proof methods of the convergence for the two algorithms are also different. In our method, we do not need to apply the Lemma 2.1 in [12] as in [4]. In addition, the assumption for α_i is weaker in our method than in [12]. In [12], the parameter α_i is closely related to the parameters λ, L and θ; see (1.14), (1.15) and Algorithm 3.1. At the same time, we obtain the strong convergence results by applying the hybrid projection step; see Step 5 in Algorithm 3.1. Finally, it is worth mentioning that we answer the open problem proposed by the authors in [12].

This paper is organized as follows. We recall some concepts and propositions in the next section and describe our algorithm formally in Section 3. Numerical experiments are reported in Section 4.

2 Preliminaries

In this section, we shall recall some notations, definitions and other results, which will be used in the sequel.

Lemma 2.1 [21] *Let C be a nonempty, closed and convex subset of real Hilbert space $\mathbb{H}$. If $T : C \to C$ is a nonexpansive mapping and the fixed point set $Fix(T)$ of T is nonempty, then $Fix(T)$ is closed and convex.*

In this paper, we adopt the following notations.

- $x_i \rightharpoonup x$ stands for the weak convergence of $\{x_i\}$ to x;
- $x_i \to x$ stands for the strong convergence of $\{x_i\}$ to x;
- $\omega_w(x_i) := \{x \in \mathbb{H} : x_{i_j} \rightharpoonup x$ for some subsequence $\{i_j\}$ of $\{i\}\}$.

It is known that $\mathbb{H}$ satisfies the Opial's condition (see [23]) that, for any sequence $\{x_i\}$ with $x_i \rightharpoonup x$, the following inequality holds

$$\liminf_{i\to\infty} \|x_i - x\| < \liminf_{i\to\infty} \|x_i - y\|, \quad \forall \ y \in \mathbb{H} \ \text{ with } \ y \neq x.$$

Lemma 2.2 [6] *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{H}$, and $T_i : C \to C (i = 1, 2, \cdots)$ be nonexpansive mapping. If $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $T = \sum_{i=1}^{\infty} k_i T_i$, where $\sum_{i=1}^{\infty} k_i = 1, k_i \in (0, 1)$, then T is a nonexpansive mapping on C and $Fix(T) = \bigcap_{i=1}^{\infty} Fix(T_i)$.*

Lemma 2.3 [24] *Let C be a nonempty closed convex subset of a Hilbert space $\mathbb{H}$. Let $x \in \mathbb{H}$ and let $\{x_i\}$ be a sequence in $\mathbb{H}$. If $\omega_w(x_i) \subseteq C$, and*

$$\|x_i - x\| \leq \|x - P_C(x)\|, \quad \forall \ i \in N,$$

then the sequence $\{x_i\}$ converges strongly to $P_C(x)$.

Lemma 2.4 [27] *Let C be a nonempty closed convex subset of a Hilbert space $\mathbb{H}$. Let $F : C \to \mathbb{H}$ be a monotone and hemicontinuous mapping and $x^* \in C$. Then*

$$x^* \in VI(F, C) \Longleftrightarrow \langle Fx, x - x^* \rangle \geq 0, \quad \forall \; x \in C.$$

Proposition 2.1 [31] *Let C be a nonempty closed convex subset of $\mathbb{H}$. For any $x, y \in \mathbb{H}$ and $z \in C$*

(i) $\|x - P_C(x)\| \leq \|x - y\|$;
(ii) $\langle x - P_C(x), z - P_C(x) \rangle \leq 0$;
(iii) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$;
(iv) $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|P_C(x) - x\|^2$.

One can easily show that

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \quad \forall \; \lambda \in [0, 1]. \tag{2.1}$$

3 Main results

Let the infinite mapping sequence $\{T_j\}_{j=1}^{\infty}$ be nonexpansive on C and

$$L_i = \sum_{j=1}^{i} (l_j / k_i) S_j, \quad k_i = \sum_{j=1}^{i} l_j, \quad i \geq 1, \tag{3.1}$$

where $\{l_j\}_{j=1}^{\infty}$ satisfying $l_j > 0$ and

$$\sum_{j=1}^{\infty} l_j = l < \infty, \quad S_j(x) = T_j P_C(x), \quad \forall \; x \in \mathbb{H}. \tag{3.2}$$

Remark 3.1 From (3.2), we know that S_j is nonexpansive for every $j \in N$.

In the following Algorithm 3.1, L denotes the modulus of Lipschitz continuous mapping F; see (1.5).

Algorithm 3.1 *Choose $x_0, x_1 \in \mathbb{H}$, the parameter $\lambda \in (0, \frac{1}{L})$, and the sequences $\{\theta_n\} \in [0, \theta)$, $\{\alpha_n\} \in [a, 1]$, $\{\beta_n\} \in [b, c]$, where $\theta \in (0, 1)$ and $a, b, c \in (0, 1)$. Set $i = 0$.*

Step 1. *Compute $w_i = x_i + \theta_i (x_i - x_{i-1})$.*

Step 2. *Compute*

$$y_i = P_C(w_i - \lambda F(w_i)). \tag{3.3}$$

Step 3. *Compute*

$$z_i = P_{H_i}(w_i - \lambda F(y_i)), \tag{3.4}$$

where

$$H_i := \{x \in \mathbb{H} : \langle w_i - \lambda F(w_i) - y_i, x - y_i\rangle \leq 0\}. \tag{3.5}$$

Step 4. *Compute*

$$t_i = (1-\alpha_i)w_i + \alpha_i[\beta_i z_i + (1-\beta_i)L_i z_i]. \tag{3.6}$$

where L_i is the mapping defined by (3.1).

Step 5. *Compute $x_{i+1} = P_{A_i \bigcap B_i}(x_0)$, where*

$$A_i := \{x \in C : \langle x_i - x, x_0 - x_i\rangle \geq 0\}, \tag{3.7}$$

$$B_i := \{x \in C : \|t_i - x\| \leq \|w_i - x\|\}. \tag{3.8}$$

Let $i := i+1$ and go to Step 1.

Remark 3.2 In view of Proposition 2.1(i) and (3.3), we have

$$\langle (w_i - \lambda F(w_i)) - y_i, x - y_i\rangle \leq 0, \quad \forall\ x \in C.$$

Therefore, $C \subseteq H_i$.

From now on, we adopt the following assumptions:

(A_1) the mapping $F : \mathbb{H} \to \mathbb{H}$ is monotone and $L-$Lipschitz continuous on $\mathbb{H}$ with constant $L > 0$;
(A_2) the mapping $T_i : C \to C$ is nonexpansive for every $i \in N$;
(A_3) $\Psi := VI(F, C) \bigcap \Gamma \neq \emptyset$, where $\Gamma := \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$ and $\text{Fix}(T_i) := \{x \in C : T_i x = x\}$.

Proposition 3.1 *Let L_i be defined by* (3.1) *and* (3.2)*, and $T = \frac{1}{l}\sum_{j=1}^{\infty} l_j S_j$, then both T and L_i are nonexpansive and $\Gamma = Fix(T) \subseteq Fix(L_i)$;*

Proof Let $\text{Fix}(T_j P_C) := \{x \in \mathbb{H} : T_j P_C(x) = x\}$ and $\text{Fix}(L_i) := \{x \in \mathbb{H} : L_i(x) = x\}$. It is obvious that $\text{Fix}(T_j) = \text{Fix}(T_j P_C)$ for every nonexpansive mapping T_j on C. According to the definition of S_j, we have $\text{Fix}(T_j) = \text{Fix}(S_j)$ for every $j \in N$ and hence $\bigcap_{j=1}^{\infty} \text{Fix}(S_j) = \Gamma$. Applying (3.2), we have

$$\frac{l_j}{l} \in (0,1), \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{l_j}{l} = 1. \tag{3.9}$$

It follows from Lemma 2.2 that the mapping T is nonexpansive and $\text{Fix}(T) = \bigcap_{j=1}^{\infty} \text{Fix}(S_j) = \Gamma$.

Next, we show that the mapping L_i is nonexpansive for every $i \in N$ and $\Gamma \subseteq \mathrm{Fix}(L_i)$. Suppose that $x \in \Gamma$. By (3.1), we have

$$L_i x = \sum_{j=1}^{i}(\frac{l_j}{k_i})S_j x = \sum_{j=1}^{i}(\frac{l_j}{k_i})x = x\frac{\sum_{j=1}^{i} l_j}{k_i} = x. \tag{3.10}$$

Therefore, $\Gamma \subseteq \mathrm{Fix}(L_i)$.

Besides, for every $x, y \in \mathbb{H}$, we have

$$\begin{aligned}\|L_i x - L_i y\| &\leq \|\sum_{j=1}^{i}(\frac{l_j}{k_i})S_j x - \sum_{j=1}^{i}(\frac{l_j}{k_i})S_j y\| \\ &\leq \sum_{j=1}^{i}(\frac{l_j}{k_i})\|S_j x - S_j y\| \\ &\leq \sum_{j=1}^{i}(\frac{l_j}{k_i})\|x - y\| \\ &\leq \|x - y\|, \end{aligned} \tag{3.11}$$

which implies that the mapping L_i is nonexpansive for every $i \in N$. This completes the proof. □

Proposition 3.2 *Suppose that the mapping $T_i (i = 1, 2, \cdots)$ is nonexpansive on C and the mapping F is monotone and $L-$Lipschitz continuous on $\mathbb{H}$. If $x^* \in \Psi$, then*

$$\begin{aligned}\|t_i - x^*\|^2 \leq\; &\|w_i - x^*\|^2 - (1 - \lambda L)\alpha_i\|w_i - y_i\|^2 - (1-\lambda L)\alpha_i\|y_i - z_i\|^2 \\ &-\alpha_i\beta_i(1-\beta_i)\|L_i z_i - z_i\|^2. \end{aligned} \tag{3.12}$$

Proof Applying Proposition 2.1 (iv), from (3.4), we have

$$\begin{aligned}\|z_i - x^*\|^2 &= \|P_{H_i}(w_i - \lambda F(y_i)) - x^*\|^2 \\ &\leq \|(w_i - \lambda F(y_i)) - x^*\|^2 - \|(w_i - \lambda F(y_i)) - z_i\|^2 \\ &= \|w_i - x^*\|^2 - \|w_i - z_i\|^2 - 2\lambda\langle F(y_i), z_i - x^*\rangle. \end{aligned} \tag{3.13}$$

Since $x^* \in VI(F, C)$, we have

$$\langle F(x^*), x - x^*\rangle \geq 0, \quad \forall\ x \in C. \tag{3.14}$$

By the monotonicity of F, we get

$$\langle F(x), x - x^*\rangle \geq 0, \quad \forall\ x \in C. \tag{3.15}$$

Since $y_i = P_C(w_i - \lambda F(w_i))$,

$$\langle F(y_i), y_i - x^*\rangle \geq 0, \tag{3.16}$$

which implies that

$$\langle F(y_i), z_i - x^*\rangle \geq \langle F(y_i), z_i - y_i\rangle. \tag{3.17}$$

Combining (3.13) and (3.17), we obtain

$$\begin{aligned}\|z_i - x^*\|^2 &\le \|w_i - x^*\|^2 - \|w_i - z_i\|^2 - 2\lambda\langle F(y_i), z_i - y_i\rangle \\ &= \|w_i - x^*\|^2 - \|w_i - y_i\|^2 - \|y_i - z_i\|^2 + 2\langle w_i - \lambda F(y_i) - y_i, z_i - y_i\rangle \\ &= \|w_i - x^*\|^2 - \|w_i - y_i\|^2 - \|y_i - z_i\|^2 + 2\langle w_i - \lambda F(w_i) - y_i, z_i - y_i\rangle \\ &\quad + 2\lambda\langle F(w_i) - F(y_i), z_i - y_i\rangle \\ &\le \|w_i - x^*\|^2 - \|w_i - y_i\|^2 - \|y_i - z_i\|^2 + 2\lambda\langle F(w_i) - F(y_i), z_i - y_i\rangle \\ &\le \|w_i - x^*\|^2 - \|w_i - y_i\|^2 - \|y_i - z_i\|^2 + 2\lambda L\|w_i - y_i\|\|z_i - y_i\| \\ &\le \|w_i - x^*\|^2 - (1-\lambda L)\|w_i - y_i\|^2 - (1-\lambda L)\|y_i - z_i\|^2, \end{aligned} \tag{3.18}$$

where the second inequality follows from (3.4) and (3.5), the third inequality follows from the $L-$Lipschitz continuity of F.

Since $x^* \in \Psi$, we have

$$\begin{aligned}\|t_i - x^*\|^2 &= \|(1-\alpha_i)w_i + \alpha_i[\beta_i z_i + (1-\beta_i)L_i z_i] - x^*\|^2 \\ &\le (1-\alpha_i)\|w_i - x^*\|^2 + \alpha_i\|\beta_i z_i + (1-\beta_i)L_i z_i - x^*\|^2 \\ &= (1-\alpha_i)\|w_i - x^*\|^2 + \alpha_i\beta_i\|z_i - x^*\|^2 + \alpha_i(1-\beta_i)\|L_i z_i - x^*\|^2 \\ &\quad - \alpha_i\beta_i(1-\beta_i)\|L_i z_i - z_i\|^2 \\ &\le (1-\alpha_i)\|w_i - x^*\|^2 + \alpha_i\|z_i - x^*\|^2 - \alpha_i\beta_i(1-\beta_i)\|L_i z_i - z_i\|^2, \end{aligned} \tag{3.19}$$

where the first inequality and the second equality follow from (2.1), the second inequality follows from Proposition 3.1. Combining (3.18) and (3.19), we get the conclusion. This completes the proof. □

Proposition 3.3 *If $\Psi \neq \emptyset$, then*

(i) x_{i+1} *is well defined;*
(ii) $\|x_i - x_0\| \le \|x_{i+1} - x_0\| \le \|x_0 - P_\Psi(x_0)\|$.

Proof (i) We only need to prove that $A_i \bigcap B_i$ is a nonempty closed and convex subset of $\mathbb{H}$.

First, we prove that $A_i \bigcap B_i$ is a closed and convex subset of $\mathbb{H}$ for every $i \in N$. It is obvious that A_i is closed and convex and B_i is closed for every $i \in N$. Since

$$B_i = \{x \in C : \langle t_i - w_i, \frac{(w_i + t_i)}{2} - x\rangle \le 0\}, \tag{3.20}$$

B_i is also convex for every $i \in N$.

Next, we prove that $\Psi \subseteq A_i \bigcap B_i$ by induction on i and hence $A_i \bigcap B_i$ is nonempty. In view of Proposition 3.2, we have that $\Psi \subseteq B_i$ for every $i \in N$. For $i = 0$, we have $A_0 = C$ and hence $\Psi \subseteq A_0 \bigcap B_0$. Now, suppose that x_i is given and $\Psi \subseteq A_i \bigcap B_i$ for some $i \in N$. Since $x_{i+1} = P_{A_i \bigcap B_i}(x_0)$ and $\Psi \subseteq A_i \bigcap B_i$, it follows from Proposition 2.1(ii) that

$$\langle x_{i+1} - x, x_0 - x_{i+1}\rangle \ge 0, \quad \forall\ x \in \Psi. \tag{3.21}$$

Thus, $\Psi \subseteq A_{i+1}$ and hence $\Psi \subseteq A_{i+1} \bigcap B_{i+1}$. Furthermore, by Assumption (A_3), we obtain that $A_{i+1} \bigcap B_{i+1} \neq \emptyset$.

(ii) Since

$$\langle x_i - x, x_0 - x_i \rangle \geq 0, \ \ \forall \ x \in A_i, \tag{3.22}$$

$x_i = P_{A_i}(x_0)$. It follows from $x_{i+1} \in A_i$ that

$$\|x_i - x_0\| = \|P_{A_i}(x_0) - x_0\| \leq \|x_{i+1} - x_0\|. \tag{3.23}$$

Since the solution set $VI(F, C)$ is closed and convex(see [28]), from Lemma 2.1 and Assumption (A_3) we know that Ψ is nonempty closed and convex. Therefore, $P_\Psi(x_0)$ is well defined. Therefore, it follows from $P_\Psi(x_0) \in \Psi \subseteq A_{i+1}$ that

$$\|x_{i+1} - x_0\| = \|P_{A_{i+1}}(x_0) - x_0\| \leq \|P_\Psi(x_0) - x_0\|. \tag{3.24}$$

This completes the proof.

□

Remark 3.3 Proposition 3.3(ii) implies that the sequence $\{\|x_i - x_0\|\}$ converges and hence the sequence $\{x_i\}$ is bounded.

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{H}$, and $\{x_i\}$ be the infinite sequence generated by Algorithm 3.1. If the assumptions (A_1), (A_2) and (A_3) hold, then the sequence $\{x_i\}$ converges strongly to $P_\Psi(x_0)$.*

Proof According to Lemma 2.3, we only need to prove that $w_w(x_i) \subseteq \Psi$, i.e., every weak limit point of $\{x_i\}$ belongs to Ψ. The proof is done in several steps.

Step 1: $\|w_i - t_i\| \to 0$, $\|w_i - y_i\| \to 0$, $\|L_i z_i - z_i\| \to 0$, $\|z_i - x_i\| \to 0$, as $i \to \infty$. By the definition of A_i, we have

$$\langle x_i - x, x_0 - x_i \rangle \geq 0, \ \ \forall \ x \in A_i. \tag{3.25}$$

Since $x_{i+1} = P_{A_i \bigcap B_i}(x_0)$,

$$\langle x_i - x_{i+1}, x_0 - x_i \rangle \geq 0. \tag{3.26}$$

Note that

$$\|x_{i+1} - x_0\|^2 = \|x_{i+1} - x_i\|^2 + \|x_i - x_0\|^2 + 2\langle x_{i+1} - x_i, x_i - x_0 \rangle. \tag{3.27}$$

Combining (3.26) and (3.27), we have

$$\|x_{i+1} - x_i\|^2 \leq \|x_{i+1} - x_0\|^2 - \|x_i - x_0\|^2. \tag{3.28}$$

In view of Proposition 3.3, taking $i \to \infty$ in (3.28), we obtain

$$\|x_{i+1} - x_i\| \to 0. \tag{3.29}$$

Since

$$\begin{aligned} \|w_i - x_i\| &= \|x_i + \theta_i(x_i - x_{i-1}) - x_i\| \\ &= \theta_i \|x_i - x_{i-1}\| \\ &\leq \theta \|x_i - x_{i-1}\|, \end{aligned} \tag{3.30}$$

it follows from (3.29) and (3.30) that

$$\|w_i - x_i\| \to 0 \text{ as } i \to \infty. \tag{3.31}$$

From (3.29) and (3.31), we have

$$\|w_i - x_{i+1}\| \to 0 \text{ as } i \to \infty. \tag{3.32}$$

Since $x_{i+1} = P_{A_i \bigcap B_i}(x_0)$, we obtain $\|t_i - x_{i+1}\| \leq \|w_i - x_{i+1}\|$. By (3.32), we have

$$\|t_i - x_{i+1}\| \to 0 \text{ as } i \to \infty. \tag{3.33}$$

Combining (3.29) and (3.33), we get

$$\|t_i - x_i\| \to 0 \text{ as } i \to \infty. \tag{3.34}$$

In view of (3.31) and (3.34), it is easy to see that

$$\|t_i - w_i\| \to 0 \text{ as } i \to \infty. \tag{3.35}$$

It follows from Proposition 3.2 that, for every $i \in N$

$$(1-\lambda L)\alpha_i\|w_i - y_i\|^2 \leq \|w_i - t_i\|(\|w_i - x^*\| + \|t_i - x^*\|), \tag{3.36}$$

$$(1-\lambda L)\alpha_i\|y_i - z_i\|^2 \leq \|w_i - t_i\|(\|w_i - x^*\| + \|t_i - x^*\|), \tag{3.37}$$

$$\alpha_i\beta_i(1-\beta_i)\|L_i z_i - z_i\|^2 \leq \|w_i - t_i\|(\|w_i - x^*\| + \|t_i - x^*\|). \tag{3.38}$$

In view of (3.31) and (3.34), from Remark 3.3, we know that both $\{w_i\}$ and $\{t_i\}$ are bounded. Since $\lambda < \frac{1}{L}$, $0 < a \leq \alpha_i$, $0 < b \leq \beta_i \leq c < 1$, from (3.35), we have, as $i \to \infty$,

$$\|w_i - y_i\| \to 0, \quad \|y_i - z_i\| \to 0, \quad \|L_i z_i - z_i\| \to 0. \tag{3.39}$$

Since $\|z_i - x_i\| \leq \|z_i - y_i\| + \|y_i - w_i\| + \|w_i - x_i\|$, it follows from (3.31) and (3.39) that

$$\|z_i - x_i\| \to 0 \text{ as } i \to \infty. \tag{3.40}$$

Since $\{x_i\}$ is bounded, there exists a subsequence $\{x_{i_j}\}$ of $\{x_i\}$ such that $x_{i_j} \rightharpoonup \hat{x}$ as $j \to \infty$.

Step 2: $\hat{x} \in VI(F, C)$. Since $C \subset H_i$, from (3.5), we have

$$\langle w_i - \lambda F(w_i) - y_i, x - y_i\rangle \leq 0, \quad \forall\ x \in C. \tag{3.41}$$

Hence,

$$\begin{aligned}\langle \lambda F(w_i), w_i - x\rangle &= \langle \lambda F(w_i), w_i - y_i\rangle + \langle \lambda F(w_i), y_i - x\rangle \\ &= \langle \lambda F(w_i), w_i - y_i\rangle + \langle w_i - \lambda F(w_i) \\ &\quad -y_i, x - y_i\rangle + \langle w_i - y_i, y_i - x\rangle \\ &\leq \langle \lambda F(w_i), w_i - y_i\rangle + \langle w_i - y_i, y_i - x\rangle \\ &\leq \|y_i - w_i\|(\lambda\|F(w_i)\| + \|y_i - x\|).\end{aligned} \tag{3.42}$$

It follows from the boundedness of $\{w_i\}$ and the continuity of F that $\{F(w_i)\}$ is bounded. Besides, by (3.39), it follows from $\lambda > 0$ that

$$\limsup_{i\to\infty}\langle F(w_i), w_i - x\rangle \leq 0. \tag{3.43}$$

Applying the monotonicity of F, we obtain

$$\limsup_{i\to\infty}\langle F(x), w_i - x\rangle \leq 0. \tag{3.44}$$

Since $x_i - w_i \to 0$ as $i \longrightarrow \infty$, we have that $w_{i_j} \rightharpoonup \hat{x}$ as $j \longrightarrow \infty$. Therefore, we have

$$\langle F(x), \hat{x} - x\rangle \leq 0. \tag{3.45}$$

In view of Lemma 2.4, we have that $\hat{x} \in VI(F, C)$.

Step 3: $\hat{x} \in \Gamma$. Since $x_i - z_i \to 0 (i \to \infty)$, we have that $z_{i_j} \rightharpoonup \hat{x} (j \to \infty)$. In view of Proposition 3.1, we only need to prove $T(\hat{x}) = \hat{x}$. Suppose that $T\hat{x} \neq \hat{x}$. Applying Opial's condition, we have

$$\begin{aligned}\liminf_{j\to\infty}\|z_{i_j} - \hat{x}\| &< \liminf_{j\to\infty}\|z_{i_j} - T(\hat{x})\| \\ &\leq \liminf_{j\to\infty}(\|z_{i_j} - L_{i_j}(z_{i_j})\| + \|L_{i_j}(z_{i_j}) - T(z_{i_j})\| + \|T(z_{i_j}) - T(\hat{x})\|) \\ &\leq \liminf_{j\to\infty}(\|z_{i_j} - L_{i_j}(z_{i_j})\| + \|L_{i_j}(z_{i_j}) - T(z_{i_j})\| + \|z_{i_j} - \hat{x}\|). \end{aligned} \tag{3.46}$$

Since $\{z_i\}$ is bounded and $\{S_i\}$ is nonexpansive, there exists some $M > 0$ such that $M = \sup_{i\geq 1}\|S_i z_i\| < \infty$. Thus, we obtain

$$\begin{aligned}\|L_i z_i - T(z_i)\| &= \|\frac{1}{k_i}\sum_{j=1}^{i} l_j S_j z_i - \frac{1}{l}\sum_{j=1}^{\infty} l_j S_j z_i\| \\ &\leq \frac{l - k_i}{lk_i}\sum_{j=1}^{i} l_j\|S_j z_i\| + \frac{1}{l}\sum_{j=i+1}^{\infty} l_j\|S_j z_i\| \\ &\leq \frac{M(l-k_i)}{lk_i}\sum_{j=1}^{i} l_j + \frac{M}{l}\sum_{j=i+1}^{\infty} l_j \\ &= 2\frac{M}{l}\sum_{j=i+1}^{\infty} l_j. \end{aligned} \tag{3.47}$$

It follows from (3.2) that $\sum_{j=i+1}^{\infty} l_j \to 0$ as $j \to \infty$. In view of (3.47), we have

$$\|L_i z_i - T(z_i)\| \to 0 \ \text{ as } \ i \to \infty. \tag{3.48}$$

Therefore,

$$\liminf_{j\to\infty}\|z_{i_j} - \hat{x}\| < \liminf_{j\to\infty}\|z_{i_j} - \hat{x}\| \tag{3.49}$$

This contradiction implies that $\hat{x} \in \Gamma$ and hence $\hat{x} \in \Psi$. This completes the proof.

□

4 Numerical experiments

In this section, we present some numerical experiments for the proposed algorithm. The MATLAB codes are run on a PC (with Intel(R) Core(TM) i3-2367M CPU @ 1.40 GHZ) under MATLAB Version 7.14.0.739(R2012a) Service Pack 1. Applying our method, we will find the common element of $VI(F, C)$ and $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$. In the following two examples, we take $T_i(x) = \frac{1}{10i}x(i = 1, 2, \cdots)$. In Example 4.1, we choose $\lambda = 0.19, \theta_i = 0.31, \beta_i = 1(i = 1, 2, \cdots)$ for the algorithm iEgA in [12] and $\lambda = 0.05, \alpha_i = 0.79, \beta_i = 0.31, \theta_i = 0.31(i = 1, 2, \cdots)$ for our Algorithm 3.1. In Example 4.2, we choose $\lambda = 0.01, \theta_i = 0.6, \beta_i = 5(i = 1, 2, \cdots)$ for the algorithm iEgA in [12] and $\lambda = 0.02, \alpha_i = 0.88, \beta_i = 0.08, \theta_i = 0.01(i = 1, 2, \cdots)$ for our Algorithm 3.1.

Furthermore, we use the sequence $\{\|x_i - x^*\|\}(i = 1, 2, \cdots)$ to check the convergence of the algorithms, where $x^* \in \Psi$. If the sequence $\{\|x_i - x^*\|\}$ tends to 0 as $i \longrightarrow \infty$, then the sequence $\{x_i\}$ generated by the algorithms converges to some element of the solution set Ψ. In Tables 1 and 2, "Iter." denotes number of iteration and "CPU" denotes the CPU time in seconds. The tolerance ε means that when $\|x_i - x^*\| \le \varepsilon$, the procedure stops. Besides,the "ISE" and "IE" in the graphical annotation denote our Algorithm 3.1 and the algorithm iEgA in [12], respectively. In the following figures, the ordinate denotes the value of $\{\|x_i - x^*\|\}(i = 1, 2, \cdots)$ while the abscissa denotes the number of iterations or the elapsed time. In addition, the graphical annotations "ISE" and "IE" denote our Algorithm 3.1 and the algorithm iEgA in [12], respectively.

Table 1 Example 4.1

	IE		ISE	
Tolerance ε	Iter.	CPU	Iter.	CPU
10	9	0.5588023	3	0.5444024
10^{-1}	22	1.2488073	13	1.0100009
10^{-2}	27	1.4376096	18	1.1888098
10^{-3}	31	1.6368103	22	1.2856101
10^{-4}	36	1.8332122	27	1.3924104
10^{-5}	41	1.9940135	31	1.5080105
10^{-6}	46	2.1600150	35	1.6204109
10^{-7}	51	2.3244174	40	1.7728113
10^{-8}	56	2.4652192	44	1.9011500
10^{-9}	61	2.6048208	48	2.0164119
10^{-10}	66	2.7802200	53	2.1788123
10^{-11}	71	2.9124229	57	2.3112127
10^{-12}	75	3.0596241	61	2.4336131

Table 2 Example 4.2

	IE		ISE	
Tolerance ε	Iter.	CPU	Iter.	CPU
10^{-1}	17	2.1216136	7	2.2056276
10^{-2}	31	3.1668203	12	3.1643190
10^{-3}	46	4.2432272	17	4.1088348
10^{-4}	60	5.2416336	22	5.0156376
10^{-5}	74	6.2712402	27	5.6912402
10^{-6}	88	7.3476471	32	6.5456426
10^{-7}	102	8.6112552	38	7.4948458
10^{-8}	116	9.6252617	43	8.1504484
10^{-9}	130	10.6860685	48	8.9304534
10^{-10}	144	11.7468753	53	9.6408618
10^{-11}	158	12.5736806	58	10.2804659
10^{-12}	172	13.4160860	63	10.9512702
10^{-13}	187	14.4924929	69	11.8404759

Example 4.1 Let

$$C := \{x = (x_1, x_2) \in \mathbb{R}^2_+ : 0 \le x_i \le 100, i = 1, 2\}, \tag{4.1}$$

and $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x) = (x_1 + x_2 + \sin x_1, -x_1 + x_2 + \sin x_2), \quad \forall\ x = (x_1, x_2) \in \mathbb{R}^2. \tag{4.2}$$

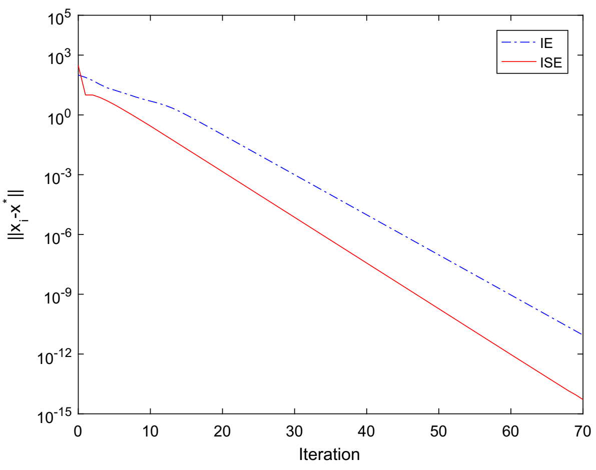

Fig. 1 $\|x_i - x^*\|$ and iterations in Example 4.1

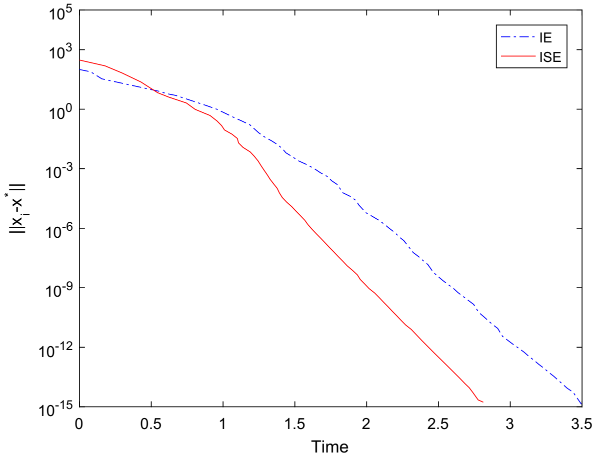

Fig. 2 $\|x_i - x^*\|$ and time in Example 4.1

Example 4.1 was tested in [12], where the mapping F is monotone and $L-$Lipschitz continuous with parameter $L = \sqrt{10}$. It is easy to verify that the point $x^* = 0$ is a common element of $VI(F, C)$ and $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$. Moreover, all the assumptions in Theorem 3.1 are satisfied. In the numerical experiment, we choose the initial point $x_0 = x_1 = (-100, 200) \in \mathbb{R}^2$ for finding an element $x^* \in \Psi$(Table 1 and Figs. 1 and 2).

Example 4.2 Let $n = 100$,

$$C := \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : -1 \le x_1 \le 3, -1 \le x_i \le 1, \ i = 2, 3, \cdots, n\}, \quad (4.3)$$

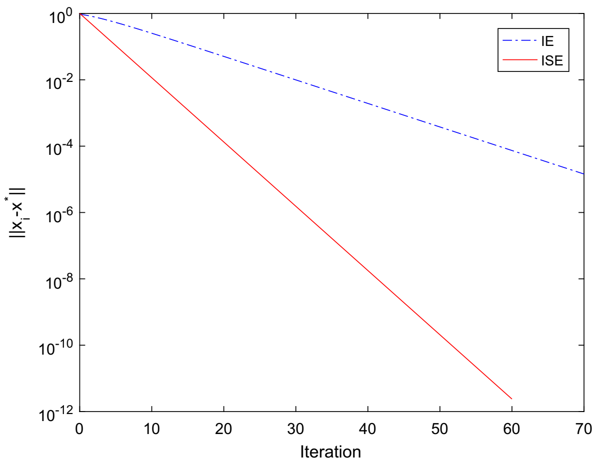

Fig. 3 $\|x_i - x^*\|$ and iterations in Example 4.2

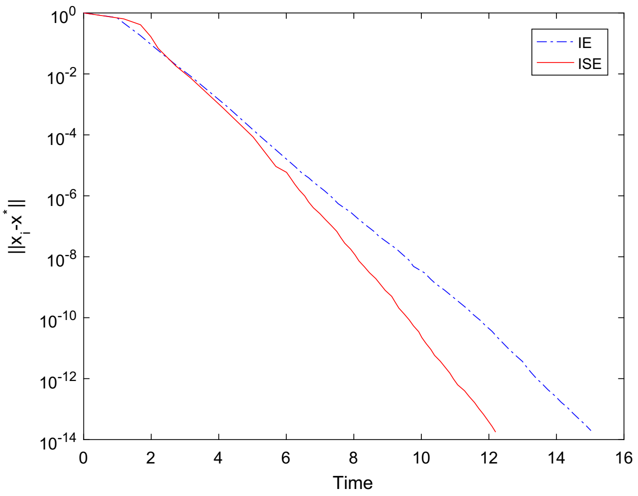

Fig. 4 $\|x_i - x^*\|$ and time in Example 4.2

and $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $F(x) = Mx + d$ with M randomly generated as follows:

$$M = AA^T + B + D, \tag{4.4}$$

where the every entry of the $n \times n$ matrix and the $n \times n$ skew-symmetric matrix B is uniformly generated from $(-5, 5)$, and every diagonal entry of the $n \times n$ diagonal matrix D is uniformly generated from $[1, 100]$ (so M is positive definite), with every entry of d uniformly generated from $[-100, 0]$.

Example 4.2 was tested in [26], where the mapping F is monotone and $L-$Lipschitz continuous with $L = \|M\|$ because of the positive definiteness of the matrix M. It is easy to verify that the point $x^* = 0$ is a common element of $VI(F, C)$ and $\bigcap_{i=1}^{\infty} \text{Fix}(T_i)$. Moreover, all the assumptions in Theorem 3.1 are satisfied. In the numerical experiment, we choose the initial point $x_0 = x_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$ for finding an element $x^* \in \Psi$(Table 2 and Figs. 3 and 4).

Funding information This work is partially supported by the National Natural Science Foundation of China (No. 11771350), Basic and Advanced Research Project of CQ CSTC (No. cstc2016jcyjA0163), Science and Technology Project of Chongqing Municipal Education Committee of China (No. KJ1600433), and Chongqing Research and Innovation Project of Graduate Students (No. CYS16173).

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