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ORIGINAL PAPER

Construction of the optimal set of two or three quadrature rules in the sense of Borges

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Abstract In this paper, we investigate a numerical method for the construction of an optimal set of quadrature rules in the sense of Borges (Numer. Math. **67**, 271–288, 1994) for two or three definite integrals with the same integrand and interval of integration, but with different weight functions, related to an arbitrary multi-index. The presented method is illustrated by numerical examples.

Keywords Multi-index · Optimal set of quadrature rules · Multiple orthogonal polynomials

Mathematics Subject Classification (2010) 65D32 · 42C05

1 Introduction

Borges [5] has considered a problem that arises in the evaluation of computer graphics illumination models. Starting with that problem, he has examined the problem of numerically evaluating a set of $r \in \mathbb{N}$, $r \geq 2$, definite integrals of the form

$$\int_E f(x)\, w_j(x)\, \mathrm{d}x, \quad j = 1, 2, \ldots, r,$$

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where w_j, $j = 1, 2, \ldots, r$, are weight functions. For such a problem, it is not efficient to use a set of r Gauss–Christoffel quadrature rules, because valuable information is wasted. Let us denote that such kind of quadrature rules were already suggested by Angelesco in 1918, in [1]. Such a problem was studied in [5, 9, 14]. The solution of that problem is connected with multiple orthogonal polynomials (see [14] and [15]). Because of that, we repeat some basic facts about multiple orthogonal polynomials.

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in which the orthogonality is distributed among a number of orthogonality weights. They arise naturally in the theory of simultaneous rational approximation, in particular in the Hermite–Padé approximation of a system of $r \in \mathbb{N}$ (Markov and Stieltjes) functions (see [6, 7, 16–18, 20, 24]). A good general introduction to multiple orthogonal polynomials is Chapter 23 of book [12]. Many of these polynomials have already been studied and applied to Diophantine number theory, rational approximation in the complex plane, spectral and scattering problems for higher-order difference equations, and corresponding dynamical systems (see [2–4, 15, 19, 22, 23]).

Let r be a positive integer and let $\mathbf{n} = (n_1, n_2, \ldots, n_r)$ be a multi–index, i.e., a vector of r nonnegative integers, with length $|\mathbf{n}| = n_1 + n_2 + \cdots + n_r$ and let $W = (w_1, w_2, \ldots, w_r)$ be weight functions on real line so that the support of each w_i is a subset of a interval E_i, $i = 1, 2, \ldots, n$.

There are two types of multiple orthogonal polynomials.

1° Type I multiple orthogonal polynomials with respect to $(W, \mathbf{n})$ are given by the vector $(A_{\mathbf{n},1}, A_{\mathbf{n},2}, \ldots, A_{\mathbf{n},r})$, where $A_{\mathbf{n},j}$ is a polynomial of degree less than or equal to $n_j - 1$, satisfying the orthogonality conditions

$$\sum_{j=1}^{r} \int_{E_j} A_{\mathbf{n},j}(x) x^k \, w_j(x) \, \mathrm{d}x = 0, \quad k = 0, 1, \ldots, |\mathbf{n}| - 2,$$

with the normalization

$$\sum_{j=1}^{r} \int_{E_j} A_{\mathbf{n},j}(x) x^{|\mathbf{n}|-1} \, w_j(x) \, \mathrm{d}x = 1.$$

2° The type II multiple orthogonal polynomial with respect to $(W, \mathbf{n})$ is the monic polynomial $P_{\mathbf{n}}$ of degree $|\mathbf{n}|$ that satisfies the following orthogonality conditions:

$$\int_{E_j} P_{\mathbf{n}}(x) x^k \, w_j(x) \, \mathrm{d}x = 0, \quad k = 0, 1, \ldots, n_j - 1, \tag{1.1}$$

for $j = 1, 2, \ldots, r$.

The orthogonality conditions (1.1) give a system of $|\mathbf{n}|$ linear equations for the $|\mathbf{n}|$ unknown coefficients of the polynomial $P_{\mathbf{n}} = \sum_{k=0}^{|\mathbf{n}|} a_{k,\mathbf{n}} x^k$, $a_{|\mathbf{n}|,\mathbf{n}} = 1$.

If the system (1.1) has a unique solution, then the multi-index $\mathbf{n}$ is normal for type II, and the polynomial $P_{\mathbf{n}}$ is unique. The type II multiple orthogonal polynomial $P_{\mathbf{n}}$ is unique if and only if the vector of the type I multiple orthogonal polynomials is unique (see [12]). If all multi-indices are normal, then we have a *perfect system*.

There are two distinct cases for which all the multi-indices are normal for multiple orthogonal polynomials:

1. Angelesco systems, where the intervals E_i, on which the weight functions are supported, are disjoint, i.e., $E_i \cap E_j = \emptyset$, for $1 \le i \ne j \le r$;
2. AT systems for the multi-index $\mathbf{n}$, where all weight functions are supported on the same interval E and the set

$$\{x^\nu w_j(x) : \ \nu = 0, 1, \ldots, n_j - 1, \ j = 1, 2, \ldots, r\}$$

form a Chebyshev system on E.

Assuming that all multi-indices are normal, the following nearest neighbor recurrence relations

$$x P_{\mathbf{n}}(x) = P_{\mathbf{n}+\mathbf{e}_k}(x) + b_{\mathbf{n},k} P_{\mathbf{n}}(x) + \sum_{j=1}^{r} a_{\mathbf{n},j} P_{\mathbf{n}-\mathbf{e}_j}(x), \quad k = 1, 2, \ldots, r, \tag{1.2}$$

hold with initial conditions $P_{\mathbf{0}}(x) = 1$ and $P_{-\mathbf{e}_j}(x) = 0$, $j = 1, 2, \ldots, r$, where $\mathbf{e}_j$ is the j-th standard unit vector with the 1 on the j-th entry and vectors $(a_{\mathbf{n},1}, a_{\mathbf{n},2}, \ldots, a_{\mathbf{n},r})$ and $(b_{\mathbf{n},1}, b_{\mathbf{n},2}, \ldots, b_{\mathbf{n},r})$ are the recurrence coefficients. These recurrence relations were derived in [12] and [25], by using different methods, and it was shown that (see [12, Theorem 23.1.11])

$$a_{\mathbf{n},j} = \frac{\displaystyle\int_{E_j} x^{n_j} P_{\mathbf{n}}(x)\, w_j(x)\, \mathrm{d}x}{\displaystyle\int_{E_j} x^{n_j-1} P_{\mathbf{n}-\mathbf{e}_j}(x)\, w_j(x)\, \mathrm{d}x},$$

$$b_{\mathbf{n},j} = \int_{E_j} x\, Q_{\mathbf{n}+\mathbf{e}_j}(x)\, \mathrm{d}x,$$

where $Q_{\mathbf{n}}(x) = \sum_{j=1}^{r} A_{\mathbf{n},j}(x) w_j(x)$.

Remark 1 It is easy to see that $P_{\mathbf{n}}(x) = 0$ whenever multi-index $\mathbf{n}$ has at least one negative entry (see [21, Remark 3.1] for an explanation). Because of that fact, in what follows we will not write polynomials with multi-indices that have at least one negative entry in the obtained equations.

The following theorem was proved in [22].

Theorem 1 *Suppose that* $\mathbf{n}$ *is a multi–index and that* $W = (w_1, w_2, \ldots, w_r)$ *is an AT system of weight functions on an interval* E *for multi-index* $\mathbf{n}$. *The type II multiple orthogonal polynomial* $P_{\mathbf{n}}(x)$ *with respect to* $(W, \mathbf{n})$ *has exactly* $|\mathbf{n}|$ *simple zeros on* E.

Now we come back to the problem of numerically evaluating a set of $r \in \mathbb{N}$, $r \ge 2$, definite integrals with the same integrand and over the same interval of integration, but related to different weight functions.

Definition 1 Let $W = (w_1, w_2, \dots, w_r)$ be an AT system on an interval E and let $\mathbf{n} = (n_1, n_2, \dots, n_r)$ be a multi-index. A set of quadrature rules of the form:

$$\int_E f(x) w_k(x)\,\mathrm{d}x \approx \sum_{i=1}^{|\mathbf{n}|} A_{k,i} f(x_i), \quad k = 1, 2, \dots, r, \tag{1.3}$$

will be called *an optimal set in Borges' sense* with respect to $(W, \mathbf{n})$ if and only if the weight coefficients $A_{k,i}$, $k = 1, 2, \dots, r$, $i = 1, 2, \dots, |\mathbf{n}|$, and the nodes x_i, $i = 1, 2, \dots, |\mathbf{n}|$, satisfy the following equations:

$$\begin{aligned}
\sum_{i=1}^{|\mathbf{n}|} A_{k,i} &= \int_E w_k(x)\,\mathrm{d}x, \\
\sum_{i=1}^{|\mathbf{n}|} A_{k,i} x_i &= \int_E x\, w_k(x)\,\mathrm{d}x, \\
&\vdots \\
\sum_{i=1}^{|\mathbf{n}|} A_{k,i} x_i^{|\mathbf{n}|+n_k-1} &= \int_E x^{|\mathbf{n}|+n_k-1}\, w_k(x)\,\mathrm{d}x,
\end{aligned}$$

for $k = 1, 2, \dots, r$.

A characterization of the optimal set of quadrature rules is given in the following theorem (see [14]), which is a generalization of the fundamental theorem of Gauss–Christoffel quadrature rules.

Theorem 2 *Let* $\mathbf{n}$ *be a multi-index and let* $W = (w_1, w_2, \dots, w_r)$ *be an AT system on interval* E. *A set of quadrature rules* (1.3) *is the optimal set in Borges' sense with respect to* $(W, \mathbf{n})$ *if and only if*

- **(i)** *all rules are exact for all polynomials of degree less than or equal to* $|\mathbf{n}| - 1$,
- **(ii)** *the polynomial* $q(x) = \prod_{i=1}^{|\mathbf{n}|}(x - x_i)$ *is the type II multiple orthogonal polynomial* $P_{\mathbf{n}}(x)$ *with respect to* $(W, \mathbf{n})$.

Remark 2 According to Theorem 1, all zeros of the type II multiple orthogonal polynomial $P_{\mathbf{n}}(x)$ with respect to $(W, \mathbf{n})$ are simple and lie on the interval E.

We want to construct square matrices for which the eigenvalues are nodes of this optimal set of quadrature rules. Then, the weights can be calculated by solving systems of linear equations. The corresponding method for calculating the nodes and weight coefficients of the optimal set of quadrature rules with respect to nearly diagonal multi-index was given in [14]. Our aim is to extend that result, i.e., to give a method for calculating the nodes and weight coefficients of the optimal set of quadrature rules for general multi-indices. In this paper, we will explain the cases of $r = 2$ and $r = 3$ weight functions in detail. The general case of $r \in \mathbb{N}$ weight functions will be given in the forthcoming paper.

Remark 3 Coussement and Van Assche considered Gaussian quadrature rules for multiple orthogonal polynomials with respect to nearly diagonal multi-indices in [8], while Lubinsky and Van Assche considered simultaneous Gaussian quadrature rules for Angelesco systems in [13].

The paper is organized as follows. In Section 2, two ways for obtaining the type II multiple orthogonal polynomials with respect to two weight functions are given, as well as the corresponding methods for calculating the nodes and weight coefficients of the optimal set of quadrature rules. The numerical construction of such quadrature rules is presented in Subsection 2.1. The method for calculating nodes and weight coefficients of an optimal set of three quadrature rules is considered in Section 3. Finally, some numerical examples are given in Section 4.

2 Optimal set of quadrature rules for $r = 2$

According to Theorem 2, the nodes of the optimal set of quadrature rules with respect to $(W, \mathbf{n})$ are the zeros of the corresponding type II multiple orthogonal polynomial $P_{\mathbf{n}}$, and our first aim is to try to calculate the nodes as eigenvalues of a certain square matrix.

In this section, we only consider the optimal set of quadrature rules (1.3) with respect to $r = 2$ weight functions. Let $P_{(n_1,n_2)}$ be the type II multiple orthogonal polynomial corresponding to a multi-index $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$ and with respect to weight functions $W = (w_1, w_2)$. Now, the nearest neighbor recurrence relations (1.2) have the following form:

$$\begin{aligned} xP_{(n_1,n_2)}(x) &= P_{(n_1+1,n_2)}(x) + c_{(n_1,n_2)}P_{(n_1,n_2)}(x) \\ &\quad + a_{(n_1,n_2)}P_{(n_1-1,n_2)}(x) + b_{(n_1,n_2)}P_{(n_1,n_2-1)}(x), \end{aligned} \tag{2.1}$$

$$\begin{aligned} xP_{(n_1,n_2)}(x) &= P_{(n_1,n_2+1)}(x) + d_{(n_1,n_2)}P_{(n_1,n_2)}(x) \\ &\quad + a_{(n_1,n_2)}P_{(n_1-1,n_2)}(x) + b_{(n_1,n_2)}P_{(n_1,n_2-1)}(x). \end{aligned} \tag{2.2}$$

Let us point out that the coefficients $a_{\mathbf{n},j}$, $j = 1, 2, \ldots, r$, in (1.2) do not depend on k, and because of that, we have the same coefficients $a_{(n_1,n_2)}$ and $b_{(n_1,n_2)}$ in (2.1)–(2.2).

From the previous relations, we obtain

$$P_{(n_1,n_2+1)}(x) = P_{(n_1+1,n_2)}(x) + (c_{(n_1,n_2)} - d_{(n_1,n_2)})P_{(n_1,n_2)}(x), \tag{2.3}$$

$$P_{(n_1+1,n_2)}(x) = P_{(n_1,n_2+1)}(x) + (d_{(n_1,n_2)} - c_{(n_1,n_2)})P_{(n_1,n_2)}(x). \tag{2.4}$$

Lemma 1 *Let $(n_1, n_2) \in \mathbb{N}^2$ be the multi-index and $P_{(n_1,n_2)}$ the type II multiple orthogonal polynomial with respect to W. Then,*

$$\begin{aligned} P_{(n_1,n_2+1)}(x) &= P_{(n_1+1,n_2)}(x) + \sum_{k=0}^{n_2-1} P_{(n_1+1,k)}(x) \prod_{i=k+1}^{n_2} (c_{(n_1,i)} - d_{(n_1,i)}) \\ &\quad + P_{(n_1,0)}(x) \prod_{i=0}^{n_2} (c_{(n_1,i)} - d_{(n_1,i)}) \end{aligned} \tag{2.5}$$

holds.

Proof We will prove by induction on ν, where $\nu = n_1+n_2$ is the length of the multi–index (n_1, n_2). For $\nu = 0$, we have the multi-index $(n_1, n_2) = (0, 0)$ and from (2.3), we obtain

$$P_{(0,1)}(x) = P_{(1,0)}(x) + (c_{(0,0)} - d_{(0,0)})P_{(0,0)}(x),$$

which is the same as (2.5) with $(n_1, n_2) = (0, 0)$.

Let us now assume that (2.5) is true for all multi-indices of length $\nu = n_1+n_2$, i.e.,

$$\begin{aligned} P_{(n_1,n_2)}(x) \;=\; & P_{(n_1+1,n_2-1)}(x) + \sum_{k=0}^{n_2-2} P_{(n_1+1,k)}(x) \prod_{i=k+1}^{n_2-1} (c_{(n_1,i)} - d_{(n_1,i)}) \\ & + P_{(n_1,0)}(x) \prod_{i=0}^{n_2-1} (c_{(n_1,i)} - d_{(n_1,i)}), \end{aligned} \tag{2.6}$$

and prove that is also true for multi-index $(n_1, n_2 + 1)$ of length $\nu + 1$. By using (2.3) and the induction hypothesis (2.6) for $P_{(n_1,n_2)}$, we obtain

$$\begin{aligned} P_{(n_1,n_2+1)}(x) =& P_{(n_1+1,n_2)}(x) + (c_{(n_1,n_2)} - d_{(n_1,n_2)}) \\ & \times \Bigg(P_{(n_1+1,n_2-1)}(x) + \sum_{k=0}^{n_2-2} P_{(n_1+1,k)}(x) \prod_{i=k+1}^{n_2-1} (c_{(n_1,i)} - d_{(n_1,i)}) \\ & + P_{(n_1,0)}(x) \prod_{i=0}^{n_2-1} (c_{(n_1,i)} - d_{(n_1,i)}) \Bigg) \\ =& P_{(n_1+1,n_2)}(x) + (c_{(n_1,n_2)} - d_{(n_1,n_2)}) P_{(n_1+1,n_2-1)}(x) \\ & + \sum_{k=0}^{n_2-2} P_{(n_1+1,k)}(x) \prod_{i=k+1}^{n_2} (c_{(n_1,i)} - d_{(n_1,i)}) \\ & + P_{(n_1,0)}(x) \prod_{i=0}^{n_2} (c_{(n_1,i)} - d_{(n_1,i)}) \\ =& P_{(n_1+1,n_2)}(x) + \sum_{k=0}^{n_2-1} P_{(n_1+1,k)}(x) \prod_{i=k+1}^{n_2} (c_{(n_1,i)} - d_{(n_1,i)}) \\ & + P_{(n_1,0)}(x) \prod_{i=0}^{n_2} (c_{(n_1,i)} - d_{(n_1,i)}), \end{aligned}$$

which completes the proof. □

In a similar way, using (2.4) and (2.7), the following lemma can be proved.

Lemma 2 *For all multi-indices $(n_1, n_2) \in \mathbb{N}^2$ and type II multiple orthogonal polynomial $P_{(n_1,n_2)}$ with respect to W, one has*

$$P_{(n_1+1,n_2)}(x) = P_{(n_1,n_2+1)}(x) + \sum_{k=0}^{n_1-1} P_{(k,n_2+1)}(x) \prod_{i=k+1}^{n_1} (d_{(i,n_2)} - c_{(i,n_2)}) \quad (2.7)$$
$$+P_{(0,n_2)}(x) \prod_{i=0}^{n_1} (d_{(i,n_2)} - c_{(i,n_2)}).$$

Remark 4 Let us notice that by using the nearest neighbor recurrence relations (2.1)–(2.2), one can obtain the polynomial $P_{(n_1,n_2)}$ by following different paths of multi-indices from $(0, 0)$ to (n_1, n_2). Our choices, which will be presented, are not the shortest paths from $(0, 0)$ to (n_1, n_2), but they imply that the zeros of the polynomial $P_{(n_1,n_2)}$ are eigenvalues of a square matrix, which gives a stable numerical technique for the construction of the optimal set of quadrature rules. Our paths are presented in Fig. 1.

Let us now consider the construction of the polynomial $P_{(n,m)}$ in the following two ways.

1° Setting $n_1 = 0, 1, \ldots, n-1$, and $n_2 = 0$ in (2.1), we get

$$\begin{aligned}
x P_{(0,0)}(x) &= P_{(1,0)}(x) + c_{(0,0)} P_{(0,0)}(x),\\
x P_{(1,0)}(x) &= P_{(2,0)}(x) + c_{(1,0)} P_{(1,0)}(x) + a_{(1,0)} P_{(0,0)}(x),\\
x P_{(2,0)}(x) &= P_{(3,0)}(x) + c_{(2,0)} P_{(2,0)}(x) + a_{(2,0)} P_{(1,0)}(x),\\
&\vdots\\
x P_{(n-1,0)}(x) &= P_{(n,0)}(x) + c_{(n-1,0)} P_{(n-1,0)}(x) + a_{(n-1,0)} P_{(n-2,0)}(x).
\end{aligned}$$

Now by applying (2.2) with $n_1 = n$ and $n_2 = 0, 1$, we obtain

$$\begin{aligned}
x P_{(n,0)}(x) &= P_{(n,1)}(x) + d_{(n,0)} P_{(n,0)}(x) + a_{(n,0)} P_{(n-1,0)}(x),\\
x P_{(n,1)}(x) &= P_{(n,2)}(x) + d_{(n,1)} P_{(n,1)}(x) + a_{(n,1)} P_{(n-1,1)}(x) + b_{(n,1)} P_{(n,0)}(x).
\end{aligned}$$

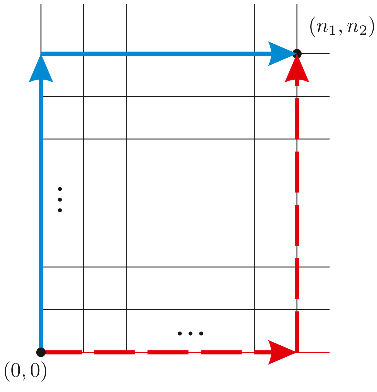

Fig. 1 Two paths of multi-indices from $(0, 0)$ to (n_1, n_2)

By using (2.5), for $(n_1, n_2) = (n-1, 0)$, we have

$$P_{(n-1,1)}(x) = P_{(n,0)}(x) + P_{(n-1,0)}(x)(c_{(n-1,0)} - d_{(n-1,0)}),$$

and hence

$$\begin{aligned} xP_{(n,1)}(x) = & P_{(n,2)}(x) + d_{(n,1)}P_{(n,1)}(x) + (a_{(n,1)} + b_{(n,1)})P_{(n,0)}(x) \\ & + a_{(n,1)}(c_{(n-1,0)} - d_{(n-1,0)})P_{(n-1,0)}(x). \end{aligned}$$

In the same way, by using (2.2) and (2.5), we obtain

$$\begin{aligned} xP_{(n,2)}(x) = & P_{(n,3)}(x) + d_{(n,2)}P_{(n,2)}(x) + (a_{(n,2)} + b_{(n,2)})P_{(n,1)}(x) \\ & + a_{(n,2)}(c_{(n-1,1)} - d_{(n-1,1)})P_{(n,0)}(x) \\ & + a_{(n,2)}(c_{(n-1,1)} - d_{(n-1,1)})(c_{(n-1,0)} - d_{(n-1,0)})P_{(n-1,0)}(x) \\ & \vdots \\ xP_{(n,m-1)}(x) = & P_{(n,m)}(x) + d_{(n,m-1)}P_{(n,m-1)}(x) \\ & + (a_{(n,m-1)} + b_{(n,m-1)})P_{(n,m-2)}(x) \\ & + a_{(n,m-1)} \sum_{k=0}^{m-3} \prod_{i=k+1}^{m-2} (c_{(n-1,i)} - d_{(n-1,i)})P_{(n,k)}(x) \\ & + a_{(n,m-1)} \prod_{i=0}^{m-2} (c_{(n-1,i)} - d_{(n-1,i)})P_{(n-1,0)}(x). \end{aligned}$$

The obtained equations can be written in the following way

$$x\mathbf{P}^{(1)}_{(n,m)}(x) = M^{(1)}_{(n,m)}\mathbf{P}^{(1)}_{(n,m)}(x) + P_{(n,m)}(x)\mathbf{e}_{(n,m)}, \tag{2.8}$$

where

$$\mathbf{P}^{(1)}_{(n,m)}(x) = \begin{bmatrix} P_{(0,0)}(x) \\ P_{(1,0)}(x) \\ \vdots \\ P_{(n,0)}(x) \\ P_{(n,1)}(x) \\ \vdots \\ P_{(n,m-1)}(x) \end{bmatrix}, \quad \mathbf{e}_{(n,m)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and $M^{(1)}_{(n,m)}$ is the following square matrix of order $n+m$

$$M^{(1)}_{(n,m)} = \begin{bmatrix} A^{(1)}_{(n,m)} \\ B^{(1)}_{(n,m)} \end{bmatrix}, \tag{2.9}$$

where

$$\begin{aligned} A^{(1)}_{(n,m)} &= \left[\widetilde{A}^{(1)}_{(n,m)} \middle| [0]_{n\times(m-1)} \right], \\ B^{(1)}_{(n,m)} &= \left[[0]_{m\times(n-1)} \middle| \widetilde{B}^{(1)}_{(n,m)} \right], \end{aligned}$$

$[0]_{p\times q}$ is the zero matrix of dimension $p \times q$,

$$\widetilde{A}^{(1)}_{(n,m)} = \begin{bmatrix} c_{(0,0)} & 1 & & & \\ a_{(1,0)} & c_{(1,0)} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{(n-1,0)} & c_{(n-1,0)} & 1 \end{bmatrix},$$

$$\widetilde{B}^{(1)}_{(n,m)} = \begin{bmatrix} a_{(n,0)} & d_{(n,0)} & 1 & & & \\ a_{(n,1)}\Pi^{(1)}_{(0,0)} & a_{(n,1)} + b_{(n,1)} & d_{(n,1)} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ a_{(n,m-2)}\Pi^{(1)}_{(0,m-3)} & a_{(n,m-2)}\Pi^{(1)}_{(1,m-3)} & a_{(n,m-2)}\Pi^{(1)}_{(2,m-3)} & \cdots & d_{(n,m-2)} & 1 \\ a_{(n,m-1)}\Pi^{(1)}_{(0,m-2)} & a_{(n,m-1)}\Pi^{(1)}_{(1,m-2)} & a_{(n,m-1)}\Pi^{(1)}_{(2,m-2)} & \cdots & a_{(n,m-1)} + b_{(n,m-1)} & d_{(n,m-1)} \end{bmatrix}$$

(where we used the notation $\Pi^{(1)}_{k,\ell} = \prod_{i=k}^{\ell}(c_{(n-1,i)} - d_{(n-1,i)})$).

Let $x_i = x_i^{(n,m)}$, $i = 1, 2, \ldots, n+m$, be the zeros of $P_{(n,m)}(x)$. Then, (2.8) reduces to the following eigenvalue problem:

$$x_i \mathbf{P}^{(1)}_{(n,m)}(x_i) = M^{(1)}_{(n,m)} \mathbf{P}^{(1)}_{(n,m)}(x_i).$$

Thus, the zeros of $P_{(n,m)}(x)$, i.e., the nodes of the optimal set of quadrature rules (1.3), are the eigenvalues of the matrix $M^{(1)}_{(n,m)}$.

2° Now we will consider the construction of the polynomial $P_{(n,m)}$ in the second way. Setting $n_1 = 0$ and $n_2 = 0, 1, \ldots, m-1$, in (2.2), we get

$$\begin{aligned} xP_{(0,0)}(x) =& P_{(0,1)}(x) + d_{(0,0)}P_{(0,0)}(x), \\ xP_{(0,1)}(x) =& P_{(0,2)}(x) + d_{(0,1)}P_{(0,1)}(x) + b_{(0,1)}P_{(0,0)}(x), \\ xP_{(0,2)}(x) =& P_{(0,3)}(x) + d_{(0,2)}P_{(0,2)}(x) + b_{(0,2)}P_{(0,1)}(x), \\ & \vdots \\ xP_{(0,m-1)}(x) =& P_{(0,m)}(x) + d_{(0,m-1)}P_{(0,m-1)}(x) + b_{(0,m-1)}P_{(0,m-2)}(x). \end{aligned}$$

Now by applying (2.1) with $n_1 = 0, 1$, and $n_2 = m$, we obtain

$$\begin{aligned} xP_{(0,m)}(x) =& P_{(1,m)}(x) + c_{(0,m)}P_{(0,m)}(x) + b_{(0,m)}P_{(0,m-1)}(x), \\ xP_{(1,m)}(x) =& P_{(2,m)}(x) + c_{(1,m)}P_{(1,m)}(x) + a_{(1,m)}P_{(0,m)}(x) + b_{(1,m)}P_{(1,m-1)}(x). \end{aligned}$$

By using (2.7), for $(n_1, n_2) = (0, m-1)$, we have

$$P_{(1,m-1)}(x) = P_{(0,m)}(x) + P_{(0,m-1)}(x)(d_{(0,m-1)} - c_{(0,m-1)}),$$

and hence

$$\begin{aligned} xP_{(1,m)}(x) =& P_{(2,m)}(x) + c_{(1,m)}P_{(1,m)}(x) + (a_{(1,m)} + b_{(1,m)})P_{(0,m)}(x) \\ & + b_{(1,m)}(d_{(0,m-1)} - c_{(0,m-1)})P_{(0,m-1)}(x). \end{aligned}$$

In the same way, by using (2.1) and (2.7), we obtain

$$
\begin{aligned}
xP_{(2,m)}(x) =& P_{(3,m)}(x) + c_{(2,m)}P_{(2,m)}(x) + (a_{(2,m)} + b_{(2,m)})P_{(1,m)}(x) \\
&+ b_{(2,m)}(d_{(1,m-1)} - c_{(1,m-1)})P_{(0,m)}(x) \\
&+ b_{(2,m)}(d_{(1,m-1)} - c_{(1,m-1)})(d_{(0,m-1)} - c_{(0,m-1)})P_{(0,m-1)}(x), \\
&\vdots \\
xP_{(n-1,m)}(x) =& P_{(n,m)}(x) + c_{(n-1,m)}P_{(n-1,m)}(x) \\
&+ (a_{(n-1,m)} + b_{(n-1,m)})P_{(n-2,m)}(x) \\
&+ b_{(n-1,m)}\sum_{k=0}^{n-3} P_{(k,m)}(x)\prod_{i=k+1}^{n-2}(d_{(i,m-1)} - c_{(i,m-1)}) \\
&+ P_{(0,m-1)}(x)b_{(n-1,m)}\prod_{i=0}^{n-2}(d_{(i,m-1)} - c_{(i,m-1)}).
\end{aligned}
$$

The obtained equations can be written in the following way

$$
x\mathbf{P}^{(2)}_{(n,m)}(x) = M^{(2)}_{(n,m)}\mathbf{P}^{(2)}_{(n,m)}(x) + P_{(n,m)}(x)\mathbf{e}_{(n,m)}, \tag{2.10}
$$

where

$$
\mathbf{P}^{(2)}_{(n,m)}(x) = \begin{bmatrix} P_{(0,0)}(x) \\ P_{(0,1)}(x) \\ \vdots \\ P_{(0,m)}(x) \\ P_{(1,m)}(x) \\ \vdots \\ P_{(n-1,m)}(x) \end{bmatrix}, \quad \mathbf{e}_{(n,m)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
$$

and $M^{(2)}_{(n,m)}$ is the following matrix of order $n+m$

$$
M^{(2)}_{(n,m)} = \begin{bmatrix} A^{(2)}_{(n,m)} \\ B^{(2)}_{(n,m)} \end{bmatrix},
$$

where

$$
\begin{aligned}
A^{(2)}_{(n,m)} &= \left[\, \widetilde{A}^{(2)}_{(n,m)} \,\middle|\, [0]_{m\times(n-1)} \right], \\
B^{(2)}_{(n,m)} &= \left[\, [0]_{n\times(m-1)} \,\middle|\, \widetilde{B}^{(2)}_{(n,m)} \right], \\
\widetilde{A}^{(2)}_{(n,m)} &= \begin{bmatrix} d_{(0,0)} & 1 & & & \\ b_{(0,1)} & d_{(0,1)} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{(0,m-1)} & d_{(0,m-1)} & 1 \end{bmatrix},
\end{aligned}
$$

$$\widetilde{B}^{(2)}_{(n,m)} = \begin{bmatrix} b_{(0,m)} & c_{(0,m)} & 1 & & & \\ b_{(1,m)}\Pi^{(2)}_{(0,0)} & a_{(1,m)} + b_{(1,m)} & c_{(1,m)} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ b_{(n-2,m)}\Pi^{(2)}_{(0,n-3)} & b_{(n-2,m)}\Pi^{(2)}_{(1,n-3)} & b_{(n-2,m)}\Pi^{(2)}_{(2,n-3)} & \cdots & c_{(n-2,m)} & 1 \\ b_{(n-1,m)}\Pi^{(2)}_{(0,n-2)} & b_{(n-1,m)}\Pi^{(2)}_{(1,n-2)} & b_{(n-1,m)}\Pi^{(2)}_{(2,n-2)} & \cdots & a_{(n-1,m)} + b_{(n-1,m)} & c_{(n-1,m)} \end{bmatrix}$$

(where we used notation $\Pi^{(2)}_{k,\ell} = \prod_{i=k}^{\ell}(d_{(i,m-1)} - c_{(i,m-1)})$).

Let $x_i = x_i^{(n,m)}$, $i = 1, 2, \ldots, n+m$, be the zeros of $P_{(n,m)}(x)$. Then (2.10) reduces to the following eigenvalue problem:

$$x_i \mathbf{P}^{(2)}_{(n,m)}(x_i) = M^{(2)}_{(n,m)} \mathbf{P}^{(2)}_{(n,m)}(x_i).$$

Thus, the zeros of $P_{(n,m)}(x)$, i.e., the nodes of the optimal set of quadrature rules (1.3), are the eigenvalues of the matrix $M^{(2)}_{(n,m)}$. For the computation of all recurrence coefficients in the matrices $M^{(2)}_{(n,m)}$, we use effective algorithms which are given in [10].

Knowing the nodes of the optimal set of quadrature formula, one can find the weight coefficients $A_{k,i}$, $k = 1, 2, \ldots, r$, $i = 1, 2, \ldots, n+m$. Here, we present the technique for obtaining weight coefficients for the given method 1° and in the similar way, the weight coefficients can be obtained for the presented method 2°.

The eigenvector associated with the eigenvalue x_i, $i = 1, 2, \ldots, n+m$, is given by $\mathbf{P}^{(1)}_{(n,m)}(x_i)$, and this fact can be used for the computation of the weight coefficients $A_{k,i}$, $k = 1, 2, \ldots, r$, $i = 1, 2, \ldots, n+m$, by requiring that each rule exactly integrates modified moments.

Let us denote by

$$V^{(1)}_{(n,m)} = \left[\mathbf{P}^{(1)}_{(n,m)}(x_1)\ \mathbf{P}^{(1)}_{(n,m)}(x_2)\ \cdots\ \mathbf{P}^{(1)}_{(n,m)}(x_{n+m}) \right] \tag{2.11}$$

the matrix of the eigenvectors of the matrix $M^{(1)}_{(n,m)}$, each normalized so that the first component is equal to 1. Then, the weight coefficients $A_{k,i}$ can be obtained by solving the systems of linear equations

$$V^{(1)}_{(n,m)} \begin{bmatrix} A_{k,1} \\ A_{k,2} \\ \vdots \\ A_{k,n+m} \end{bmatrix} = \begin{bmatrix} \mu^{*(k)}_{(0,0)} \\ \mu^{*(k)}_{(1,0)} \\ \vdots \\ \mu^{*(k)}_{(n,0)} \\ \mu^{*(k)}_{(n,1)} \\ \vdots \\ \mu^{*(k)}_{(n,m-1)} \end{bmatrix}, \quad k = 1, 2, \ldots, r, \tag{2.12}$$

where

$$\mu^{*(k)}_{(i,j)} = \int_E P_{(i,j)}(x)\, w_k(x)\, \mathrm{d}x, \ \ i = 0, 1, \ldots, n, \ \ j = 0, 1, \ldots, m-1,$$

and $k = 1, 2, \ldots, r$.

2.1 Numerical construction

In this subsection, we present an effective numerical method for constructing the matrix $M^{(1)}_{(n,m)}$ (in the same way, one can construct the matrix $M^{(2)}_{(n,m)}$), the corresponding type II multiple orthogonal polynomial $P_{(n,m)}$ and the optimal set of quadrature rules. For the numerical construction of the matrix $M^{(1)}_{(n,m)}$, we will use algorithms from [10] and because of that, we need to introduce the marginal and nearly diagonal recurrence relations.

For the (monic) orthogonal polynomials with respect to the each weight function w_j, $j = 1, 2, \ldots, r$,

$$\int_{E_j} P_n(x; w_j) P_m(x; w_j)\, w_j(x)\, \mathrm{d}x = 0, \quad m = 0, 1, \ldots, n-1,$$

the recurrence relation

$$x P_n(x; w_j) = P_{n+1}(x; w_j) + b_n(w_j) P_n(x; w_j) + a_n^2(w_j) P_{n-1}(x; w_j)$$

and the corresponding coefficients will be called marginal.

A multi-index of the form

$$\mathbf{d}(n) = (\underbrace{\ell+1, \ell+1, \ldots, \ell+1}_{j \text{ times}}, \underbrace{\ell, \ell, \ldots, \ell}_{r-j \text{ times}}),$$

where $n = r\ell + j$, $\ell = [n/r]$, $0 \le j \le r$, is called nearly diagonal. The corresponding type II multiple orthogonal polynomial $P_{\mathbf{d}(n)}$ satisfies the following nearly diagonal recurrence relation

$$x P_{\mathbf{d}(n)}(x) = P_{\mathbf{d}(n+1)}(x) + \sum_{i=0}^{r} \alpha_{n,r-i} P_{\mathbf{d}(n-i)}(x), \ \ n \ge 0,$$

with initial conditions $P_{\mathbf{d}(0)}(x) = 1$, and $P_{\mathbf{d}(i)}(x) = 0$, for $i = -1, -2, \ldots, -r$ (see [14, 26]).

Our first aim is to obtain the nearest neighbor recurrence coefficients, i.e., the elements of matrix $M^{(1)}_{(n,m)}$. We can do that by starting with the corresponding coefficients of marginal or nearly diagonal recurrence relations, which can be obtained by using the Stieltjes procedure (see [14]) or the discretized Stieltjes–Gautschi procedure (see [11]), respectively. Then, one of the algorithms given in [10] can be used for obtaining the coefficients of the nearest neighbor recurrence relations, i.e., the elements of matrix $M^{(1)}_{(n,m)}$. More precisely, if one starts with the coefficients of the marginal measures then the nearest neighbor recurrence coefficients can be obtained by using [10, Theorem 3.1.], and if the starting coefficients are the coefficients of nearly diagonal recurrence relations then [10, Theorem 2.3.] can be used for obtaining the nearest neighbor recurrence coefficients. The eigenvalues of the matrix $M^{(1)}_{(n,m)}$ are the nodes of the optimal set of quadrature rules (1.3). The type II multiple orthogonal polynomial $P_{(n,m)}(x)$ can be obtained by using the previously given method. Finally, when we have the type II multiple orthogonal polynomials and eigenvectors of the matrix $M^{(1)}_{(n,m)}$, i.e., the matrix $V^{(1)}_{(n,m)}$ given by (2.11), the weight coefficients of the optimal set of quadrature rules (1.3) can be found by solving system (2.12).

3 Optimal set of quadrature rules for $r = 3$

In this section, we will consider the optimal set of quadrature rules (1.3) with respect to $r = 3$ weight functions. Let $P_{(n_1,n_2,n_3)}$ be the type II multiple orthogonal polynomial corresponding to a multi-index $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$ and with respect to weight functions $W = (w_1, w_2, w_3)$. Now, the nearest neighbor recurrence relations (1.2) have the following form:

$$\begin{aligned} xP_{(n_1,n_2,n_3)}(x) &= P_{(n_1+1,n_2,n_3)}(x) + d_{(n_1,n_2,n_3)}P_{(n_1,n_2,n_3)}(x) \\ &\quad + a_{(n_1,n_2,n_3)}P_{(n_1-1,n_2,n_3)}(x) + b_{(n_1,n_2,n_3)}P_{(n_1,n_2-1,n_3)}(x) \\ &\quad + c_{(n_1,n_2,n_3)}P_{(n_1,n_2,n_3-1)}(x), \end{aligned} \tag{3.1}$$

$$\begin{aligned} xP_{(n_1,n_2,n_3)}(x) &= P_{(n_1,n_2+1,n_3)}(x) + e_{(n_1,n_2,n_3)}P_{(n_1,n_2,n_3)}(x) \\ &\quad + a_{(n_1,n_2,n_3)}P_{(n_1-1,n_2,n_3)}(x) + b_{(n_1,n_2,n_3)}P_{(n_1,n_2-1,n_3)}(x) \\ &\quad + c_{(n_1,n_2,n_3)}P_{(n_1,n_2,n_3-1)}(x), \end{aligned} \tag{3.2}$$

$$\begin{aligned} xP_{(n_1,n_2,n_3)}(x) &= P_{(n_1,n_2,n_3+1)}(x) + f_{(n_1,n_2,n_3)}P_{(n_1,n_2,n_3)}(x) \\ &\quad + a_{(n_1,n_2,n_3)}P_{(n_1-1,n_2,n_3)}(x) + b_{(n_1,n_2,n_3)}P_{(n_1,n_2-1,n_3)}(x) \\ &\quad + c_{(n_1,n_2,n_3)}P_{(n_1,n_2,n_3-1)}(x). \end{aligned} \tag{3.3}$$

In the same way as in the case $r = 2$, one can prove the following three lemmas by using mathematical induction. We will prove only Lemma 4.

Lemma 3 *Let $(n_1, n_2, n_3) \in \mathbb{N}^3$ be the multi-index and $P_{(n_1,n_2,n_3)}$ type II multiple orthogonal polynomial with respect to W. Then,*

$$\begin{aligned} P_{(n_1,n_2+1,0)}(x) &= P_{(n_1+1,n_2,0)}(x) \\ &\quad + \sum_{k=0}^{n_2-1} P_{(n_1+1,k,0)}(x) \prod_{i=k+1}^{n_2} (d_{(n_1,i,0)} - e_{(n_1,i,0)}) \\ &\quad + P_{(n_1,0,0)}(x) \prod_{i=0}^{n_2} (d_{(n_1,i,0)} - e_{(n_1,i,0)}) \end{aligned} \tag{3.4}$$

holds.

Lemma 4 *Let $(n_1, n_2, n_3) \in \mathbb{N}^3$ be the multi-index and $P_{(n_1,n_2,n_3)}$ type II multiple orthogonal polynomial with respect to W. Then,*

$$\begin{aligned} P_{(n_1,n_2,n_3+1)}(x) &= P_{(n_1+1,n_2,n_3)}(x) \\ &\quad + \sum_{k=0}^{n_3-1} P_{(n_1+1,n_2,k)}(x) \prod_{i=k+1}^{n_3} (d_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}) \\ &\quad + P_{(n_1,n_2,0)}(x) \prod_{i=0}^{n_3} (d_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}) \end{aligned} \tag{3.5}$$

holds.

Proof We prove this by induction on $\nu = n_1 + n_2 + n_3$. From (3.1) and (3.3), we obtain

$$P_{(n_1,n_2,n_3+1)}(x) = P_{(n_1+1,n_2,n_3)}(x) + (d_{(n_1,n_2,n_3)} - f_{(n_1,n_2,n_3)})P_{(n_1,n_2,n_3)}(x). \quad (3.6)$$

Now, from (3.6) for $(n_1, n_2, n_3) = (0, 0, 0)$ we obtain

$$P_{(0,0,1)}(x) = P_{(1,0,0)}(x) + (d_{(0,0,0)} - f_{(0,0,0)})P_{(0,0,0)}(x),$$

which is the same as (3.5) for $\nu = 0$, i.e., for $(n_1, n_2, n_3) = (0, 0, 0)$.

Let us now assume that (3.5) is true for all multi-indices (n_1, n_2, n_3) of length $\nu = n_1 + n_2 + n_3$, i.e.,

$$\begin{aligned} P_{(n_1,n_2,n_3)}(x) &= P_{(n_1+1,n_2,n_3-1)}(x) \\ &\quad + \sum_{k=0}^{n_3-2} P_{(n_1+1,n_2,k)}(x) \prod_{i=k+1}^{n_3-1} (d_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}) \\ &\quad + P_{(n_1,n_2,0)}(x) \prod_{i=0}^{n_3-1} (d_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}). \end{aligned} \quad (3.7)$$

We will now prove that it is true for the multi-index $(n_1, n_2, n_3 + 1)$. By using (3.6) and the induction hypothesis (3.7), we obtain

$$\begin{aligned} P_{(n_1,n_2,n_3+1)}(x) =& P_{(n_1+1,n_2,n_3)}(x) \\ &+ (d_{(n_1,n_2,n_3)} - f_{(n_1,n_2,n_3)})P_{(n_1+1,n_2,n_3-1)}(x) \\ &+ (d_{(n_1,n_2,n_3)} - f_{(n_1,n_2,n_3)}) \\ &\times \left(\sum_{k=0}^{n_3-2} P_{(n_1+1,n_2,k)}(x) \prod_{i=k+1}^{n_3-1} (d_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}) \right. \\ &\left. + P_{(n_1,n_2,0)}(x) \prod_{i=0}^{n_3-1} (d_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}) \right) \\ =& P_{(n_1+1,n_2,n_3)}(x) \\ &+ \sum_{k=0}^{n_3-1} P_{(n_1+1,n_2,k)}(x) \prod_{i=k+1}^{n_3} (d_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}) \\ &+ P_{(n_1,n_2,0)}(x) \prod_{i=0}^{n_3} (d_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}). \end{aligned}$$

By the mathematical induction principle, (3.5) is correct for all multi-indices (n_1, n_2, n_3). □

Lemma 5 *Let* $(n_1, n_2, n_3) \in \mathbb{N}^3$ *be the multi-index and* $P_{(n_1,n_2,n_3)}$ *type II multiple orthogonal polynomial with respect to* W*. Then,*

$$P_{(n_1,n_2,n_3+1)}(x) = P_{(n_1,n_2+1,n_3)}(x) + \sum_{k=0}^{n_3-1} P_{(n_1,n_2+1,k)}(x) \prod_{i=k+1}^{n_3} (e_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}) + P_{(n_1,n_2,0)}(x) \prod_{i=0}^{n_3} (e_{(n_1,n_2,i)} - f_{(n_1,n_2,i)}) \tag{3.8}$$

holds.

In the case $r = 3$, we consider one way for the construction of the polynomial $P_{(n,m,p)}$ which has the property that the zeros of the polynomial $P_{(n,m,p)}$ are eigenvalues of a square matrix, and which gives a stable numerical method for the construction of the optimal set of quadrature rules, similarly as in Subsection 2.1.

Setting $n_1 = 0, 1, \ldots, n-1$, and $n_2 = n_3 = 0$ in (3.1), we get

$$\begin{aligned}
xP_{(0,0,0)}(x) &= P_{(1,0,0)}(x) + d_{(0,0,0)}P_{(0,0,0)}(x),\\
xP_{(1,0,0)}(x) &= P_{(2,0,0)}(x) + d_{(1,0,0)}P_{(1,0,0)}(x) + a_{(1,0,0)}P_{(0,0,0)}(x),\\
xP_{(2,0,0)}(x) &= P_{(3,0,0)}(x) + d_{(2,0,0)}P_{(2,0,0)}(x) + a_{(2,0,0)}P_{(1,0,0)}(x),\\
&\vdots\\
xP_{(n-1,0,0)}(x) &= P_{(n,0,0)}(x) + d_{(n-1,0,0)}P_{(n-1,0,0)}(x) + a_{(n-1,0,0)}P_{(n-2,0,0)}(x).
\end{aligned}$$

Now by applying (3.2) with $n_1 = n$, $n_2 = 0, 1$ and $n_3 = 0$, we obtain

$$\begin{aligned}
xP_{(n,0,0)}(x) &= P_{(n,1,0)}(x) + e_{(n,0,0)}P_{(n,0,0)}(x) + a_{(n,0,0)}P_{(n-1,0,0)}(x),\\
xP_{(n,1,0)}(x) &= P_{(n,2,0)}(x) + e_{(n,1,0)}P_{(n,1,0)}(x) + a_{(n,1,0)}P_{(n-1,1,0)}(x)\\
&\quad + b_{(n,1,0)}P_{(n,0,0)}(x).
\end{aligned}$$

By using (3.4), for $(n_1, n_2, n_3) = (n-1, 0, 0)$, we have

$$P_{(n-1,1,0)}(x) = P_{(n,0,0)}(x) + P_{(n-1,0,0)}(x)(d_{(n-1,0,0)} - e_{(n-1,0,0)}),$$

and hence

$$\begin{aligned}
xP_{(n,1,0)}(x) &= P_{(n,2,0)}(x) + e_{(n,1,0)}P_{(n,1,0)}(x) + (a_{(n,1,0)} + b_{(n,1,0)})P_{(n,0,0)}(x)\\
&\quad + a_{(n,1,0)}(d_{(n-1,0,0)} - e_{(n-1,0,0)})P_{(n-1,0,0)}(x).
\end{aligned}$$

In the same way, by using (3.2) and (3.4), we obtain

$$\begin{aligned}
xP_{(n,2,0)}(x) =& P_{(n,3,0)}(x) + e_{(n,2,0)}P_{(n,2,0)}(x) + (a_{(n,2,0)} + b_{(n,2,0)})P_{(n,1,0)}(x) \\
&+ a_{(n,2,0)}(d_{(n-1,1,0)} - e_{(n-1,1,0)})P_{(n,0,0)}(x) \\
&+ a_{(n,2,0)}\prod_{i=0}^{1}(d_{(n-1,i,0)} - e_{(n-1,i,0)})P_{(n-1,0,0)}(x) \\
&\vdots \\
xP_{(n,m-1,0)}(x) =& P_{(n,m,0)}(x) + e_{(n,m-1,0)}P_{(n,m-1,0)}(x) \\
&+ (a_{(n,m-1,0)} + b_{(n,m-1,0)})P_{(n,m-2,0)}(x) \\
&+ a_{(n,m-1,0)}\sum_{k=0}^{m-3} P_{(n,k,0)}(x)\prod_{i=k+1}^{m-2}(d_{(n-1,i,0)} - e_{(n-1,i,0)}) \\
&+ a_{(n,m-1,0)}\prod_{i=0}^{m-2}(d_{(n-1,i,0)} - e_{(n-1,i,0)})P_{(n-1,0,0)}(x).
\end{aligned}$$

Now by applying (3.3) with $(n_1, n_2, n_3) = (n, m, 0)$, we obtain

$$\begin{aligned}
xP_{(n,m,0)}(x) =& P_{(n,m,1)}(x) + f_{(n,m,0)}P_{(n,m,0)}(x) + a_{(n,m,0)}P_{(n-1,m,0)}(x) \\
&+ b_{(n,m,0)}P_{(n,m-1,0)}(x).
\end{aligned}$$

By using (3.4), for $(n_1, n_2, n_3) = (n-1, m-1, 0)$, we have

$$\begin{aligned}
P_{(n-1,m,0)}(x) =& P_{(n,m-1,0)}(x) \\
&+ \sum_{k=0}^{m-2} P_{(n,k,0)}(x)\prod_{i=k+1}^{m-1}(d_{(n-1,i,0)} - e_{(n-1,i,0)}) \\
&+ \prod_{i=0}^{m-1}(d_{(n-1,i,0)} - e_{(n-1,i,0)})P_{(n-1,0,0)},
\end{aligned} \tag{3.9}$$

and hence

$$\begin{aligned}
xP_{(n,m,0)}(x) =& P_{(n,m,1)}(x) + f_{(n,m,0)}P_{(n,m,0)}(x) \\
&+ (a_{(n,m,0)} + b_{(n,m,0)})P_{(n,m-1,0)}(x) \\
&+ a_{(n,m,0)}\sum_{k=0}^{m-2} P_{(n,k,0)}(x)\prod_{i=k+1}^{m-1}(d_{(n-1,i,0)} - e_{(n-1,i,0)}) \\
&+ a_{(n,m,0)}\prod_{i=0}^{m-1}(d_{(n-1,i,0)} - e_{(n-1,i,0)})P_{(n-1,0,0)}(x).
\end{aligned}$$

In a similar way, by applying (3.3) with $(n_1, n_2, n_3) = (n, m, 1)$, we obtain

$$\begin{aligned} xP_{(n,m,1)}(x) =& P_{(n,m,2)}(x) + f_{(n,m,1)}P_{(n,m,1)}(x) + a_{(n,m,1)}P_{(n-1,m,1)}(x) \\ &+ b_{(n,m,1)}P_{(n,m-1,1)}(x) + c_{(n,m,1)}P_{(n,m,0)}(x). \end{aligned}$$

By using (3.5), for $(n_1, n_2, n_3) = (n-1, m, 0)$, and (3.4), for $(n_1, n_2, n_3) = (n-1, m-1, 0)$, we have

$$\begin{aligned} P_{(n-1,m,1)}(x) =& P_{(n,m,0)}(x) + (d_{(n-1,m,0)} - f_{(n-1,m,0)})P_{(n-1,m,0)}(x) \\ =& P_{(n,m,0)}(x) + (d_{(n-1,m,0)} - f_{(n-1,m,0)})P_{(n,m-1,0)}(x) \\ &+ (d_{(n-1,m,0)} - f_{(n-1,m,0)}) \\ &\times \sum_{k=0}^{m-2} P_{(n,k,0)}(x) \prod_{i=k+1}^{m-1} (d_{(n-1,i,0)} - e_{(n-1,i,0)}) \\ &+ (d_{(n-1,m,0)} - f_{(n-1,m,0)}) \prod_{i=0}^{m-1} (d_{(n-1,i,0)} - e_{(n-1,i,0)}) \\ &\times P_{(n-1,0,0)}, \end{aligned}$$

and by using (3.8), for $(n_1, n_2, n_3) = (n, m-1, 0)$, we have

$$P_{(n,m-1,1)}(x) = P_{(n,m,0)}(x) + (e_{(n,m-1,0)} - f_{(n,m-1,0)})P_{(n,m-1,0)}(x),$$

and hence

$$\begin{aligned} xP_{(n,m,1)}(x) =& P_{(n,m,2)}(x) + f_{(n,m,1)}P_{(n,m,1)}(x) \\ &+ (a_{(n,m,1)} + b_{(n,m,1)} + c_{(n,m,1)})P_{(n,m,0)}(x) \\ &+ (a_{(n,m,1)}(d_{(n-1,m,0)} - f_{(n-1,m,0)}) \\ &+ b_{(n,m,1)}(e_{(n,m-1,0)} - f_{(n,m-1,0)}))P_{(n,m-1,0)} \\ &+ a_{(n,m,1)}(d_{(n-1,m,0)} - f_{(n-1,m,0)}) \\ &\times \sum_{k=0}^{m-2} P_{(n,k,0)}(x) \prod_{i=k+1}^{m-1} (d_{(n-1,i,0)} - e_{(n-1,i,0)}) \\ &+ a_{(n,m,1)}(d_{(n-1,m,0)} - f_{(n-1,m,0)}) \\ &\times \prod_{i=0}^{m-1} (d_{(n-1,i,0)} - e_{(n-1,i,0)})P_{(n-1,0,0)}(x). \end{aligned}$$

Continuing in this manner, by using (3.3) for $(n_1, n_2, n_3) = (n, m, p-1)$, (3.5) for $(n_1, n_2, n_3) = (n-1, m, p-2)$, (3.9), and (3.8) for $(n_1, n_2, n_3) = (n, m-1, p-2)$, we obtain

$$\begin{aligned}
xP_{(n,m,p-1)}(x) =&P_{(n,m,p)}(x) + f_{(n,m,p-1)}P_{(n,m,p-1)}(x)\\
&+ (a_{(n,m,p-1)} + b_{(n,m,p-1)} + c_{(n,m,p-1)})P_{(n,m,p-2)}(x)\\
&+ \sum_{k=0}^{p-3} P_{(n,m,k)} \left(a_{(n,m,p-1)} \prod_{i=k+1}^{p-2} (d_{(n-1,m,i)} - f_{(n-1,m,i)}) \right.\\
&\left. + b_{(n,m,p-1)} \prod_{i=k+1}^{p-2} (e_{(n,m-1,i)} - f_{(n,m-1,i)}) \right)\\
&+ P_{(n,m-1,0)} \left(a_{(n,m,p-1)} \prod_{i=0}^{p-2} (d_{(n-1,m,i)} - f_{(n-1,m,i)}) \right.\\
&\left. + b_{(n,m,p-1)} \prod_{i=0}^{p-2} (e_{(n,m-1,i)} - f_{(n,m-1,i)}) \right)\\
&+ a_{(n,m,p-1)} \prod_{i=0}^{p-2} (d_{(n-1,m,i)} - f_{(n-1,m,i)})\\
&\times \sum_{k=0}^{m-2} P_{(n,k,0)} \prod_{i=k+1}^{m-1} (d_{(n-1,i,0)} - e_{(n-1,i,0)})\\
&+ P_{(n-1,0,0)} a_{(n,m,p-1)} \prod_{i=0}^{p-2} (d_{(n-1,m,i)} - f_{(n-1,m,i)})\\
&\times \prod_{i=0}^{m-1} (d_{(n-1,i,0)} - e_{(n-1,i,0)}).
\end{aligned}$$

The obtained equations can be written in the following way

$$x\mathbf{P}_{(n,m,p)}(x) = M_{(n,m,p)}\mathbf{P}_{(n,m,p)}(x) + P_{(n,m,p)}(x)\mathbf{e}_{(n,m,p)}, \tag{3.10}$$

where

$$\mathbf{P}_{(n,m,p)}(x) = \begin{bmatrix} P_{(0,0,0)}(x) \\ P_{(1,0,0)}(x) \\ \vdots \\ P_{(n,0,0)}(x) \\ P_{(n,1,0)}(x) \\ \vdots \\ P_{(n,m,0)}(x) \\ P_{(n,m,1)}(x) \\ \vdots \\ P_{(n,m,p-1)}(x) \end{bmatrix}, \quad \mathbf{e}_{(n,m,p)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and $M_{(n,m,p)}$ is the following matrix of order $n+m+p$

$$M_{(n,m,p)} = \begin{bmatrix} A_{(n,m,p)} \\ B_{(n,m,p)} \\ C_{(n,m,p)} \end{bmatrix},$$

where

$$\begin{aligned}
A_{(n,m,p)} &= \left[\, \widetilde{A}_{(n,m,p)} \middle| \, [0]_{n\times(m+p-1)} \right], \\
B_{(n,m,p)} &= \left[[0]_{m\times(n-1)} \middle| \widetilde{B}_{(n,m,p)} \middle| \, [0]_{m\times(p-1)} \right], \\
C_{(n,m,p)} &= \left[[0]_{m\times(n-1)} \middle| \widetilde{C}_{(n,m,p)} \right], \\
\widetilde{A}_{(n,m,p)} &= \begin{bmatrix} d_{(0,0,0)} & 1 & & & \\ a_{(1,0,0)} & d_{(1,0,0)} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & d_{(n-1,0,0)} & 1 \end{bmatrix},
\end{aligned}$$

$$\widetilde{B}_{(n,m,p)} = \begin{bmatrix} a_{(n,0,0)} & e_{(n,0,0)} & 1 & & & \\ a_{(n,1,0)}\Pi^{(3,3)}_{0,0} & a_{(n,1,0)} + b_{(n,1,0)} & e_{(n,1,0)} & 1 & & \\ \vdots & \vdots & & \ddots & \ddots & \\ a_{(n,m-2,0)}\Pi^{(3,3)}_{0,m-3} & a_{(n,m-2,0)}\Pi^{(3,3)}_{1,m-3} & \cdots & \cdots & 1 & \\ a_{(n,m-1,0)}\Pi^{(3,3)}_{0,m-2} & a_{(n,m-1,0)}\Pi^{(3,3)}_{1,m-2} & \cdots & \cdots & e_{(n,m-1,0)} & 1 \end{bmatrix},$$

$$\widetilde{C}_{(n,m,p)} = \left[\begin{matrix} a_{(n,m,0)}\Pi^{(3,3)}_{0,m-1} & a_{(n,m,0)}\Pi^{(3,3)}_{1,m-1} & \cdots & a_{(n,m,0)} + b_{(n,m,0)} \\ a_{(n,m,1)}\Pi^{(3,3)}_{0,m-1}\Pi^{(3,1)}_{0,0} & a_{(n,m,1)}\Pi^{(3,3)}_{1,m-1}\Pi^{(3,1)}_{0,0} & \cdots & \begin{matrix} a_{(n,m,1)}\Pi^{(3,1)}_{0,0} \\ +b_{(n,m,1)}\Pi^{(3,2)}_{0,0} \end{matrix} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(n,m,p-2)}\Pi^{(3,3)}_{0,m-1}\Pi^{(3,1)}_{0,p-3} & a_{(n,m,p-2)}\Pi^{(3,3)}_{1,m-1}\Pi^{(3,1)}_{0,p-3} & \cdots & \begin{matrix} a_{(n,m,p-2)}\Pi^{(3,1)}_{0,p-3} \\ +b_{(n,m,p-2)}\Pi^{(3,2)}_{0,p-3} \end{matrix} \\ a_{(n,m,p-1)}\Pi^{(3,3)}_{0,m-1}\Pi^{(3,1)}_{0,p-2} & a_{(n,m,p-1)}\Pi^{(3,3)}_{1,m-1}\Pi^{(3,1)}_{0,p-2} & \cdots & \begin{matrix} a_{(n,m,p-1)}\Pi^{(3,1)}_{0,p-2} \\ +b_{(n,m,p-1)}\Pi^{(3,2)}_{0,p-2} \end{matrix} \end{matrix} \right.$$

$$\left. \begin{matrix} f_{(n,m,0)} & 1 & & & \\ \begin{matrix} a_{(n,m,1)} + b_{(n,m,1)} \\ +c_{(n,m,1)} \end{matrix} & f_{(n,m,1)} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \begin{matrix} a_{(n,m,p-2)}\Pi^{(3,1)}_{1,p-3} \\ +b_{(n,m,p-2)}\Pi^{(3,2)}_{1,p-3} \end{matrix} & \cdots & & \ddots & 1 \\ \begin{matrix} a_{(n,m,p-1)}\Pi^{(3,1)}_{1,p-2} \\ +b_{(n,m,p-1)}\Pi^{(3,2)}_{1,p-2} \end{matrix} & \cdots & & \cdots & f_{(n,m,p-1)} \end{matrix} \right]$$

(where we used notation $\Pi^{(3,1)}_{k,\ell} = \prod_{i=k}^{\ell}(d_{(n-1,m,i)} - f_{(n-1,m,i)})$, $\Pi^{(3,2)}_{k,\ell} = \prod_{i=k}^{\ell}(e_{(n,m-1,i)} - f_{(n,m-1,i)})$, and $\Pi^{(3,3)}_{k,\ell} = \prod_{i=k}^{\ell}(d_{(n-1,i,0)} - e_{(n-1,i,0)})$).

If $x_i = x_i^{(n,m,p)}$, $i = 1, 2, \ldots, n+m+p$, are the zeros of $P_{(n,m,p)}(x)$, then (3.10) reduces to the following eigenvalue problem:

$$x_i \mathbf{P}_{(n,m,p)}(x_i) = M_{(n,m,p)} \mathbf{P}_{(n,m,p)}(x_i).$$

Thus, the zeros of $P_{(n,m,p)}(x)$, i.e., the nodes of the optimal set of quadrature rules (1.3), are the eigenvalues of the matrix $M_{(n,m,p)}$. For the computation of all recurrence coefficients in the matrices $M_{(n,m,p)}$, we use again algorithms given in [10].

By requiring that each rule exactly integrates modified moments, the weight coefficients $A_{k,i}$, $k = 1, 2, \ldots, r$, $i = 1, 2, \ldots, n+m+p$, can be computed. Let us denote by

$$V_{(n,m,p)} = \left[\, \mathbf{P}_{(n,m,p)}(x_1)\ \mathbf{P}_{(n,m,p)}(x_2)\ \cdots\ \mathbf{P}_{(n,m,p)}(x_{n+m+p})\,\right] \tag{3.11}$$

the matrix of the eigenvectors of the matrix $M_{(n,m,p)}$, each normalized so that the first component is equal to 1. Then, the weight coefficients $A_{k,i}$ can be obtained by solving the following systems of linear equations

$$V_{(n,m,p)} \begin{bmatrix} A_{k,1} \\ A_{k,2} \\ \vdots \\ A_{k,n+m+p} \end{bmatrix} = \begin{bmatrix} \mu^{*(k)}_{(0,0,0)} \\ \mu^{*(k)}_{(1,0,0)} \\ \vdots \\ \mu^{*(k)}_{(n,0,0)} \\ \mu^{*(k)}_{(n,1,0)} \\ \vdots \\ \mu^{*(k)}_{(n,m,0)} \\ \mu^{*(k)}_{(n,m,1)} \\ \vdots \\ \mu^{*(k)}_{(n,m,p-1)} \end{bmatrix}, \quad k = 1, 2, \ldots, r, \tag{3.12}$$

where

$$\mu^{*(k)}_{(i,j,\ell)} = \int_E P_{(i,j,\ell)}(x)\, w_k(x)\, \mathrm{d}x,$$

$i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$, $\ell = 0, 1, \ldots, p-1$ and $k = 1, 2, \ldots, r$.

4 Numerical examples

In this section, we give numerical examples to illustrate our results obtained in the previous sections.

Example 1 Let us consider the optimal set of quadrature rules with respect to the multi-index $\mathbf{n} = (5, 3)$ and $W = (w_1, w_2)$, where $w_1(x) = (1-x)^2(1+x)$, $w_2(x) = \sqrt{1-x^2}$, $x \in (-1, 1)$.

For obtaining the nodes and weight coefficients of the optimal set of quadrature rules, we will use the numerical construction described in Subsection 2.1. By starting with the marginal coefficients of the recurrence relations for the weight functions w_1 and w_2 and using the algorithm given in [10, Section 3], we obtain the coefficients of the nearest neighbor recurrence relation, i.e., the elements of the matrix $M^{(1)}_{(5,3)}$. Thus, the matrix $M^{(1)}_{(5,3)}$ given by (2.9) is

$$M^{(1)}_{(5,3)} = \begin{bmatrix} -\frac{1}{5} & 1 & & & & & & \\ \frac{4}{25} & -\frac{3}{35} & 1 & & & & & \\ & \frac{10}{49} & -\frac{1}{21} & 1 & & & & \\ & & \frac{2}{9} & -\frac{1}{33} & 1 & & & \\ & & & \frac{28}{121} & -\frac{3}{143} & 1 & & \\ & & & & \frac{40}{169} & \frac{53}{104} & 1 & \\ & & & & -\frac{21}{169} & \frac{759}{4160} & -\frac{299}{600} & 1 \\ & & & & -\frac{196}{4225} & \frac{196}{1625} & \frac{1316}{5625} & \frac{29}{100} \end{bmatrix}.$$

The eigenvalues of the matrix $M^{(1)}_{(5,3)}$, i.e., the nodes of the corresponding optimal set of quadrature rules, are given in the Table 1.

Table 1 The nodes x_i and the weight coefficients $A_{k,i}$, $k = 1, 2$, $i = 1, 2, \ldots, 8$, of the optimal set of quadrature rules with respect to $W = ((1-x)^2(1+x), \sqrt{1-x^2})$ and $\mathbf{n} = (5, 3)$

i	x_i	$A_{1,i}$	$A_{2,i}$
1	−0.9493304705920284	0.0197078273304	0.0326709063623
2	−0.7886628776462804	0.146501264081	0.132969969692
3	−0.5248625748575822	0.337025835132	0.259812205396
4	−0.1921623705288847	0.405091137455	0.346127182775
5	0.1650040445478995	0.287118405877	0.348761739542
6	0.4991945368064593	0.115654000545	0.266342952132
7	0.7673682547519254	0.0213877867540	0.143723361760
8	0.9401181241851581	0.000847076159763	0.0403880091370

Table 2 The nodes x_i and the weight coefficients $A_{k,i}$, $k = 1, 2, 3$, $i = 1, 2, \ldots, 4$, of the optimal set of quadrature rules with respect to the multi-index $\mathbf{n} = (1, 2, 1)$ and the weights $W = ((1-x)^{-1/2}(1+x), (1-x)^{1/2}(1+x)^{-1/4}, (1-x)^{-1/2}(1+x)^{1/2})$

i	x_i	$A_{1,i}$	$A_{2,i}$	$A_{3,i}$
1	−0.9114416355780501	0.017876803419132	0.69824047538114	0.057439205425096
2	−0.4168874368020922	0.33745505055855	0.93771530042234	0.44266015401279
3	0.3450910487187731	1.24856299056426	0.56475072571780	1.07634156393954
4	0.9176442132164949	2.1673413217863	0.079032525548488	1.56515173021236

Now, by using the method 1° given in Section 2, we obtain the type II multiple orthogonal polynomials

$$P_{(0,0)}(x) = 1,$$
$$P_{(1,0)}(x) = x + \frac{1}{5},$$
$$P_{(2,0)}(x) = x^2 + \frac{2x}{7} - \frac{1}{7},$$
$$P_{(3,0)}(x) = x^3 + \frac{x^2}{3} - \frac{x}{3} - \frac{1}{21},$$
$$P_{(4,0)}(x) = x^4 + \frac{4x^3}{11} - \frac{6x^2}{11} - \frac{4x}{33} + \frac{1}{33},$$
$$P_{(5,0)}(x) = x^5 + \frac{5x^4}{13} - \frac{10x^3}{13} - \frac{30x^2}{143} + \frac{15x}{143} + \frac{5}{429},$$
$$P_{(5,1)}(x) = x^6 - \frac{x^5}{8} - \frac{125x^4}{104} + \frac{5x^3}{52} + \frac{15x^2}{44} - \frac{15x}{1144} - \frac{15}{1144},$$
$$P_{(5,2)}(x) = x^7 + \frac{28x^6}{75} - \frac{217x^5}{150} - \frac{35x^4}{78} + \frac{112x^3}{195} + \frac{7x^2}{55} - \frac{7x}{130} - \frac{7}{1430}.$$

By solving the system (2.12), we obtain the weight coefficients, which are given in Table 1.

Example 2 Let us consider the optimal set of quadrature rules with respect to the multi-index $\mathbf{n} = (1, 2, 1)$ and $W = (w_1, w_2, w_3)$, where the weight functions are the following: $w_1(x) = (1-x)^{-1/2}(1+x)$, $w_2(x) = (1-x)^{1/2}(1+x)^{-1/4}$, $w_3(x) = (1-x)^{-1/2}(1+x)^{1/2}$, $x \in (-1, 1)$.

The nodes and the weight coefficients of the optimal set of quadrature rules obtained by using method given in Section 3 are given in Table 2.

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