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A new single-step iteration method for solving complex symmetric linear systems

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Abstract For solving a class of complex symmetric linear systems, we introduce a new single-step iteration method, which can be taken as a fixed-point iteration adding the asymptotical error (FPAE). In order to accelerate the convergence, we further develop the parameterized variant of the FPAE (PFPAE) iteration method. Each iteration of the FPAE and the PFPAE methods requires the solution of only one linear system with a real symmetric positive definite coefficient matrix. Under suitable conditions, we derive the spectral radius of the FPAE and the PFPAE iteration matrices, and discuss the quasi-optimal parameters which minimize the above spectral radius. Numerical tests support the contention that the PFPAE iteration method has comparable advantage over some other commonly used iteration methods, particularly when the experimental optimal parameters are not used.

Keywords Complex linear system · Positive definite · HSS iteration · Spectral radius · Convergence analysis

Mathematics subject classification (2010) 65F10 · 65F50

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1 Introduction and motivations

We consider to find the iterative solution of complex linear systems of the form

$$Ax := (W + iT)x = b, \tag{1}$$

where $W, T \in \mathbb{R}^{n\times n}$ are symmetric matrices and at least one of them, e.g., W is positive definite. Throughout the paper, the vector $b \in \mathbb{C}^n$ is given, $x \in \mathbb{C}^n$ is unknown and $i = \sqrt{-1}$ is the imaginary unit. This kind of complex symmetric linear systems comes from many problems in scientific computing and applied engineering branches, we refer to [1, 7, 8, 23, 24] and the references therein.

As we all know, a simple Hermitian and skew-Hermitian splitting (HSS) has been commonly used so that we can solve the problem (1) iteratively, where the coefficient matrix is rewritten as

$$A = H + S,$$

with

$$H = \frac{1}{2}(A + A^*) = W \quad \text{and} \quad S = \frac{1}{2}(A - A^*) = iT.$$

Here, A^* denotes the conjugate transpose of the matrix A. Notice that the coefficient matrix A is non-Hermitian, but positive definite.

When the matrix T is positive semi-definite, a number of iteration methods have been developed to solve the system (1). Bai et al. [14] first proposed the HSS iteration method by the scheme

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - iT)x^{(k)} + b, \\ (\alpha I + iT)x^{(k+1)} = (\alpha I - W)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where $\alpha > 0$ is a given constant and I is the identity matrix. For more information about the HSS iteration method and its variants, we refer to [4, 5, 13, 15–17, 22].

Later, a modified HSS (MHSS) iteration method was introduced in [8]. To speed up the convergence, Bai et al. [9, 10] proposed a preconditioned variant of the MHSS (PMHSS) iteration method and applied it to solve distributed control problems. The PMHSS iteration method is constructed by the scheme

$$\begin{cases} (\alpha V + W)x^{(k+\frac{1}{2})} = (\alpha V - iT)x^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} = (\alpha V + iW)x^{(k+\frac{1}{2})} - ib, \end{cases}$$

where $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix. Furthermore, a lopsided PMHSS iteration method was generalized by Li et al. [29] when the real part of the coefficient matrix is dominant. A new HSS (NHSS) iteration method was proposed by Pour and Goughery [30], and a parameterized variant of the NHSS iteration method was developed in [35].

Recently, Li and Wu [27] presented a single-step HSS (SHSS) iteration method by the scheme

$$(\alpha I + W)x^{(k+1)} = (\alpha I - iT)x^{(k)} + b.$$

Later, a parameterized variant of the SHSS (PSHSS) method was constructed by Zeng and Ma [36], which is confirmed to be powerful according to their numerical experiments. The PSHSS iteration method is given by

$$(\alpha I + \omega W + T)x^{(k+1)} = [\alpha I - i(\omega T - W)]x^{(k)} + (\omega - i)b,$$

where α, ω are given positive constants. In addition, a preconditioned variant of the PSHSS iteration method was developed in [34].

Note that the HSS-like iteration methods for complex linear systems have been extended in many literatures, and some of them have been used for solving other systems of equations, see [2, 3, 6, 11, 12, 18–21, 28, 32, 33, 37–39] for more details.

On the other hand, let $x = u + iv$ and $b = p + iq$ with $u, v, p, q \in \mathbb{R}^n$, then problem (1) can be reduced to a 2-by-2 block real linear system

$$\begin{bmatrix} W & -T \\ T & W \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{2}$$

with u, v being unknown vectors. To solve this system, Salkuyeh et al. [31] proposed a generalized successive overrelaxation (GSOR) iterative method. Hezari et al. [26] developed a preconditioned variant of the GSOR (PGSOR) iterative method by the scheme

$$\begin{cases} (\omega W + T)u^{(k+1)} = (1-\alpha)(\omega W + T)u^{(k)} + \alpha(\omega T - W)v^{(k)} + \alpha(\omega p + q), \\ (\omega W + T)v^{(k+1)} = -\alpha(\omega T - W)u^{(k+1)} + (1-\alpha)(\omega W + T)v^{(k)} + \alpha(\omega q - p), \end{cases}$$

with α, ω being given positive constants.

In this paper, we introduce a new single-step iteration method which can be taken as a fixed-point iteration adding the asymptotical error (FPAE). In particular, if W, $T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite, we may use the parameterized variant of the FPAE (PFPAE) iteration method because of its higher efficiency.

This paper is structured as follows. In Section 2, the FPAE iteration method is constructed and its convergence properties are analyzed, including the convergence conditions, the spectral radius of the iterative matrix and the quasi-optimal parameter. In Section 3, the PFPAE iteration method is developed and its convergence properties are considered. In Section 4, some numerical examples are presented to show the computational efficiencies of the FPAE and the PFPAE iteration methods by comparing with several other iteration methods. Finally, a brief conclusion is made in Section 5.

2 Convergence analysis of a new single-step iteration method

In this section, suppose that W is symmetric positive definite and T is symmetric, then we introduce a new iteration method to solve the complex linear system (1). For any symmetric positive definite matrix V and any $\alpha > 0$, the fact that

$$Vx = Vx - \alpha[(W + iT)x - b],$$

inspires us to construct

$$Vx^{(k+1)} = Vx^{(k)} - \alpha[(W + iT)x^{(k)} - b],$$

which can be taken as a fixed-point iteration adding the asymptotical error (FPAE), thus we have the following iteration method.

Algorithm 1 The FPAE Iteration Method

Given an initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ by the scheme

$$Vx^{(k+1)} = [V - \alpha W - i\alpha T]x^{(k)} + \alpha b. \tag{3}$$

We can rewrite (3) as

$$x^{(k+1)} = M(\alpha, V)x^{(k)} + N(\alpha, V)b,$$

where

$$\begin{aligned} M(\alpha, V) &= I - \alpha V^{-1}W - i\alpha V^{-1}T \\ &= V^{-\frac{1}{2}}[I - V^{-\frac{1}{2}}WV^{-\frac{1}{2}} - i\alpha V^{-\frac{1}{2}}TV^{-\frac{1}{2}}]V^{\frac{1}{2}}, \end{aligned} \tag{4}$$

and $N(\alpha, V) = \alpha V^{-1}$. Let

$$F(\alpha, V) = \frac{1}{\alpha}V, \quad \text{and} \quad G(\alpha, V) = \frac{1}{\alpha}V - (W + iT),$$

then we have

$$A = F(\alpha, V) - G(\alpha, V), \quad \text{and} \quad M(\alpha, V) = F(\alpha, V)^{-1}G(\alpha, V).$$

Thus, the splitting matrix $F(\alpha, V)$ can be used as a preconditioner for the complex matrix $A \in \mathbb{C}^{n\times n}$, which is referred as the FPAE preconditioner.

In order to study the convergence properties of the FPAE iteration method, we use the following notations

$$\begin{aligned} \tilde{\lambda}^V_{min} &= \min_{\tilde{\lambda}_j \in sp(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})} \{\tilde{\lambda}_j\}, \qquad \tilde{\sigma}^V_{max} = \max_{\tilde{\sigma}_j \in sp(V^{-\frac{1}{2}}TV^{-\frac{1}{2}})} \{|\tilde{\sigma}_j|\}, \\ \tilde{\lambda}^V_{max} &= \max_{\tilde{\lambda}_j \in sp(V^{-\frac{1}{2}}WV^{-\frac{1}{2}})} \{\tilde{\lambda}_j\}, \qquad \tilde{\sigma}^V_{min} = \min_{\tilde{\sigma}_j \in sp(V^{-\frac{1}{2}}TV^{-\frac{1}{2}})} \{|\tilde{\sigma}_j|\}, \end{aligned} \tag{5}$$

where $sp(X)$ is the spectral set of the matrix X.

Theorem 1 *For the FPAE iteration method, the spectral radius*

$$\rho(M(\alpha, V)) \le \sqrt{\max\left\{(1 - \alpha\tilde{\lambda}^V_{min})^2, (1 - \alpha\tilde{\lambda}^V_{max})^2\right\} + \alpha^2(\tilde{\sigma}^V_{max})^2} =: \delta^V(\alpha). \tag{6}$$

Moreover, if

$$0 < \alpha < \min\left\{\frac{2\tilde{\lambda}_{min}^{V}}{(\tilde{\lambda}_{min}^{V})^2+(\tilde{\sigma}_{max}^{V})^2}, \frac{2\tilde{\lambda}_{max}^{V}}{(\tilde{\lambda}_{max}^{V})^2+(\tilde{\sigma}_{max}^{V})^2}\right\}, \tag{7}$$

then $\delta^V(\alpha) < 1$, *i.e., the iteration converges.*

Proof It is easy to obtain (6) in terms of (4) and the fact that

$$\rho(M(\alpha, V)) = \rho(I - V^{-\frac{1}{2}} W V^{-\frac{1}{2}} - i\alpha V^{-\frac{1}{2}} T V^{-\frac{1}{2}}).$$

After simple discussions and direct calculations, we have

$$(\delta^V(\alpha))^2 = \begin{cases} (1-\alpha\tilde{\lambda}_{min}^{V})^2+\alpha^2(\tilde{\sigma}_{max}^{V})^2, & 0 < \alpha \le \frac{2}{\tilde{\lambda}_{max}^{V}+\tilde{\lambda}_{min}^{V}}, \\ (\alpha\tilde{\lambda}_{max}^{V}-1)^2+\alpha^2(\tilde{\sigma}_{max}^{V})^2, & \alpha > \frac{2}{\tilde{\lambda}_{max}^{V}+\tilde{\lambda}_{min}^{V}}. \end{cases} \tag{8}$$

Therefore, a concrete analysis of (8) makes it clear that we have $\delta^V(\alpha) < 1$ if $\tilde{\lambda}_{min}^{V}\tilde{\lambda}_{max}^{V} \le (\tilde{\sigma}_{max}^{V})^2$ and $0 < \alpha < \frac{2\tilde{\lambda}_{min}^{V}}{(\tilde{\lambda}_{min}^{V})^2+(\tilde{\sigma}_{max}^{V})^2}$, or if $\tilde{\lambda}_{min}^{V}\tilde{\lambda}_{max}^{V} > (\tilde{\sigma}_{max}^{V})^2$ and $0 < \alpha < \frac{2\tilde{\lambda}_{max}^{V}}{(\tilde{\lambda}_{max}^{V})^2+(\tilde{\sigma}_{max}^{V})^2}$, while those conditions can be reduced to the inequality (7). □

Corollary 1 *For the FPAE iteration method, if*

$$(\tilde{\lambda}_{min}^{V})^2 + 2(\tilde{\sigma}_{max}^{V})^2 \ge \tilde{\lambda}_{max}^{V}\tilde{\lambda}_{min}^{V}, \tag{9}$$

then the optimal parameter α_*^V *which minimizes the upper bound* $\delta^V(\alpha)$ *of the spectral radius* $\rho(M(\alpha, V))$ *is given by*

$$\alpha_*^V = \frac{\tilde{\lambda}_{min}^{V}}{(\tilde{\lambda}_{min}^{V})^2+(\tilde{\sigma}_{max}^{V})^2}, \tag{10}$$

and

$$\delta^V(\alpha_*^V) = \frac{\tilde{\sigma}_{max}^{V}}{\sqrt{(\tilde{\lambda}_{min}^{V})^2+(\tilde{\sigma}_{max}^{V})^2}}. \tag{11}$$

Proof Since the two parts in the right-hand side of (8) are parabolas, then the upper bound $\delta^V(\alpha)$ of the spectral radius $\rho(M(\alpha, V))$ achieves its minimum $\delta^V(\alpha_*^V)$ given by

$$\delta^V(\alpha_*^V) = \begin{cases} \frac{\tilde{\sigma}_{max}^{V}}{\sqrt{(\tilde{\lambda}_{min}^{V})^2+(\tilde{\sigma}_{max}^{V})^2}}, & \alpha_*^V = \frac{\tilde{\lambda}_{min}^{V}}{(\tilde{\lambda}_{min}^{V})^2+(\tilde{\sigma}_{max}^{V})^2} \le \frac{2}{\tilde{\lambda}_{max}^{V}+\tilde{\lambda}_{min}^{V}}, \\ \frac{\tilde{\sigma}_{max}^{V}}{\sqrt{(\tilde{\lambda}_{max}^{V})^2+(\tilde{\sigma}_{max}^{V})^2}}, & \alpha_*^V = \frac{\tilde{\lambda}_{max}^{V}}{(\tilde{\lambda}_{max}^{V})^2+(\tilde{\sigma}_{max}^{V})^2} > \frac{2}{\tilde{\lambda}_{max}^{V}+\tilde{\lambda}_{min}^{V}}. \end{cases}$$

However, the inequality $\frac{\tilde{\lambda}_{max}^{V}}{(\tilde{\lambda}_{max}^{V})^2+(\tilde{\sigma}_{max}^{V})^2} > \frac{2}{\tilde{\lambda}_{max}^{V}+\tilde{\lambda}_{min}^{V}}$ never holds true, since $\tilde{\lambda}_{max}^{V}(\tilde{\lambda}_{max}^{V}+\tilde{\lambda}_{min}^{V}) \le 2(\tilde{\lambda}_{max}^{V})^2$. Hence, if the condition (9) is satisfied, we obtain (10) and (11). □

Remark 1 According to Theorem 1, the FPAE iteration can be made convergent if the parameter α is properly chosen. Moreover, under the condition (9), theoretically the optimal symmetric positive definite matrix V_* minimizing $\delta^V(\alpha_*^V)$ is given by

$$V_* = \underset{V}{argmin}\left\{\frac{\tilde{\sigma}_{max}^V}{\sqrt{(\tilde{\lambda}_{min}^V)^2+(\tilde{\sigma}_{max}^V)^2}}\right\} = \underset{V}{argmin}\left\{\frac{\tilde{\sigma}_{max}^V}{\tilde{\lambda}_{min}^V}\right\},$$

since $f(x) := \frac{x}{\sqrt{1+x^2}}$ is a strictly monotonic increasing function. While practically we usually take the positive definite matrix W to substitute the matrix V.

In fact, when $V = W$, the FPAE iteration becomes

$$Wx^{(k+1)} = [(1-\alpha)W - i\alpha T]x^{(k)} + \alpha b. \tag{12}$$

We can rewrite (12) as

$$x^{(k+1)} = M(\alpha)x^{(k)} + N(\alpha)b,$$

where $M(\alpha) = (1-\alpha)I - i\alpha W^{-1}T$ and $N(\alpha) = \alpha W^{-1}$. If we define

$$F(\alpha) = \frac{1}{\alpha}W, \quad \text{and} \quad G(\alpha) = \frac{1-\alpha}{\alpha}W - iT,$$

then it holds that

$$A = F(\alpha) - G(\alpha), \quad \text{and} \quad M(\alpha) = F(\alpha)^{-1}G(\alpha).$$

Therefore, the splitting matrix $F(\alpha)$ can be taken as a preconditioner for the complex matrix $A \in \mathbb{C}^{n\times n}$.

Since the spectral radius $\rho(M(\alpha)) = \rho((1-\alpha)I - i\alpha W^{-1}T)$, then we can easily get the following result without proof.

Theorem 2 *For the FPAE iteration method using $V = W$, the spectral radius $\rho(M(\alpha))$ is given by*

$$\delta(\alpha) = \sqrt{(1-\alpha)^2 + \alpha^2\rho^2(W^{-1}T)}. \tag{13}$$

If $0 < \alpha < \frac{2}{1+\rho^2(W^{-1}T)}$, then $\delta(\alpha) < 1$ i.e., the iteration converges. Moreover, the optimal parameter α_* minimizing the spectral radius $\delta(\alpha)$ is given by $\alpha_* = \frac{1}{1+\rho^2(W^{-1}T)} \in (0,1)$ which leads to

$$\delta(\alpha_*) = \frac{\rho(W^{-1}T)}{\sqrt{1+\rho^2(W^{-1}T)}}. \tag{14}$$

Mention that under the conditions of Theorem 2, if $\rho(W^{-1}T) \le 1$, then it holds that

$$\rho(M(\alpha_*)) = \delta(\alpha_*) \le \frac{1}{\sqrt{2}}.$$

Remark 2 According to Corollary 2.1 in [27] for the SHSS iteration method, the minimal upper bound δ_α of the spectral radius $\rho((\alpha I + W)^{-1}(\alpha I - iT))$ is given by

$$\delta_{\alpha_*} = \frac{\sigma_{max}}{\sqrt{\lambda_{min}^2 + \sigma_{max}^2}} = \frac{\frac{\sigma_{max}}{\lambda_{min}}}{\sqrt{1 + (\frac{\sigma_{max}}{\lambda_{min}})^2}}, \tag{15}$$

where λ_{min} is the smallest eigenvalue of matrix W and σ_{max} is the largest singular-value of matrix iT. From (14) and (15), using the fact that

$$\rho(W^{-1}T) \leq \|W^{-1}T\|_2 \leq \|W^{-1}\|_2\|T\|_2 = \frac{\sigma_{max}}{\lambda_{min}},$$

we obtain

$$\delta(\alpha_*) \leq \delta_{\alpha_*}.$$

Therefore, the FPAE method using $V = W$ would be more efficient than the SHSS method when the optimal parameters are used, respectively.

3 Convergence analysis of the parameterized variant of the FPAE iteration method

In this section, we suppose $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite. Multiplying a parameter $\omega - i$ by the two sides of the original linear system (1) leads to

$$(\omega - i)Ax := [(\omega W + T) + i(\omega T - W)]x = (\omega - i)b. \tag{16}$$

Let

$$\tilde{A} = (\omega - i)A, \quad \tilde{W} = \omega W + T, \quad \tilde{T} = \omega T - W, \quad \tilde{b} = (\omega - i)b. \tag{17}$$

Then, (16) is rewritten as

$$\tilde{A}x := (\tilde{W} + i\tilde{T})x = \tilde{b}.$$

By comparing it with the original linear system (1), we directly obtain the following variant of the FPAE iteration method for $V = \tilde{W}$

$$\tilde{W}x^{(k+1)} = [(1 - \alpha)\tilde{W} - i\alpha\tilde{T}]x^{(k)} + \alpha\tilde{b}. \tag{18}$$

Substituting (17) into (18), we get the following parameterized variant of the FPAE (PFPAE) iteration method.

Algorithm 2 The PFPAE Iteration Method

Given an initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ by the scheme

$$(\omega W + T)x^{(k+1)} = [(1 - \alpha)(\omega W + T) - i\alpha(\omega T - W)]x^{(k)} + \alpha(\omega - i)b, \tag{19}$$

where α, ω are given positive constants.

Next, we discuss the convergence properties of the PFPAE iteration method and search for the optimal parameters. First, we rewrite the PFPAE iteration scheme (19) as

$$x^{(k+1)} = M(\alpha,\omega)x^{(k)} + N(\alpha,\omega)(\omega - i)b, \quad k = 0, 1, 2, \ldots,$$

where

$$M(\alpha,\omega) = (1-\alpha)I - i\alpha(\omega W + T)^{-1}(\omega T - W), \quad N(\alpha,\omega) = \alpha(\omega W + T)^{-1}.$$

Let

$$F(\alpha,\omega) = \frac{1}{\alpha}(\omega W + T), \quad \text{and} \quad G(\alpha,\omega) = \frac{1-\alpha}{\alpha}(\omega W + T) - i(\omega T - W),$$

then we obtain

$$(\omega - i)A = F(\alpha,\omega) - G(\alpha,\omega), \quad \text{and} \quad M(\alpha,\omega) = F(\alpha,\omega)^{-1}G(\alpha,\omega).$$

Thus, the splitting matrix $F(\alpha,\omega)$ can be taken as a preconditioner for the complex symmetric matrix $(\omega - i)A \in \mathbb{C}^{n\times n}$, which is referred as the PFPAE preconditioner.

Since $\tilde{W} = \omega W + T$ is a symmetric positive definite matrix, then according to the convergence analysis about the FPAE iteration method given in Section 3, we immediately obtain the following results.

Theorem 3 *If $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices with at least one of them being positive definite, then the spectral radius $\rho(M(\alpha,\omega))$ is given by*

$$\delta(\alpha,\omega) = \sqrt{(1-\alpha)^2 + \alpha^2\rho^2(\tilde{W}^{-1}\tilde{T})}. \tag{20}$$

If $0 < \alpha < \frac{2}{1+\rho^2(\tilde{W}^{-1}\tilde{T})}$, then $\delta(\alpha,\omega) < 1$ i.e. the iteration converges. In addition, the optimal parameter $\alpha_*(\omega)$ minimizing the spectral radius $\delta(\alpha,\omega)$ is given by $\alpha_*(\omega) = \frac{1}{1+\rho^2(\tilde{W}^{-1}\tilde{T})}$ which leads to $\delta(\alpha_*(\omega),\omega) = \frac{\rho(\tilde{W}^{-1}\tilde{T})}{\sqrt{1+\rho^2(\tilde{W}^{-1}\tilde{T})}}$.

Remark 3 According to Theorem 3, by similar analysis as in Remark 2, for comparing the PFPAE method and the PSHSS method [36], we have

$$\delta(\alpha_*(\omega),\omega) \le \delta_{(\alpha_*(\omega),\omega)}.$$

Therefore, the PFPAE method would be superior to the PSHSS method when the optimal parameters $\alpha_*(\omega)$ are used, respectively, for the same ω.

Note that when $\rho(\tilde{W}^{-1}\tilde{T}) \le 1$, if we take the optimal parameter $\alpha_*(\omega)$ for the PFPAE method, then it holds that

$$\rho(M(\alpha_*(\omega),\omega)) = \delta(\alpha_*(\omega),\omega) \le \frac{1}{\sqrt{2}}.$$

Since the minimal spectral radius $\delta(\alpha_*(\omega),\omega) = \frac{\rho(\tilde{W}^{-1}\tilde{T})}{\sqrt{1+\rho^2(\tilde{W}^{-1}\tilde{T})}}$ is strictly monotonic increasing about $\rho(\tilde{W}^{-1}\tilde{T})$, then the parameter ω should be chosen such that the spectral radius $\rho(\tilde{W}^{-1}\tilde{T})$ is as small as possible.

Theorem 4 *Under the conditions of Theorem 3, let μ_{min} and μ_{max} be the smallest and largest eigenvalues of $W^{-1}T$, respectively. Then for any given $\theta \in (0, 1]$, we can make the spectral radius $\rho(\tilde{W}^{-1}\tilde{T}) < \theta$, if any of the following conditions holds:*

(1) $\omega > \frac{1-\theta\mu_{min}}{\theta+\mu_{min}}$ *for* $\mu_{min} \leq \mu_{max} \leq \theta$;

(2) $\frac{1-\theta\mu_{min}}{\theta+\mu_{min}} < \omega < \frac{\theta\mu_{max}+1}{\mu_{max}-\theta}$ *for* $\mu_{min} < \frac{1}{\theta}$, $\mu_{max} > \theta$ *and* $\frac{1+\mu_{max}\mu_{min}}{\mu_{max}-\mu_{min}} > \frac{1-\theta^2}{2\theta}$;

(3) $0 < \omega < \frac{\theta\mu_{max}+1}{\mu_{max}-\theta}$ *for* $\mu_{max} \geq \mu_{min} \geq \frac{1}{\theta}$.

Proof According to Lemma 2.2 in [26], the spectral radius $\rho(\tilde{W}^{-1}\tilde{T})$ is defined by

$$\rho(\tilde{W}^{-1}\tilde{T}) = \max\left\{\frac{1-\omega\mu_{min}}{\omega+\mu_{min}}, \frac{\omega\mu_{max}-1}{\omega+\mu_{max}}\right\}, \tag{21}$$

then $\rho(\tilde{W}^{-1}\tilde{T}) < \theta$ is equivalent to

$$\begin{cases} 1-\omega\mu_{min} < \theta(\omega+\mu_{min}), \\ \omega\mu_{max}-1 < \theta(\omega+\mu_{max}). \end{cases}$$

Solving this system of inequalities, we get

$$\begin{cases} \omega > \frac{1-\theta\mu_{min}}{\theta+\mu_{min}}, \\ 0 < \omega < \frac{\theta\mu_{max}+1}{\mu_{max}-\theta}, \quad \text{or} \quad \omega > 0 \quad \text{for} \quad \mu_{max} \leq \theta. \end{cases}$$

Since $\frac{1+\mu_{max}\mu_{min}}{\mu_{max}-\mu_{min}} > \frac{1-\theta^2}{2\theta}$ is equivalent to $\frac{1-\theta\mu_{min}}{\theta+\mu_{min}} < \frac{\theta\mu_{max}+1}{\mu_{max}-\theta}$, then we complete the proof by simple discussions. □

In particular, when $\theta = 1$, we can make $\delta(\alpha_*(\omega), \omega) < \frac{1}{\sqrt{2}}$ if any of the conditions in Theorem 4 holds.

Remark 4 According to Lemma 2.3 in [26], when W is symmetric positive and T is symmetric semi-definite, to make the minimal spectral radius $\delta(\alpha_*(\omega), \omega)$ for the PFPAE iteration method smaller than $\delta(\alpha_*)$ for the FPAE iteration method, we should choose the parameter ω such that the spectral radius $\rho(\tilde{W}^{-1}\tilde{T}) \leq \rho(W^{-1}T)$ which is equivalent to

$$\omega > \max\left\{0, \frac{1-\mu_{min}\mu_{max}}{\mu_{min}+\mu_{max}}\right\}. \tag{22}$$

On the other hand, according to Lemma 2.4 in [26], we can obtain the optimal parameter ω_* minimizing the spectral radius $\rho(\tilde{W}^{-1}\tilde{T})$ which is given by

$$\omega_* = \frac{1-\mu_{min}\mu_{max}+\sqrt{(1+\mu_{min}^2)(1+\mu_{max}^2)}}{\mu_{min}+\mu_{max}}. \tag{23}$$

In addition, from the proof of Corollary 2.2 in [26], we obtain

$$\rho(M(\alpha_*(\omega_*), \omega_*)) = \delta(\alpha_*(\omega_*), \omega_*) < \frac{1}{\sqrt{2}},$$

since the spectral radius $\rho(\tilde{W}^{-1}\tilde{T}) < 1$ for ω_*.

Therefore, if the parameter ω satisfies (22), then the PFPAE iteration method with the optimal parameter $\alpha_*(\omega)$ may be more efficient than the FPAE iteration method with the optimal parameter α_*. Theorem 3 tells us that the PFPAE iteration method converges for any $\omega > 0$ if $0 < \alpha < \frac{2}{1+\rho^2(\tilde{W}^{-1}\tilde{T})}$. While Remark 4 implies that the PFPAE iteration method converges for any $0 < \alpha \leq 1$ if the optimal parameter ω_* is chosen.

4 Numerical results

In this section, we perform some numerical experiments to test the effectiveness of the FPAE using $V = W$ and the PFPAE iteration methods. Numerical comparisons with the SHSS [27], the PSHSS [36], the PMHSS [9], and the PGSOR [26] iteration methods are made to show the advantage of the FPAE and the PFPAE iteration methods in terms of the number of iterations (denoted as IT) and the CPU times (in seconds, denoted as CPU).

All experiments are carried out in MATLAB (version 8.1.0.604, R2013a) on a personal computer with Intel(R) Core(TM) CPU 1.8 × 2Ghz and 4.00 GB of RAM. The CPU time is recorded by the command "tic-toc."

In our computations, all iteration schemes are started from the zero vector and terminated if

$$\frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} \leq 10^{-6}$$

where $x^{(k)} = u^{(k)} + iv^{(k)}$ is the current approximation.

Example 1 Consider the complex Helmholtz equation [24, 26, 29, 36]

$$-\Delta u + \sigma_1 u + i\sigma_2 u = f,$$

with σ_1, σ_2 being real coefficient functions. Here, u satisfies Dirichlet boundary conditions in the square $D = [0, 1] \times [0, 1]$. By discretizing this equation with finite differences on an $m \times m$ grid with mesh size $h = 1/(m + 1)$, we obtain a complex linear system

$$[(K + \sigma_1 I) + i\sigma_2 I]x = b,$$

where the matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form

$$K = I \otimes B_m + B_m \otimes I \quad \text{with} \quad B_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

Actually, K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$. In our tests, let the right-hand side vector $b = (1+i)A\mathbf{1}$ with $\mathbf{1}$ being the vector of all entries equal to 1. In addition, we normalize the complex linear system by multiplying both sides by h^2.

Example 2 Consider a complex system of the form [8]

$$[(K - \varpi^2 M) + i(C_H + \varpi C_V)]x = b,$$

which arises from direct frequency domain analysis of an n-degree-of-freedom linear system. Here K and M are the stiffness and the inertia matrices, C_H and C_V are the hysteretic and the viscous damping matrices, respectively, and the parameter ϖ is called the driving circular frequency. See [23, 25] for more details.

Similar as [8, 26, 31], we are going to solve the linear system

$$\left[\left(K - \varpi^2 I\right) + i\left(\beta K + 10\varpi I\right)\right]x = b,$$

with ϖ and β being given real constants. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form

$$K = I \otimes B_m + B_m \otimes I \quad \text{with} \quad B_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

Table 1 Numerical results of Example 1 for different iteration methods when $(\sigma_1, \sigma_2) = (1, 10)$

Method		Grid				
		16×16	32×32	64×64	96×96	125×125
PMHSS	α_*	0.90	0.90	0.90	0.90	0.90
	IT	38	40	40	40	40
	CPU(s)	0.0403	0.1445	0.7326	2.3314	4.2471
PGSOR	α_*	0.985	0.985	0.985	0.985	0.985
	ω_*	4.5	5	6	6	6
	IT	4	4	4	4	4
	CPU(s)	0.0113	0.0332	0.1516	0.4032	0.7612
PSHSS	α_*	0.001	0.001	0.001	0.001	0.001
	ω_*	4.5	5	6	6	6
	IT	9	9	8	8	8
	CPU(s)	0.0118	0.0424	0.2481	0.9183	2.4764
SHSS	α_*	0.01	0.001	0.0005	0.0005	0.0005
	IT	15	14	13	12	11
	CPU(s)	0.0146	0.0496	0.3019	1.0541	2.5054
FPAE	α_*	0.90	0.80	0.80	0.80	0.80
	IT	15	13	12	12	11
	CPU(s)	0.0139	0.0422	0.2038	0.5192	0.9013
PFPAE	α_*	0.985	0.985	0.985	0.985	0.985
	ω_*	4.5	5	6	6	6
	IT	9	9	8	8	8
	CPU(s)	0.0114	0.0337	0.1545	0.4197	0.7668

In our tests, the right-hand side vector b with its j-th entry b_j is given by

$$b_j = \frac{(1+i)j}{h^2(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

As before, we normalize the system by multiplying both sides by h^2.

In Tables 1–4, we solve $x^{(k+1)}$ in the system $Bx^{(k+1)} = C^{(k)}$ by computing $x^{(k+1)} = B \setminus C^{(k)}$ for the sparse matrices B and $C^{(k)}$ in each step of iteration for all those considered methods. Tables 1 and 2 show the numerical results of several iteration methods with respect to different problem sizes for Examples 1–2 by using the experimental optimal parameters α_* and ω_*, respectively. It is apparent that the experimental optimal parameters of the considered methods are stable, except for the SHSS method, which brings great convenience for us to find the optimal parameters for large systems. We can conclude that the FPAE method has a little advantage over the SHSS method from the point of view of iterations and CPU times by using the experimental optimal parameters, and it is justified that the minimal value of spectral radius $\rho(M(\alpha))$ of the iteration matrix of the FPAE method is less than that of

Table 2 Numerical results of Example 2 for different iteration methods when $(\varpi, \beta) = (1, 1)$

Method		Grid				
		16×16	32×32	64×64	96×96	125×125
PMHSS	α_*	1.1	1.1	1.1	1.1	1.1
	IT	21	21	20	20	20
	CPU(s)	0.0238	0.0802	0.4158	1.2707	2.3372
PGSOR	α_*	0.99	0.99	0.99	0.99	0.99
	ω_*	0.85	0.85	0.85	0.85	0.85
	IT	3	3	3	3	3
	CPU(s)	0.0122	0.0291	0.1440	0.3479	0.6192
PSHSS	α_*	0.0001	0.0001	0.0001	0.0001	0.0001
	ω_*	0.85	0.85	0.85	0.85	0.85
	IT	7	7	6	6	6
	CPU(s)	0.0126	0.0413	0.2205	0.9263	2.2216
SHSS	α_*	0.58	0.32	0.168	0.115	0.090
	IT	145	250	457	660	842
	CPU(s)	0.0674	0.4227	4.0014	18.6412	42.6365
FPAE	α_*	0.28	0.28	0.28	0.28	0.28
	IT	76	72	67	65	63
	CPU(s)	0.0401	0.1361	0.6557	1.9775	3.5391
PFPAE	α_*	0.99	0.99	0.99	0.99	0.99
	ω_*	0.85	0.85	0.85	0.85	0.85
	IT	7	7	6	6	6
	CPU(s)	0.0125	0.0312	0.1457	0.3512	0.6258

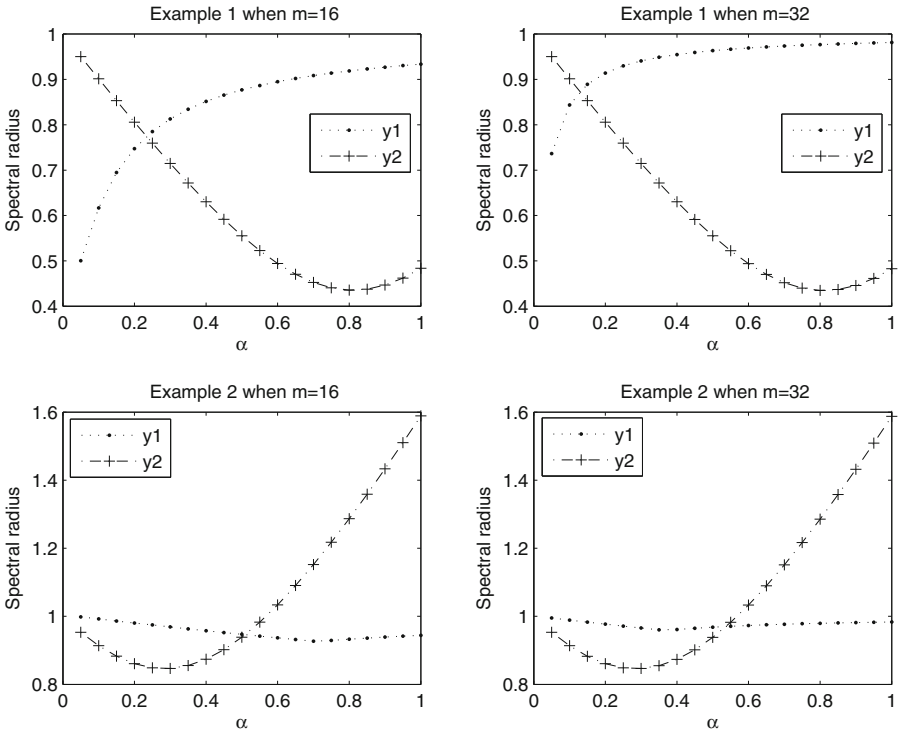

Fig. 1 The spectral radius $\rho(M(\alpha))$ of the iteration matrices of SHSS (y1) and FPAE (y2) methods for Example 1 when $(\sigma_1, \sigma_2) = (1, 10)$ and Example 2 when $(\varpi, \beta) = (1, 1)$

the SHSS method by Fig. 1. Similarly, the PFPAE iteration method is a little more efficient than the PSHSS iteration method by using the experimental optimal parameters from the point of view of CPU times, and the minimal value of spectral radius $\rho(M(\alpha, \omega))$ of the iteration matrix of the PFPAE method is less than that of the PSHSS method from Fig. 2. Moreover, the PFPAE method is much more efficient than the FPAE method since $\rho(\tilde{W}^{-1}\tilde{T}) \leq \rho(W^{-1}T)$ which is verified by Fig. 3.

In order to make further comparisons among the PSHSS, the PGSOR and the PFPAE iteration methods, we also perform numerical tests for these three methods without using the experimental optimal parameters in Tables 3 and 4, where we always take the non-optimal parameter $\alpha = 0.5$ for the sake of fairness, because the distances between 0.5 and the experimental optimal parameters α_* are almost the same for all these three iteration methods. We can see that the convergence speed of the PFPAE iteration method is mildly affected by changes of the parameters, and the PFPAE method is the most efficient from the point of view of CPU times.

Moreover, according to the results of Tables 1 and 2, the computational efficiency of the PFPAE iteration method is very close to that of the PGSOR iteration method from the point of view of CPU times when the experimental optimal parameters are used. Thus we make further comparison by computations with the preconditioned GMRES methods and the numerical results are given in Tables 5 and 6, from which

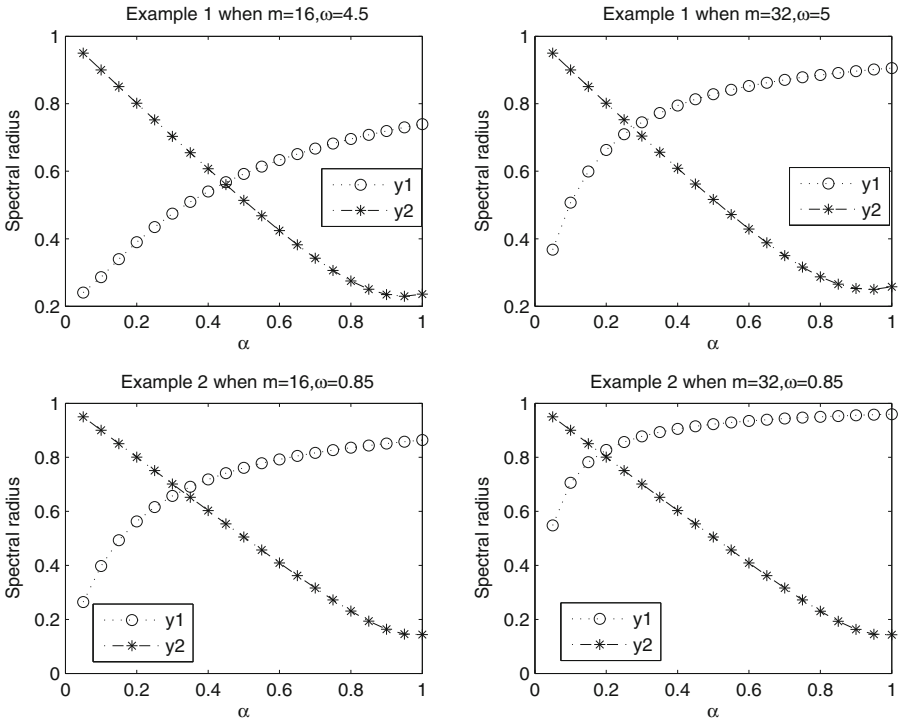

Fig. 2 The spectral radius $\rho(M(\alpha,\omega))$ of the iteration matrices of PSHSS (y1) and PFPAE (y2) methods for Example 1 when $(\sigma_1,\sigma_2)=(1,10)$ and Example 2 when $(\varpi,\beta)=(1,1)$

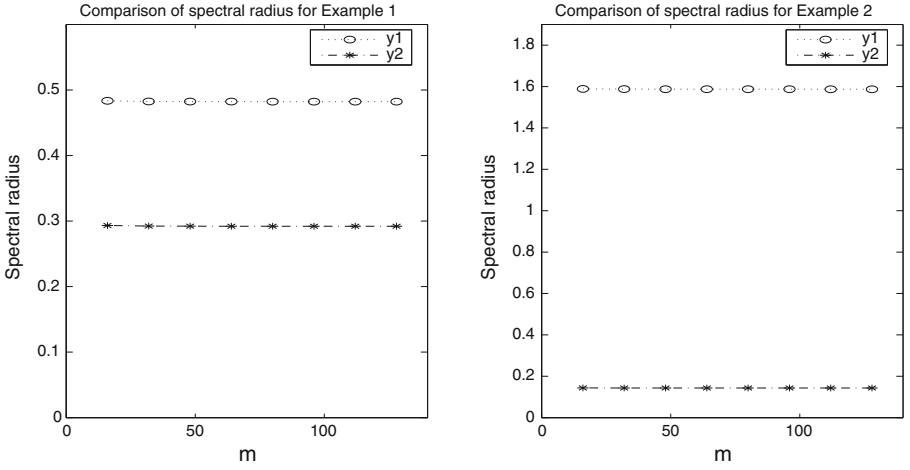

Fig. 3 Comparison of the spectral radius $y1=\rho(W^{-1}T)$ and $y2=\rho(\tilde{W}^{-1}\tilde{T})$ for Example 1 when $(\sigma_1,\sigma_2,\omega)=(1,10,6)$ and Example 2 when $(\varpi,\beta,\omega)=(1,1,0.85)$

Table 3 Numerical results of Example 1 for different iteration methods without using the experimental optimal parameters when $(\sigma_1, \sigma_2) = (1, 10)$

Method		Grid				
		16×16	32×32	64×64	96×96	125×125
PSHSS	IT	15	34	101	204	326
$(\alpha, \omega) = (0.5, 10)$	CPU(s)	0.0138	0.0786	1.0241	6.2993	16.9201
PGSOR	IT	20	20	20	20	20
$(\alpha, \omega) = (0.5, 10)$	CPU(s)	0.0250	0.08549	0.4072	1.2148	2.2329
PFPAE	IT	21	21	21	21	21
$(\alpha, \omega) = (0.5, 10)$	CPU(s)	0.0175	0.0534	0.2702	0.7924	1.3486
PSHSS	IT	48	148	508	1056	1702
$(\alpha, \omega) = (0.5, 1.5)$	CPU(s)	0.0284	0.2594	4.3530	29.2957	80.8415
PGSOR	IT	21	20	20	20	20
$(\alpha, \omega) = (0.5, 1.5)$	CPU(s)	0.0254	0.0811	0.4066	1.1989	2.1006
PFPAE	IT	27	27	28	28	28
$(\alpha, \omega) = (0.5, 1.5)$	CPU(s)	0.0194	0.0645	0.3365	0.9602	1.6897

we can conclude that the PFPAE method is superior to the PGSOR method from the point of view of iterations and CPU times.

In a word, the numerical results confirm that the PFPAE iteration method is very useful whether the experimental optimal parameters are used or not.

Table 4 Numerical results of Example 2 for different iteration methods without using the experimental optimal parameters when $(\varpi, \beta) = (1, 1)$

Method		Grid				
		16×16	32×32	64×64	96×96	125×125
PSHSS	IT	40	68	204	421	684
$(\alpha, \omega) = (0.5, 5)$	CPU(s)	0.0282	0.1349	1.8872	12.1528	34.6258
PGSOR	IT	20	20	20	20	20
$(\alpha, \omega) = (0.5, 5)$	CPU(s)	0.0242	0.0799	0.4143	1.1855	2.0733
PFPAE	IT	40	38	35	34	33
$(\alpha, \omega) = (0.5, 5)$	CPU(s)	0.0246	0.0793	0.3879	1.1441	1.9622
PSHSS	IT	66	212	752	1603	2631
$(\alpha, \omega) = (0.5, 0.1)$	CPU(s)	0.0461	0.3598	6.7128	44.2157	132.2548
PGSOR	IT	20	20	20	20	20
$(\alpha, \omega) = (0.5, 0.1)$	CPU(s)	0.0247	0.0792	0.4154	1.1863	2.0445
PFPAE	IT	31	32	32	32	32
$(\alpha, \omega) = (0.5, 0.1)$	CPU(s)	0.0206	0.0706	0.3738	1.0801	1.9297

Table 5 Numerical results of Example 1 by computing with the preconditioned GMRES methods when $(\sigma_1, \sigma_2) = (10, 1)$

Method		Grid				
		16×16	32×32	48×48	64×64	75×75
PGSOR	α_*	0.99	0.99	0.99	0.99	0.99
	ω_*	6	6	6	6	6
	IT	3	3	3	3	3
	CPU(s)	0.0829	0.2125	0.5479	1.2218	2.1841
PFPAE	α_*	0.99	0.99	0.99	0.99	0.99
	ω_*	6	6	6	6	6
	IT	3	3	3	3	3
	CPU(s)	0.0608	0.0729	0.1228	0.2249	0.3756
PGSOR	IT	10	10	10	10	10
$(\alpha, \omega) = (0.1, 0.1)$	CPU(s)	0.1106	0.3595	0.9290	2.0596	3.3027
PFPAE	IT	4	5	5	5	5
$(\alpha, \omega) = (0.1, 0.1)$	CPU(s)	0.0629	0.0761	0.1274	0.2390	0.3966
PGSOR	IT	5	5	5	5	5
$(\alpha, \omega) = (0.1, 10)$	CPU(s)	0.0870	0.2345	0.6005	1.3602	2.3272
PFPAE	IT	3	3	3	3	3
$(\alpha, \omega) = (0.1, 10)$	CPU(s)	0.0608	0.0731	0.1232	0.2254	0.3767

Table 6 Numerical results of Example 2 by computing with the preconditioned GMRES methods when $(\varpi, \beta) = (0.5, 2)$

Method		Grid				
		16×16	32×32	48×48	64×64	75×75
PGSOR	α_*	0.99	0.99	0.99	0.99	0.99
	ω_*	0.5	0.5	0.5	0.5	0.5
	IT	3	3	3	3	3
	CPU(s)	0.0698	0.1558	0.3940	0.9352	1.6579
PFPAE	α_*	0.99	0.99	0.99	0.99	0.99
	ω_*	0.5	0.5	0.5	0.5	0.5
	IT	3	3	3	3	3
	CPU(s)	0.0616	0.0737	0.1241	0.2273	0.3781
PGSOR	IT	6	6	6	6	6
$(\alpha, \omega) = (0.1, 0.1)$	CPU(s)	0.0868	0.2764	0.7135	1.6107	2.6469
PFPAE	IT	3	3	3	3	3
$(\alpha, \omega) = (0.1, 0.1)$	CPU(s)	0.0621	0.0748	0.1259	0.2311	0.3817
PGSOR	IT	8	8	8	8	8
$(\alpha, \omega) = (0.1, 10)$	CPU(s)	0.1004	0.3167	0.8243	1.8235	3.1785
PFPAE	IT	4	4	4	4	4
$(\alpha, \omega) = (0.1, 10)$	CPU(s)	0.0629	0.0755	0.1270	0.2335	0.3856

5 Conclusions

In this work, we propose a new single-step iteration method, which can be taken as a fixed-point iteration adding the asymptotical error (FPAE), and then develop an efficient parameterized FPAE (PFPAE) iteration method for solving complex symmetric linear systems. Theoretical analysis confirms that, under suitable conditions on the parameters, the iterative sequences converge to the unique solution of the linear system for any initial guess. Furthermore, the quasi-optimal values of the iteration parameters for the FPAE and the PFPAE iteration methods are also derived by detailed discussions. Numerical results show that the FPAE iteration method is a little superior to the SHSS iteration method. In addition, the PFPAE iteration method outperforms several commonly used iteration methods from the point of view of iterations and CPU times. In particular, the PFPAE iteration method is the most efficient by comparing with the PSHSS and the PGSOR iteration methods in the case that the experimental optimal parameters are not used.

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