Numer Algor (2018) 78:1061–1086
DOI 10.1007/s11075-017-0413-y

ORIGINAL PAPER

Fully discrete second-order backward difference method for Kelvin-Voigt fluid flow model

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Received: 25 April 2017 / Accepted: 1 September 2017 / Published online: 9 September 2017

Abstract In this article, based on a second-order backward difference method, a completely discrete scheme is discussed for a Kelvin-Voigt viscoelastic fluid flow model with nonzero forcing function, which is either independent of time or in $\mathbf{L}^{\infty}(\mathbf{L}^2)$. After deriving some a priori bounds for the solution of a semidiscrete Galerkin finite element scheme, a second-order backward difference method is applied for temporal discretization. Then, a priori estimates in Dirichlet norm are derived, which are valid uniformly in time using a combination of discrete Gronwall's lemma and Stolz-Cesaro's classical result on sequences. Moreover, an existence of a discrete global attractor for the discrete problem is established. Further, optimal a priori error estimates are obtained, whose bounds may depend exponentially in time. Under uniqueness condition, these estimates are shown to be uniform in time. Finally, several numerical experiments are conducted to confirm our theoretical findings.

Keywords Viscoelastic fluids · Kelvin-Voigt model · A priori bounds · Second-order backward difference scheme · Existence of discrete attractor · Optimal error estimates

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1 Introduction

In this paper, we consider a fully discrete method which is based on a second-order backward difference scheme for the following Kelvin-Voigt viscoelastic fluid flow model (see, [20, 21]):

$$\frac{\partial \mathbf{u}}{\partial t} - \kappa \Delta \mathbf{u}_t - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}(x,t), \quad x \in \Omega, \quad t > 0 \tag{1.1}$$

with incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \quad x \in \Omega, \ t > 0, \tag{1.2}$$

and initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \partial\Omega, \ t \geq 0. \tag{1.3}$$

Here, Ω be a bounded convex polygonal or polyhedron domain in $\mathbb{R}^d$ ($d = 2$ or 3) with boundary $\partial\Omega$, $\mathbf{u} = \mathbf{u}(x,t)$ and $p = p(x,t)$ denote the velocity vector and the pressure, respectively, $\nu > 0$ represents kinematic viscosity coefficient and κ is the time of relaxation of deformations or the retardation in time parameter. For some applications, we refer to [7–9] and references, therein.

Now, we present a quick review of some related literature on the Kelvin-Voigt model. Based on the proof technique of Ladyzenskaya [19], Oskolkov [20, 21] has proved an existence of a unique global “almost” classical solution for the initial and boundary value problem (1.1)–(1.3) in finite time interval. Further, investigations on existence and uniqueness results for all time $t > 0$ have been continued by him and his collaborators under various conditions on the forcing function $\mathbf{f}$, see [23] and [24].

For earlier results on numerical methods applied to Kelvin-Voigt viscoelastic fluid flow problem, we refer to [3, 4, 22, 28] and [29]. Under the assumption that the solution is asymptotically stable as $t \to \infty$, Oskolkov [22] has proved that the spectral Galerkin approximation to the problem (1.1)–(1.3) is convergent in semitime axis $t \geq 0$. Later on, Pani et al. [27] have employed a variant of nonlinear semidiscrete spectral Galerkin method and derived optimal error estimates. Recently, Bajpai et al. [4] have analyzed both backward Euler scheme and backward difference scheme for the completely discretization of the problem (1.1)–(1.2), when the forcing function $\mathbf{f} = 0$. Firstly, the authors have shown existence of solution for the discrete nonlinear problem using a variant of Brouwer fixed point theorem and then, proved optimal error estimates which reflect exponential decay property. Note that their error bounds contain term like $\frac{1}{\kappa^r}$, for $r \geq 2$, which may blow up as $\kappa \mapsto 0$. For related articles on Navier-Stokes equations, see [13] and on Oldroyd model, refer to [2, 11, 12, 14, 25–28, 31–34].

When the forcing function ($\mathbf{f} \neq 0$) with $\mathbf{f} \in L^\infty(\mathbf{L}^2)$, which is important in the study of dynamical system, Pany et al. [28] have employed semidiscrete finite element method for the problem (1.1)–(1.3) and have proved an existence of a global attractor. New regularity results for the exact solution are established which are valid uniformly in time as $t \mapsto \infty$ and also uniformly in κ as $\kappa \mapsto 0$. They have also derived a priori optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$-norm as well as velocity in $L^\infty(\mathbf{H}^1)$-norm and for the pressure term in $L^\infty(L^2)$-norm. Under uniqueness assumption, it is shown that error bounds are valid uniformly in time. It is, further, established that quasi-optimal error estimates are valid for small κ. In continuation to the investigation in [28] on semidiscrete problem, Pany et al. [29] have employed a backward Euler method along with its linearized version for the time discretization of the problem (1.1)–(1.2), which are first order in time schemes. A priori bounds for the discrete solution, specially in the Dirichlet norm are shown using a combination of discrete Gronwall's lemma and Stolz-Cesaro theorem. It is, further, derived that the discrete problem has a global discrete attractor and then, optimal error estimates are established. Under uniqueness assumption, it is also proved that error bounds are valid uniformly in time.

In this article, we continue our investigation further and a second-order backward difference scheme for the time discretization is analyzed. A priori estimates in Dirichlet norm for the fully discrete scheme are obtained, which are valid uniformly in time using a combination of discrete Gronwall's lemma and Stolz-Cesaro's classical result for sequences. Moreover, an existence of a discrete global attractor for the discrete problem is established and a priori error estimates are derived. More precisely, the following estimates are obtained for the fully discrete solution $(\mathbf{U}^n, P^n)$:

$$\|\mathbf{u}(t_n) - \mathbf{U}^n\| \le C\left(\frac{h^2}{\sqrt{\kappa}} + \frac{k^2}{\kappa}\right)$$

and

$$\|(p(t_n) - P^n)\| \le C\left(\frac{h}{\sqrt{\kappa}} + \frac{k^{2-\gamma}}{\kappa^{3/2}}\right),$$

where the pair $(\mathbf{U}^n, P^n)$ is the fully discrete solution of the second-order backward difference scheme and

$$\gamma = \begin{cases} 0 \ \text{ if } n \ge 2; \\ 1 \ \text{ if } n = 1. \end{cases}$$

Since constants in these error bounds depend e^{Ct}, these results as in the Navier-Stokes case are valid locally. But, under the uniqueness condition, it is further shown that error estimates are valid uniformly in time. Finally, we obtain error estimates for the fully discrete scheme.

This article is organized as follows. Section 2 deals with some assumptions and discusses the weak formulation. Section 3 focuses on some a priori estimates for the solution of the semidiscrete scheme, which are valid uniformly in time. In Section 4, we discuss a second-order backward finite difference method and show that discrete solution is bounded in Dirichlet norm. Moreover, we prove an existence of solution for the discrete nonlinear system using a variant of Brouwer's fixed point theorem and also derive existence of a discrete global attractor. In Section 5, we establish optimal error estimates for the velocity and the pressure term. Section 6 deals with some numerical experiments, which confirm our theoretical findings.

2 Preliminaries and weak formulation

We denote by bold face letters the $\mathbb{R}^d$, $(d = 2, 3)$-valued function spaces such as

$$\mathbf{H}_0^1 = \left(H_0^1(\Omega)\right)^d, \quad \mathbf{L}^2 = (L^2(\Omega))^d \text{ and } \mathbf{H}^m = (H^m(\Omega))^d,$$

where $H^m(\Omega)$ is the usual Sobolev space of order m with norm $\|\cdot\|_m$. Here, $\mathbf{H}_0^1$ is equipped with a norm

$$\|\nabla \mathbf{v}\| = \left(\sum_{i=1}^{d}(\nabla v_i, \nabla v_i)\right)^{1/2}.$$

Let $H^m/\mathbb{R}$ be the quotient space consisting of equivalence classes of elements of H^m differing by constants, with norm $\|p\|_{H^m/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|p + c\|_m$.
Now, introduce the following vector valued function spaces :

$$\mathbf{J}_1 = \{\boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla\cdot\boldsymbol{\phi} = 0\},$$
$$\mathbf{J} = \{\boldsymbol{\phi} \in \mathbf{L}^2 : \nabla\cdot\boldsymbol{\phi} = 0 \text{ in } \Omega, \quad \boldsymbol{\phi}\cdot\mathbf{n}|_{\partial\Omega} = 0 \text{ holds weakly}\},$$

where $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$ and $\boldsymbol{\phi}\cdot\mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$, see [18, 30]. For any Banach space $\mathbf{X}$, let $L^p(0, T; X)$ be the space of measurable $\mathbf{X}$-valued functions ϕ on $(0, T)$ such that

$$\int_0^T \|\phi(t)\|_X^p \, dt < \infty \quad if \quad 1 \le p \le \infty,$$

and for $p = \infty$,

$$\operatorname*{ess\,sup}_{0<t<T} \|\phi(t)\|_X < \infty.$$

Let P be the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{J}$.

Throughout this article, we following assumptions are made:

(**A1**) For $\mathbf{g} \in \mathbf{L}^2$, let the unique pair of solution $\{\mathbf{v} \in \mathbf{J}_1, q \in L^2/\mathbb{R}\}$ to the steady state Stokes problem, see [30],

$$-\Delta\mathbf{v} + \nabla q = \mathbf{g},$$
$$\nabla\cdot\mathbf{v} = 0 \quad \text{in} \quad \Omega, \qquad \mathbf{v}|_{\partial\Omega} = 0$$

satisfying the following regularity result

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/\mathbb{R}} \le C\|\mathbf{g}\|. \tag{2.1}$$

Setting the Stokes operator as

$$-\tilde{\Delta} = -P\Delta : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \to \mathbf{J},$$

then the assumption (**A1**) gives rise to the estimates

$$\|\mathbf{v}\|_2 \le C\|\tilde{\Delta}\mathbf{v}\| \quad \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2. \tag{2.2}$$

Note that, the following estimates holds

$$\|\mathbf{v}\|^2 \le \lambda_1^{-1}\|\nabla\mathbf{v}\|^2 \;\; \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \;\; \text{and } \|\nabla\mathbf{v}\|^2 \le \lambda_1^{-1}\|\tilde{\Delta}\mathbf{v}\|^2 \;\; \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2, \tag{2.3}$$

where λ_1^{-1} is the best possible positive constant, depends on the domain Ω in the Poincaré inequality.

(A2) There exists a positive constant M_0 such that the initial velocity $\mathbf{u}_0$ and the external force $\mathbf{f}$ satisfy

$$\mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{J}_1, \mathbf{f}, \mathbf{f}_t \in L^\infty(0, \infty; \mathbf{L}^2) \quad \text{with}$$
$$\|\mathbf{u}_0\|_2 \le M_0, \sup_{0<t<\infty} (\|\mathbf{f}\|, \|\mathbf{f}_t\|_{-1}, \|\mathbf{f}_{tt}\|_{-1}) \le M_0.$$

Moreover, set a bilinear form $a(\cdot,\cdot)$ on $\mathbf{H}_0^1 \times \mathbf{H}_0^1$ as

$$a(\mathbf{v}, \boldsymbol{\phi}) = (\nabla \mathbf{v}, \nabla \boldsymbol{\phi}) \quad \forall \mathbf{v},\ \boldsymbol{\phi} \in \mathbf{H}_0^1, \tag{2.4}$$

and the trilinear form $b(\cdot,\cdot,\cdot)$ on $\mathbf{H}_0^1 \times \mathbf{H}_0^1 \times \mathbf{H}_0^1$ by

$$b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = \frac{1}{2}(\mathbf{v}\cdot\nabla\mathbf{w}, \boldsymbol{\phi}) - \frac{1}{2}(\mathbf{v}\cdot\nabla\boldsymbol{\phi}, \mathbf{w}) \ \ \forall \mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1. \tag{2.5}$$

Now, the weak formulation of problem (1.1)–(1.3) is to seek a pair of functions $(\mathbf{u}(t), p(t)) \in \mathbf{H}_0^1 \times Ł^2/\mathbb{R}$ with $\mathbf{u}(0) = \mathbf{u}_0$ such that for all $t > 0$

$$\begin{aligned}(\mathbf{u}_t, \boldsymbol{\phi}) + \kappa(\nabla\mathbf{u}_t, \nabla\boldsymbol{\phi}) \ + \ \nu(\nabla\mathbf{u}, \nabla\boldsymbol{\phi}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}) \\ +\ (p, \nabla\cdot\boldsymbol{\phi}) = (\mathbf{f}, \boldsymbol{\phi}) \ \ \forall\boldsymbol{\phi} \in \mathbf{H}_0^1, \\ (\nabla\cdot\mathbf{u}, \chi) = 0 \qquad \forall \chi \in L^2\end{aligned} \tag{2.6}$$

Equivalently, find $\mathbf{u}(t) \in \mathbf{J}_1$ such that for $t > 0$

$$\begin{aligned}(\mathbf{u}_t, \boldsymbol{\phi}) + \kappa\, a(\mathbf{u}_t, \boldsymbol{\phi}) + \nu\, a(\mathbf{u}, \boldsymbol{\phi}) + b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}) \ &= \ (\mathbf{f}, \boldsymbol{\phi}) \quad \forall\boldsymbol{\phi} \in \mathbf{J}_1, \\ \mathbf{u}(0) \ &= \ \mathbf{u}_0.\end{aligned} \tag{2.7}$$

Throughout this article, C denotes a generic positive constant, which is valid uniformly with respect to time t and with respect to the parameter κ, but may depend on ν, M_0, and λ_1.

3 Finite element approximation

Let $\mathbf{H}_h$ and L_h, $0 < h < 1$ be finite dimensional subspaces of $\mathbf{H}_0^1$ and L^2, respectively, where $h > 0$ is a spatial discretization parameter, satisfying the following approximation properties:

(B1) For $\mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2$ and $q \in H^1/\mathbb{R}$, there are approximations $i_h\mathbf{w} \in \mathbf{J_h}$ and $j_h q \in L_h$ such that

$$\|\mathbf{w} - i_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - i_h\mathbf{w})\| \le K_0 h^2 \|\mathbf{w}\|_2, \quad \|q - j_h q\|_{L^2/\mathbb{R}} \le K_0 h \|q\|_{H^1/\mathbb{R}}.$$

Now, set the subspace $\mathbf{J}_h$ of $\mathbf{H}_h$ as

$$\mathbf{J}_h = \{\mathbf{v}_h \in \mathbf{H}_h : (\chi_h, \nabla\cdot\mathbf{v}_h) = 0 \quad \forall \chi_h \in L_h\}.$$

The semidiscrete formulation of (2.6) is to seek $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for all $t > 0$

$$\begin{aligned}(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) \\ - (p_h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{f}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ (\nabla \cdot \mathbf{u}_h, \chi_h) = 0 \quad \forall \chi_h \in L_h.\end{aligned} \tag{3.1}$$

Equivalently, seek $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h, \boldsymbol{\phi}_h) = -b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) + (\mathbf{f}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{J}_h. \tag{3.2}$$

First, we compute $\mathbf{u}_h(t) \in J_h$, then, $p_h(t) \in L_h$ approximation to the pressure $p(t)$ can be computed out by solving the following system

$$\begin{aligned}(p_h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \kappa\, a(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \nu\, a(\mathbf{u}_h, \boldsymbol{\phi}_h) \\ + b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) + (\mathbf{f}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h.\end{aligned} \tag{3.3}$$

For solvability of the above systems (3.2) and (3.3), see [28]. Uniqueness is obtained in the quotient space L_h/N_h with norm given by

$$\|q_h\|_{L^2/N_h} = \inf_{\chi_h \in N_h} \|q_h + \chi_h\|,$$

where

$$N_h = \{q_h \in L_h : (q_h, \nabla \cdot \boldsymbol{\phi}_h) = 0 \ \forall \boldsymbol{\phi}_h \in \mathbf{H}_h\}.$$

Moreover, assume that the pair $(\mathbf{H}_h, L_h/N_h)$ satisfies the following uniform inf-sup condition:

(**B2**) For every $q_h \in L_h$, there is a positive constant K_1 and a nontrivial function $\boldsymbol{\phi}_h \in \mathbf{H_h}$, independent of h, such that

$$|(q_h, \nabla \cdot \boldsymbol{\phi}_h)| \geq K_1 \|\nabla \boldsymbol{\phi}_h\| \|q_h\|_{L^2/N_h}.$$

As a consequence of (**B1**), the following properties of the L^2 projection $P_h : \mathbf{L}^2 \to \mathbf{J}_h$ hold: For $\boldsymbol{\phi} \in \mathbf{J}_1$, we note that, see [10, 15],

$$\|\boldsymbol{\phi} - P_h \boldsymbol{\phi}\| + h \|\nabla P_h \boldsymbol{\phi}\| \leq Ch \|\nabla \boldsymbol{\phi}\|, \tag{3.4}$$

and for $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$,

$$\|\boldsymbol{\phi} - P_h \boldsymbol{\phi}\| + h \|\nabla(\boldsymbol{\phi} - P_h \boldsymbol{\phi})\| \leq Ch^2 \|\tilde{\Delta} \boldsymbol{\phi}\|. \tag{3.5}$$

Now, define the discrete operator $\Delta_h : \mathbf{H}_h \to \mathbf{H}_h$ via the bilinear form $a(\cdot, \cdot)$ as

$$a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (-\Delta_h \mathbf{v}_h, \boldsymbol{\phi}_h) \quad \forall \mathbf{v}_h, \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{3.6}$$

Then, the discrete analog of the Stokes operator $\tilde{\Delta} = P\Delta$ is given as $\tilde{\Delta}_h = P_h \Delta_h$.

Further, the trilinear form satisfies see page 360 of [16].

$$b(\mathbf{v}_h, \mathbf{w}_h, \mathbf{w}_h) = 0 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h. \tag{3.7}$$

Examples of subspaces $\mathbf{H_h}$ satisfying assumptions (**B1**) and (**B2**) can be found in [5, 6] and [15].

Below, we recall a couple of Lemmas on a priori estimates for the semidiscrete solution $\mathbf{u}_h$ of (3.2) whose prove can be found in [28, 29].

Lemma 3.1 *With* $0 \le \alpha < \dfrac{\nu\lambda_1}{4(1+\kappa\lambda_1)}$, *and* $\boldsymbol{u}_{0h} = P_h\boldsymbol{u}_0$, *suppose assumptions* (**A1**)–(**A2**) *hold true. Then, there exists a positive constant* $C = C(\nu, \alpha, \lambda_1, M_0)$ *such that for all* $t > 0$ *the solution* $\boldsymbol{u}_h$ *of* (3.2) *satisfies*

$$\|\boldsymbol{u}_h(t)\|^2 + \|\nabla \boldsymbol{u}_h(t)\|^2 + \kappa\|\tilde{\Delta}_h\boldsymbol{u}_h(t)\|^2$$
$$+2\beta e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\nabla \boldsymbol{u}_h(s)\|^2 + \|\tilde{\Delta}_h\boldsymbol{u}_h(s)\|^2)\,ds \le C(\nu, \alpha, \lambda_1, M_0) \quad t > 0,$$

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\boldsymbol{u}_{ht}(s)\|^2 + 2\kappa\|\nabla \boldsymbol{u}_{ht}(s)\|^2)\,ds + \nu\|\nabla \boldsymbol{u}_h(t)\|^2 \le C,$$

where $\beta = (\nu/2) - \alpha(\lambda_1^{-1} + \kappa) \ge \nu/4 > 0$.
Further,

$$\|\boldsymbol{u}_{ht}(t)\|^2 + \kappa\|\nabla \boldsymbol{u}_{ht}(t)\|^2 + \nu e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla \boldsymbol{u}_{ht}(s)\|^2 ds \le C.$$

Lemma 3.2 *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let assumptions* (**A1**)–(**A2**) *hold true. Then, there is a positive constant* $C = C(\nu, \alpha, \lambda_1, M_0)$ *such that for all* $t > 0$,

$$\|\boldsymbol{u}_{htt}(t)\|_{-1,h} + \kappa\|\nabla \boldsymbol{u}_{htt}\| \le \frac{C}{\sqrt{\kappa}}.$$

Moreover, there holds

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\boldsymbol{u}_{htt}(s)\|_{-1,h}^2 + \kappa\|\nabla \boldsymbol{u}_{htt}(s)\|^2)ds \le C.$$

For our subsequent use, we also derive the following estimates.

Lemma 3.3 *Let* $0 \le \alpha < \dfrac{\nu\lambda_1}{2(1+\kappa\lambda_1)}$ *and let assumptions* (**A1**)–(**A2**) *hold true. Then, there is a positive constant* $C = C(\nu, \alpha, \lambda_1, M_0)$ *such that for all* $t > 0$

$$\|\boldsymbol{u}_{httt}(t)\|_{-1,h} + \kappa\|\nabla \boldsymbol{u}_{httt}\| \le \frac{C}{\kappa^{3/2}}$$

Further,

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}(\|\boldsymbol{u}_{httt}(s)\|_{-1,h}^2 + \kappa\|\nabla \boldsymbol{u}_{httt}(s)\|^2)ds \le \frac{C}{\kappa^2}$$

Proof Differentiate twice (3.2) with respect to time and obtain

$$(\mathbf{u}_{httt}, \boldsymbol{\phi}_h) = -\kappa a(\mathbf{u}_{httt}, \boldsymbol{\phi}_h) - \nu a(\mathbf{u}_{htt}, \boldsymbol{\phi}_h) - b(\mathbf{u}_{htt}, \mathbf{u}_h, \boldsymbol{\phi}_h)$$
$$-b(\mathbf{u}_{ht}, \mathbf{u}_{ht}, \boldsymbol{\phi}_h) - b(\mathbf{u}_h, \mathbf{u}_{htt}, \boldsymbol{\phi}_h) + (\mathbf{f}_{tt}, \boldsymbol{\phi}_h) \quad \forall\ \boldsymbol{\phi}_h \in \mathbf{J}_h. \quad (3.8)$$

An application of the Ladyzhenskaya inequality yields

$$(\mathbf{u}_{httt}, \boldsymbol{\phi}_h) \le \Big(\kappa\|\nabla\mathbf{u}_{httt}\| + \nu\|\nabla\mathbf{u}_{htt}\| + C\Big(\|\nabla\mathbf{u}_{htt}\|\|\nabla\mathbf{u}_h\| + \|\nabla\mathbf{u}_{ht}\|^2\Big) + \|\mathbf{f}_{tt}\|_{-1}\Big)\|\nabla\boldsymbol{\phi}_h\|. \tag{3.9}$$

Note that, choose $\boldsymbol{\phi} = \mathbf{u}_{httt}$ in (3.8) drop the first term from the left-hand side, we arrive at

$$\kappa\|\nabla\mathbf{u}_{httt}\|^2 \le \Big(\nu\|\nabla\mathbf{u}_{htt}\| + C(\|\nabla\mathbf{u}_{htt}\|\|\nabla\mathbf{u}_h\| + \|\nabla\mathbf{u}_{ht}\|^2) + \|\mathbf{f}_{tt}\|_{-1}\Big)\|\nabla\mathbf{u}_{httt}\|. \tag{3.10}$$

An application of Lemmas 3.1 and 3.2 in (3.10) yields

$$\kappa\|\nabla\mathbf{u}_{httt}\| \le \frac{C}{\kappa^{3/2}}. \tag{3.11}$$

Now, dividing by $\|\nabla\boldsymbol{\phi}_h\|$ in (3.8) and taking supremum over $\boldsymbol{\phi}_h \in \mathbf{H}_h$.

$$\|\mathbf{u}_{httt}\|_{-1,h} \le \Big(\kappa\|\nabla\mathbf{u}_{httt}\| + \|\nabla\mathbf{u}_{htt}\| + C(\|\nabla\mathbf{u}_{htt}\|\|\nabla\mathbf{u}_h\| + \|\nabla\mathbf{u}_{ht}\|^2) + \|\mathbf{f}_{tt}\|_{-1}\Big). \tag{3.12}$$

Substitute (3.11) and estimates from Lemmas 3.1 and 3.2 in 3.8 to establish

$$\|\mathbf{u}_{httt}\|_{-1,h} \le \frac{C}{\kappa^{3/2}}. \tag{3.13}$$

Squaring (3.11), multiply by $e^{2\alpha t}$ and then, integrate from 0 to t. Again, multiply the resulting inequality by $e^{-2\alpha t}$ to obtain

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\kappa\|\nabla\mathbf{u}_{httt}\|^2\,ds \le e^{-2\alpha t}\int_0^t e^{2\alpha s}\Big(\nu\|\nabla\mathbf{u}_{htt}\|^2 + C(\|\nabla\mathbf{u}_{htt}\|^2\|\nabla\mathbf{u}_h\|^2 + \|\nabla\mathbf{u}_{ht}\|^2) + \|\mathbf{f}_{tt}\|_{-1}^2\Big)\,ds \le \tfrac{C}{\kappa^2}. \tag{3.14}$$

Similarly, from (3.12) we find that

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_{httt}\|_{-1,h}^2\,ds \le \frac{C}{\kappa^2}. \tag{3.15}$$

This completes the rest of the proof. □

We now recall the following error estimates of semidiscrete solutions of (3.2), which are proved in [28].

Theorem 3.1 *Let conditions* (**A1**)–(**A2**) *and* (**B1**)–(**B2**) *be satisfied and let the discrete initial velocity* $\boldsymbol{u}_{0h} = P_h\boldsymbol{u}_0$. *Then, there exists a positive constant* $C(\lambda_1, \nu, \alpha, M_0)$ *such that for all* $t > 0$ *and for* $0 \le \alpha < \dfrac{\nu\lambda_1}{4(1+\lambda_1\kappa)}$,

$$\|(\boldsymbol{u} - \boldsymbol{u}_h)(t)\| + h\Big(\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)(t)\| + \|(p - p_h)(t)\|\Big) \le \frac{C}{\sqrt{\kappa}}h^2e^{Ct}.$$

Moreover, under the assumption of the uniqueness condition, that is,

$$\frac{N_0}{\nu}\|\boldsymbol{f}\|_{L^\infty(\boldsymbol{H}^{-1})} < 1 \quad \textit{and} \quad N_0 = \sup_{u,v,w\in \boldsymbol{H}_0^1(\Omega)} \frac{b(u,v,w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|}, \tag{3.16}$$

the following uniform in time estimate holds

$$\|(\boldsymbol{u}-\boldsymbol{u}_h(t))\| + h\|(p-p_h)(t)\| \le \frac{C}{\sqrt{\kappa}}h^2.$$

4 Second-order backward difference scheme

In this section, a second-order backward difference scheme is analyzed.

Let $\boldsymbol{\phi}$ be a smooth function defined on $[0, T]$, set $\boldsymbol{\phi}^n = \boldsymbol{\phi}(t_n)$ where $t_n = nk$ for time step size $k > 0$ and $\bar{\partial}_t\boldsymbol{\phi}^n = (\boldsymbol{\phi}^n - \boldsymbol{\phi}^{n-1})/k$. Define

$$D_t^{(2)}\mathbf{U}^n = \frac{1}{2k}(3\mathbf{U}^n - 4\mathbf{U}^{n-1} + \mathbf{U}^{n-2}), \tag{4.1}$$

Now the second-order backward difference scheme applied to (3.1) is to find $(\mathbf{U}^n, P^n) \in (\mathbf{H}_h, L_h)$ such that for all $n \ge 1$

$$\begin{aligned}\left(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h\right) &+ \kappa a\left(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h\right) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) + b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) \qquad (4.2)\\ &- (P^n, \nabla\cdot\boldsymbol{\phi}_h) = (\mathbf{f}^n, \boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{H}_h, \quad n \ge 2,\\ (\bar{\partial}_t\mathbf{U}^1, \boldsymbol{\phi}_h) &+ \kappa a(\partial_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^1, \boldsymbol{\phi}_h) + b(\mathbf{U}^1, \mathbf{U}^1, \boldsymbol{\phi}_h) \qquad (4.3)\\ &- (P^1, \nabla\cdot\boldsymbol{\phi}_h) = (\mathbf{f}^1, \boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{H}_h,\\ &\quad (\nabla\cdot\mathbf{U}^n, \chi_h) = 0 \quad \forall\chi_h \in L_h,\\ &\quad \mathbf{U}^0 = \mathbf{u}_{0h}.\end{aligned}$$

Equivalently, seek $\{\mathbf{U}^n\}_{n\ge1} \subset \mathbf{J}_h$ such that

$$\begin{aligned}\left(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h\right) &+ \kappa a\left(D_t^{(2)}\mathbf{U}^n, \boldsymbol{\phi}_h\right) + \nu a(\mathbf{U}^n, \boldsymbol{\phi}_h) \qquad (4.4)\\ &+ b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) = (\mathbf{f}^n, \boldsymbol{\phi}_h) \quad \forall n \ge 2 \quad \forall\boldsymbol{\phi}_h \in \mathbf{J}_h,\\ (\bar{\partial}_t\mathbf{U}^1, \boldsymbol{\phi}_h) &+ \kappa a(\partial_t\mathbf{U}^1, \boldsymbol{\phi}_h) + \nu a(\mathbf{U}^1, \boldsymbol{\phi}_h)\\ &+ b(\mathbf{U}^1, \mathbf{U}^1, \boldsymbol{\phi}_h) = (\mathbf{f}^1, \boldsymbol{\phi}_h) \quad \forall\boldsymbol{\phi}_h \in \mathbf{J}_h, \qquad (4.5)\\ &\quad \mathbf{U}^0 = \mathbf{u}_{0h}.\end{aligned}$$

Let recall the following identity from [1, 4]:

$$\begin{aligned}\left(\hat{a}^n, 3\hat{a}^n - 4\hat{a}^{n-1} + \hat{a}^{n-2}\right) &= \|\hat{a}^n\|^2 - \|\hat{a}^{n-1}\|^2 + (1-e^{2\alpha k})\left(\|\hat{a}^n\|^2 + \|\hat{a}^{n-1}\|^2\right) \qquad (4.6)\\ &+ \|\delta^2\hat{a}^{n-1}\|^2 + \|2\hat{a}^n - e^{\alpha k}\hat{a}^{n-1}\|^2 - \|2\hat{a}^{n-1} - e^{\alpha k}\hat{a}^{n-2}\|^2,\end{aligned}$$

where

$$\delta^2\hat{a}^{n-1} = e^{\alpha k}\hat{a}^n - 2\hat{a}^{n-1} + e^{\alpha k}\hat{a}^{n-2}.$$

Before obtaining a priori estimates for the discrete problem (4.4), we recall the following result for sequences.

Theorem 4.1 *(Stolz-Cesaro Theorem). Let $\{\phi^n\}_{n=0}^{\infty}$ be a sequence of real numbers. Further, let $\{\psi^n\}_{n=0}^{\infty}$ be a strictly monotone and divergent sequence. If*

$$\lim_{n\longrightarrow\infty}\Big(\frac{\phi^n-\phi^{n-1}}{\psi^n-\psi^{n-1}}\Big)=\ell,$$

then

$$\lim_{n\longrightarrow\infty}\Big(\frac{\phi^n}{\psi^n}\Big)=\ell$$

holds.

A use of (4.6) yields the following result.

Lemma 4.1 *With $0\leq\alpha<\dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$, choose k_0 so that for $0<k\leq k_0$*

$$\left(\frac{\nu k\lambda_1}{\kappa\lambda_1+1}+1\right)>e^{2\alpha k}. \tag{4.7}$$

Then, the discrete solution $\boldsymbol{U}^N$, $N\geq 1$ of (4.4) satisfies

$$(\|\boldsymbol{U}^N\|^2+\kappa\|\nabla\boldsymbol{U}^N\|^2)+\beta_2 e^{-2\alpha t_N}\,k\sum_{n=1}^{N}e^{2\alpha t_n}\|\nabla\boldsymbol{U}^n\|^2$$
$$\leq C(\alpha,\nu,\lambda_1)e^{-2\alpha t_N}(\|\boldsymbol{U}^0\|^2+\kappa\|\nabla\boldsymbol{U}^0\|^2)+\frac{e^{2\alpha k}}{\lambda\nu}\|\boldsymbol{f}\|^2_{L^\infty(\boldsymbol{H}^{-1})} \tag{4.8}$$

where, $2\beta_2=\left(2\nu e^{-2\alpha k}-2\big(\frac{1-e^{-2\alpha k}}{k}\big)\big(\kappa+\frac{1}{\lambda_1}\big)\right)>2\nu e^{-2\alpha k}>0$. Moreover, the following estimate holds:

$$\limsup_{N\longrightarrow\infty}\|\nabla U^N\|^2\leq\frac{1}{\nu^2}\|\boldsymbol{f}\|^2_{L^\infty(H^{-1})}. \tag{4.9}$$

Proof Choose $\boldsymbol{\phi}_h=\mathbf{U}^n$ in (4.4) and multiply by $e^{2\alpha t_n}$. Then, an application of (4.6) shows

$$\frac{1}{4}\bar\partial_t(\|\hat{\mathbf{U}}^n\|^2+\kappa\|\nabla\hat{\mathbf{U}}^n\|^2)+\nu\|\nabla\hat{\mathbf{U}}^n\|^2+\left(\frac{1-e^{2\alpha k}}{4k}\right)\left(\|\hat{\mathbf{U}}^n\|^2+\kappa\|\nabla\hat{\mathbf{U}}^n\|^2\right)$$
$$+\left(\frac{1-e^{2\alpha k}}{4k}\right)\left(\|\hat{\mathbf{U}}^{n-1}\|^2+\kappa\|\nabla\hat{\mathbf{U}}^{n-1}\|^2\right)+\frac{1}{4k}\left(\|\delta^2\hat{\mathbf{U}}^{n-1}\|^2+\kappa\|\delta^2\nabla\hat{\mathbf{U}}^{n-1}\|^2\right)$$
$$+\frac{1}{4}\bar\partial_t\left(\|2\hat{\mathbf{U}}^n-e^{\alpha k}\hat{\mathbf{U}}^{n-1}\|^2+\kappa\;\|2\nabla\hat{\mathbf{U}}^n-e^{\alpha k}\nabla\hat{\mathbf{U}}^{n-1}\|^2\right)$$
$$=(\hat{\mathbf{f}}^n,\hat{\mathbf{U}}^n). \tag{4.10}$$

The right-hand side of (4.10) can be estimated as

$$\frac{1}{2}\nu\|\nabla\hat{\mathbf{U}}^{n}\|^2 + \frac{1}{2\nu\lambda_1}\|\hat{\mathbf{f}}^{n}\|^2_{L^\infty(\mathbf{H}^{-1})}$$

As the fifth term on the left-hand side of (4.10) is positive, we can drop this term. Now, multiply the resulting on by $4ke^{-2\alpha k}$ and sum up from $n = 2$ to N. Then, a use of (2.3) yields

$$\begin{aligned}
&\|\hat{\mathbf{U}}^{N}\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^{N}\|^2 + k\left(2\nu e^{-2\alpha k} - 2\left(\frac{1-e^{-2\alpha k}}{k}\right)\left(\kappa+\frac{1}{\lambda_1}\right)\right)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{U}}^{n}\|^2\\
&+\|2e^{-\alpha k}\hat{\mathbf{U}}^{N} - \hat{\mathbf{U}}^{N-1}\|^2 + \kappa\|2e^{-\alpha k}\nabla\hat{\mathbf{U}}^{N} - \nabla\hat{\mathbf{U}}^{N-1}\|^2 \le \left(\|\hat{\mathbf{U}}^{1}\|^2 + \kappa\|\nabla\hat{\mathbf{U}}^{1}\|^2\right)\\
&+\left(\|2e^{-\alpha k}\hat{\mathbf{U}}^{1} - \mathbf{U}^0\|^2 + \kappa\|2e^{-\alpha k}\nabla\hat{\mathbf{U}}^{1} - \nabla\mathbf{U}^0\|^2\right)\\
&+\frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}e^{-2\alpha k}k\sum_{n=2}^{N}e^{2\alpha t_n}. \qquad (4.11)
\end{aligned}$$

For $n = 1$, that is, (4.5), we easily obtain as in the estimates of backward Euler scheme (Lemma 4.1 of [29])

$$\begin{aligned}
e^{-2\alpha k}\left(\|\mathbf{U}^1\|^2 + \kappa\|\nabla\mathbf{U}^1\|^2\right) &\le C\, e^{-2\alpha k}\,(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2)\\
&\quad+\frac{1}{\nu\lambda_1}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})} \qquad (4.12)
\end{aligned}$$

A use of the Cauchy-Schwarz's inequality with the Young's inequality and (4.12) yields a bound for the second term on the right-hand side of (4.11) as

$$\begin{aligned}
\|2e^{-\alpha k}\hat{\mathbf{U}}^{1} - \mathbf{U}^0\|^2 + \kappa\|2e^{-\alpha k}\nabla\hat{\mathbf{U}}^{1} - \nabla\mathbf{U}^0\|^2 &\le C(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2)\\
&\quad+\frac{e^{2\alpha k}}{\nu\lambda_1}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}. \qquad (4.13)
\end{aligned}$$

Using (4.12) and (4.13) in (4.11), we arrive after multiplying the resulting on by $e^{-2\alpha t_n}$ at

$$\begin{aligned}
\|\mathbf{U}^N\|^2 + \kappa\|\nabla\mathbf{U}^N\|^2 + \beta_2 ke^{-2\alpha t_N}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{U}}^{n}\|^2 &\le Ce^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2)\\
&\quad+\frac{e^{2\alpha k}}{\nu\lambda_1}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}, \qquad (4.14)
\end{aligned}$$

and this completes the first part of the proof.

For the remaining part, drop first two terms from (4.14) and then, to apply Stolz-Cesaro Theorem, (see [29]) to the resulting inequality, we observe that

$$\phi^N = 2\nu e^{-2\alpha k}k\sum_{n=1}^{N}\|\nabla\hat{\mathbf{U}}^{n}\|^2 \quad \text{and} \quad \psi^N = e^{2\alpha t_N}.$$

Note that the sequence $\{\psi^n\}$ is monotonically strictly increasing with $\psi^N \longrightarrow \infty$ and $N \longrightarrow \infty$. Hence, an appeal to Stolz-Cesaro Theorem yields

$$\frac{2\nu}{(1-e^{-2\alpha k})} e^{-2\alpha k} k \limsup_{N\longrightarrow\infty} \|\nabla U^N\|^2 \leq k \frac{1}{\nu(1-e^{-2\alpha k})} \|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}.$$

This concludes the rest of the proof. □

As in [4], we now appeal to a variant of Brouwer's fixed point theorem to prove existence of solution to the discrete problem (4.4)

Theorem 4.2 *(Brouwer's fixed point theorem)* [18]. *Let $\boldsymbol{H}$ be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)$ and $\|\cdot\|$. Let $\mathbb{G}:\boldsymbol{H}\to\boldsymbol{H}$ be a continuous function. If there exists a positive real number R such that $(\mathbb{G}(z), z) > 0\ \forall z$ with $\|z\| = R$, then there exists $z^* \in \boldsymbol{H}$ such that $\|z\| \leq R$ and $\mathbb{G}(z^*) = 0$.*

Theorem 4.3 *Given a sequence of discrete solution $\{\boldsymbol{U}^j\}_{j=0}^{n-1}$, there exists a unique discrete solution $\boldsymbol{U}^n$ of* (4.4) *for $n \geq 1$.*

Proof Assuming that $\mathbf{U}^m$, $m = 0, 1, \ldots n-1$ are known, we need to show the existence of $\mathbf{U}^n$ to the problem (4.4). Now, define a function $\mathbb{G}:\mathbf{J}_h \to \mathbf{J}_h$ for a fixed n by

$$\begin{aligned}(\mathbb{G}(\mathbf{w}), \boldsymbol{\phi}_h) =& 3(\mathbf{w}, \boldsymbol{\phi}_h) + 3\kappa(\nabla\mathbf{w}, \nabla\boldsymbol{\phi}_h) + k\nu(\nabla\mathbf{w}, \nabla\boldsymbol{\phi}_h) + 2k\, b(\mathbf{w}, \mathbf{w}, \boldsymbol{\phi}_h) \\ &-4(\mathbf{U}^{n-1}, \boldsymbol{\phi}_h) - 4\kappa(\nabla\mathbf{U}^{n-1}, \nabla\boldsymbol{\phi}_h) + (\mathbf{U}^{n-2}, \boldsymbol{\phi}_h) + \kappa(\nabla\mathbf{U}^{n-2}, \nabla\boldsymbol{\phi}_h) - 2k(\mathbf{f}^n, \boldsymbol{\phi}_h).\end{aligned} \quad (4.15)$$

Set a norm on $\mathbf{J}_h$ as

$$|||\mathbf{w}||| = (\|\mathbf{w}\|^2 + \kappa\|\nabla\mathbf{w}\|^2)^{\frac{1}{2}}. \quad (4.16)$$

It is easy to show that $\mathbb{G}$ is continuous. Now, after choosing $\boldsymbol{\phi}_h = \mathbf{w}$ in (4.15), we use (3.7), (4.16), the Cauchy-Schwarz's inequality and the Young's inequality to obtain

$$(\mathbb{G}(\mathbf{w}), \mathbf{w}) \geq \Big(3|||\mathbf{w}||| - 4|||\mathbf{U}^{n-1}||| + |||\mathbf{U}^{n-2}||| - 2k\|\mathbf{f}^n\|\Big)|||\mathbf{w}|||.$$

Choose R such a way that for $|||\mathbf{w}||| = R$, $\big(3R - 4|||\mathbf{U}^{n-1}||| + |||\mathbf{U}^{n-2}||| - 2k\|\mathbf{f}^n\|\big) > 0$ and hence,

$$(\mathbb{G}(\mathbf{w}), \mathbf{w}) > 0.$$

An appeal to Theorem 4.2 concludes an existence of the discrete solution $\{\mathbf{U}^n\}_{n\geq 1}$ of (4.4).

The part of uniqueness is quite similar to the proof of uniqueness problem in [4], so we skip the proof and this completes the rest of the proof. □

Remark 1 From the Theorem 4.4, we note that for a given $\mathbf{U}^{n-1} \in \mathbf{J}_h$, there exists a unique discrete solution $\mathbf{U}^n \in \mathbf{J}_h$. Thus, it defines a map $S_h^n : \mathbf{J}_h \to \mathbf{J}_h$ such that $S_h^n(\mathbf{U}^{n-1}) = U^n$, which is continuous and globally defined.

Theorem 4.4 *As a consequence of* (4.14)*, there exists a bounded absorbing set*

$$\mathbf{B}_{\rho_2}(\mathbf{0}) : \left\{(\|\boldsymbol{U}^N\|^2 + \kappa\|\nabla \boldsymbol{U}^N\|^2) \leq \rho_2^2\right\},$$

where ρ_0 *is given by*

$$\rho_2^2 = \frac{2e^{2\alpha k}}{\nu\lambda_1}\|\boldsymbol{f}\|_{L^\infty(L^2)}.$$

Moreover, the discrete problem (4.4) *has a global attractor.*

Proof To prove the first part of the Theorem 4.4, now, we claim that if $\left(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2\right)^{1/2} \in B_{\rho_1}(0)$, there exists $t_{n^*} = n^*k$ depending on $\left(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2\right)^{1/2}$ such that for $t_N \geq t_{n^*}$, the discrete solution $\mathbf{U}^N$ satisfies

$$\left(\|\mathbf{U}^N\|^2 + \kappa\|\nabla\mathbf{U}^N\|^2\right)^{1/2} \in B_{\rho_2}(0).$$

To prove we observe from the estimate (4.14) that

$$\|\mathbf{U}^N\|^2 + \kappa\|\nabla\mathbf{U}^N\|^2 \leq e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2) + \frac{\rho_0^2}{2}. \tag{4.17}$$

To complete the first part of the proof, it is enough to claim that

$$e^{-2\alpha t_N}(\|\mathbf{U}^0\|^2 + \kappa\|\nabla\mathbf{U}^0\|^2) \leq \frac{\rho_0^2}{2}. \tag{4.18}$$

A use of the fact that $2(a^2 + b^2) \geq (a+b)^2$ yields

$$\frac{1}{\rho_2}\|\mathbf{U}^0\| + \kappa\|\nabla\mathbf{U}^0\| \leq e^{\alpha t_N}.$$

That means, there is $t_n^\star = n^\star k \geq \frac{1}{\alpha}\log(\frac{\|\mathbf{U}^0\|+\kappa\|\nabla\mathbf{U}^0\|}{\rho_2})$ such that the above holds for $\rho_1 > \frac{\rho_2}{2}$ and $t_N \geq t_{n^\star}$ $B_{\rho_1}(0) \subset B_{\rho_2}(0)$. For $\rho_1 < \frac{\rho_2}{2}$, the result trivially holds for any $t_n \geq 0$. Therefore, $B_{\rho_2}(0)$ is an absorbing ball. Now, S^n possess a global attractor, say $A_{n,k}$, by mimicking the proof of existence of an attractor in the continuous case, see Titi et al. [17]. This concludes the rest of the proof. □

Lemma 4.2 *With* $0 \leq \alpha < \dfrac{\nu\lambda_1}{2(1+\lambda_1\kappa)}$*, choose* k_0 *so that for* $0 < k \leq k_0$ *the estimate* (4.7) *is satisfied. Then, there is a positive constant* K *depending on* $M_0, \nu, \alpha, \lambda_1$ *such that the discrete solution* $\boldsymbol{U}^N$*,* $N \geq 1$ *of* (4.4) *satisfies*

$$(\|\nabla\boldsymbol{U}^N\|^2 + \kappa\|\tilde{\Delta}_h\boldsymbol{U}^N\|^2) + 2\beta_2 e^{-2\alpha t_N}\, k\sum_{n=1}^{N} e^{2\alpha t_n}\|\tilde{\Delta}_h\boldsymbol{U}^n\|^2 \leq K(M_0, \nu, \alpha, \lambda_1) \tag{4.19}$$

where, $2\beta_2 = \left(2\nu e^{-2\alpha k} - 2\left(\frac{1-e^{-2\alpha k}}{k}\right)\left(\kappa + \frac{1}{\lambda_1}\right)\right) > 2\nu e^{-2\alpha k} > 0.$

Proof Put $\boldsymbol{\phi}_h = \tilde{\Delta}_h \mathbf{U}^n$ in (4.4) and multiply by $e^{2\alpha t_n}$. Then, an application of (4.6) shows

$$\frac{1}{4}\bar{\partial}_t(\|\nabla\hat{\mathbf{U}}^n\|^2+\kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2)+\nu\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2+\left(\frac{1-e^{2\alpha k}}{4k}\right)\left(\|\nabla\hat{\mathbf{U}}^n\|^2+\kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2\right)$$
$$+\left(\frac{1-e^{2\alpha k}}{4k}\right)\left(\|\nabla\hat{\mathbf{U}}^{n-1}\|^2+\kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^{n-1}\|^2\right)+\frac{1}{4k}\left(\|\delta^2\nabla\hat{\mathbf{U}}^{n-1}\|^2+\kappa\|\delta^2\tilde{\Delta}_h\hat{\mathbf{U}}^{n-1}\|^2\right)$$
$$+\frac{1}{4}\,\bar{\partial}_t\left(\|(2\nabla\hat{\mathbf{U}}^n-e^{\alpha k}\nabla\hat{\mathbf{U}}^{n-1})\|^2+\kappa\,\|2\tilde{\Delta}_h\hat{\mathbf{U}}^n-e^{\alpha k}\tilde{\Delta}_h\hat{\mathbf{U}}^{n-1}\|^2\right)$$
$$=-e^{-\alpha t_n}b(\hat{\mathbf{U}}^n,\hat{\mathbf{U}}^n,\tilde{\Delta}_h\hat{\mathbf{U}}^n)+(\hat{\mathbf{f}}^n,\tilde{\Delta}_h\hat{\mathbf{U}}^n)=I_1+I_2. \tag{4.20}$$

For I_1, use of the generalized Holder's inequality that

$$|I_1|\le Ce^{-\alpha t_n}\|\hat{\mathbf{U}}^n\|_{L^4}\|\nabla\hat{\mathbf{U}}^n\|_{L^4}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\| \tag{4.21}$$

Recall the following Ladyzhenskaya's inequality for ($d = 2, 3$) to our subsequent use:

For $d = 2$,

$$\|\hat{\mathbf{U}}^n\|_{L^4}\le C\|\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\quad\text{and}\quad\|\nabla\hat{\mathbf{U}}^n\|_{L^4}\le C\|\nabla\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^{\frac{1}{2}}. \tag{4.22}$$

In (4.21), an application of the Young's inequality with $p = 4$, $q = 4/3$, $\epsilon^p = \frac{2\nu}{9}$ shows

$$|I_1|\le Ce^{-\alpha t_n}\|\hat{\mathbf{U}}^n\|^{\frac{1}{2}}\|\nabla\hat{\mathbf{U}}^n\|\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^{\frac{3}{2}}\le C\left(\frac{1}{\nu}\right)^3e^{2\alpha t_n}\|\mathbf{U}^n\|^2\|\nabla\mathbf{U}^n\|^4+\frac{\nu}{6}\|\tilde{\Delta}\hat{\mathbf{U}}^n\|^2. \tag{4.23}$$

For $d = 3$,

$$\|\hat{\mathbf{U}}^n\|_{L^4}\le C\|\hat{\mathbf{U}}^n\|^{\frac{1}{4}}\|\nabla\hat{\mathbf{U}}^n\|^{\frac{3}{4}}\quad\text{and}\quad\|\nabla\hat{\mathbf{U}}^n\|_{L^4}\le C\|\nabla\hat{\mathbf{U}}^n\|^{\frac{1}{4}}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^{\frac{3}{4}}. \tag{4.24}$$

In (4.21), a use of the Young's inequality with $p = 8/7$, $q = 8$, $\epsilon^p = \frac{4\nu}{21}$ yields

$$|I_1|\le Ce^{-\alpha t_n}\|\hat{\mathbf{U}}^n\|^{\frac{1}{4}}\|\nabla\hat{\mathbf{U}}^n\|\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^{\frac{7}{4}}\le C\left(\frac{1}{\nu}\right)^7e^{2\alpha t_n}\|\mathbf{U}^n\|^2\|\nabla\mathbf{U}^n\|^8+\frac{\nu}{6}\|\tilde{\Delta}\hat{\mathbf{U}}^n\|^2. \tag{4.25}$$

For I_2, an application of the Cauchy-Schwarz inequality with Young's inequality leads to

$$|I_2|=(\hat{\mathbf{f}}^n,\tilde{\Delta}_h\hat{\mathbf{U}}^n)\le\|\hat{\mathbf{f}}^n\|\;\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|\le\frac{3}{2\nu}\|\hat{\mathbf{f}}^n\|+\frac{\nu}{3}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|. \tag{4.26}$$

Note that,

$$\sum_{n=2}^{N}(\|\nabla\hat{\mathbf{U}}^{n-1}\|^2+\kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^{n-1}\|^2)=(\|\nabla\hat{\mathbf{U}}^1\|^2+\kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^1\|^2)$$
$$+\sum_{n=2}^{N}(\|\nabla\hat{\mathbf{U}}^n\|^2+\kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2)$$
$$-(\|\nabla\hat{\mathbf{U}}^N\|^2+\kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^N\|^2). \tag{4.27}$$

For $d = 2$, drop the fifth term on the left-hand side of (4.20) as it is positive. Multiply the resulting on by $4ke^{-2\alpha k}$ and sum up from $n = 2$ to N. Then, a use of (2.3), (4.23), (4.26) and (4.27) yields

$$\|\nabla\hat{\mathbf{U}}^N\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^N\|^2 + k\left(2\nu e^{-2\alpha k} - 2\left(\frac{1-e^{-2\alpha k}}{k}\right)\left(\kappa + \frac{1}{\lambda_1}\right)\right)\sum_{n=2}^{N}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2 \quad (4.28)$$

$$+\|2e^{-\alpha k}\nabla\hat{\mathbf{U}}^N - \nabla\hat{\mathbf{U}}^{N-1}\|^2 + \kappa\|2e^{-\alpha k}\tilde{\Delta}_h\hat{\mathbf{U}}^N - \tilde{\Delta}_h\hat{\mathbf{U}}^{N-1}\|^2 \leq \big(\|\nabla\hat{\mathbf{U}}^1\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^1\|^2\big)$$

$$+\|2e^{-\alpha k}\nabla\hat{\mathbf{U}}^1 - \nabla\mathbf{U}^0\|^2 + \kappa\|2e^{-\alpha k}\tilde{\Delta}_h\hat{\mathbf{U}}^1 - \tilde{\Delta}_h\mathbf{U}^0\|^2\big) + C(\nu)\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}e^{-2\alpha k}k\sum_{n=2}^{N}e^{2\alpha t_n}.$$

$$+C(\nu)e^{-2\alpha k}k\sum_{n=2}^{N}\|\mathbf{U}^n\|^2\|\|\nabla\mathbf{U}^n\|^2\nabla\hat{\mathbf{U}}^n\|^2. \quad (4.29)$$

From a priori estimates of backward Euler scheme (Lemma 4.2 of [29]), we find that

$$\|\hat{\nabla}\mathbf{U}^1\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^1\|^2 \leq C(M_0, \nu, \alpha, \lambda_1). \quad (4.30)$$

An application of the Cauchy-Schwarz's inequality with the Young's inequality and (4.30) yields a bound for the second term on the right-hand side of (4.28) as follows:

$$\|2e^{-\alpha k}\nabla\hat{\mathbf{U}}^1 - \nabla\mathbf{U}^0\|^2 + \kappa\|2e^{-\alpha k}\tilde{\Delta}_h\hat{\mathbf{U}}^1 - \tilde{\Delta}_h\mathbf{U}^0\|^2 \leq C(M_0, \nu, \alpha, \lambda_1). \quad (4.31)$$

Using (4.30) and (4.31) in (4.28), we arrive at

$$\|\nabla\hat{\mathbf{U}}^N\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^N\|^2 + 2\beta_2 k\sum_{n=2}^{N}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2$$

$$\leq C(\|\nabla\mathbf{U}^0\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{U}^0\|^2) + \frac{e^{2\alpha k}}{\nu\lambda_1}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}e^{2\alpha t_n}$$

$$+C(\nu)e^{-2\alpha k}k\sum_{n=2}^{N}\|\mathbf{U}^n\|^2\|\|\nabla\mathbf{U}^n\|^2\nabla\hat{\mathbf{U}}^n\|^2. \quad (4.32)$$

An Application of Gronwall's lemma leads to

$$\|\nabla\hat{\mathbf{U}}^N\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^N\|^2 + 2\beta_2 k\sum_{n=2}^{N}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2$$

$$\leq C(\|\nabla\mathbf{U}^0\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{U}^0\|^2) + \frac{e^{2\alpha k}}{\nu\lambda_1}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}e^{2\alpha t_n}$$

$$\times exp\left(Cke^{-2\alpha k}\sum_{n=2}^{N}\|\mathbf{U}^n\|^2\|\|\nabla\mathbf{U}^n\|^2\right). \quad (4.33)$$

Apply assumption **(A2)** in (4.33) to arrive at

$$\|\nabla\hat{\mathbf{U}}^N\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^N\|^2 + 2\beta_2 k\sum_{n=2}^{N}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2$$
$$\leq C(\nu,\alpha,M_0)exp\left(Cke^{-2\alpha k}\sum_{n=2}^{N}\|\mathbf{U}^n\|^2\|\|\nabla\mathbf{U}^n\|^2\right). \tag{4.34}$$

Multiplying (4.34) by $e^{-2\alpha t_n}$, use (4.8) along with (4.9) to obtain

$$\|\nabla\mathbf{U}^N\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{U}^N\|^2 + 2\nu e^{-2\alpha k}e^{-2\alpha t_n}k\sum_{n=2}^{N}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2 \leq C(\nu,\alpha,M_0). \tag{4.35}$$

This concludes the proof for $d = 2$.

Now, for $d = 3$ substitute (4.23) and (4.26)in (4.20) and under the similar lines of proof for $d = 2$, we obtain

$$\|\nabla\hat{\mathbf{U}}^N\|^2 + \kappa\|\tilde{\Delta}_h\hat{\mathbf{U}}^N\|^2 + 2\beta_2 k\sum_{n=2}^{N}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2$$
$$\leq C(\|\nabla\mathbf{U}^0\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{U}^0\|^2) + \frac{e^{2\alpha k}}{\nu\lambda_1}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})}e^{2\alpha t_n}$$
$$+C(\nu)\ k\ e^{-2\alpha k}e^{2\alpha t_n}\sum_{n=2}^{N}\|\mathbf{U}^n\|^2\|\|\nabla\mathbf{U}^n\|^8. \tag{4.36}$$

Multiplying (4.36) by $e^{-2\alpha t_n}$ to obtain

$$\|\nabla\mathbf{U}^N\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{U}^N\|^2\ +\ 2e^{-2\alpha t_n}\beta_2 k\sum_{n=2}^{N}\|\tilde{\Delta}_h\hat{\mathbf{U}}^n\|^2 \leq e^{-2\alpha t_n}C(\|\nabla\mathbf{U}^0\|^2 + \kappa\|\tilde{\Delta}_h\mathbf{U}^0\|^2)$$
$$+\ \frac{e^{2\alpha k}}{\nu\lambda_1}\|\mathbf{f}\|^2_{L^\infty(\mathbf{H}^{-1})} + C(\nu)e^{-2\alpha k}\sum_{n=2}^{N}\|\mathbf{U}^n\|^2\|\|\nabla\mathbf{U}^n\|^8$$
$$\leq\quad C_1(M_0,\alpha,\lambda,\nu) + C_2(M_0,\alpha,\nu)k\sum_{n=2}^{N}\|\nabla\mathbf{U}^n\|^8. \tag{4.37}$$

Now, under smallness assumption on both initial data and forcing function the boundedness of $\|\nabla\mathbf{U}^N\|$ is proved for all $t_N > 0$. This completes the rest of the proof. □

5 Error analysis for second-order backward difference method

This section deals with error analysis of our second-order difference scheme (4.4)–(4.5).

Set, for a fixed n, $\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}_h(t_n) = \mathbf{U}^n - \mathbf{u}_h^n$. Now, we derive the following error estimates for second-order backward difference method.

Theorem 5.1 *Let* $0 \leq \alpha < \dfrac{\nu\lambda_1}{4(1+\kappa\lambda_1)}$ *and* $k_0 \geq 0$ *such that for* $0 < k \leq k_0$,

$$\frac{\nu k\lambda_1}{1+\kappa\lambda_1} + 1 > e^{2\alpha k}.$$

Then, there exist a positive constant $C = C(\nu, \alpha, \lambda_1, M_0)$ *such that*

$$\|\boldsymbol{e}^n\|^2 + \kappa\|\nabla \boldsymbol{e}^n\|^2 + ke^{-2\alpha t_n}\sum_{i=2}^{n} e^{2\alpha t_i}\|\nabla \boldsymbol{e}_i\|^2 \leq \frac{C}{\kappa}k^4 e^{Ct_N}, \tag{5.1}$$

and for $n = 2, \cdots, N$,

$$\|D_t^2\boldsymbol{e}^n\|^2 + \kappa\|D_t^2\nabla\boldsymbol{e}^n\|^2 \leq \frac{C}{\kappa^{3/2}}k^4 e^{Ct_n}. \tag{5.2}$$

Proof Rewrite (3.2) at $t = t_n$ and substract it from (4.4) to obtain

$$\begin{aligned} \left(D_t^{(2)}\mathbf{e}^n, \boldsymbol{\phi}_h\right) \;+\; \kappa a\left(D_t^{(2)}\mathbf{e}^n, \boldsymbol{\phi}_h\right) + \nu a(\mathbf{e}^n, \boldsymbol{\phi}_h) \\ ;= \quad E_1^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) + \Lambda_h^n(\boldsymbol{\phi}_h), \end{aligned} \tag{5.3}$$

where,

$$E_1^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) = \left(\mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{u}_h^n, \boldsymbol{\phi}_h\right) + \kappa a\left(\mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{u}_h^n, \boldsymbol{\phi}_h\right) \tag{5.4}$$

and

$$\begin{aligned} \Lambda_h(\boldsymbol{\phi}_h) &= b\left(\mathbf{u}_h^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h\right) - b(\mathbf{U}^n, \mathbf{U}^n, \boldsymbol{\phi}_h) \\ &= -b\left(\mathbf{u}_h^n, \mathbf{e}^n, \boldsymbol{\phi}_h\right) - b(\mathbf{e}^n, \mathbf{U}^n, \boldsymbol{\phi}_h). \\ &= -b\left(\mathbf{u}_h^n, \mathbf{e}^n, \boldsymbol{\phi}_h\right) + b(\mathbf{e}^n, \mathbf{e}^n, \boldsymbol{\phi}_h) - b(\mathbf{e}^n, \mathbf{u}_h^n, \boldsymbol{\phi}_h). \end{aligned} \tag{5.5}$$

Further,

$$\begin{aligned} |\Lambda_h(\boldsymbol{\phi}_h)| &= |b\left(\mathbf{u}_h^n, \mathbf{e}^n, \boldsymbol{\phi}_h\right) + b(\mathbf{e}^n, \mathbf{U}^n, \boldsymbol{\phi}_h)| \\ &\leq C(\lambda_1)\Big(\|\nabla\mathbf{u}_h^n\| + \|\nabla\mathbf{U}^n\|\Big)\|\nabla\mathbf{e}^n\|\|\nabla\boldsymbol{\phi}_h\|. \end{aligned} \tag{5.6}$$

With the help of Lemmas 3.2 and 4.2, we find that

$$|\Lambda_h(\boldsymbol{\phi}_h)| \leq C(\nu, \alpha, \lambda_1, M_0)\|\nabla\mathbf{e}^n\|\;\|\nabla\boldsymbol{\phi}_h\|. \tag{5.7}$$

Multiplying (5.3) by $4ke^{\alpha t_n}$ and choose $\boldsymbol{\phi}_h = \hat{\mathbf{e}}^n$. A use of identity (4.6) yields

$$\begin{aligned} k\bar{\partial}_t(\|\hat{\mathbf{e}}^n\|^2 \;&+\; \kappa\|\nabla\hat{\mathbf{e}}^n\|^2) + \|\delta^2\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2 + 4k\nu\|\nabla\hat{\mathbf{e}}^n\|^2 \\ &+\; (1-e^{2\alpha k})(\|\hat{\mathbf{e}}^n\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^n\|^2) + (1-e^{2\alpha k})(\|\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|\nabla\hat{\mathbf{e}}^{n-1}\|^2) \\ &+\; k\;\bar{\partial}_t\Big(\|2\hat{\mathbf{e}}^n - e^{\alpha k}\hat{\mathbf{e}}^{n-1}\|^2 + \kappa\|2\nabla\hat{\mathbf{e}}^n - e^{\alpha k}\nabla\hat{\mathbf{e}}^{n-1}\|^2\Big) = 4k\,e^{\alpha t_n}E_1^n(\mathbf{u}_h)(\hat{\mathbf{e}}^n) \\ &+\; 4k\,e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n). \end{aligned} \tag{5.8}$$

Summing (5.8) over $n=2$ to N and multiplying by $e^{-2\alpha k}$, we arrive at

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+e^{-2\alpha k}\sum_{n=2}^{N}(\|\delta^2\hat{\mathbf{e}}^{n-1}\|^2+\kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2)+\|2e^{-\alpha k}\hat{\mathbf{e}}^N-\hat{\mathbf{e}}^{N-1}\|^2$$
$$+\kappa\|2e^{-\alpha k}\nabla\hat{\mathbf{e}}^N-\nabla\hat{\mathbf{e}}^{N-1}\|^2+k\left(4\nu e^{-2\alpha k}-2\left(\frac{1-e^{-2\alpha k}}{k}\right)\left(\kappa+\frac{1}{\lambda_1}\right)\right)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2$$
$$\leq \|\hat{\mathbf{e}}^1\|^2+\kappa\|\nabla\hat{\mathbf{e}}^1\|^2+\|2e^{-\alpha k}\hat{\mathbf{e}}^1-\mathbf{e}^0\|^2+\kappa\|2e^{-\alpha k}\nabla\hat{\mathbf{e}}^1-\nabla\mathbf{e}^0\|^2$$
$$+4e^{-2\alpha k}k\sum_{n=2}^{N}e^{\alpha t_n}E_1^n(\mathbf{u}_h)(\boldsymbol{\phi}_h)+4ke^{-2\alpha k}\sum_{n=2}^{N}e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n)$$
$$\leq C(\|\hat{\mathbf{e}}^1\|^2+\kappa\|\nabla\hat{\mathbf{e}}^1\|^2)+4e^{-2\alpha k}k\sum_{n=2}^{N}e^{\alpha t_n}E_1^n(\mathbf{u}_h)(\hat{\mathbf{e}}^n)+4ke^{-2\alpha k}\sum_{n=2}^{N}e^{\alpha t_n}\Lambda_h(\hat{\mathbf{e}}^n). \tag{5.9}$$

To estimate the second term of the right-hand side of (5.9), we find that

$$4e^{-2\alpha k}k\sum_{n=2}^{N}e^{\alpha t_n}E_1^n(\mathbf{u}_h)(\boldsymbol{\phi}_h) = 4e^{-2\alpha k}k\sum_{n=2}^{N}e^{\alpha t_n}\left(\mathbf{u}_{ht}^n-D_t^2\mathbf{U}^n,\hat{\mathbf{e}}^n\right) \tag{5.10}$$
$$+4e^{-2\alpha k}\kappa k\sum_{n=2}^{N}e^{\alpha t_n}a\left(\mathbf{u}_{ht}^n-D_t^2\mathbf{U}^n,\hat{\mathbf{e}}^n\right)=I_1+I_2(say).$$

Now, a use of the Cauchy-Schwarz's inequality, (2.3) and the Young's inequality, we bound $|I_1|$ as

$$|I_1|\leq 4e^{-2\alpha k}k\left(\sum_{n=2}^{N}\|e^{\alpha t_n}(\mathbf{u}_{ht}^n-D_t^2\mathbf{U}^n)\|_{-1}^2\right)^{1/2}\left(\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\right)^{1/2}$$
$$\leq C(\epsilon,\lambda_1)ke^{-2\alpha k}\sum_{n=2}^{N}\|e^{\alpha t_n}\left(\mathbf{u}_{ht}^n-D_t^2\mathbf{U}^n\right)\|_{-1}^2+\epsilon ke^{-2\alpha k}\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2. \tag{5.11}$$

A Use of $\|e^{\alpha t_n}\left(\mathbf{u}_{ht}^n-D_t^2\mathbf{U}^n\right)\|_{-1}^2 \leq \frac{k^3}{2}\int_{t_{n-2}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{httt}(t)\|_{-1}^2\,dt$ ([1]) and Lemma 3.2, we obtain

$$k\sum_{n=2}^{N}\|e^{\alpha t_n}\left(\mathbf{u}_{ht}^n-D_t^2\mathbf{U}^n\right)\|_{-1}^2 \leq \frac{k^4}{2}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n}e^{2\alpha t_n}\|\mathbf{u}_{httt}(t)\|_{-1}^2dt$$
$$=\frac{k^4}{2}e^{4\alpha k}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n}e^{2\alpha t_{n-2}}\|\mathbf{u}_{httt}(t)\|_{-1}^2dt$$
$$\leq\frac{k^4}{2}e^{4\alpha k}\sum_{n=2}^{N}\int_{t_{n-2}}^{t_n}e^{2\alpha t}\|\mathbf{u}_{httt}(t)\|_{-1}^2dt$$
$$\leq k^4e^{4\alpha k}\int_0^{t_N}e^{2\alpha t}\|\mathbf{u}_{httt}(t)\|_{-1}^2dt$$
$$\leq\frac{C(\nu,\alpha,\lambda_1,M_0)}{\kappa^2}\,k^4e^{2\alpha(n+2)k}. \tag{5.12}$$

Using (5.12) in (5.11), yields

$$|I_1| \leq \frac{C(\nu, \alpha, \lambda_1, M_0, \epsilon)}{\kappa^2}\, k^4 e^{2\alpha t_n} + \epsilon k e^{-2\alpha k} \sum_{n=2}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2. \tag{5.13}$$

Similarly, as for bound of $|I_1|$ and using (5.13), we find that

$$|I_2| \leq \frac{C(\nu, \alpha, \lambda_1, M_0, \epsilon)}{\kappa^2}\, k^4 e^{2\alpha t_n} + \epsilon k e^{-2\alpha k} \sum_{n=2}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2. \tag{5.14}$$

A use of anti-symmetric property for the second term of the right-hand side of (5.9) shows

$$\begin{aligned} e^{\alpha t_n}|\Lambda_h(\hat{\mathbf{e}}^n)| &\leq e^{-\alpha t_n}|b(\hat{\mathbf{e}}^n, \nabla \hat{\mathbf{u}}_h^n), \hat{\mathbf{e}}^n| \\ &\leq C\ \|\nabla \hat{\mathbf{u}}_h^n\| \|\hat{\mathbf{e}}^n\| \|\nabla \hat{\mathbf{e}}^n\|. \end{aligned} \tag{5.15}$$

A use of (5.15) yields

$$\begin{aligned} |4ke^{-2\alpha k} \sum_{n=2}^{N} e^{\alpha t_n} \Lambda_h(\hat{\mathbf{e}}^n)| \leq\ & C(\epsilon) \sum_{n=2}^{N} k e^{-2\alpha k} e^{-2\alpha t_n} \|\nabla \hat{\mathbf{u}}_h^n\|^2 \|\hat{\mathbf{e}}^n\|^2 \\ & + \epsilon k e^{-2\alpha k} \sum_{n=2}^{N} \|\nabla \hat{\mathbf{e}}^n\|^2. \end{aligned} \tag{5.16}$$

For $n = 1$, we now observe that

$$\begin{aligned} &\frac{1}{2}\bar{\partial}_t\big(\|\hat{\mathbf{e}}^1\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^1\|^2\big) + \left(\nu e^{-\alpha k} - \left(\frac{1-e^{-\alpha k}}{k}\right)\left(\kappa + \frac{1}{\lambda_1}\right)\right)\|\nabla \hat{\mathbf{e}}^1\|^2 \\ &= e^{-\alpha k}(e^{\alpha k}(E^1(\mathbf{u}_h)(\hat{\mathbf{e}}^1)), \hat{\mathbf{e}}^1) + e^{-\alpha k} e^{\alpha k} \Lambda_h(\hat{\mathbf{e}}^1). \end{aligned} \tag{5.17}$$

Multiply (5.17) by $2k$,and the, with a help of the Young's inequality, the Cauchy-Schwarz's inequality and (2.3) with the estimates (5.15) (for $n = 1$ and $\epsilon = \nu$), we arrive at

$$\begin{aligned} &\|\hat{\mathbf{e}}^1\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^1\|^2 + 2k\left(\nu e^{-\alpha k} - \left(\frac{1-e^{-\alpha k}}{k}\right)\left(\kappa + \frac{1}{\lambda_1}\right)\right)\|\nabla \hat{\mathbf{e}}^1\|^2 \\ &\leq 2ke^{-\alpha k}(e^{\alpha k}(E^1(\mathbf{u}_h)(\hat{\mathbf{e}}^1)), \hat{\mathbf{e}}^1) + 2ke^{-\alpha k} e^{\alpha k} \Lambda_h(\hat{\mathbf{e}}^1) \\ &\leq Ck^2 e^{-2\alpha k}\big(\|e^{\alpha k}(E^1(\mathbf{u}_h)(\hat{\mathbf{e}}^1))\|^2\big) + \frac{1}{2}(\|\hat{\mathbf{e}}^1\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^1\|^2) + \nu k e^{-\alpha k}\|\nabla \hat{\mathbf{e}}^1\|^2 \\ &\quad + C(\nu) k e^{-\alpha k} e^{-2\alpha k} \|\nabla \hat{\mathbf{u}}_h^1\|^2 \|\hat{\mathbf{e}}^1\|^2, \end{aligned} \tag{5.18}$$

and hence, we similarly obtain

$$\begin{aligned} &\|\hat{\mathbf{e}}^1\|^2 + \kappa \|\nabla \hat{\mathbf{e}}^1\|^2 + k\left(\nu e^{-\alpha k} - 2\left(\frac{1-e^{-\alpha k}}{k}\right)\left(\kappa + \frac{1}{\lambda_1}\right)\right)\|\nabla \hat{\mathbf{e}}^1\|^2 \\ &\leq C(\nu, \alpha, \lambda_1, M_0)\frac{k^4}{\kappa^2} e^{2\alpha t_n} + C(\nu) k e^{-\alpha k} e^{-2\alpha k} \|\nabla \hat{\mathbf{u}}_h^1\|^2 \|\hat{\mathbf{e}}^1\|^2. \end{aligned} \tag{5.19}$$

A use of (5.13), (5.14) and (5.16) with $\epsilon = \frac{2\nu}{3}$, (5.19), $\mathbf{e}^o = 0$ and results from Lemma 4.1 in (5.9), yields

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+e^{-2\alpha k}\sum_{n=2}^{N}(\|\delta^2\hat{\mathbf{e}}^{n-1}\|^2+\kappa\|\delta^2\nabla\hat{\mathbf{e}}^{n-1}\|^2)+\|2e^{-\alpha k}\hat{\mathbf{e}}^N-\hat{\mathbf{e}}^{N-1}\|^2$$
$$+\kappa\|2e^{-\alpha k}\nabla\hat{\mathbf{e}}^N-\nabla\hat{\mathbf{e}}^{N-1}\|^2+2k\left(\nu e^{-2\alpha k}-\left(\frac{1-e^{-2\alpha k}}{k}\right)\left(\kappa+\frac{1}{\lambda_1}\right)\right)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2$$
$$\leq C(\nu,\alpha,\lambda_1,M_0)\frac{k^4}{\kappa^2}e^{2\alpha t_n}+C(\nu)\sum_{n=2}^{N}ke^{-2\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{u}}_h^n\|^2\|\hat{\mathbf{e}}^n\|^2$$
$$+C(\nu)ke^{-\alpha k}e^{-2\alpha k}\|\nabla\hat{\mathbf{u}}_h^1\|^2\|\hat{\mathbf{e}}^1\|^2$$
$$\leq C(\nu,\alpha,\lambda_1,M_0)\frac{k^4}{\kappa^2}e^{2\alpha t_n}+C(\nu)\sum_{n=0}^{N-1}ke^{-\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{u}}_h^n\|^2\|\hat{\mathbf{e}}^n\|^2$$
$$+C(\nu)ke^{-2\alpha k}e^{-2\alpha t_N}\|\nabla\hat{\mathbf{u}}_h^N\|^2\|\hat{\mathbf{e}}^N\|^2$$
$$\leq C(\nu,\alpha,\lambda_1,M_0)\frac{k^4}{\kappa^2}e^{2\alpha t_n}+C(\nu)\sum_{n=0}^{N-1}ke^{-\alpha k}e^{-2\alpha t_n}\|\nabla\hat{\mathbf{u}}_h^n\|^2\|\hat{\mathbf{e}}^n\|^2$$
$$+Cke^{-2\alpha k}(\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2). \tag{5.20}$$

Now, select k_0, so that (4.7) is satisfied and $(1-Cke^{-2\alpha k})>0$ for $0<k\leq k_0$. Then, an application of the discrete Gronwall's Lemma yields

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+k\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\leq\frac{C(\nu,\alpha,\lambda_1,M_0)}{\kappa^2}k^4\exp\left(k\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{u}}_h^n\|^2\right). \tag{5.21}$$

With the help of Lemma 4.1, we bound

$$k\sum_{n=0}^{N-1}\|\nabla\hat{\mathbf{u}}_h^n\|^2\leq Ct_N, \tag{5.22}$$

and hence, using (5.22) in (5.21), we now arrive at

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+k\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\leq\frac{C(\nu,\alpha,\lambda_1,M_0)}{\kappa^2}k^4e^{Ct_N}. \tag{5.23}$$

This concludes the proof of (5.1) for $n\geq 2$.

For $n=1$, we similarly obtain

$$\|\mathbf{e}^1\|^2+\kappa\|\nabla\mathbf{e}^1\|^2+k\|\nabla\mathbf{e}^1\|^2\leq\frac{C(\nu,\alpha,\lambda_1,M_0)}{\kappa^2}k^4e^{2\alpha k}. \tag{5.24}$$

To complete the rest of the proof, we now, substitute $\boldsymbol{\phi}_h = D_t^{(2)}\mathbf{e}^n$ in (5.3) to arrive at

$$\|D_t^{(2)}\mathbf{e}^n\|^2 + \kappa\|\nabla D_t^{(2)}\mathbf{e}^n\|^2 = -\nu a\left(\mathbf{e}^n, D_t^{(2)}\mathbf{e}^n\right) \tag{5.25}$$
$$+\left(\left(\mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{U}^n\right), D_t^{(2)}\mathbf{e}^n\right) + \kappa a\left(\left(\mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{U}^n\right), D_t^{(2)}\mathbf{e}^n\right) + \Lambda_h\left(D_t^{(2)}\mathbf{e}^n\right).$$

Using (5.7), we obtain

$$|\Lambda_h\left(D_t^{(2)}\mathbf{e}^n\right)| \leq C(\nu, \alpha, \lambda_1, M_0)\|\nabla\mathbf{e}^n\|\|D_t^{(2)}\nabla\mathbf{e}^n\|. \tag{5.26}$$

A use of the Cauchy-Schwarz's inequality with the Young's inequality (2.3) and (5.26) in (5.25), yields

$$\|D_t^{(2)}\ \mathbf{e}^n\|^2 + \kappa\|\nabla D_t^{(2)}\mathbf{e}^n\|^2 \leq C(\nu, \alpha, \lambda_1, M_0)\left(\|\nabla\mathbf{e}^n\|^2 + \kappa^2\|\nabla\left(\mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{U}^n\right)\|_{-1}^2\right). \tag{5.27}$$

We apply (5.12) and Lemma 3.3 to estimate the second term on the right-hand side of (5.27) as

$$\begin{aligned}\kappa^2\|e^{\alpha t_n}\nabla\left(\mathbf{u}_{ht}^n - D_t^{(2)}\mathbf{U}^n\right)\|_{-1}^2 &\leq \tfrac{k^3}{2}\int_{t_{n-2}}^{t_n} e^{2\alpha t_n}\kappa^2\|\nabla\mathbf{u}_{httt}(t)\|^2 dt\\ &\leq \tfrac{C(\nu,\alpha,\lambda_1,M_0)}{\kappa^3}k^3\, e^{2\alpha t_n}\int_{t_{n-2}}^{t_n} dt\\ &\leq \tfrac{C(\nu,\alpha,\lambda_1,M_0)}{\kappa^3}k^4 e^{2\alpha t_n},\end{aligned} \tag{5.28}$$

where $k^* \in (0, k)$. In view of (5.1) and (5.28), (5.27) implies (5.2). This completes the rest of the proof. □

Theorem 5.2 *A use of the uniqueness condition , that is,*

$$\frac{N_0}{\nu^2}\|\boldsymbol{f}\|_{L^\infty(\boldsymbol{H}^{-1})} < 1 \quad \textit{and} \quad N_0 = \sup_{u,v,w\in\boldsymbol{H}_0^1(\Omega)} \frac{b(u, v, w)}{\|\nabla u\|\|\nabla v\|\|\nabla w\|}$$

the following estimate holds true for all $n \in [N_0, \infty)$

$$\|\boldsymbol{e}^n\|^2 + \kappa\|\nabla\boldsymbol{e}^n\|^2 \leq \frac{C}{\kappa}k^4.$$

Proof The outline of this proof is bit similar to Theorem 5.3 (see, [29]). Note that, apply Taylor's series expansion of $u_h(t)$ at t_n in the interval (t_{n-1}, t_n) and then use the Cauchy-Schwarz inequality to obtain

$$\|\nabla\left(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right)\|^2 \leq k\int_{t_{n-1}}^{t_n}\|\nabla\mathbf{u}_{ht}(s)\|^2\, ds. \tag{5.29}$$

Now, a use of (5.29) in (5.20) yields

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+2k\left(\nu e^{-2\alpha k}-\left(\frac{1-e^{-2\alpha k}}{k}\right)\left(\kappa+\frac{1}{\lambda_1}\right)\right)\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2$$
$$\leq\frac{C(\nu,\alpha,\lambda_1,M_0)}{\kappa}k^4e^{2\alpha t_n}+C(\nu,\lambda)ke^{-\alpha k}\sum_{n=0}^{N-1}\frac{N}{\nu^2}\|\mathbf{f}\|_{L^\infty(\mathbf{H}^{-1})}\|\hat{\mathbf{e}}^n\|^2$$
$$+Cke^{-2\alpha k}(\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2). \tag{5.30}$$

Rewrite (5.30) as:

$$\|\hat{\mathbf{e}}^N\|^2+\kappa\|\nabla\hat{\mathbf{e}}^N\|^2+2k\left(\nu e^{-2\alpha k}-\left(\frac{1-e^{-2\alpha k}}{k}\right)\left(\kappa+\frac{1}{\lambda_1}\right)-e^{-\alpha k}\frac{N}{\nu^2}\|\mathbf{f}\|_{L^\infty(\mathbf{H}^{-1})}\|\right)$$
$$\times\sum_{n=2}^{N}\|\nabla\hat{\mathbf{e}}^n\|^2\leq\frac{C(\nu,\alpha,\lambda_1,M_0)}{\kappa}k^4e^{2\alpha t_n}. \tag{5.31}$$

Using uniqueness condition, it is easy to show that the coefficient of third term of the (5.31) becomes positive. Multiply $e^{-2\alpha tn}$ in (5.31) to complete the rest of the proof. □

Now, we only need to prove the error estimates for the pressure P^n. Define $\boldsymbol{\rho}^n = P^n - p_h(t_n)$ and consider (3.1) at $t = t_n$ and subtract it from (4.2) to obtain

$$(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h) = \left(D_t^2\mathbf{e}^n,\boldsymbol{\phi}_h\right)+\kappa a\left(D_t^2\mathbf{e}^n,\boldsymbol{\phi}_h\right)+\nu a(\mathbf{e}^n,\boldsymbol{\phi}_h)$$
$$-\ E_1^n(\mathbf{u}_h)(\boldsymbol{\phi}_H)-\Lambda_h(\boldsymbol{\phi}_h).$$

An application of (2.3) with the Cauchy-Schwarz's inequality and (5.7) shows

$$(\boldsymbol{\rho}^n,\nabla\cdot\boldsymbol{\phi}_h)\leq C(\kappa,\nu,\lambda_1)\Big(\|D_t^2\mathbf{e}^n\|+\kappa\|D_t^2\nabla\mathbf{e}^n\|+\nu\|\nabla\mathbf{e}^n\|+\|\nabla(\mathbf{u}_{ht}^n-\mathbf{U}^n)\|\Big)\|\nabla\boldsymbol{\phi}_h\|. \tag{5.32}$$

From (5.28) and the Theorem 5.1 in (5.32), we now arrive at

$$\|\boldsymbol{\rho}^n\|\leq\frac{1}{\kappa^{3/2}}C(\nu,\alpha,\lambda_1,M_0)k^2. \tag{5.33}$$

Altogether, a use of Theorems 3.1, 5.1, 5.2, (5.24) and (5.33) would conclude the proof of our main theorem below.

Theorem 5.3 *Under the assumption of Theorems 3.1 and 5.1, the following holds true:*

$$\|\boldsymbol{u}(t_n)-\boldsymbol{U}^n\|\leq C\left(\frac{h^2}{\kappa^{1/2}}+\frac{k^2}{\kappa}\right) \tag{5.34}$$

and

$$\|(p(t_n)-P^n)\|\leq C\left(\frac{h}{\sqrt{\kappa}}+\frac{k^{2-\gamma}}{\kappa^{3/2}}\right),$$

where

$$\gamma = \begin{cases} 0 & if\, n \geq 2; \\ 1 & if\, n = 1. \end{cases}$$

Remark 2 The error estimates in Theorems 3.1 and 5.3 are not optimal with respect to κ as shown in the numerical experiments in Section 6. However, with higher regularity results, that is, $u_0 \in H^3 \cap H_0^1$ with some compatibility, it is possible to derive error estimates which are independent of κ. Since one of our objectives in this paper is to prove error bounds with minimal assumption **A2**, we refrain from pursuing it further with higher regularity assumption.

6 Numerical experiments

In this section, we focus on several numerical experiments with varying κ, using (P_2-P_0) mixed finite element space (see, [6]) for spatial discretization . Below, we implement a second-order backward difference method for time discretization and compute the order of convergence, which would confirm our theoretical findings in Section 5.

Now, consider the following finite dimensional approximating spaces $\mathbf{H}_h$ and L_h as:

$$\mathbf{H}_h = \left\{ \mathbf{v} \in \left(H_0^1(\Omega)\right)^2 \cap \left(C(\bar{\Omega})\right)^2 : \mathbf{v}|_K \in (P_2(K))^2, K \in \mathcal{T}_h \right\},$$
$$L_h = \{q \in L^2(\Omega) : q|_K \in P_0(K), K \in \mathcal{T}_h\},$$

where $\mathcal{T}_h$ denotes the regular triangulation of the domain $\bar{\Omega}$. Then, apply the second-order backward difference approximation to (3.1) is as follows: given $\mathbf{U}^{n-2}$ and $\mathbf{U}^{n-1}$, find the pair $(\mathbf{U}^n, P^n)$ satisfying:

$$\begin{aligned} &(3\mathbf{U}^n, \mathbf{v}_h) + (\kappa + 2\nu\Delta t)\, a(\mathbf{U}^n, \mathbf{v}_h) + 2\Delta t\, c(\mathbf{U}^n, \mathbf{U}^n, \mathbf{v}_h) \\ &+ 2\Delta t\, b(\mathbf{v}_h, P^n) = 4(\mathbf{U}^{n-1}, \mathbf{v}_h) + 4\,\kappa a(\mathbf{U}^{n-1}, \mathbf{v}_h) - (\mathbf{U}^{n-2}, \mathbf{v}_h) \\ &- \kappa\, a(\mathbf{U}^{n-2}, \mathbf{v}_h) + \Delta t\, (\mathbf{f}(t_n), \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ &\qquad b(\mathbf{U}^n, w_h) = 0 \qquad \forall w_h \in W_h. \end{aligned} \tag{6.1}$$

Table 1 Errors and convergence rates for backward difference scheme with $k = \mathbb{O}(h)$

h	$\Vert\mathbf{u}(t_n) - \mathbf{U}^n\Vert_{\mathbf{L}^2}$	Rate	$\Vert\mathbf{u}(t_n) - \mathbf{U}^n\Vert_{\mathbf{H}^1}$	Rate	$\Vert p(t_n) - P^n\Vert$	Rate
1/2	0.0112643		0.078272		0.161085	
1/4	0.0038994	1.530408	0.045709	0.776018	0.076906	1.066641
1/8	0.0012055	1.693566	0.025378	0.848892	0.037627	1.031309
1/16	0.0003351	1.846867	0.013375	0.923988	0.018442	1.028795

Table 2 Numerical convergence rates for velocity in $\mathbf{L}^2$-norm with variation in κ for Example 1

S No.	h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L}^2}$ $\kappa = 0.01$	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L}^2}$ $\kappa = 0.0001$	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{L}^2}$ $\kappa = 0.00000001$
1	1/4	1.5304088	1.5038179	1.5034958
2	1/8	1.6935666	1.6919373	1.6919256
3	1/16	1.8468671	1.8464987	1.8464952
3	1/32	1.91154251	1.9106039	1.9105942

Using basis functions, we approximate the velocity and pressure as

$$\mathbf{U}^n = \sum_{j=1}^{ng} \begin{pmatrix} \mathbf{u}_j^{nx} \\ \mathbf{u}_j^{hy} \end{pmatrix} \boldsymbol{\phi}_j^{\mathbf{u}}(\mathbf{x}), \quad P^n = \sum_{j=1}^{ne} p_j^n \phi_j^p(\mathbf{x}), \tag{6.2}$$

where $\boldsymbol{\phi}_j^{\mathbf{u}}(\mathbf{x})$ and $\phi_j^p(\mathbf{x})$ form bases for $\mathbf{H}_h$ and L_h with cardinality ng and ne, respectively. Here, $\mathbf{u}_j^{nx}$ and $\mathbf{u}_j^{ny}$ represent the x and y component of the approximate velocity field, respectively, at time $t = t_n$. Using (6.2), the basis functions for $\mathbf{H}_h$ and L_h in (6.1), we obtain a system of nonlinear algebraic equations, which is solved using Newton's method.

Example 1 Choose the forcing function f in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ is

$$u_1 = .01e^{-t^2}x^2(x-1)^2(y^3 - 2y^2 + y), u_2 = -0.01e^{-t^2}y^2(y-1)^2(x^3 - 2x^2 + x),$$
$$p = -3.84e^{-t^2}xy^2.$$

We choose $\kappa = 0.01$, $\nu = 1$, with $\Omega = (0, 1) \times (0, 1)$ and time $t = [0, 1]$. Here, $\bar{\Omega}$ is subdivided into triangles with mesh size h.

The theoretical analysis provides a convergence rate of $\mathcal{O}(h^2)$ in $\mathbf{L}^2$-norm, for the velocity convergence rate of $\mathcal{O}(h)$ in $\mathbf{H}^1$-norm and for the pressure term, the velocity convergence rate of $\mathcal{O}(h)$ in L^2-norm. Table 1 presents numerical errors and computed convergence rates obtained on successively refined meshes for backward

Table 3 Numerical convergence rates for velocity in $\mathbf{H}^1$-norm with variation in κ for Example 1

S No.	h	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$ $\kappa = 0.01$	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$ $\kappa = 0.0001$	$\|\mathbf{u}(t_n) - \mathbf{U}^n\|_{\mathbf{H}^1}$ $\kappa = 0.00000001$
1	1/4	0.7760188	0.7484161	0.7480840
2	1/8	0.8488925	0.8473675	0.8473556
3	1/16	0.9239889	0.9236156	0.9236121
3	1/32	0.9630824	0.9630052	0.9630045

Table 4 Numerical convergence rates for pressure in L^2-norm with variation in κ for Example 1

S No.	h	$\|p(t_n) - P^n\|$ $\kappa = 0.01$	$\|p(t_n) - P^n\|$ $\kappa = 0.0001$	$\|p(t_n) - P^n\|$ $\kappa = 0.00000001$
1	1/4	1.0666418	1.0641700	1.0641470
2	1/8	1.0313096	1.0314830	1.0314819
3	1/16	1.0287957	1.0288386	1.0288391
3	1/32	1.0169415	1.0169587	1.0169589

difference scheme, respectively. These computational results agree with optimal convergence rates obtained in Theorem 5.3, respectively. Further, when $\kappa \to 0$ the order of convergence for velocity and pressure terms are given through Tables 2, 3 and 4 which again confirm our theoretical results given in Theorem 5.3.

Acknowledgments The author would like to express his gratitude to anonymous referees for their constructive and valuable suggestions, which help to improve the paper. The author also gratefully acknowledges the financial support from IRCC project No. 13IRAWD007 of IIT Bombay during his visit to IIT Bombay in 2016.

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