Numer Algor (2018) 79:529–553
https://doi.org/10.1007/s11075-017-0449-z

ORIGINAL PAPER

Iterative algorithms for solving fixed point problems and variational inequalities with uniformly continuous monotone operators

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Received: 21 February 2017 / Accepted: 20 November 2017 / Published online: 27 November 2017

Abstract Using the double projection and Halpern methods, we prove two strong convergence results for finding a solution of a variational inequality problem involving uniformly continuous monotone operator which is also a fixed point of a quasi-nonexpansive mapping in a real Hilbert space. In our proposed methods, only two projections onto the feasible set in each iteration are performed, rather than one projection for each tentative step during the Armijo-type search, which represents a considerable saving especially when the projection is computationally expensive. We also give some numerical results which show that our proposed algorithms are efficient and implementable from the numerical point of view.

Keywords Monotone mappings · Double projection Method · Halpern method · Quasi-nonexpansive mapping · Strong convergence · Hilbert spaces

Mathematics Subject Classification (2010) 47H06 · 47H09 · 47J05 · 47J25

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1 Introduction

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$. Let C be a nonempty, closed and convex subset of H and A be a mapping of C into H. Then A is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in C. \tag{1.1}$$

We say that A is L-Lipschitz continuous if there exists a positive constant L such that

$$\|Ax - Ay\| \leq L\|x - y\|, \forall x, y \in C.$$

Let us consider the following variational inequality (for short, VI(A,C)): find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \forall y \in C. \tag{1.2}$$

Let Γ be the set of solutions of VI(A,C) (1.2). It is well known that x solves the VI(A,C) (1.2) if and only if x solves the fixed point equation (see [12, 13, 33] for the details):

$$x = P_C(x - \gamma Ax), \gamma > 0 \text{ and } r_\gamma(x) := x - P_C(x - \gamma Ax) = 0.$$

Variational inequality theory is an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework, e.g., see [3, 4, 13, 20, 22, 33]. This field is dynamic and is experiencing an explosive growth in both theory and applications. Several numerical methods have been developed for solving variational inequality and related optimization problems, see books [5, 12, 22] and the references therein.

The extragradient method, introduced in 1976 by Korpelevich [21] and Antipin [1] for a finite-dimensional space, provides an iterative process converging to a solution of VI(A,C) by only assuming that $C \subset \mathbb{R}^n$ is nonempty, closed and convex and $A : C \to \mathbb{R}^n$ is monotone and L-Lipschitz continuous. Some other methods have been introduced in the literature for finding a solution to VI(A,C) (1.2) when the monotone operator A is continuous in $\mathbb{R}^n$ (see, for example, [14, 36]). Quite recently, Mainge [27] introduced the following projected reflected gradient-type method in $\mathbb{R}^n$ for VI(A,C) (1.2) by incorporating a linesearch procedure that does not require any additional evaluation of P_C when A is *monotone and continuous mapping* in $\mathbb{R}^n$. Similarly, the extragradient method has been further extended to infinite dimensional spaces by many authors; see for instance, [2, 8–10, 15–17, 23, 24, 26, 28, 29, 32, 36, 37].

A mapping $S : C \to C$ is called

- *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C;$$

  and
- *quasi-nonexpansive* if

$$\|Sx - p\| \leq \|x - p\|, \quad \forall x \in C, p \in F(S),$$

where $F(S)$ denotes its fixed point set, i.e.,

$$F(S) := \{x \in C : Sx = x\}.$$

In [31], Nadezhkina and Takahashi obtained weak convergence result for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping and in [32], they introduced an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem (1.2) for a monotone, Lipschitz-continuous mapping using the two well-known methods of hybrid and extragradient and obtained a strong convergence theorem for the sequence generated by this process. Similarly, weak and strong convergence results have been obtained for finding a common element of the set of fixed points of a nonexpansive mapping (or quasi-nonexpansive) and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping using the subgradient extragradient method in [9, 10, 25].

Inspired by the subgradient extragradient method studied by Censor et al. in [9, 10], Kraikaew and Saejung [25] proved the strong convergence of the iterative sequence generated by a combination of subgradient extragradient method and Halpern method for the problem of finding a common element of the solution set of a variational inequality and the fixed-point set of a quasi-nonexpansive mapping in real Hilbert spaces. In particular, they proved the following theorem.

Theorem 1.1 *Let $S : H \to H$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $A : H \to H$ a monotone and L-Lipschitz mapping on C. Let λ be a positive real number such that $\lambda L < 1$. Suppose that $F(S) \cap \Gamma \neq \emptyset$. Let $\{x_n\} \subset H$ be a sequence generated by $x_1 \in H$,*

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n := \{w \in H : \langle x_n - \lambda A x_n - y_n, w - y_n \rangle \leq 0\}, \\ z_n = \alpha_n x_1 + (1 - \alpha_n) P_{T_n}(x_n - \lambda A y_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S z_n, \end{cases}$$

where $\{\beta_n\} \subset [a, b] \subset]0, 1[$ for some $a, b \in]0, 1[$ and $\{\alpha_n\}$ is a sequence in $]0, 1[$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{F(S)\cap\Gamma} x_1$.

We remark here that the framework presented by Kraikaew and Saejung [25] requires the Lipschitz constant of A as an input parameter. Thus, the result cannot be applied to the case when either A is L-Lipschitz continuous but the Lipschitz constant L is unknown or A is uniformly continuous monotone mapping.

Weak convergence result for variational inequality problem (1.2) involving uniformly continuous monotone operator in infinite dimensional Banach spaces was obtained in [16] and its strong convergence result in infinite dimensional Hilbert spaces using the Haugazeau method was given in [6]. We remark that in order to implement the iterative method introduced in [6] one has to calculate, at each iteration step, the metric projection onto the intersection of two half spaces and the feasible set C. This is a drawback to the iterative algorithm introduced in [6] since the projection onto this intersection can be very difficult to compute as the intersection changes from iteration to iteration.

It is our aim in this paper to establish strong convergence results for approximating a solution of VI(A,C) (1.2) when A is a uniformly continuous monotone operator and the solution is also a fixed point of a quasi-nonexpansive mapping in real Hilbert spaces. We propose two convergence methods and prove strong convergence of the sequences generated by our proposed methods. Our proposed algorithms are based on known processes of double projection and Halpern methods and our results extend most of the existing known results on this subject, including [9, 10, 25, 31, 32] from Lipschitz monotone variational inequality to uniformly continuous variational inequality with only two projections onto feasible set C per iteration, unlike other variants, e.g., [11, 35] with projections onto C inside the inner loop for the search. Our scheme and method of proof is different from the method of proof given in [6]. We also give some numerical implementation of our results.

The paper is therefore organized as follows: We first recall some basic results which will be used in the sequel in Section 2 and the main contribution of the paper is given in Section 3. In Section 4, we give some numerical examples of our result and finally in Section 5, we conclude with some final remarks on our next focus on monotone variational inequalities.

2 Preliminaries

Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{2.1}$$

for all $x, y \in H$. Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$,

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{2.2}$$

for all $y \in C$ and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.3}$$

for all $x \in H$ and $y \in C$.

We state the following well-known lemmas which will be used in the sequel.

Lemma 2.1 *Let H be a real Hilbert space. Then there holds the following well-known results:*

$$(i)\;\; \|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H.$$

$$(ii)\;\; \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H.$$

(iii) $\|tx+sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x-y\|^2, \forall x, y \in H, \forall t, s \in \mathbb{R}.$

$$(iv)\ \ 2\langle x-y, x-z\rangle = \|x-y\|^2 + \|x-z\|^2 - |y-z\|^2 \forall x, y, z \in H.$$

Lemma 2.2 (Xu [38]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \ \ n \geq 1,$$

where

(i) $\{a_n\} \subset [0, 1], \ \ \sum \alpha_n = \infty;$
(ii) $\limsup \sigma_n \leq 0$*;*
(iii) $\gamma_n \geq 0; \ \ (n \geq 1), \ \ \Sigma\gamma_n < \infty.$

Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.3 ([18]) *For all $0 \neq v \in H$, $\tilde{y} \in H$, $x \in D^+$ and $\bar{x} \in D^-$, we have that $\|\bar{x} - x\|^2 \geq \|\bar{x} - z\|^2 + \|z - x\|^2$, where z is the unique minimizer of $\frac{1}{2}\|. - x\|^2$ on D where $D := \{y \in H : \langle v, y - \tilde{y}\rangle = 0\}$, $D^+ := \{y \in H : \langle v, y - \tilde{y}\rangle \geq 0\}$, $D^- := \{y \in H : \langle v, y - \tilde{y}\rangle \leq 0\}$.*

Lemma 2.4 ([18]) *Let H_1 and H_2 be two real Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then $A(M)$ is bounded.*

Lemma 2.5 (See Lemma 7.1.7 of [39]) *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $A : C \to H$ be a monotone and continuous mapping and $z \in C$. Then*

$$z \in VI(C, A) \Leftrightarrow \langle Ax, x - z\rangle \geq 0 \ \ for \ \ all \ \ x \in C.$$

3 Main result

3.1 The first Halpern type double projection method

In this subsection, we propose our first Halpern type double projection algorithm and prove that the sequences generated by the proposed method converge strongly to an element of Γ which is also a fixed point of a quasi-nonexpansive mapping. Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S : C \to C$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at the origin (i.e., if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$ and $Sx_n - x_n \to 0$, as $n \to 0$, then $x = Sx$). Let $A : C \to C$ be a monotone mapping which is uniformly continuous on bounded subsets of C and $F(S) \cap \Gamma \neq \emptyset$. Suppose $\{x_n\}_{n=1}^{\infty}$ is generated in the following manner:

Algorithm 3.1

1: Choose $\gamma \in (0, 1)$, $\sigma \in (0, 1)$, and $\rho > 0$, and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ real sequences in (0,1)

2: Given $x_1 \in C$, starting point

3: Compute: $r(x_n) := x_n - P_C(x_n - Ax_n)$

4: Compute:

$$y_n := (1 - \eta_n)x_n + \eta_n P_C(x_n - Ax_n),$$

where the stepsize $\eta_n := \rho\gamma^{m_n}$ and
m_n is the smallest nonnegative integer m satisfying

$$\langle Ay_n, r(x_n)\rangle \geq \frac{\sigma}{2}\|r(x_n)\|^2.$$

5: Compute:

$$z_n = \alpha_n x_1 + (1 - \alpha_n)P_C(x_n - \lambda_n Ay_n)$$

where $\lambda_n := \frac{\langle Ay_n, x_n - y_n\rangle}{\|Ay_n\|^2}$

6: Then compute:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Sz_n, \quad n \geq 1,$$

7: Set $n \leftarrow n + 1$ and **goto 3**.

We first show that Algorithm 3.1 is well defined and implementable. This was done in [19] but we include the proof for the sake of completeness.

Lemma 3.1 *The stepsize procedure in Step 4 of Algorithm 3.1 is well-defined, i.e. it terminates after finitely many inner loops.*

Proof Consider an arbitrary index $n \in \mathbb{N}$. Observe that we assume implicitly that $r(x_n) \neq 0$. Assume that the stepsize rule does not terminate finitely at this iteration n. Then we have

$$\left\langle A((1 - \rho\gamma^m)x_n + \rho\gamma^m P_C(x_n - Ax_n)), r(x_n)\right\rangle < \frac{\sigma}{2}\|r(x_n)\|^2, \quad \forall m \geq 1.$$

Since A is continuous, we obtain for $m \to \infty$ that

$$\left\langle Ax_n, x_n - P_C(x_n - Ax_n)\right\rangle \leq \frac{\sigma}{2}\|x_n - P_C(x_n - Ax_n)\|^2.$$

Let $w_n := x_n - Ax_n$. Then we get

$$2\left\langle x_n - w_n, x_n - P_C(x_n - Ax_n)\right\rangle \leq \sigma\|x_n - P_C(x_n - Ax_n)\|^2.$$

Using Lemma 2(iv), we obtain from the previous inequality

$$\|P_C(x_n - Ax_n) - x_n\|^2 + \|x_n - w_n\|^2 - \|P_C(x_n - Ax_n) - w_n\|^2$$
$$\leq \sigma\|P_C(x_n - Ax_n) - x_n\|^2.$$

Since $\|P_C(x_n - Ax_n) - x_n\| = \|r(x_n)\| > 0$ and $\sigma \in (0, 1)$, we obtain

$$\begin{aligned}\|P_C(x_n - Ax_n) - x_n\|^2 + \|x_n - w_n\|^2 - \|P_C(x_n - Ax_n) - w_n\|^2 \\ \leq \sigma \|P_C(x_n - Ax_n) - x_n\|^2 \\ < \|P_C(x_n - Ax_n) - x_n\|^2.\end{aligned}$$

Hence, $\|x_n - w_n\| < \|P_C(x_n - Ax_n) - w_n\|$. Since $w_n = x_n - Ax_n$ by definition and $x_n \in C$, this contradicts the definition of a metric projection. □

A direct consequence of the previous result is that the scalar λ_n in Step 5 and Step 6 of Algorithm 3.1 are also well-defined.

Corollary 3.2 *We have $\langle Ay_n, x_n - y_n\rangle > 0$; in particular, $Ay_n \neq 0$ and, therefore λ_n is well-defined and positive.*

Proof Consider once again a fixed iteration index $n \in \mathbb{N}$, and recall that $\|x_n - P_C(x_n - Ax_n)\| = \|r(x_n)\| > 0$ holds due to our implicit assumption regarding termination of the algorithm. Since the stepsize rule in Step 4 is well-defined by Lemma 3.1, the definition of y_n yields

$$\langle Ay_n, x_n - y_n\rangle = \eta_n \big\langle Ay_n, x_n - P_C(x_n - Ax_n)\big\rangle \geq \frac{\sigma\eta_n}{2}\|x_n - P_C(x_n - Ax_n)\|^2 > 0,$$

so the statements follow. □

We now prove the following theorem.

Theorem 3.3 *Assume that*

(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < a \leq \beta_n \leq b < 1$.

Then the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ generated by Algorithm 3.1 strongly converge to $z \in F(S) \cap \Gamma$, where $z = P_{F(S)\cap\Gamma} x_1$.

Proof Let $z = P_{F(S)\cap\Gamma} x_1$. Let us define for each n,

$$D_n^- := \{x \in H : \langle Ay_n, x - y_n\rangle \leq 0\},$$

$$D_n := \{x \in H : \langle Ay_n, x - y_n\rangle = 0\},$$

and

$$D_n^+ := \{x \in H : \langle Ay_n, x - y_n\rangle \geq 0\},$$

where $\{y_n\}$ is generated by Algorithm 3.1.

Since A is monotone, we have that

$$\langle Ax, x - z\rangle \geq 0, \quad \forall x \in C.$$

This implies that $z \in D_n^-$. Also, observe that if the Algorithm 3.1 does not stop at iteration n, then

$$\begin{aligned}\langle Ay_n, x_n - y_n\rangle &= \langle Ay_n, x_n - P_C(x_n - Ax_n)\rangle \\ &\geq \frac{\eta_n \sigma}{2}\|r(x_n)\|^2 > 0.\end{aligned}$$

Therefore, $x_n \in D_n^+$ and $x_n \notin D_n^-$. Let $u_n := x_n - \lambda_n Ay_n$. Using the definition of λ_n, we have that

$$\begin{aligned}u_n &= x_n - \lambda_n Ay_n \\ &= x_n - \frac{\langle Ay_n, x_n - y_n\rangle}{\|Ay_n\|^2} Ay_n \\ &= P_{D_n}(x_n),\end{aligned}$$

where $D_n := \{y \in H : \langle Ay_n, y - y_n\rangle = 0\}$. Thus, $u_n \in D_n$. Furthermore, by Lemma 2.3, we get that

$$\|x_n - z\|^2 \geq \|u_n - z\|^2 + \|u_n - x_n\|^2, \tag{3.1}$$

By Lemma 2.1 (iv), (2.2) and the fact that $v_n := P_C(x_n - \lambda_n Ay_n)$, we obtain

$$\begin{aligned}&\|v_n - z\|^2 + \|v_n - u_n\|^2 - \|u_n - z\|^2 \\ &= 2\langle v_n - u_n, v_n - z\rangle \leq 0.\end{aligned}$$

This implies that

$$\|u_n - z\|^2 \geq \|v_n - z\|^2 + \|v_n - u_n\|^2. \tag{3.2}$$

It then follows from (3.1) and (3.2) that

$$\begin{aligned}\|x_n - z\|^2 \geq\ & \|v_n - z\|^2 + \|v_n - u_n\|^2 \\ & + \|u_n - x_n\|^2.\end{aligned}$$

Therefore,

$$\|v_n - z\|^2 \leq \|x_n - z\|^2 - \|v_n - u_n\|^2 - \|u_n - x_n\|^2. \tag{3.3}$$

Thus,

$$\|v_n - z\| \leq \|x_n - z\|.$$

We then obtain from Step 6 of Algorithm 3.1 and (3.3) that

$$\begin{aligned}\|x_{n+1} - z\| &\leq \beta_n\|x_n - z\| + (1-\beta_n)\|Sz_n - z\| \\ &\leq \beta_n\|x_n - z\| + (1-\beta_n)\|z_n - z\| \\ &= \beta_n\|x_n - z\| + (1-\beta_n)\|\alpha_n(x_1 - z) + (1-\alpha_n)(v_n - z)\| \\ &\leq \beta_n\|x_n - z\| + (1-\beta_n)(\alpha_n\|x_1 - z\| + (1-\alpha_n)\|v_n - z\|) \\ &\leq \beta_n\|x_n - z\| + (1-\beta_n)(\alpha_n\|x_1 - z\| + (1-\alpha_n)\|x_n - z\|) \\ &\leq \max\left\{\|x_n - z\|, \|x_1 - z\|\right\} \\ &\ \vdots \\ &\leq \max\left\{\|x_1 - z\|, \|x_1 - z\|\right\}.\end{aligned}$$

This shows that $\{x_n\}$ is bounded. Furthermore, by the fact that A is uniformly continuous on bounded subsets of C (see Lemma 2.4), we have that $\{P_C(x_n - Ax_n\}$, $\{y_n\}$ and $\{Ay_n\}$ are all bounded.

Then, using Lemma 2.1 (ii), (iii), and (3.3), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\beta_n(x_n-z)+(1-\beta_n)(Sz_n-z)\|^2\\ &= \beta_n\|x_n-z\|^2+(1-\beta_n)\|Sz_n-z\|^2-\beta_n(1-\beta_n)\|x_n-Sz_n\|^2\\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|z_n-z\|^2-\beta_n(1-\beta_n)\|x_n-Sz_n\|^2\\ &= \beta_n\|x_n-z\|^2+(1-\beta_n)\|\alpha_n(x_1-z)+(1-\alpha_n)(t_n-z)\|^2\\ &\quad -\beta_n(1-\beta_n)\|x_n-Sz_n\|^2\\ &\le \beta_n\|x_n-z\|^2-\beta_n(1-\beta_n)\|x_n-Sz_n\|^2\\ &\quad +(1-\beta_n)((1-\alpha_n)^2\|v_n-z\|^2+2\alpha_n\langle x_1-z,z_n-z\rangle)\\ &\le \beta_n\|x_n-z\|^2-\beta_n(1-\beta_n)\|x_n-Sz_n\|^2\\ &\quad +(1-\beta_n)((1-\alpha_n)\|v_n-z\|^2+2\alpha_n\langle x_1-z,z_n-z\rangle)\\ &\le (1-\alpha_n(1-\beta_n))\|x_n-z\|^2+2\alpha_n(1-\beta_n)\langle x_1-z,z_n-z\rangle\\ &\quad -\beta_n(1-\beta_n)\|x_n-Sz_n\|^2.\end{aligned} \tag{3.4}$$

Furthermore, we obtain

$$\|x_{n+1}-z\|^2 \le (1-\alpha_n(1-\beta_n))\|x_n-z\|^2+2\alpha_n(1-\beta_n)\langle x_1-z,z_n-z\rangle \tag{3.5}$$

The rest of the proof will be divided into two parts.

Case 1

Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{\|x_n - z\|\}_{n=1}^{\infty}$ converges and $\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0$, $n \to \infty$. From (3.4), we have that

$$\beta_n(1-\beta_n)\|x_n-Sz_n\|^2 \le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n M_1, \tag{3.6}$$

for some $M_1 > 0$. Thus,

$$\|x_n - Sz_n\| \to 0, \quad n \to \infty.$$

Furthermore, we have from Step 6 of Algorithm 3.1 and (3.3) that

$$\begin{aligned}0 &= \liminf_{n\to\infty}(\|x_{n+1}-z\|-\|x_n-z\|)\\ &\le \liminf_{n\to\infty}(\beta_n\|x_n-z\|+(1-\beta_n)\|Sz_n-z\|-\|x_n-z\|)\\ &\le \liminf_{n\to\infty}(1-\beta_n)(\alpha_n\|x_1-z\|+(1-\beta_n)\|v_n-z\|-\|x_n-z\|)\\ &= \liminf_{n\to\infty}(1-\beta_n)(\|v_n-z\|-\|x_n-z\|)\\ &\le (1-a)\liminf_{n\to\infty}(\|v_n-z\|-\|x_n-z\|)\\ &\le (1-a)\limsup_{n\to\infty}(\|v_n-z\|-\|x_n-z\|)\\ &\le 0.\end{aligned}$$

So,

$$\limsup_{n\to\infty}(\|v_n - z\| - \|x_n - z\|) = 0.$$

We obtain from (3.3) that

$$\begin{aligned}\|u_n - x_n\|^2 &\leq \|x_n - z\|^2 - \|v_n - z\|^2 \\ &= (\|x_n - z\| - \|t_n - z\|)(\|x_n - z\| + \|v_n - z\|) \\ &\leq (\|x_n - z\| - \|v_n - z\|)M_2,\end{aligned}$$

for some $M_2 > 0$. Thus

$$\limsup_{n\to\infty}\|x_n - u_n\| = 0$$

and this implies that

$$\|x_n - u_n\| \to 0, \quad n \to \infty.$$

From (3.3) again, we have

$$\begin{aligned}\|u_n - v_n\|^2 &\leq \|x_n - z\|^2 - \|v_n - z\|^2 \\ &= (\|x_n - z\| - \|v_n - z\|)(\|x_n - z\| + \|v_n - z\|) \\ &\leq (\|x_n - z\| - \|v_n - z\|)M_2, \end{aligned} \tag{3.7}$$

from which we have

$$\|u_n - v_n\| \to 0, \quad n \to \infty.$$

Furthermore,

$$\|x_n - v_n\| \leq \|x_n - u_n\| + \|u_n - t_n\| \to 0, \quad n \to \infty$$

and from Step 5 of Algorithm 3.1, we get

$$\|z_n - v_n\| = \alpha_n\|u - v_n\| \to 0, \quad n \to \infty$$

and

$$\|x_n - z_n\| \leq \|x_n - v_n\| + \|z_n - v_n\| \to 0, \quad n \to \infty.$$

Also

$$\|z_n - Sz_n\| \leq \|z_n - v_n\| + \|x_n - Sz_n\| + \|x_n - v_n\| \to 0, \quad n \to \infty$$

and

$$\|x_{n+1} - x_n\| = (1 - \beta_n)\|x_n - Sz_n\| \to 0, \quad n \to \infty.$$

Furthermore, we get (since $u_n \in D_n$) that

$$0 = \langle Ay_n, u_n - y_n\rangle = \langle Ay_n, u_n - w_n\rangle + \langle Ay_n, w_n - y_n\rangle.$$

Hence,

$$\begin{aligned}|\langle Ay_n, x_n - y_n\rangle| &= |\langle Ay_n, x_n - u_n\rangle| \\ &\leq \|Ay_n\|\|x_n - u_n\| \to 0, \quad n \to \infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\langle Ay_n, x_n - y_n\rangle = 0. \tag{3.8}$$

Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges weakly to some $p \in C$ and $\limsup_{n\to\infty}\langle x_1 - z, x_n - z\rangle = \lim_{k\to\infty}\langle x_1 - z, x_{n_k} - z\rangle$.

We next claim that there exists at least a subsequence $\{x_{n_k}\}$ such that $0 \leq \liminf_{k\to\infty}\langle Ax_{n_k}, x - x_{n_k}\rangle, \forall x \in C$.

To achieve this, let us define $s_{n_k} := P_C(x_{n_k} - Ax_{n_k})$. Observe that since $\{\eta_n\}$ is a bounded sequence of real numbers, it has a convergent subsequence and this gives rise to the following two sub-cases to be considered:

Sub-case 1: Suppose that there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ which converges to zero. In this case, we first show that $\limsup_{k\to\infty}\|s_{n_k} - x_{n_k}\| = 0$. Assume the contrary that $\limsup_{k\to\infty}\|s_{n_k} - x_{n_k}\| = \delta > 0$. Let $\bar{y}_k := \frac{1}{\gamma}\eta_{n_k}s_{n_k} + (1 - \frac{1}{\gamma}\eta_{n_k})x_{n_k}$ or equivalently $\bar{y}_k - x_{n_k} = \frac{1}{\gamma}\eta_{n_k}(s_{n_k} - x_{n_k})$. Since $\{s_{n_k} - x_{n_k}\}$ is bounded and $\limsup_{k\to\infty}\eta_{n_k} = 0$, it follows that

$$\limsup_{k\to\infty}\|\bar{y}_k - x_{n_k}\| = 0.$$

This implies that

$$\lim_{k\to\infty}\|\bar{y}_k - x_{n_k}\| = 0. \tag{3.9}$$

From the line search and the definition of $\bar{y}_k$, we have

$$\langle A\bar{y}_k, x_{n_k} - s_{n_k}\rangle < \frac{\sigma}{2}\|x_{n_k} - s_{n_k}\|^2, \forall k \in \mathbb{N}.$$

Since A is uniformly continuous on bounded subsets of C and $\sigma \in (0, 1)$, we obtain from (3.9) that there exists $N \in \mathbb{N}$ such that

$$2\langle Ax_{n_k}, x_{n_k} - s_{n_k}\rangle < \|x_{n_k} - s_{n_k}\|^2, \forall k \in \mathbb{N}.$$

Therefore,

$$2\langle x_{n_k} - t_{n_k}, x_{n_k} - s_{n_k}\rangle < \|x_{n_k} - s_{n_k}\|^2, \forall k \in \mathbb{N},$$

where $t_{n_k} := x_{n_k} - Ax_{n_k}$. Using Lemma 2.1 (b) in the last inequality, we obtain

$$\|x_{n_k} - s_{n_k}\|^2 + \|x_{n_k} - t_{n_k}\|^2 - \|s_{n_k} - t_{n_k}\|^2 < \|x_{n_k} - s_{n_k}\|^2.$$

Thus,

$$\|x_{n_k} - t_{n_k}\| < \|s_{n_k} - t_{n_k}\|.$$

This is a contradiction to the definition of $s_{n_k} = P_C(t_{n_k}) = P_C(x_{n_k} - Ax_{n_k})$. Therefore, $\limsup_{k\to\infty}\|s_{n_k} - x_{n_k}\| = 0$.

Furthermore, observe that from Algorithm 3.1 and (2.2) that

$$\langle x_{n_k} - Ax_{n_k} - s_{n_k}, x - s_{n_k}\rangle \leq 0, \forall x \in C,$$

which implies that

$$\langle x_{n_k} - s_{n_k}, x - s_{n_k}\rangle \leq \langle Ax_{n_k}, x - s_{n_k}\rangle, \forall z \in C.$$

Hence,

$$\begin{aligned}&\langle x_{n_k} - s_{n_k}, x - s_{n_k}\rangle + \langle Ax_{n_k}, s_{n_k} - x_{n_k}\rangle\\ &\leq \langle Ax_{n_k}, x - x_{n_k}\rangle,\end{aligned} \tag{3.10}$$

for all $x \in C$. Fix $x \in C$ and let $k \to \infty$ in (3.10) (noting that $\lim_{i\to\infty} \|s_{n_k} - x_{n_k}\| = 0$), we have

$$0 \leq \liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k}\rangle, \forall x \in C.$$

Sub-case 2: Suppose that $\{\eta_{n_k}\}$ is any subsequence of $\{\eta_n\}$ that is bounded away from zero. Then, we have $\eta_{n_k} \geq \mu > 0$. It follows from the line search in Algorithm 3.1 that

$$\langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle \geq \frac{\sigma}{2}\eta_{n_k}\|x_{n_k} - s_{n_k}\|^2. \tag{3.11}$$

Therefore, by (3.8) we get that $\lim_{k\to\infty} \|s_{n_k} - x_{n_k}\| = 0$. Following the same line of arguments in (3.10) above, we can show that

$$0 \leq \liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k}\rangle, \forall x \in C.$$

Since A is monotone, we have for an arbitrary $x \in C$ that

$$\langle Ax, x - x_{n_k}\rangle \geq \langle Ax_{n_k}, x - x_{n_k}\rangle, \forall k \in \mathbb{N}. \tag{3.12}$$

Taking lim inf on both sides of (3.12), we have

$$\liminf_{k\to\infty} \langle Ax, x - x_{n_k}\rangle \geq \liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k}\rangle \geq 0.$$

Since $x_{n_k} \rightharpoonup p$, we have for all $x \in C$ that

$$\begin{aligned}\langle Ax, x - p\rangle &= \lim_{k\to\infty} \langle Ax, x - x_{n_k}\rangle \\ &= \liminf_{k\to\infty} \langle Ax, x - x_{n_k}\rangle \geq 0.\end{aligned}$$

This implies by Lemma 2.5 that $p \in \Gamma$.

Since $\{x_{n_k}\}$ converges weakly to some $p \in C$ and $x_n - z_n \to 0, \quad n \to \infty$, we have that $\{z_{n_k}\}$ converges weakly to some $p \in C$. By demiclosedness of $I - S$ at origin and the fact that $\|z_n - Sz_n\| \to 0, \quad n \to \infty$, we have that $p \in F(S)$. Hence, $p \in F(S) \cap \Gamma$.

Since $z = P_{F(S)\cap\Gamma} x_1$, we have that

$$\begin{aligned}\limsup_{n\to\infty} \langle x_1 - z, z_n - z\rangle &= \lim_{k\to\infty} \langle x_1 - z, z_{n_k} - z\rangle \\ &= \langle x_1 - z, p - z\rangle \\ &\leq 0.\end{aligned}$$

Using Lemma 2.2 in (3.5), we obtain $\lim_{n\to\infty} \|x_n - z\| = 0$. Thus, $x_n \to z, n \to \infty$.

Case 2

Assume that $\{\|x_n - z\|\}$ is not monotonically decreasing sequence. Set $\Gamma_n = \|x_n - z\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough)by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0.$$

This implies that $\|x_{\tau(n)} - z\| \leq \|x_{\tau(n)+1} - z\|, \forall n \geq n_0$. Thus $\lim_{n\to\infty} \|x_{\tau(n)} - z\|$ exists. Following the arguments in Case 1, we can show that

$$\|x_{\tau(n)} - Sz_{\tau(n)}\| \to 0, \quad n \to \infty,$$

$$\limsup_{n\to\infty}(\|v_{\tau(n)} - z\| - \|x_n - \tau(n)\|) = 0,$$

$$\|x_{\tau(n)} - v_{\tau(n)}\| \to 0, \quad n \to \infty,$$

$$\|z_{\tau(n)} - v_{\tau(n)}\| \to 0, \quad n \to \infty,$$

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0, \quad n \to \infty$$

and

$$\|z_{\tau(n)} - Sz_{\tau(n)}\| \to 0, \quad n \to \infty.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$ which converges weakly to p. Observe that since $\lim_{n\to\infty} \|x_{\tau(n)} - z_{\tau(n)}\| = 0$, we also have $z_{\tau(n)} \rightharpoonup p$. By similar argument in Case 1, we can show that $w \in F(S) \cap \Gamma$ and

$$\limsup_{n\to\infty}\langle x_1 - z, z_{\tau(n)} - z\rangle \leq 0.$$

By (3.5), we obtain that

$$\|x_{\tau(n)+1} - z\|^2 \leq (1 - \alpha_{\tau(n)}(1 - \beta_{\tau(n)}))\|x_{\tau(n)} - z\|^2 + 2\alpha_{\tau(n)}(1 - \beta_{\tau(n)})\langle x_1 - z, z_{\tau(n)} - z\rangle.$$

which implies that (noting that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)}(1 - \beta_{\tau(n)}) > 0$)

$$\|x_{\tau(n)} - z\|^2 \leq 2\langle x_1 - z, z_{\tau(n)} - z\rangle.$$

This implies that

$$\limsup_{n\to\infty}\|x_{\tau(n)} - z\| \leq 0.$$

Thus,

$$\lim_{n\to\infty} \|x_{\tau(n)} - z\| = 0.$$

and

$$\|x_{\tau(n)+1} - z\| \leq \|x_{\tau(n)} - z\| + \|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0, n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim \Gamma_n = 0$, that is, $\lim_{n\to\infty} \|x_n - z\| = 0$. Hence, $\{x_n\}$ converges strongly to z. Similarly, $y_n \to z$. This completes the proof. □

Remark 3.4 Our strong convergence result using Algorithm 3.1 extends the strong convergence results of Kraikaew and Saejung [25] (see also [26, 29, 32]) from monotone mapping A which is Lipschitz continuous to uniformly continuous monotone mapping in infinite dimensional Hilbert space.

3.2 The second Halpern type double projection method

In this subsection, we present another Halpern type double projection method which finds a solution of the Variational inequality for a uniformly continuous monotone operator which is also a fixed point of a given quasi-nonexpansive mapping. Then, we establish a strong convergence theorem of the sequence generated by our scheme.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $S : C \to C$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at the origin and denote by $F(S)$ its fixed point set. Let $A : C \to C$ be a monotone and uniformly continuous on bounded subsets of C and $F(S) \cap \Gamma \neq \emptyset$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences generated by the following manner:

Algorithm 3.2

1: Choose $\gamma \in (0, 1)$, $\sigma \in (0, 1)$, $\rho > 0$, and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ and $\{\omega_n\}_{n=1}^{\infty}$ real sequences in (0,1) such that $\alpha_n + \beta_n + \gamma_n = 1$

2: Given $x_1 \in C$, starting point.

3: Compute: $r(x_n) := x_n - P_C(x_n - Ax_n)$

4: Compute:

$$y_n := (1 - \eta_n)x_n + \eta_n P_C(x_n - Ax_n),$$

where the stepsize $\eta_n := \rho\gamma^{m_n}$ and
m_n is the smallest nonnegative integer m satisfying

$$\langle Ay_n, r(x_n)\rangle \geq \frac{\sigma}{2}\|r(x_n)\|^2.$$

5: Compute:

$$x_{n+1} = \alpha_n x_1 + \beta_n x_n + \gamma_n(\omega_n Sx_n + (1 - \omega_n)P_C(x_n - \lambda_n Ay_n)), \quad n \geq 1,$$

where $\lambda_n := \frac{\langle Ay_n, x_n - y_n\rangle}{\|Ay_n\|^2}$

6: Set $n \leftarrow n + 1$ and **goto 3**.

Remark 3.5 By Lemma 3.1, Algorithm 3.2 is well defined and implementable.

Theorem 3.6 *Assume that*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\beta_n \geq \epsilon_1 > 0, \quad \gamma_n \geq \epsilon_2 > 0$;

(d) $0 < c \leq \omega_n \leq d < 1$.

Then the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ generated by Algorithm 3.2 strongly converge to $z \in F(S) \cap \Gamma$, where $z = P_{F(S)\cap\Gamma}x_1$.

Proof Let $z = P_{F(S)\cap\Gamma}x_1$. Then following the method of proof in Theorem 3.3, we can show that

$$\|v_n - z\|^2 \le \|x_n - z\|^2 - \|v_n - u_n\|^2 - \|u_n - x_n\|^2.$$

Let $z_n := \omega_n S x_n + (1-\omega_n)v_n, \quad \forall n \ge 1$. Then

$$\begin{aligned}\|z_n - z\| &\le \omega_n\|Sx_n - z\| + (1-\omega_n)\|v_n - z\| \\ &\le \omega_n\|x_n - z\| + (1-\omega_n)\|x_n - z\| \\ &= \|x_n - z\|.\end{aligned}$$

Furthermore, by (5), we have

$$\begin{aligned}\|x_{n+1} - z\| &\le \alpha_n\|x_1 - z\| + \beta_n\|x_n - z\| + \gamma_n\|z_n - z\| \\ &\le \alpha_n\|x_1 - z\| + \beta_n\|x_n - z\| + \gamma_n\|x_n - z\| \\ &= \alpha_n\|x_1 - z\| + (1-\alpha_n)\|x_n - z\| \\ &\le \max\Big\{\|x_n - z\|, \|x_1 - z\|\Big\},\end{aligned}$$

which by induction implies that $\{x_n\}$ is bounded. So also is $\{z_n\}$. By Lemma 2.1 (ii) and (iii), we get

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n(x_1 - z) + \beta_n(x_n - z) + \gamma_n(z_n - z)\|^2 \\ &\le \|\beta_n(x_n - z) + \gamma_n(z_n - z)\|^2 + 2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle \\ &= \beta_n(\beta_n + \gamma_n)\|x_n - z\|^2 + \gamma_n(\beta_n + \gamma_n)\|z_n - z\|^2 \\ &\quad -\beta_n\gamma_n\|z_n - x_n\|^2 + 2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle \\ &\le \beta_n(\beta_n + \gamma_n)\|x_n - z\|^2 + \gamma_n(\beta_n + \gamma_n)\|x_n - z\|^2 \\ &\quad -\beta_n\gamma_n\|z_n - x_n\|^2 + 2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle \\ &= (\beta_n + \gamma_n)^2\|x_n - z\|^2 - \beta_n\gamma_n\|z_n - x_n\|^2 \\ &\quad +2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle \\ &\le (1-\alpha_n)\|x_n - z\|^2 - \beta_n\gamma_n\|z_n - x_n\|^2 \\ &\quad +2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle. \end{aligned} \tag{3.13}$$

We now distinguish two cases

Case 1

Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - z||\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{||x_n - z||\}_{n=1}^{\infty}$ converges and $||x_n - z||^2 - ||x_{n+1} - z||^2 \to 0, \quad n \to \infty$. By the boundedness of $\{x_n\}$, we have from (3.13) that

$$\beta_n\gamma_n\|z_n - x_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M, \tag{3.14}$$

for some $M > 0$. By Condition (c), we have that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

Observe that

$$\begin{aligned} x_{n+1}-x_n &= \alpha_n x_1+\beta_n x_n+\gamma_n z_n-(\alpha_n x_n+\beta_n x_n+\gamma_n x_n)\\ &= \alpha_n(u-x_n)+\gamma_n(z_n-x_n). \end{aligned}$$

This implies that

$$\|x_{n+1}-x_n\| \le \alpha_n\|x_1-x_n\|+\gamma_n\|z_n-x_n\| \to 0, \quad n\to\infty.$$

Also,

$$\|x_{n+1}-z_n\| \le \|x_{n+1}-x_n\|+\|x_n-z_n\| \to 0, \quad n\to\infty.$$

By (5), we have

$$\begin{aligned} \|x_{n+1}-z\|^2 &\le \alpha_n\|x_1-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\|z_n-z\|^2\\ &\le \alpha_n\|x_1-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\Big(\omega_n\|Sx_n-z\|^2\\ &\quad +(1-\omega_n)\|v_n-z\|^2\Big)\\ &\le \alpha_n\|x_1-z\|^2+\beta_n\|x_n-z\|^2+\omega_n\gamma_n\|x_n-z\|^2\\ &\quad +\gamma_n(1-\omega_n)\|v_n-z\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} -\|v_n-z\|^2 \le\ & \frac{1}{\gamma_n(1-\omega_n)}\Big[\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2\\ &+\omega_n\gamma_n\|x_n-z\|^2-\|x_{n+1}-z\|^2\Big]. \end{aligned} \tag{3.15}$$

Using (3.15) in (3.3), we have

$$\begin{aligned} \|u_n-x_n\|^2 &\le \|x_n-z\|^2-\|v_n-z\|^2\\ &\le \|x_n-z\|^2-\frac{1}{\gamma_n(1-\omega_n)}\|x_{n+1}-z\|^2+\frac{\alpha_n}{\gamma_n(1-\omega_n)}\|x_1-z\|^2\\ &\quad +\frac{\beta_n+\omega_n\gamma_n}{\gamma_n(1-\omega_n)}\|x_n-z\|^2\\ &= \frac{1-\alpha_n}{\gamma_n(1-\omega_n)}\|x_n-z\|^2-\frac{1}{\gamma_n(1-\omega_n)}\|x_{n+1}-z\|^2\\ &\quad +\frac{\alpha_n}{\gamma_n(1-\omega_n)}\|x_1-z\|^2\\ &= \frac{1}{\gamma_n(1-\omega_n)}\Big[\|x_n-z\|^2-\|x_{n+1}-z\|^2\Big]\\ &\quad +\frac{\alpha_n}{\gamma_n(1-\omega_n)}\Big[\|x_1-z\|^2-\|x_n-z\|^2\Big]. \end{aligned}$$

This implies that

$$\|x_n-u_n\| \to 0, \quad n\to\infty.$$

Similarly, by (3.15) and (3.3), we can show that

$$\|u_n-v_n\| \to 0, \quad n\to\infty.$$

Hence,

$$\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - v_n\| \to 0, \quad n \to \infty.$$

Now,

$$\|z_n - v_n\| \le \|x_n - v_n\| + \|x_n - z_n\| \to 0, \quad n \to \infty.$$

From $z_n = \omega_n Sx_n + (1 - \omega_n)v_n$, we get

$$\|Sx_n - v_n\| = \frac{1}{\omega_n}\|z_n - v_n\| \to 0, \quad n \to \infty.$$

Furthermore,

$$\|x_n - Sx_n\| \le \|Sx_n - v_n\| + \|x_n - v_n\| \to 0, \quad n \to \infty. \tag{3.16}$$

Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges weakly to some $p \in C$ and $\limsup_{n\to\infty}\langle x_1 - z, x_n - z\rangle = \lim_{k\to\infty}\langle x_1 - z, x_{n_k} - z\rangle$. Following the method of proof in Theorem 3.3, we can show that $p \in \Gamma$. Also, by the demiclosedness principle of $I - S$ and (3.16), we have that $p \in F(S)$. Hence, $p \in F(S) \cap \Gamma$. Consequently,

$$\begin{aligned}\limsup_{n\to\infty}\langle x_1 - z, x_n - z\rangle &= \lim_{k\to\infty}\langle x_1 - z, x_{n_k} - z\rangle \\ &= \langle x_1 - z, p - z\rangle \\ &\le 0.\end{aligned}$$

Since $\|x_{n+1} - x_n\| \to 0, \quad n \to \infty$, we have that

$$\limsup_{n\to\infty}\langle x_1 - z, x_{n+1} - z\rangle \le 0.$$

From (3.13) we have

$$\|x_{n+1} - z\|^2 \le (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle x_1 - z, x_{n+1} - z\rangle. \tag{3.17}$$

Using Lemma 2.2 in (3.17), we obtain $\lim_{n\to\infty}\|x_n - z\| = 0$. Thus, $x_n \to z, n \to \infty$.

Case 2

Assume that $\{\|x_n - z\|\}$ is not monotonically decreasing sequence. Set $\Gamma_n = \|x_n - z\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough)by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}.$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \le \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \forall n \ge n_0.$$

This implies that $\|x_{\tau(n)} - z\| \le \|x_{\tau(n)+1} - z\|, \forall n \ge n_0$. Thus $\lim_{n\to\infty}\|x_{\tau(n)} - z\|$ exists. By using similar arguments as in Case 1, we obtain

$$\|x_{\tau(n)} - u_{\tau(n)}\| \to 0, \quad n \to \infty, \|u_{\tau(n)} - v_{\tau(n)}\| \to 0, \quad n \to \infty.$$

and

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0, \quad n \to \infty.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$ which converges weakly to p. Observe that since $\lim_{n\to\infty} \|x_{\tau(n)} - u_{\tau(n)}\| = 0$, we also have $u_{\tau(n)} \rightharpoonup w$. By similar argument in Case 1, we can show that $p \in \Gamma$ and

$$\limsup_{n\to\infty} \langle x_1 - z, x_{\tau(n)} - z\rangle \leq 0.$$

Since $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0, \;\; n \to \infty$ and $\limsup_{n\to\infty} \langle x_1 - z, x_{\tau(n)} - z\rangle \leq 0$, we can show that

$$\limsup_{n\to\infty} \langle x_1 - z, x_{\tau(n)+1} - z\rangle \leq 0.$$

By (3.13), we have

$$\|x_{\tau(n)+1} - z\|^2 \leq (1 - \alpha_{\tau(n)})\|x_{\tau(n)} - z\|^2 + \langle x_1 - z, x_{\tau(n)+1} - z\rangle,$$

which implies that (noting that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$)

$$\|x_{\tau(n)} - z\|^2 \leq \langle x_1 - z, x_{\tau(n)+1} - z\rangle.$$

This implies that

$$\limsup_{n\to\infty} \|x_{\tau(n)} - z\| \leq 0.$$

Thus,

$$\lim_{n\to\infty} \|x_{\tau(n)} - z\| = 0.$$

and

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - z\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim \Gamma_n = 0$, that is, $\lim_{n\to\infty} \|x_n - z\| = 0$. Hence, $\{x_n\}$ converges strongly to z. This completes the proof. □

4 Numerical example

In this section, we provide some concrete example including numerical results of the problem considered in Section 3 of this paper. All codes were written in Matlab 2012b and run on Hp $i - 5$ Dual-Core 8.00 GB (7.78 GB usable) RAM laptop.

Suppose that $H = L^2([0, 1])$ with norm $\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)dt, \;\; x, y \in H, t \in [0, 1]$. Furthermore, let us take

$$C := \{x \in L^2([0, 1]) : \langle a, x\rangle \leq b\},$$

where $0 \neq a \in L^2([0,1])$ and $b \in \mathbb{R}$, then (see [7] and a projection formula for a half-space)

$$P_C(x) = \begin{cases} \frac{b-\langle a,x\rangle}{\|a\|}a + x, & \langle a,x\rangle > b \\ x, \langle a,x\rangle \leq b, \end{cases}$$

We consider the following VI(A,C) problem:
find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C \text{ and } Sx = x, \tag{4.1}$$

where $C := \{x \in L^2([0,1]) : \int_0^1 (t^2+1)x(t)dt \leq 1\}$ and $S : C \to C$ is a quasi-nonexpansive mapping. Define operator $A : C \to C$ by $(Ax)(t) := \max\{0, x(t)\}$ for all $x \in C$. Then it can be easily verified that A is monotone and uniformly continuous on bounded subsets of C.

Define a function $S : C \to C$ by $(Sx)(t) = \frac{1}{t+2}x(t)$. Then it is clear that S is a quasi-nonexpansive mapping. Observe that $0 \in F(S) \cap \Omega$ and so $F(S) \cap \Omega \neq \emptyset$. Take $\alpha_n = \frac{1}{n+1}$, $\beta_n = \gamma_n = \omega_n = \frac{n}{2(n+1)}$. Then our Algorithm 3.1 and Algorithm 3.2 respectively become:

Algorithm 4.3

1: Choose $\gamma \in (0,1)$, $\sigma \in (0,1)$ and $\rho > 0$

2: Given $x_1 \in C$, starting point

3: Compute: $r(x_n(t)) := x_n - P_C(x_n(t) - Ax_n(t))$

4: Compute:

$$y_n(t) := (1-\eta_n)x_n(t) + \eta_n P_C(x_n(t) - Ax_n(t)),$$

where the stepsize $\eta_n := \rho\gamma^{m_n}$ and
m_n is the smallest nonnegative integer m satisfying

$$\langle Ay_n(t), r(x_n(t))\rangle \geq \frac{\sigma}{2}\|r(x_n(t))\|^2.$$

5: Compute:

$$z_n(t) = \frac{1}{n+1}x_1(t) + \left(1 - \frac{1}{n+1}\right) P_C(x_n(t) - \lambda_n Ay_n(t))$$

where $\lambda_n := \frac{\langle Ay_n(t), x_n(t) - y_n(t)\rangle}{\|Ay_n(t)\|^2}$

6: Then compute:

$$x_{n+1}(t) = \frac{n}{2(n+1)}x_n(t) + \left(1 - \frac{n}{2(n+1)}\right)\frac{1}{t+2}z_n(t), \quad n \geq 1,$$

7: Set $n \leftarrow n+1$ and **goto 3**.

Algorithm 4.4

1: Choose $\gamma \in (0,1)$, $\sigma \in (0,1)$ and $\rho > 0$
2: Given $x_1 \in C$, starting point.
3: Compute: $r(x_n(t)) := x_n - P_C(x_n(t) - Ax_n(t))$
4: Compute:

$$y_n(t) := (1-\eta_n)x_n(t) + \eta_n P_C(x_n(t) - Ax_n(t)),$$

where the stepsize $\eta_n := \rho\gamma^{m_n}$ and
m_n is the smallest nonnegative integer m satisfying

$$\langle Ay_n(t), r(x_n(t))\rangle \geq \frac{\sigma}{2}\|r(x_n(t))\|^2.$$

5: Compute:

$$x_{n+1} = \frac{1}{n+1}x_1(t) + \frac{n}{2(n+1)}x_n(t) + \frac{n}{2(n+1)}\left[\frac{n}{2(n+1)}\frac{1}{t+2}x_n(t) + \left(1 - \frac{n}{2(n+1)}\right)P_C(x_n(t) - \lambda_n Ay_n(t))\right], \quad n \geq 1,$$

where $\lambda_n := \frac{\langle Ay_n(t), x_n(t) - y_n(t)\rangle}{\|Ay_n(t)\|^2}$
6: Set $n \leftarrow n+1$ and **goto 3**.

The parameters in Algorithm 4.1 and Algorithm 4.2 are chosen as $\sigma = 10^{-3}$ and $\rho = 1$. We carry out our computation using different choices of $x_1(t)$ with different choices of γ. We terminate the iteration if $\frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|} < 10^{-2}$.

The comparison between the iterative Algorithm 4.1 and Algorithm 4.2 is done using the following cases:

Case I: $x_1 = \frac{2}{5}t^2\sin(3t)e^{2t}$ and $\gamma = 0.1$ & 0.8.
Case II: $x_1 = \frac{1}{85}(t^3+1)e^{5t}$ and $\gamma = 0.1$ & 0.8

Remark 4.1 (1) Over all, we could see from Table 1 and Figs. 1, 2, 3, and 4 that both Algorithm 4.1 and Algorithm 4.2 are consistent, efficient and easy to implement.

Table 1 Comparison between Algorithm (4.1) and Algorithm (4.1) with different Cases

		$\gamma = 0.1$		$\gamma = 0.8$	
		Case I	Case II	Case I	Case II
Algorithm 4.1	No. of Iterations	13	12	13	12
	CPU (Time)	0.008572	0.0072221	0.0085952	0.0081592
Algorithm 4.2	No. of Iterations	126	122	124	121
	CPU (Time)	0.071375	0.071602	0.072572	0.068813

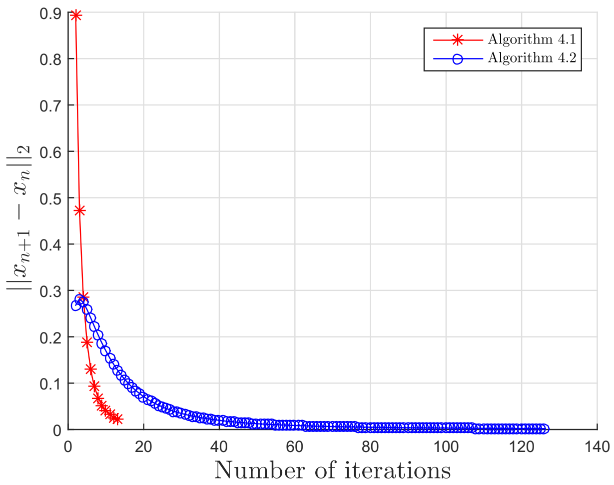

Fig. 1 The L^2-norm of the residual: case I with $\gamma = 0.1$

(2) We observe from Table 1 and Figs. 1, 2, 3, and 4 that Algorithm 4.1 terminates successfully after about 12 iterations while Algorithm 4.2 terminates successfully after about 122 iterations. So, Algorithm 4.1 performs better than Algorithm 4.2 in terms of number of iterations requires to terminate.

(3) Similarly, Algorithm 4.1 is about eight times faster than Algorithm 4.2 in terms of CPU time used before termination. However, both Algorithms used relatively small CPU time before termination.

Remark 4.2 We remark here that our results carry over for the case when S is a β-demicontractive mapping on a real Hilbert space H with $F(S) \neq \emptyset$ (i.e., there exists

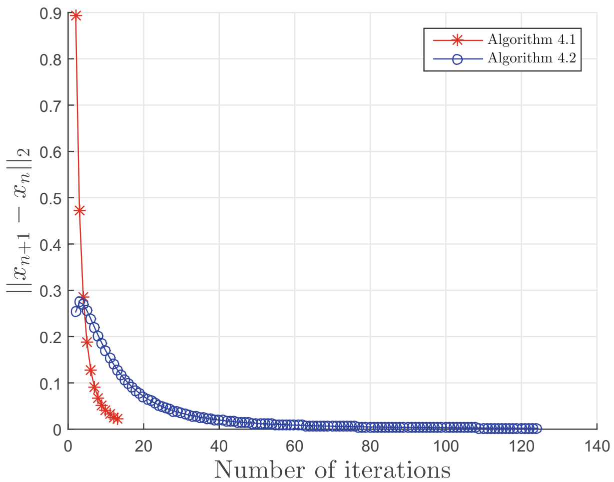

Fig. 2 The L^2-norm of the residual: case I with $\gamma = 0.8$

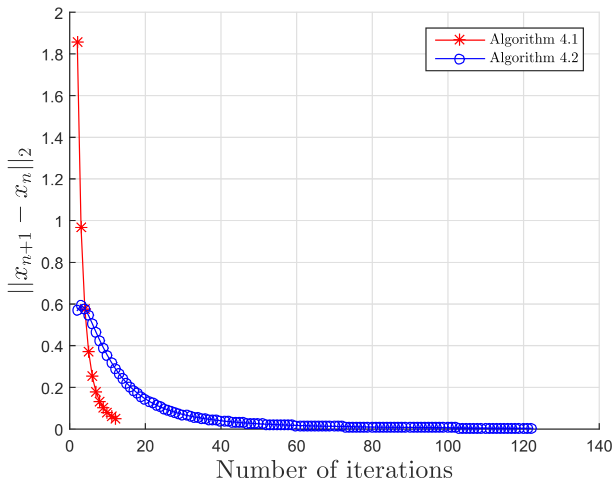

Fig. 3 The L^2-norm of the residual: case II with $\gamma = 0.1$

$\beta \in [0, 1)$ such that $\|Sx - q\|^2 \leq \|x - q\|^2 + \beta\|x - Sx\|^2, \forall x \in H, q \in F(S)$). It is known that if S is a β-demicontractive mapping on a real Hilbert space H with $F(S) \neq \emptyset$ and $S_\omega := (1 - \omega)I + \omega S$ for $\omega \in (0, 1]$, then S_ω is quasi-nonexpansive mapping and $F(S) = F(S_\omega)$, where $\omega \in (0, 1 - \beta)$ (e.g., see [28]).

Remark 4.3 1. In a way, the problem considered in this paper can be re-casted as a common fixed point problem but caution must be applied here that the sequence of iterates generated by $x_{n+1} = [TP_c(I - \lambda A)]x_n$ cannot converge to the solution of the common fixed point problem considered in this paper. Take for example, A to be a rotation map in $\mathbb{R}^2$ and $T \equiv I$, where I is the identity map in $\mathbb{R}^2$.

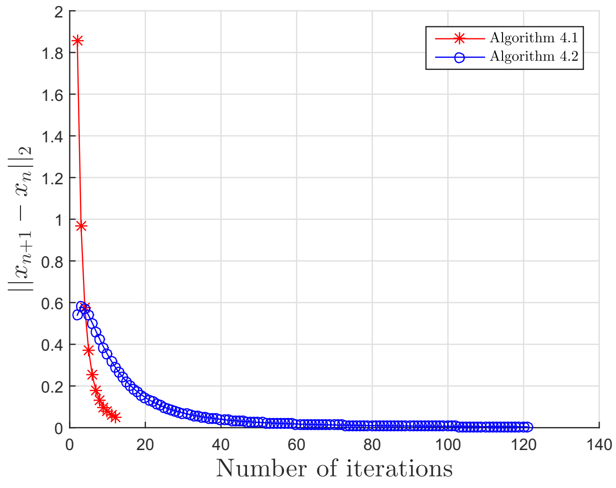

Fig. 4 The L^2-norm of the residual: case II with $\gamma = 0.8$

2. Our proposed algorithms solve fixed point problem and variational inequality problem involving uniformly continuous monotone operator simultaneously. The problem considered here generalizes fixed point problem and variational inequality problem.
3. Our results can also hold for any finite number of related problems, of one kind or mixed, and the results can be obtained by Pierra's [34] product space arguments if appropriate conditions are imposed such that the conditions in our convergence analysis are satisfied.
4. In the next project, we shall study the extension of the results in this paper to certain Banach spaces.
5. Our result can still be obtained if one uses the Moudafi's viscosity approximation approach, see, e. g., [30], which uses a contraction instead of the identity operator I in Halpern idea in our algorithms.

5 Final remarks

In this paper, we proposed two double projection methods for solving variational inequality and fixed point problem for a quasi-nonexpansive mapping and the underline monotone operator is uniformly continuous on bounded subsets of C. Furthermore, we established strong convergence results for the two methods and give numerical results regarding their implementation. Our results in this paper improves on the results in [11, 35], where one projection for each tentative step during the Armijo-type search is considered and computationally expensive. Also, our results extend several other results where common solution to fixed point problem for quasi-nonexpansive mapping and variational inequality problem for Lipschitz continuous monotone operator is studied (see, [8, 25, 31, 32]). In the future, we shall designing new algorithms including inexact or perturbed methods as well as inertial-type extrapolation for the problems considered in this paper.

Acknowledgments The research was carried out when the first author was an Alexander von Humboldt Postdoctoral Fellow at the Institute of Mathematics, University of Wurzburg, Germany. He is grateful to the Alexander von Humboldt Foundation, Bonn for the fellowship and the Institute of Mathematics, University of Wurzburg, Germany for the hospitality and facilities.

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