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ORIGINAL PAPER

Proximal point algorithms for solving convex minimization problem and common fixed points problem of asymptotically quasi-nonexpansive mappings in CAT(0) spaces with convergence analysis

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Abstract In this paper, we introduce the modified proximal point algorithm for common fixed points of asymptotically quasi-nonexpansive mappings in CAT(0) spaces and also prove some convergence theorems of the proposed algorithm to a common fixed point of asymptotically quasi-nonexpansive mappings and a minimizer of a convex function. The main results in this paper improve and generalize the corresponding results given by some authors. Moreover, we then give numerical examples to illustrate and show efficiency of the proposed algorithm for supporting our main results.

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1 Introduction

The proximal point method was initiated by Martinet [1] and henceforth developed by Rockafellar [2]. These original constructions were meant to seek for points in the preimage of zero under a certain (set-valued) map. It is known [2] in the context of Hilbert space that under typical criteria, the proximal point method converges in the weak topology to such solution. In particular, if the subdifferential of a proper convex lower semi-continuous (for short, l.s.c.) function (of course, defined on a Hilbert space, with probably infinite values) is taken into account, then such weak limits are minimizers for that function. In this case, the proximal point method resolves into solving the minimizer of Yosida-Moreau envelopes.

It is known [3] that a minimization problem can be written in the form of variational inequality with respect to the (sub)gradient of the corresponding objective function. In general, variational inequalities have its own role outside optimization, especially in nonlinear analysis.

Apart from nonlinear spaces like Riemannian (or Hadamard) manifolds, there are still a varieties of structures that lies within a more general and more useful class of CAT(0) spaces (also known as the non-positively curved spaces). Simple examples of CAT(0) spaces that is not a Riemannian manifold are are spiders, booklets, metric trees, Hilbert cubes, and Euclidean Bruhat-Tits buildings.

The proximal point method in Hadamard spaces, as was pioneered by Bačák [4], was for minimization. His results exploit the developments of convex analysis in Hadamard spaces, which became extensive during the past two decades. Note that the method has been modified so that it converges in the metric by Cholamjiak [5] using the Halpern procedure. However, the concept of subdifferential in a Hadamard space was not available until [6], where it is also proved that a particular point minimizes a proper convex l.s.c. function if and only if the corresponding subdifferential at that point contains zero.

Recently, some proximal point algorithms (shortly, the PPA) for solving optimization problems in classical linear spaces, such as Hilbert spaces and Banach spaces, have been extended to the setting of manifolds [7–11].

A geodesic metric space (X, d) is a CAT(0) space if each geodesic triangle is at least as 'thin' as its comparison triangle in $\mathbb{R}^2$. A complete CAT(0) space is then called a *Hadamard space*. Let C be a nonempty closed subset of a CAT(0) space X and let $T : C \to C$ be a mapping. The set of fixed point of T is denote by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. Recall that T is said to be:

(1) *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$;
(2) *quasi-nonexpansive* if $d(Tx, p) \leq d(x, p)$ for all $x \in C$ and $p \in F(T)$;

(3) *asymptotically nonexpansive* if there exists a sequence $\{\nu_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \nu_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + \nu_n) d(x, y)$$

for all $x, y \in C$ and $n \geq 1$;

(4) *asymptotically quasi-nonexpansive* if there exists a sequence $\{\nu_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \nu_n = 0$ and $p \in F(T)$ such that

$$d(T^n x, p) \leq (1 + \nu_n) d(x, p)$$

for all $x \in C$ and $n \geq 1$;

(5) *uniformly L-Lipschitz* if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x, y)$$

for all $x, y \in C$ and $n \geq 1$.

Nowadays, there have been many iterative methods constructed and proposed to find approximating fixed points of nonlinear mappings. The *Halpern iteration process* is defined as follows: $x_1 \in C$ and

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T x_n \tag{1.1}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a real sequence in (0,1). The *S-iteration process* is defined as follows:

$$\begin{cases} y_n = (1 - \alpha_n) x_n + \alpha_n T x_n, \\ x_{n+1} = (1 - \beta_n) T x_n + \beta_n T y_n \end{cases} \tag{1.2}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in (0,1). The *Noor iteration process* is defined as follows:

$$\begin{cases} z_n = (1 - \alpha_n) x_n + \alpha_n T x_n, \\ y_n = (1 - \beta_n) x_n + \beta_n T z_n, \\ x_{n+1} = (1 - \gamma_n) x_n + \gamma_n T y_n \end{cases} \tag{1.3}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in (0,1) that the process (1.3) can reduce to be both the *Ishikawa iteration process* (see [12]) and the *Mann iteration process* (see [13]). In 2011, Phuengrattana and Suantai [14] introduced following the *SP-iteration process* as follows:

$$\begin{cases} w_n = (1 - \alpha_n) x_n + \alpha_n T x_n, \\ y_n = (1 - \beta_n) w_n + \beta_n T w_n, \\ x_{n+1} = (1 - \gamma_n) y_n + \gamma_n T y_n \end{cases} \tag{1.4}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in (0,1). Subsequently, in 2016, Kitkuan and Padcharoen [15] studied and applied the process (1.4) in CAT(0) spaces as follows:

$$\begin{cases} w_n = (1 - \alpha_n) x_n \oplus \alpha_n T^n x_n, \\ y_n = (1 - \beta_n) w_n \oplus \beta_n T^n w_n, \\ x_{n+1} = (1 - \gamma_n) y_n \oplus \gamma_n T^n y_n \end{cases} \tag{1.5}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in (0,1). They also proved some strong convergence theorems for generalized asymptotically quasi-nonexpansive mappings in a such space under some conditions.

Let (X, d) be a geodesic space and $f : X \to (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization theory is to solve $x \in X$ such that

$$f(x) = \min_{y\in X} f(y).$$

We denote by $\arg\min_{y\in X} f(y)$ by the set of a minimizer of a convex function. A powerful tool for solving this problem is the well-known PPA which was introduced by Martinet [1] in 1970. In 1976, Rockafellar [2] studied, by the PPA, the convergence to a solution of the convex minimization problem in the framework of a Hilbert space. Also, he proved that the sequence $\{x_n\}$ converges weakly to a minimizer of a convex function f such that $\sum_{n=1}^{\infty} \lambda_n = \infty$.

In 2013, Bačák [4] introduced the PPA in a CAT(0) space (X, d) as follows: $x_1 \in X$ and

$$x_{n+1} = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)]$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$, and he showed that, if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer (see also [16]).

In 2015, Cholamjiak [5] modified the PPA by using the process (1.1) in CAT(0) spaces (X, d) as follows:

$$\begin{cases} y_n = \arg\min_{y\in X}[f(y) + \frac{1}{2r} d^2(y, x_n)], \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) y_n \end{cases}$$

for each $n \in \mathbb{N}$, where $r > 0$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} = \infty$. He proved the sequence $\{x_n\}$ converges to its minimizer. Moreover, he illustrated the numerical example for supporting the main result.

In the same year, Cholamjiak et al. [17] introduced the following PPA with the process (1.2) in CAT(0) spaces as follows:

$$\begin{cases} z_n = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ y_n = (1 - \beta_n) x_n \oplus \beta_n T_1 z_n, \\ x_{n+1} = (1 - \alpha_n) T_1 x_n \oplus \alpha_n T_2 y_n \end{cases}$$

for each $n \in \mathbb{N}$ and established some strong convergence theorems of the proposed algorithm to common fixed points of nonexpansive mappings and to minimizers of a convex function in such spaces (see also [18]). Recently, Chang et al. [19] established some strong convergence theorems of the PPA with process (1.2) to common fixed point of asymptotically nonexpansive mappings and to minimizers of a convex function in CAT(0) spaces.

Motivated and inspired by the above results, in this paper, we propose the modified proximal point algorithm with the process (1.5) for three asymptotically quasi-nonexpansive mappings in CAT(0) spaces and under suitable conditions, we also prove some convergence theorems of the proposed processes. Furthermore, we provide numerical examples to illustrate and show efficiency of the proposed algorithm for supporting our main results.

2 Preliminaries

In this section, we will mention basic concepts, definitions, notations, and some useful lemmas for use in the next sections.

Recall that a metric space (X, d) is called a *CAT(0) space* if it is geodesically connected and every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane.

A subset C of a CAT(0) space X is said to be *convex* if, for any $x, y \in K$, we have $[x, y] \subset C$, where

$$[x, y] := \{\alpha x \oplus (1-\alpha)y : 0 \leq \alpha \leq 1\}$$

is the unique geodesic joining x and y. For more details, see [20–25]. In this paper, we can write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x, z) = td(x, y), \quad d(y, z) = (1-t)d(x, y)$$

where $t \in [0, 1]$. It is well known that a geodesic space (X, d) is a CAT(0) space if and only if

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y) \tag{2.1}$$

for all $x, y, z \in X$ and $t \in [0, 1]$. In particular, if x, y, and z are points in a CAT(0) space (X, d) and $t \in [0, 1]$, then we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z). \tag{2.2}$$

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d). For any $x \in X$, we put

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

(1) The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\};$$

(2) The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is well known that, in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point (cf. [20]).

Definition 2.1 A sequence $\{x_n\}$ in a CAT(0) space X is said to be Δ-*convergent* to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ of $\{x_n\}$ and denote $W_\Delta(x_n) := \cup\{A(\{u_n\})\}$, where the union is sum over all subsequences $\{u_n\}$ of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be *regular* if $r(\{u_n\}) = r(\{x_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. It is well known that every bounded sequence in X has a Δ-convergent subsequence [26].

Lemma 2.2 [20] *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.3 [27] *Assume that a subset of a complete CAT(0) space (X, d) is closed, convex and $T : C \to C$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\Delta - \lim x_n = p$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $Tp = p$.*

Recall that a function $f : C \to (-\infty, \infty]$ is said to be *convex* if, for any a geodesic $[x, y] := \{\gamma_{x,y}(\alpha) : 0 \leq \alpha \leq 1\} := \{\alpha x \oplus (1-\alpha)y : 0 \leq \alpha \leq 1\}$ joining $x, y \in C$, the function $f \circ \gamma$ is convex, i.e.,

$$f(\gamma_{x,y}(\alpha)) := f(\alpha x \oplus (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y).$$

Now, we give some examples of a convex function in CAT(0) space X as follows:

Example 2.1 For a nonempty, closed and convex subset $C \subset X$, the *indicator function* $\delta_C : X \to \mathbb{R}$ defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

be a proper, convex and lower semi-continuous function.

Example 2.2 The function $f : X \to [0, \infty)$ defined by $f(y) = d(x, y)$ for all $y \in X$ is convex.

Example 2.3 The function $f : X \to [0, \infty)$ defined by $f(y) = d^2(x, y)$ for all $y \in X$ is convex.

For all $\lambda > 0$, define the *Moreau-Yosida resolvent* of f in a complete CAT(0) space X as follows:

$$J_\lambda(x) = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda} d^2(y, x)] \tag{2.3}$$

for all $x \in X$.

Let $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. It was shown in [16] that the set $F(J_\lambda)$ of the fixed point of the resolvent J_λ associated with f coincides with the set $\arg\min_{y\in X} f(y)$ of minimizers of f. Also, for any $\lambda > 0$, the resolvent J_λ of f is nonexpansive [28].

Lemma 2.4 [29] *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be proper, convex and lower semi-continuous function. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}d^2(J_\lambda x, y) - \frac{1}{2\lambda}d^2(x, y) + \frac{1}{2\lambda}d^2(x, J_\lambda x) + f(J_\lambda x) \leq f(y).$$

Lemma 2.5 [30] *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous function. Then, the following identity holds:*

$$J_\lambda x = J_\mu\Big(\frac{\lambda - \mu}{\lambda}J_\lambda x \oplus \frac{\mu}{\lambda}x\Big)$$

for all $x \in X$ and $\lambda > \mu > 0$.

Lemma 2.6 [31] *Let $\{\mu_n\}$ and $\{\nu_n\}$ are two sequences of non-negative real numbers such that:*

$$\mu_{n+1} \leq (1 + \nu_n)\mu_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \nu_n < \infty$, then $\lim_{n\to\infty} \mu_n$ exists.

3 Convergence theorems

3.1 Some Δ-convergence theorems

Now, we construct and prove the main results in this paper.

Theorem 3.1 *Let (X, d) be a complete CAT(0) space and C be a nonempty closed convex subset of X. Let $f : X \to (\infty, \infty]$ be a proper convex and lower semi-continuous function and $R, S, T : C \to C$ are three asymptotically quasi-nonexpansive mappings with $F(R) \cap F(S) \cap F(T) \neq \emptyset$ and*

$$\omega := F(R) \cap F(S) \cap F(T) \cap \arg\min_{y\in C} f(y) \neq \emptyset.$$

Let $\{\nu_n\}$ be a non-negative real sequence with $\sum_{n=1}^{\infty} \nu_n < \infty$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0, 1]$ with $0 < a \leq \alpha_n, \gamma_n, \delta_n \leq c < 1$ for all $n \in \mathbb{N}$ and for some a, c are positive constants in $(0, 1)$ and $\{\lambda_n\}$ be a sequence with $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be the sequence generated in the following manner:

$$\begin{cases} z_n = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)], \\ w_n = (1 - \alpha_n)z_n \oplus \alpha_n R^n z_n, \\ y_n = (1 - \beta_n)w_n \oplus \beta_n S^n w_n, \\ x_{n+1} = (1 - \gamma_n)y_n \oplus \gamma_n T^n y_n \end{cases} \tag{3.1}$$

for each $n \in \mathbb{N}$. Then, the following statements hold:

(1) $\lim_{n\to\infty} d(x_n, \tilde{p})$ *exists for all $\tilde{p} \in \omega$;*
(2) $\lim_{n\to\infty} d(x_n, z_n) = 0$*;*
(3) $\lim_{n\to\infty} d(x_n, Rx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0.$

Proof Let $\tilde{p} \in \omega$. Then, $\tilde{p} = R\tilde{p} = S\tilde{p} = T\tilde{p}$ and $f(\tilde{p}) \le f(y)$ for any $y \in C$. Hence, we have

$$f(\tilde{p}) + \frac{1}{2\lambda_n} d^2(\tilde{p}, \tilde{p}) \le f(y) + \frac{1}{2\lambda_n} d^2(y, \tilde{p})$$

for each $y \in C$ and so $\tilde{p} = J_{\lambda_n}\tilde{p}$ for each $n \in \mathbb{N}$.

(1) Now, we show that $\lim_{n\to\infty} d(x_n, \tilde{p})$ exists. Indeed, $z_n = J_{\lambda_n} x_n$ and J_{λ_n} is nonexpansive [28]. Hence, we have

$$d(z_n, \tilde{p}) = d(J_{\lambda_n} x_n, J_{\lambda_n}\tilde{p}) \le d(x_n, \tilde{p}). \tag{3.2}$$

Also, by (3.1), (3.2) and (2.2), we have

$$\begin{aligned} d(w_n, \tilde{p}) &= d((1-\alpha_n)z_n \oplus \alpha_n R^n z_n, \tilde{p}) \\ &\le (1-\alpha_n)d(z_n, \tilde{p}) + \alpha_n d(R^n z_n, \tilde{p}) \\ &\le (1-\alpha_n)d(z_n, \tilde{p}) + \alpha_n(1+\nu_n)d(z_n, \tilde{p}) \\ &\le (1+\alpha_n\nu_n)d(z_n, \tilde{p}) \\ &\le (1+\alpha_n\nu_n)d(x_n, \tilde{p}) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(y_n, \tilde{p}) &= d((1-\beta_n)w_n \oplus \beta_n S^n w_n, \tilde{p}) \\ &\le (1-\beta_n)d(w_n, \tilde{p}) + \beta_n d(S^n w_n, \tilde{p}) \\ &\le (1-\beta_n)d(w_n, \tilde{p}) + \beta_n(1+\nu_n)d(w_n, \tilde{p}) \\ &= (1+\beta_n\nu_n)d(w_n, \tilde{p}) \\ &\le (1+\beta_n\nu_n)(1+\alpha_n\nu_n)d(z_n, \tilde{p}) \\ &\le (1+\alpha_n\nu_n+\beta_n\nu_n+\alpha_n\beta_n\nu_n^2)d(x_n, \tilde{p}). \end{aligned} \tag{3.4}$$

Similarly, by (3.3) and (3.4), we have

$$\begin{aligned} d(x_{n+1}, \tilde{p}) &= d((1-\gamma_n)y_n \oplus \gamma_n T^n y_n, \tilde{p}) \\ &\le (1-\gamma_n)d(y_n, \tilde{p}) + \gamma_n d(T^n y_n, \tilde{p}) \\ &\le (1-\gamma_n)d(y_n, \tilde{p}) + \gamma_n(1+\nu_n)d(y_n, \tilde{p}) \\ &\le (1+\gamma_n\nu_n)d(y_n, \tilde{p}) \\ &\le (1+\alpha_n\nu_n+\beta_n\nu_n+\alpha_n\beta_n\nu_n^2+\gamma_n\nu_n \\ &\quad +\alpha_n\gamma_n\nu_n^2+\beta_n\gamma_n\nu_n^2+\alpha_n\beta_n\gamma_n\nu_n^3)d(x_n, \tilde{p}) \\ &= (1+(\alpha_n+\beta_n+\alpha_n\beta_n\nu_n+\gamma_n \\ &\quad +\alpha_n\gamma_n\nu_n+\beta_n\gamma_n\nu_n+\alpha_n\beta_n\gamma_n\nu_n^2)\nu_n)d(x_n, \tilde{p}) \end{aligned} \tag{3.5}$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$, by Lemma 2.6, it follows that $\lim_{n\to\infty} d(x_n, \tilde{p})$ exists and so we assume that

$$\lim_{n\to\infty} d(x_n, \tilde{p}) = c \ge 0. \tag{3.6}$$

Therefore, $\{x_n\}$ is bounded and so the sequences $\{z_n\}$, $\{w_n\}$, $\{y_n\}$, $\{R^n x_n\}$, $\{S^n x_n\}$ and $\{T^n z_n\}$ are bounded.

(2) Next, we show that $\lim_{n\to\infty} d(x_n, z_n) = 0$. Indeed, by Lemma 2.4, we have

$$\frac{1}{2\lambda_n}\{d^2(z_n, \tilde{p}) - d^2(x_n, \tilde{p}) + d^2(x_n, z_n)\} \leq f(\tilde{p}) - f(z_n).$$

Since $f(\tilde{p}) \leq f(z_n)$ for each $n \in \mathbb{N}$, it follows that

$$d^2(x_n, z_n) \leq d^2(x_n, \tilde{p}) - d^2(z_n, \tilde{p}). \tag{3.7}$$

Furthermore, from (3.5), we have

$$d(x_{n+1}, \tilde{p}) \leq (1 + \gamma_n \nu_n) d(y_n, \tilde{p}) \tag{3.8}$$

and

$$\liminf_{n\to\infty} d(y_n, \tilde{p}) \geq c. \tag{3.9}$$

On the other hand, it follows from (3.4) that

$$\limsup_{n\to\infty} d(y_n, \tilde{p}) \leq c \tag{3.10}$$

and so

$$\lim_{n\to\infty} d(y_n, \tilde{p}) = c. \tag{3.11}$$

Similarly, from (3.4), it follows that

$$d(y_n, \tilde{p}) \leq (1 + \beta_n \nu_n) d(w_n, \tilde{p}), \tag{3.12}$$

which yields

$$\liminf_{n\to\infty} d(w_n, \tilde{p}) \geq c. \tag{3.13}$$

Also, by (3.3), we have

$$\limsup_{n\to\infty} d(w_n, \tilde{p}) \leq c. \tag{3.14}$$

Hence, by (3.13) and (3.14), we obtain

$$\lim_{n\to\infty} d(w_n, \tilde{p}) = c. \tag{3.15}$$

Since

$$d(w_n, \tilde{p}) \leq (1 + \beta_n \nu_n) d(z_n, \tilde{p}), \tag{3.16}$$

which yields

$$\liminf_{n\to\infty} d(z_n, \tilde{p}) \geq c. \tag{3.17}$$

Also, by (3.3), we have

$$\limsup_{n\to\infty} d(z_n, \tilde{p}) \leq c \tag{3.18}$$

and so

$$\lim_{n\to\infty} d(z_n, \tilde{p}) = c. \tag{3.19}$$

This shows that

$$\lim_{n\to\infty} d(x_n, z_n) = 0 \tag{3.20}$$

and so we prove (2).

(3) Now, we show that $\lim_{n\to\infty} d(x_n, Rx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0$. Observe that

$$\begin{aligned} d^2(w_n, \tilde{p}) &= d^2((1-\alpha_n)z_n \oplus \alpha_n R^n z_n, \tilde{p}) \\ &\leq (1-\alpha_n)d^2(z_n, \tilde{p}) + \alpha_n d^2(R^n z_n, \tilde{p}) - \alpha_n(1-\alpha_n)d^2(z_n, R^n z_n) \\ &\leq (1-\alpha_n)d^2(z_n, \tilde{p}) + \alpha_n(1+\nu_n)d^2(z_n, \tilde{p}) - \alpha_n(1-\alpha_n)d^2(z_n, R^n z_n) \\ &= (1+\alpha_n\nu_n)d^2(z_n, \tilde{p}) - \alpha_n(1-\alpha_n)d^2(z_n, R^n z_n), \end{aligned}$$

which implies that

$$\begin{aligned} d^2(z_n, R^n z_n) &\leq \frac{1}{\alpha_n(1-\alpha_n)}\left[(d^2(z_n, \tilde{p}) - d^2(w_n, \tilde{p})) + \alpha_n\nu_n d^2(z_n, \tilde{p})\right] \\ &\leq \frac{1}{a(1-c)}\left[(d^2(z_n, \tilde{p}) - d^2(w_n, \tilde{p})) + \alpha_n\nu_n d^2(z_n, \tilde{p})\right] \\ &\to 0 \end{aligned} \tag{3.21}$$

as $n \to \infty$. Using the triangle inequality, by (3.20) and (3.21), we have

$$\begin{aligned} d(R^n x_n, x_n) &\leq d(R^n x_n, R^n z_n) + d(R^n z_n, z_n) + d(z_n, x_n) \\ &\leq Ld(x_n, z_n) + d(R^n z_n, z_n) + d(z_n, x_n) \\ &\to 0 \end{aligned} \tag{3.22}$$

as $n \to \infty$. Thus, we have

$$\begin{aligned} d(w_n, x_n) &= d((1-\alpha_n)z_n \oplus \alpha_n R^n z_n, x_n) \\ &\leq (1-\alpha_n)d(x_n z_n) + \alpha_n d(R^n z_n, x_n) \\ &\leq (1-\alpha_n)d(x_n, z_n) + \alpha_n\{d(R^n z_n, z_n) + d(z_n, x_n)\} \\ &\to 0 \end{aligned} \tag{3.23}$$

as $n \to \infty$. Similarly, it follows that

$$\begin{aligned} d^2(y_n, \tilde{p}) &= d^2((1-\beta_n)w_n \oplus \beta_n S^n w_n, \tilde{p}) \\ &\leq (1-\beta_n)d^2(w_n, \tilde{p}) + \beta_n d^2(S^n w_n, \tilde{p}) - \beta_n(1-\beta_n)d^2(w_n, S^n w_n) \\ &\leq (1-\beta_n)d^2(w_n, \tilde{p}) + \beta_n(1+\nu_n)d^2(w_n, p) - \beta_n(1-\beta_n)d^2(w_n, S^n w_n) \\ &= (1+\beta_n\nu_n)d^2(w_n, \tilde{p}) - \beta_n(1-\beta_n)d^2(w_n, S^n w_n), \end{aligned}$$

which implies that

$$\begin{aligned} d^2(w_n, S^n w_n) &\leq \frac{1}{\beta_n(1-\beta_n)}\{(d^2(w_n, \tilde{p}) - d^2(y_n, \tilde{p})) - \beta_n\nu_n d^2(w_n, \tilde{p})\} \\ &\leq \frac{1}{a(1-c)}\{(d^2(w_n, \tilde{p}) - d^2(y_n, \tilde{p})) - \beta_n\nu_n d^2(w_n, \tilde{p})\} \\ &\to 0 \end{aligned} \tag{3.24}$$

as $n \to \infty$. Again, by the triangle inequality, (3.23) and (3.24), we have

$$\begin{aligned} d(S^n x_n, x_n) &\leq d(S^n x_n, S^n w_n) + d(S^n w_n, w_n) + d(w_n, x_n) \\ &\leq d(x_n, w_n) + d(S^n w_n, w_n) + d(w_n, x_n) \\ &\to 0 \end{aligned} \tag{3.25}$$

as $n \to \infty$, which implies that

$$\begin{aligned} d(y_n, x_n) &= d((1-\beta_n)w_n \oplus \beta_n S^n w_n, x_n) \\ &\leq (1-\beta_n)d(w_n, x_n) + \beta_n\{d(S^n w_n, w_n) + d(w_n, x_n)\} \\ &\to 0 \end{aligned} \tag{3.26}$$

as $n \to \infty$. Similarly, observe that

$$\begin{aligned} d^2(x_{n+1}, \tilde{p}) &= d^2((1-\gamma_n)y_n \oplus \gamma_n T^n y_n, \tilde{p}) \\ &\leq (1-\gamma_n)d^2(y_n, \tilde{p}) + \gamma_n d^2(T^n y_n, \tilde{p}) - \gamma_n(1-\gamma_n)d^2(y_n, T^n y_n) \\ &\leq (1-\gamma_n)d^2(y_n, \tilde{p}) + \gamma_n(1+\nu_n)d^2(y_n, \tilde{p}) - \gamma_n(1-\gamma_n)d^2(y_n, T^n y_n) \end{aligned}$$

and so

$$\begin{aligned} d^2(y_n, T^n y_n) &\leq \frac{1}{\gamma_n(1-\gamma_n)}\{(d^2(y_n, \tilde{p}) - d^2(x_{n+1}, \tilde{p})) - \gamma_n \nu_n d^2(y_n, \tilde{p})\} \\ &\leq \frac{1}{a(1-c)}\{(d^2(y_n, \tilde{p}) - d^2(x_{n+1}, \tilde{p})) - \gamma_n \nu_n d^2(y_n, \tilde{p})\} \\ &\to 0 \end{aligned} \tag{3.27}$$

as $n \to \infty$, which implies that

$$\begin{aligned} d(T^n x_n, x_n) &\leq d(T^n x_n, T^n y_n) + d(T^n y_n, y_n) + d(y_n, x_n) \\ &\leq Ld(x_n, y_n) + d(T^n y_n, y_n) + d(y_n, x_n) \\ &\to 0 \end{aligned} \tag{3.28}$$

as $n \to \infty$. It follows that

$$\begin{aligned} d(x_{n+1}, x_n) &= d((1-\gamma_n)y_n \oplus \gamma_n T^n y_n, x_n) \\ &\leq (1-\gamma_n)d(y_n, x_n) + \gamma_n d(T^n y_n x_n) \\ &\leq (1-\gamma_n)d(y_n, x_n) + \gamma_n\{d(T^n y_n, y_n) + d(y_n, x_n)\} \\ &\to 0 \end{aligned} \tag{3.29}$$

as $n \to \infty$, we have

$$\begin{aligned} d(x_n, Rx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, R^{n+1}x_{n+1}) \\ &\quad + d(R^{n+1}x_{n+1}, R^{n+1}x_n) + d(R^{n+1}x_n, Rx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, R^{n+1}x_{n+1}) + Ld(x_{n+1}, x_n) + Ld(R^n x_n, x_n) \\ &\to 0 \end{aligned} \tag{3.30}$$

as $n \to \infty$. Similarly, we can conclude that

$$\lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

Therefore, we prove (3). This completes the proof. □

Theorem 3.2 *Under the hypothesis of Theorem 3.1, the sequence* $\{x_n\}$ *defined by* (3.1) Δ*-converges to a common element of* ω.

Proof In fact, it follows from (3.20) and Lemma 2.5 that

$$\begin{aligned} d(J_\lambda x_n, x_n) &\leq d(J_\lambda x_n, z_n) + d(z_n, x_n) \\ &= d(J_\lambda x_n, J_{\lambda n} x_n) + d(z_n, x_n) \\ &= d\Big(J_\lambda x_n, J_\lambda\Big(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\Big)\Big) + d(z_n, x_n) \\ &\leq d\Big(x_n, \Big(1 - \frac{\lambda}{\lambda_n}\Big) J_{\lambda n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\Big) + d(z_n, x_n) \\ &\leq \Big(1 - \frac{\lambda}{\lambda_n}\Big) d(x_n, J_{\lambda n} x_n) + \frac{\lambda}{\lambda_n} d(x_n, x_n) + d(z_n, x_n) \\ &= \Big(1 - \frac{\lambda}{\lambda_n}\Big) d(x_n, z_n) + d(z_n, x_n) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. By Theorem 3.1 (1), it follows that $\lim_{n\to\infty} d(x_n, \tilde{p})$ exists for all $\tilde{p} \in \omega$ and, by Theorem 3.1 (3), we have $\lim_{n\to\infty} d(x_n, Rx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Next, we prove that

$$W_\Delta(x_n) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\}) \subset \omega.$$

Let $u \in W_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Definition 2.1, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some $v \in C$. By Lemma 2.3, $v \in \omega$. So, $u = v$ by Lemma 2.2. This shows that $W_\Delta(x_n) \subset \omega$.

Finally, we will prove that the sequence $\{x_n\}$ Δ-converges to a point in ω. To this end, it suffices to prove that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in W_\Delta(x_n) \subset \omega$ and $\{d(x_n, u)\}$ converges, by Lemma 2.2, we have $x = u$. Thus, $W_\Delta(x_n) = \{x\}$. This completes the proof. □

We know that every real Hilbert space H is a complete CAT(0) space. The following result can be obtained from Theorem 3.1.

Corollary 3.3 *Let C be a nonempty closed and convex subset of real Hilbert spaces H. Suppose that R, S, T, $\{k_n\}$, f, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, λ and ω satisfy all the hypothesis in Theorem* 3.1. *If $\{x_n\}$ is the sequence generated in the following manner:*

$$\begin{cases} z_n = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n}\|y - x_n\|^2], \\ w_n = (1-\alpha_n)z_n + \alpha_n R^n z_n, \\ y_n = (1-\beta_n)w_n + \beta_n S^n w_n, \\ x_{n+1} = (1-\gamma_n)y_n + \gamma_n T^n y_n \end{cases}$$

for each $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges weakly to a common element in ω.

3.2 Some strong convergence theorems

In this subsection, under mild conditions, we construct and prove strong convergence theorems.

Let C be a nonempty closed convex subset of a CAT(0) space (X, d). A family $\{P, Q, R, S\}$ of mappings is said to satisfy the *condition* (ω) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Px) \geq f(d(x, F))$$

or

$$d(x, Qx) \geq f(d(x, F))$$

or

$$d(x, Rx) \geq f(d(x, F))$$

or

$$d(x, Sx) \geq f(d(x, F))$$

for all $x \in X$, where $F = F(P) \cap F(Q) \cap F(R) \cap F(S)$.

Theorem 3.4 *Under the hypothesis of Theorem* 3.1*, suppose that the family* $\{R, S, T, J_\lambda\}$ *satisfy the condition* (ω)*. Then, the sequence* $\{x_n\}$ *defined by* (3.1) *strongly converges to a common element of* ω*.*

Proof From Theorem 3.1 (1), we have $\lim_{n\to\infty} d(x_n, \tilde{p})$ exists for all $\tilde{p} \in \omega$. Also, it follows that $\lim_{n\to\infty} d(x_n, \omega)$ exists. On the other hand, by the condition (ω), we have

$$\lim_{n\to\infty} f(d(x_n, \omega)) \geq \lim_{n\to\infty} d(x_n, Rx_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, \omega)) \geq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, \omega)) \geq \lim_{n\to\infty} d(x_n, Sx_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, \omega)) \geq \lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0.$$

Thus, we have $\lim_{n\to\infty} f(d(x_n, \omega)) = 0$. By using the property of f, we have $\lim_{n\to\infty} d(x_n, \omega) = 0$. Thus, following the proof of Theorem 3.3 of [32], we can show that $\{x_n\}$ is a Cauchy sequence in X and so $\{x_n\}$ converges to a point p^* in X and hence $d(p^*, \omega) = 0$. Since ω is closed, we have $p^* \in \omega$. This completes the proof. □

A mapping $T : C \to C$ is said to be *semi-compact* if any sequence $\{x_n\}$ in C satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 3.5 *Under the hypothesis of Theorem* 3.1*, suppose that* R *or* S *or* T *or* J_λ *is semi-compact. Then the sequence* $\{x_n\}$ *defined by* (3.1) *strongly converges to a common element of* ω*.*

Proof Suppose that R is semi-compact. By Theorem 3.1 (3), we have $\lim_{n\to\infty} d(x_n, Rx_n) = 0$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p^* \in X$. Since

$$\lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0,$$

we have $d(p^*, Sp^*) = d(p^*, Tp^*) = 0$ and $d(p^*, J_\lambda p^*) = 0$, which show that $p^* \in \omega$. For other mappings, we also prove the strong convergence of $\{x_n\}$ to a common element of ω. This completes the proof. □

Remark 3.6 Our main results extends the results of Bačák [4], Chang et al. [19] and Cholamjiak et al. [17] in the framework of CAT(0) spaces. Moreover, our results generalize the results of Cholamjiak et al. [17] from two nonexpansive mappings to three asymptotically quasi-nonexpansive mappings involving the convex and lower semi-continuous function in such a framework. In fact, we present the modified proximal point algorithm with the process (1.5) for solving the convex minimization problem as well as the common fixed points problem.

4 Numerical results

In this section, we provide numerical examples to illustrate reckoning the convergence of modified proximal point algorithm with SP-type iteration (3.1) for three asymptotically quasi-nonexpansive mappings by numerical experiment results for supporting main results.

Example 4.1 Let $X = \mathbb{R}$ be a Euclidean metric space and $C = [1, 10]$. Let $R, S, T : \mathbb{R} \to \mathbb{R}$ be mappings defined by

$$Rx = \frac{x+3}{2}, \qquad Sx = \sqrt[3]{18 + x^2}, \qquad Tx = \sqrt{x^2 - 6x + 18}.$$

For all $x \in C$, let $f : X \to (-\infty, \infty]$ defined by

$$f(x) = \|x\|_1 + \frac{1}{2}\|x\|_2^2 - 4x + 5.$$

It is easy to check that R, S, T are continuous uniformly L-Lipschitzian and asymptotically quasi-nonexpansive mappings with $F(R) = F(S) = F(T) = \{3\}$ and f is proper convex and lower semi-continuous function.

Let $\alpha_n = \frac{4n-3}{8n}$, $\beta_n = \frac{6n-1}{10n}$, $\gamma_n = \frac{12n-7}{15n}$ and also we put $x_1 = 10$ is the initial value, using the soft thresholding operator [33] and the proximity operator [34]. Then, we obtain numerical results in Table 1 with the values error (see Fig. 1).

Table 1 Numerical results of Example 4.1

Number of iterates	x_n	$f(x_n)$	$\|x_n - x_{n-1}\|_2$
1	10.0000000000	25.0000000000	–
2	4.59492461474	1.77189226335	5.40507538526
3	3.12672761382	0.50802994405	1.46819700092
4	3.00665392713	0.50002213737	0.12007368669
5	3.00027813747	0.50000003868	0.00637578966
6	3.00001008672	0.50000000005	2.6805075e−04
7	3.00000033239	0.50000000000	9.7543334e−06
8	3.00000001023	0.50000000000	3.2216186e−07
9	3.00000000030	0.50000000000	9.9293258e−09
10	3.00000000001	0.50000000000	2.9060310e−10
11	3.00000000000	0.50000000000	8.1703533e−12
12	3.00000000000	0.50000000000	2.2248869e−13
13	3.00000000000	0.50000000000	5.7731597e−15
14	3.00000000000	0.50000000000	0.00000000000
15	3.00000000000	0.50000000000	0.00000000000

From Table 1 and Fig. 2, we see that the sequence $\{x_n\}$ converges to 3 which is a common fixed point of three asymptotically quasi-nonexpansive mappings and also a minimizer of a function f. That is a solution of constrained convex minimization problems as follows:

$$\min_{x \in C \subseteq \mathbb{R}} \|x\|_1 + \frac{1}{2}\|x\|_2^2 - 4x + 5.$$

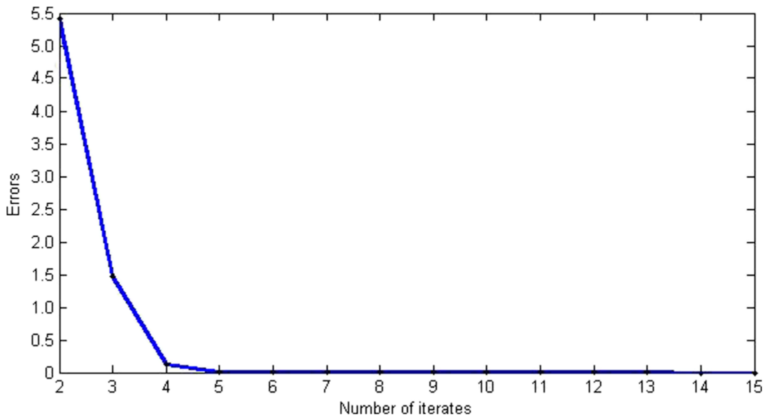

Fig. 1 The values of $\|x_n - x_{n-1}\|_2$ plotting in Table 1

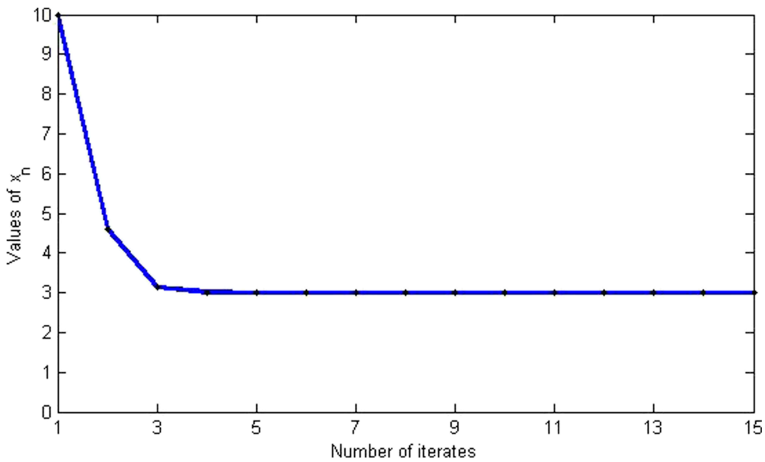

Fig. 2 The values of $\{x_n\}$ plotting in Table 1

Example 4.2 Let $X = \mathbb{R}^2$ be a Euclidean metric space and $C = \{x | x = (x^{\widehat{1}}, x^{\widehat{2}}) \in \mathbb{R}^2 : 1 \leq x^{\widehat{1}}, x^{\widehat{2}} \leq 100\}$. Let $R, S, T : \mathbb{R}^2 \to \mathbb{R}^2$ be mappings defined by

$$Rx = \left(\frac{5 + x^{\widehat{1}}}{6}, \frac{4 + 3x^{\widehat{2}}}{2} \right),$$

Table 2 Numerical results of Example 4.2

Number of iterates	$x_n = (x^{\widehat{1}}, x^{\widehat{2}})$	$f(x_n)$	$\|x_n - x_{n-1}\|_2$
1	(50.0000000000, 100.0000000000)	6008.0000000000	–
2	(11.1524232136, 21.19083961382)	241.18001109574	87.8636328805
3	(2.14191795123, 4.977335166394)	10.584250650198	18.5490412569
4	(1.07590173580, 2.262240265625)	5.53726551520641	2.91687005054
5	(1.00417100495, 2.017159422807)	5.50015592153666	0.25536232546
6	(1.00020889593, 2.000956075306)	5.50000047885875	0.01668073074
7	(1.00000984348, 2.000047980177)	5.50000000119950	9.2965512e−04
8	(1.00000044408, 2.000002244761)	5.50000000000262	4.6691296e−05
9	(1.00000001939, 2.000000100056)	5.50000000000001	2.1863483e−06
10	(1.00000000083, 2.000000004307)	5.50000000000000	9.7531631e−08
11	(1.00000000003, 2.000000000181)	5.50000000000000	4.2018489e−09
12	(1.00000000000, 2.000000000007)	5.50000000000000	1.7636027e−10
13	(1.00000000000, 2.000000000000)	5.50000000000000	7.2531323e−12
14	(1.00000000000, 2.000000000000)	5.50000000000000	2.9334437e−13
15	(1.00000000000, 2.000000000000)	5.50000000000000	1.1757886e−14
16	(1.00000000000, 2.000000000000)	5.50000000000000	4.4408921e−16
17	(1.00000000000, 2.000000000000)	5.50000000000000	0.00000000000
18	(1.00000000000, 2.000000000000)	5.50000000000000	0.00000000000

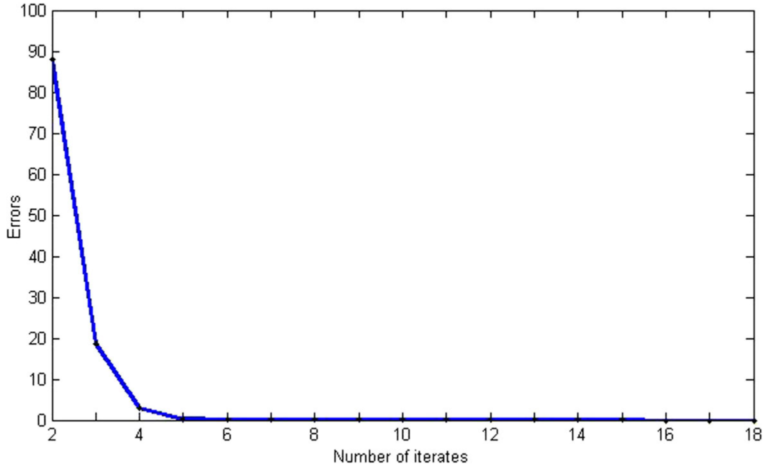

Fig. 3 The values of $\|x_n - x_{n-1}\|_2$ plotting in Table 2

$$Sx = \left(\sqrt{1 - \widehat{x^1} + (\widehat{x^1})^2}, \sqrt[3]{6 + \widehat{x^2}}\right),$$

$$Tx = \left(\sqrt[3]{\frac{1 + \widehat{x^1}}{2}}, \sqrt{4 - 2\widehat{x^2} + (\widehat{x^2})^2}\right).$$

For all $x \in C$, let $f : X \to (-\infty, \infty]$ defined by

$$f(x) = \|x\|_1 + \frac{1}{2}\|x\|_2^2 + (-2, -3)x + 8.$$

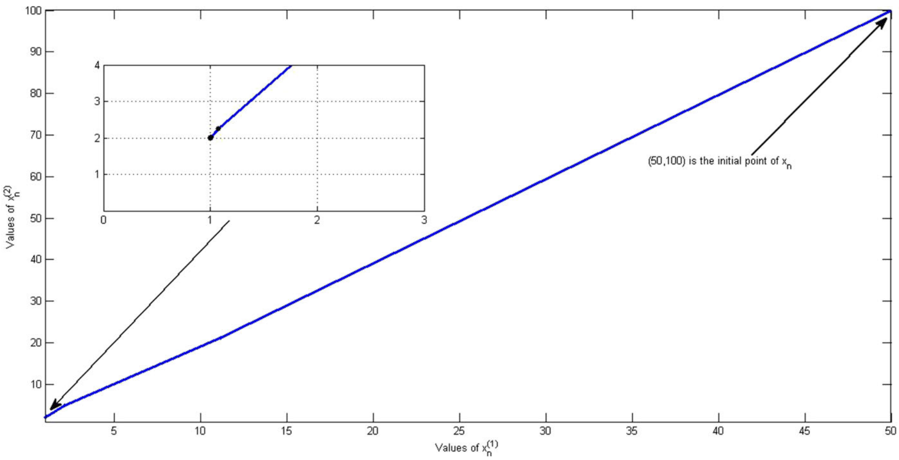

Fig. 4 The values of $\{x_n\} = (\widehat{x^1}, \widehat{x^2})$ plotting in Table 2

It is easy to check that R, S, T are continuous uniformly L-Lipschitzian and asymptotically quasi-nonexpansive mappings with $F(R) = F(S) = F(T) = (1, 2)$ and f is proper convex and lower semi-continuous function.

Let $\alpha_n = \frac{160n+47}{400n}$, $\beta_n = \frac{390n+131}{1000n}$, $\gamma_n = \frac{490n-323}{600n}$ and also we put (50, 100) is the initial point. Then, we obtain numerical results in Table 2 with values of error (see Fig. 3).

From Table 2 and Fig. 4, we see that the sequence $\{x_n\}$ converges to a point (1,2) which is a common fixed point of three asymptotically quasi-nonexpansive mappings and also a minimizer of a function f. That is a solution of constrained convex minimization problems as follows:

$$\min_{x \in C \subseteq \mathbb{R}^2} \|x\|_1 + \frac{1}{2}\|x\|_2^2 + (-2, -3)x + 8.$$

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