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A modified generalized shift-splitting preconditioner for nonsymmetric saddle point problems

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Abstract For the nonsymmetric saddle point problems with nonsymmetric positive definite (1,1) parts, the modified generalized shift-splitting (MGSS) preconditioner as well as the MGSS iteration method is derived in this paper, which generalize the modified shift-splitting (MSS) preconditioner and the MSS iteration method newly developed by Huang and Su (J. Comput. Appl. Math. **317**:535–546, 2017), respectively. The convergent and semi-convergent analyses of the MGSS iteration method are presented, and we prove that this method is unconditionally convergent and semi-convergent. Meanwhile, some spectral properties of the preconditioned matrix are carefully analyzed. Numerical results demonstrate the robustness and effectiveness of the MGSS preconditioner and the MGSS iteration method and also illustrate that the MGSS iteration method outperforms the generalized shift-splitting (GSS) and the generalized modified shift-splitting (GMSS) iteration methods, and the MGSS preconditioner is superior to the shift-splitting (SS), GSS, modified SS (M-SS), GMSS

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and MSS preconditioners for the generalized minimal residual (GMRES) method for solving the nonsymmetric saddle point problems.

Keywords Nonsymmetric saddle point problem · Modified generalized shift-splitting · Convergence · Semi-convergence · Spectral properties

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1 Introduction

In a wide variety of scientific and engineering applications, such as mixed finite element approximation of elliptic partial differential equations, the image reconstruction and registration, computational fluid dynamics, weighted least-squares problems, networks computer graphics, and constrained optimization [2, 16, 27], we need to solve the following nonsymmetric saddle point problems of the form

$$\mathcal{A}u = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$ is nonsymmetric positive definite, $B \in \mathbb{R}^{m\times n}$ is a rectangular matrix, $f \in \mathbb{R}^m$ and $g \in \mathbb{R}^n$ are given vectors, with $n \le m$. Here, B^T denotes the transpose of B. The system of linear (1) is also termed as a Karush-Kuhn-Tucker (KKT) system, or an augmented system [25, 29]. For a wider class of saddle point problems, the readers can refer to [13].

Since the matrices A and B in $\mathcal{A}$ are large and sparse in general, the solution of (1) is suitable for being solved by the iterative method. In the case that the matrix B is of full column rank, a large amount of effective iterative methods have been proposed to solve the saddle point problems in the literature, for example, the successive overrelaxation (SOR)-like methods [10, 30, 31, 40], the Uzawa-type methods [10, 11, 15, 23, 26, 38], the Hermitian and skew-Hermitian splitting (HSS) methods [7] and its variants [5, 6, 8, 9], and the Krylov subspace methods [28, 41] with high-quality preconditioners such as the structured preconditioners [2], the shift-splitting (SS) preconditioner [17], and its variants [18, 20, 33, 34, 47]. For more details, we refer the readers to [13] for a comprehensive survey of existing approaches for solving the saddle point problems.

If B in (1) is rank deficient, then the coefficient matrix $\mathcal{A}$ in (1) is singular, and we call (1) the singular saddle point problem. Some iteration methods and preconditioning techniques for solving singular saddle point problems have been proposed in the recent literature, see, e.g., [35, 36, 43, 44]. Zheng et al. [45] proposed some sufficient conditions for the semi-convergence of the generalized SOR (GSOR) method and determined the optimal iteration parameters. Bai [3] derived some necessary and sufficient conditions to assure the semi-convergence of the HSS method. Chen et al. [21] and Cao et al. [19] investigated the generalized shift-splitting (GSS) iteration method for singular saddle point problems. Very recently, Dou et al. [24] introduced the modified parameterized inexact Uzawa (MPIU) for singular saddle point problems, and Zheng and Lu [46] proved the semi-convergence of the upper and lower triangular (ULT) splitting iterative method for singular saddle point problems.

In [12], Bai et al. proposed an efficient shift-splitting (SS) preconditioner to accelerate the convergence rates of the Krylov subspace methods for a class of non-Hermitian positive definite linear systems. Recently, Cao et al. [17] generalized the idea in [12] and presented the SS preconditioner of the form

$$\mathcal{P}_{SS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B \\ -B^T & \alpha I \end{pmatrix}$$

for the saddle point problem (1), where α is a positive constant and I is the identity matrix with appropriate dimension. The authors also proved that the corresponding SS iteration method is unconditionally convergent.

After that, on the basis of the SS preconditioner [17], Chen and Ma [20] and Cao et al. [18] replaced the parameter α in (2,2)-block of the SS preconditioner by another parameter β and proposed the generalized SS (GSS) preconditioner of the form:

$$\mathcal{P}_{GSS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B \\ -B^T & \beta I \end{pmatrix},$$

where $\alpha \geq 0$, $\beta > 0$, and I is the identity matrix with appropriate dimension. It is easy to see that $\mathcal{P}_{SS}$ is a special case of $\mathcal{P}_{GSS}$ when $\alpha = \beta$. It is shown in [20, 21] that the GSS preconditioner is more efficient than the SS preconditioner.

Very recently, based on the well-known Hermitian and skew-Hermitian splitting (HSS) of the matrix A: $A = H + S$, where $H = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$, and inspired by the ideas in [12, 17], Zhou et al. in [47] proposed the modified shift-splitting (M-SS) preconditioner for nonsymmetric saddle point problem (1), and investigated the convergence properties of the M-SS iteration method.

In the sequel, by replacing the parameter α in (2,2)-block of the M-SS preconditioner by another parameter β, Huang et al. [33] established the generalized M-SS (GMSS) preconditioner. They proved that, under proper conditions, the corresponding GMSS iteration method is convergent and semi-convergent, respectively, for the nonsingular and singular saddle point problems. Numerical results showed that the GMSS iteration method and the GMSS preconditioner outperform the M-SS iteration method and the M-SS preconditioner, respectively.

In order to increase the convergence rate of the GSS iteration method for the nonsingular saddle point problems with symmetric positive definite (1,1) parts, Huang and Su [34] newly developed the modified shift-splitting (MSS) preconditioner of the form:

$$\mathcal{P}_{MSS} = \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \alpha I \end{pmatrix}$$

with $\alpha > 0$ being a constant and I being the identity matrix with appropriate dimension, which derived from the following modified shift-splitting of the saddle point matrix $\mathcal{A}$:

$$\mathcal{A} = \mathcal{P}_{MSS} - \mathcal{Q}_{MSS} = \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \alpha I \end{pmatrix} - \begin{pmatrix} \alpha I + A & B \\ -B^T & \alpha I \end{pmatrix}.$$

The authors in [34] theoretically verified the unconditional convergence of the corresponding MSS iteration method and estimated the bounds of the eigenvalues of the iteration matrix of the MSS iteration method. Numerical experiments illustrated that the MSS preconditioner outperforms the SS and the GSS preconditioners for the nonsingular saddle point problems with symmetric positive definite (1,1) parts.

To further improve the efficiency of the GSS and the GMSS preconditioned GMRES methods for the saddle point problems with nonsymmetric positive definite (1,1) parts, a new preconditioner which is referred to as the modified generalized shift-splitting (MGSS) preconditioner is developed for the nonsymmetric saddle point problems in this paper. Theoretical analysis also shows that the corresponding splitting iteration method is convergent and semi-convergent unconditionally. Besides, we investigate the spectral properties of the corresponding preconditioned matrix and show that it has clustered eigenvalue distribution by choosing proper parameters. Numerical experiments are carried out to validate the effectiveness of the MGSS iteration method and the MGSS preconditioned GMRES method for solving the nonsymmetric saddle point problems.

The framework of this paper is organized as follows. Section 2 introduces the MGSS iteration method and analyzes the implementation aspects of the MGSS preconditioner induced by the MGSS iteration method. The unconditionally convergent and semi-convergent properties of the MGSS iteration method will be proved in Sections 3 and 4, respectively. The spectral properties of the MGSS preconditioned matrix will be investigated in detail in Section 5. We examine the feasibility and effectiveness of the MGSS iteration method and the MGSS preconditioned GMRES method for solving the nonsymmetric nonsingular and singular saddle point problems by numerical experiments in Section 6. Finally in Section 7, some conclusions will be given to end this work.

Throughout this paper, $\lambda_{\min}(A)$ and $\rho(A)$ represent the minimum eigenvalue and the spectral radius of the matrix A, respectively. $(.)^*$ denotes the conjugate transpose of either a vector or a matrix.

2 The modified generalized shift-splitting preconditioner and its implementation

In this section, motivated by the ideas of [18, 20, 34], we develop a new splitting called the modified generalized shift-splitting (MGSS) of the nonsymmetric saddle point matrix $\mathcal{A}$ by combining the generalized shift-splitting and the modified shift-splitting of the saddle point matrix $\mathcal{A}$ as follows:

$$\mathcal{A} = \mathcal{P}_{MGSS} - \mathcal{Q}_{MGSS} = \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix} - \begin{pmatrix} \alpha I + A & B \\ -B^T & \beta I \end{pmatrix}, \tag{2}$$

where $\alpha \geq 0$ and $\beta > 0$ are the two constants and I is the unit matrix with appropriate dimension. Then, the modified generalized shift-splitting iteration method based on the splitting (2) can be derived as follows:

The modified generalized shift-splitting (MGSS) iteration method Let $\alpha \geq 0$ and $\beta > 0$ be two given constants. Given an initial guess $(x^{(0)^T}, y^{(0)^T})^T$. For $k = 0, 1, 2, \cdots$, until $(x^{(k)^T}, y^{(k)^T})^T$ converges, compute

$$\begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix} \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} \alpha I + A & B \\ -B^T & \beta I \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} f \\ -g \end{pmatrix},$$

which can be rewritten as the following fixed point form

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \mathcal{T}(\alpha, \beta) \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix}^{-1} \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{3}$$

where

$$\mathcal{T}(\alpha, \beta) = \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I + A & B \\ -B^T & \beta I \end{pmatrix}$$

is the iteration matrix of the MGSS iteration method.

As a matter of fact, any matrix splitting not only can automatically lead to a splitting iteration method, but also can naturally induce a splitting preconditioner for the Krylov subspace methods. The splitting preconditioner corresponding to the MGSS iteration (2) is given by

$$\mathcal{P}_{MGSS} = \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix}, \tag{4}$$

which is called the MGSS preconditioner for the nonsymmetric saddle point matrix $\mathcal{A}$.

At each step of the MGSS iteration (3) or applying the MGSS preconditioner $\mathcal{P}_{MGSS}$ within a Krylov subspace method, a linear system with $\mathcal{P}_{MGSS}$ as the coefficient matrix needs to be solved. That is to say, we need to solve a linear system of the form

$$\begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix} z = r,$$

where $z = (z_1^T, z_2^T)^T$ and $r = (r_1^T, r_2^T)^T$ with $z_1, r_1 \in \mathbb{R}^m$ and $z_2, r_2 \in \mathbb{R}^n$. It is not difficult to check that

$$\mathcal{P}_{MGSS} = \begin{pmatrix} I & \frac{2}{\beta} B \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha I + 2A + \frac{4}{\beta} BB^T & 0 \\ 0 & \beta I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\frac{2}{\beta} B^T & I \end{pmatrix}. \tag{5}$$

It follows from the decomposition of $\mathcal{P}_{MGSS}$ in (5) that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ \frac{2}{\beta} B^T & I \end{pmatrix} \begin{pmatrix} \alpha I + 2A + \frac{4}{\beta} BB^T & 0 \\ 0 & \beta I \end{pmatrix}^{-1} \begin{pmatrix} I & -\frac{2}{\beta} B \\ 0 & I \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{6}$$

Therefore, we can derive the following algorithmic version of the MGSS iteration method.

Algorithm 2.1 For a given vector $r = (r_1^T, r_2^T)^T$, the vector $z = (z_1^T, z_2^T)^T$ can be computed by (6) according to the following steps:

(1) compute $t_1 = r_1 - \frac{2}{\beta} B r_2$;
(2) solve $(\alpha I + 2A + \frac{4}{\beta} B B^T) z_1 = t_1$;
(3) compute $z_2 = \frac{1}{\beta}(2B^T z_1 + r_2)$.

From Algorithm 2.1, it is known that at each iteration, it is required to solve a linear system with the coefficient matrix $\alpha I + 2A + \frac{4}{\beta} B B^T$. Since the matrix $\alpha I + 2A + \frac{4}{\beta} B B^T$ is positive definite for all $\alpha \geq 0$ and $\beta > 0$, in inexact manner, we can employ the GMRES method to solve the sub-linear systems with the coefficient matrix $\alpha I + 2A + \frac{4}{\beta} B B^T$ by a prescribed accuracy. In addition, they also can be solved exactly by the LU factorization in combination with AMD or column AMD reordering [17]; however, using the direct methods may be time consuming, so what we want to pose here is that we always use the GMRES method to solve this problem in our paper.

3 Convergence of the MGSS iteration method for nonsingular saddle point problems

The main purpose of this section is to study the convergent properties of the MGSS iteration method by analyzing the spectral properties of the iteration matrix. To this end, we start with some lemmas which will be useful in our proofs.

Lemma 3.1 [11] *Both roots of the complex quadratic equation* $x^2 - \phi x + \psi = 0$ *are less than one in modulus if and only if* $|\phi - \bar{\phi}\psi| + |\psi|^2 < 1$, *where* $\bar{\phi}$ *denotes the conjugate complex of* ϕ.

Lemma 3.2 *Let* $A \in \mathbb{R}^{m\times m}$ *be a positive definite matrix,* $B \in \mathbb{R}^{m\times n}$ *be of full column rank, and* $\alpha \geq 0$ *and* $\beta > 0$ *be two given constants. If* λ *is an eigenvalue of the iteration matrix* $\mathcal{T}(\alpha, \beta)$, *then* $\lambda \neq \pm 1$.

Proof Let λ be an eigenvalue of the iteration matrix $\mathcal{T}(\alpha, \beta)$ of the MGSS iteration method, and $(u^*, v^*)^* \in \mathbb{C}^{m+n}$ be the corresponding eigenvector. Then, it holds that

$$\begin{pmatrix} \alpha I + A & B \\ -B^T & \beta I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

which can be equivalently written as

$$\begin{cases} (\alpha I + A)u + Bv = \lambda(\alpha I + 2A)u + 2\lambda Bv, \\ -B^T u + \beta v = -2\lambda B^T u + \lambda \beta v. \end{cases} \tag{7}$$

If $\lambda = 1$, then from (7), it has $Au + Bv = 0$ and $B^T u = 0$, which lead to $u = -A^{-1}Bv$ and $B^T A^{-1} B v = 0$. Thus, we get $Bv = 0$ by the positive definiteness of A^{-1}, and therefore, $v = 0$ and $u = -A^{-1}Bv = 0$, which contradicts with the assumption that $(u^*, v^*)^*$ is an eigenvector. In addition, if $\lambda = -1$, then it follows from the second equation of (7) that $v = \frac{3B^T u}{2\beta}$. Substituting this relation into the first equation of (7) gives $\bar{A}u = \left(2\alpha I + 3A + \frac{9BB^T}{2\beta}\right)u = 0$, then $u = 0$ is due to the fact that $\bar{A}$ is nonsingular, which yields that $v = \frac{3B^T u}{2\beta} = 0$, a contradiction. □

Lemma 3.3 *Assume that the conditions in Lemma 3.2 are satisfied. Let λ be an eigenvalue of the iteration matrix $\mathcal{T}(\alpha, \beta)$ of the MGSS iteration method and $\mathbf{u} = (u^*, v^*)^* \in \mathbb{C}^{m+n}$, with $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$, be the corresponding eigenvector. Then $u \neq 0$. Moreover, if $v = 0$, then $|\lambda| < 1$.*

Proof If $u = 0$, then from the second equation of (7), we have $(\lambda - 1)\beta v = 0$. Inasmuch as $\lambda \neq 1$ and $\beta > 0$, we derive $v = 0$. This contradicts to the assumption that $\mathbf{u} = (u^*, v^*)^*$ is an eigenvector. Furthermore, if $v = 0$, then it follows from the first equation of (7) that

$$(\alpha I + A)u = \lambda(\alpha I + 2A)u. \tag{8}$$

Since $u \neq 0$, the definition $\frac{u^*}{u^* u}$ does make sense. Premultiplying (8) with $\frac{u^*}{u^* u}$ gives

$$\lambda = \frac{(\alpha + a) + ib}{(\alpha + 2a) + 2ib}, \tag{9}$$

where $a + ib = \frac{u^* A u}{u^* u}$. Since A is positive definite, $a > 0$. It follows from (9) that

$$|\lambda| = \sqrt{\frac{(\alpha + a)^2 + b^2}{(\alpha + 2a)^2 + 4b^2}} < 1.$$

Thus, the proof of Lemma 3.3 is completed. □

Theorem 3.1 *Assume that the conditions in Lemma 3.2 are satisfied. Let λ be an eigenvalue of the iteration matrix $\mathcal{T}(\alpha, \beta)$ of the MGSS iteration method and $\mathbf{u} = (u^*, v^*)^* \in \mathbb{C}^{m+n}$, with $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$, be the corresponding eigenvector. Then the MGSS iteration method converges to the exact solution of the saddle point problem (1) for all $\alpha \geq 0$ and $\beta > 0$.*

Proof By making use of Lemma 3.2, we have $\lambda \neq 1$, then from the second equation of (7), it has

$$v = \frac{(2\lambda - 1)B^T u}{(\lambda - 1)\beta},$$

substituting it into the first equation of (7) results in

$$\lambda^2(\alpha\beta I + 2\beta A + 4BB^T)u - \lambda(2\alpha\beta I + 3\beta A + 4BB^T)u + (\alpha\beta I + \beta A + BB^T)u = 0. \tag{10}$$

By making use of Lemma 3.3, it holds that $u \neq 0$. Denote

$$a + ib = \frac{u^* A u}{u^* u}, \quad c = \frac{u^* BB^T u}{u^* u} \geq 0.$$

By multiplying $\frac{u^*}{u^*u}$ on (10) from the left, we have

$$\lambda^2(\alpha\beta+2\beta a+4c+2\beta bi)-\lambda(2\alpha\beta+3\beta a+4c+3\beta bi)+(\alpha\beta+\beta a+c+\beta bi)=0. \quad (11)$$

Having in mind that A is positive definite, we get $a > 0$ and $c \geq 0$, which lead to $\alpha\beta + 2\beta a + 4c + 2\beta bi \neq 0$ by $\alpha \geq 0$ and $\beta > 0$. Hence, (11) can be rewritten as $\lambda^2 - \phi\lambda + \psi = 0$, where

$$\phi = \frac{2\alpha\beta + 3\beta a + 4c + 3\beta bi}{\alpha\beta + 2\beta a + 4c + 2\beta bi}, \ \psi = \frac{\alpha\beta + \beta a + c + \beta bi}{\alpha\beta + 2\beta a + 4c + 2\beta bi}.$$

If $c = 0$, then (11) can be expressed as

$$\lambda^2 - \lambda\frac{2\alpha + 3a + 3bi}{\alpha + 2a + 2bi} + \frac{\alpha + a + bi}{\alpha + 2a + 2bi} = 0. \quad (12)$$

Solving the two roots of (12), we obtain

$$\lambda = 1 \text{ or } \lambda = \frac{\alpha + a + bi}{\alpha + 2a + 2bi}.$$

Lemma 3.2 implies that $\lambda \neq 1$, then

$$|\lambda| = \left|\frac{\alpha + a + bi}{\alpha + 2a + 2bi}\right| = \sqrt{\frac{(\alpha + a)^2 + b^2}{(\alpha + 2a)^2 + 4b^2}} < 1.$$

Now, we turn to prove $|\lambda| < 1$ under the condition $c > 0$. According to Lemma 3.1, we know that $|\lambda| < 1$ if and only if $|\phi - \bar{\phi}\psi| + |\psi|^2 < 1$. After some manipulations, we derive

$$\phi - \bar{\phi}\psi = \frac{2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2 + 3\beta bci}{(\alpha\beta + 2\beta a + 4c)^2 + 4\beta^2 b^2}$$

and

$$1 - |\psi|^2 = \frac{2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 14\beta ac + 15c^2 + 3\beta^2 b^2}{(\alpha\beta + 2\beta a + 4c)^2 + 4\beta^2 b^2}.$$

Hence, $|\phi - \bar{\phi}\psi| + |\psi|^2 < 1$ is valid if and only if

$$\begin{aligned} & |2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2 + 3\beta bci| \\ = & \sqrt{(2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2)^2 + 9\beta^2 b^2 c^2} \\ < & \ 2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 14\beta ac + 15c^2 + 3\beta^2 b^2, \end{aligned}$$

which is equivalent to

$$\begin{aligned} (2\alpha\beta^2 a \ + \ 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2)^2 + 9\beta^2 b^2 c^2 < (2\alpha\beta^2 a \\ + \ 6\alpha\beta c + 3\beta^2 a^2 + 14\beta ac + 15c^2 + 3\beta^2 b^2)^2. \end{aligned} \quad (13)$$

In terms of $a > 0$, $c > 0$, $b^2 \geq 0$, $\alpha \geq 0$ and $\beta > 0$, it holds that

$$\begin{aligned}
& (2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 14\beta ac + 15c^2 + 3\beta^2 b^2)^2 \\
= & [(2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2 + (\beta ac + 3c^2)]^2 \\
= & (2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2)^2 + (\beta ac + 3c^2)^2 \\
& +2(2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2)(\beta ac + 3c^2) \\
> & (2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2)^2 \\
& +(2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2)(\beta ac + 3c^2) \\
> & (2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2)^2 + 3\beta^2 b^2(\beta ac + 3c^2) \\
\geq & (2\alpha\beta^2 a + 6\alpha\beta c + 3\beta^2 a^2 + 13\beta ac + 12c^2 + 3\beta^2 b^2)^2 + 9\beta^2 b^2 c^2,
\end{aligned}$$

which implies that (13) holds true, i.e., $|\phi - \bar{\phi}\psi| + |\psi|^2 < 1$, and therefore, $|\lambda| < 1$. Hence, the MGSS iteration method is convergent for any $\alpha \geq 0$ and $\beta > 0$. This proof is completed. □

4 Semi-convergence of the MGSS iteration method for singular saddle point problems

When the saddle point matrix $\mathcal{A}$ is nonsingular, the MGSS iteration scheme (3) converges to the exact solution of (1) for any initial vector if and only if $\rho(\mathcal{T}(\alpha, \beta)) < 1$, whereas for the singular matrix $\mathcal{A}$, we have $\rho(\mathcal{T}(\alpha, \beta)) \geq 1$. In this section, we assume that the sub-matrix B in (1) is rank deficient and discuss the semi-convergence of the MGSS iteration method for solving the singular saddle point problems.

To analyze the semi-convergent properties of the MGSS iteration method, we first present the following lemma which describes the semi-convergent property about the iteration scheme (3) when $\mathcal{A}$ is singular.

Lemma 4.1 [14] *The iteration scheme* (3) *is semi-convergent if and only if the following two conditions are satisfied:*
(i) $index(I - T) = 1$, *or equivalently,* $rank((I - T)^2) = rank(I - T)$, *where* $T = I - GM$ *is the iteration matrix;*
(ii) the pseudo-spectral radius of T *is less than 1, i.e.,*

$$\gamma(T) = \max\{|\lambda| : \lambda \in \sigma(T), \lambda \neq 1\} < 1,$$

where $\sigma(T)$ *is the spectral set of the matrix* T. *Here, we denote the null space, the index and the rank of* A *by* $null(A)$, $index(A)$ *and* $rank(A)$, *respectively.*

Lemma 4.1 describes the semi-convergence property about the iteration scheme (3) when $\mathcal{A}$ is singular. Therefore, to get the semi-convergent properties of the MGSS iteration method, only the two conditions in Lemma 4.1 need to be verified. We consider these two conditions in Lemmas 4.2 and 4.3, respectively.

Lemma 4.2 *Let A be nonsymmetric positive definite, B be rank deficient and $\alpha \geq 0$, $\beta > 0$ be given constants. Then, the iteration matrix $\mathcal{T}(\alpha, \beta)$ of the MGSS iteration method satisfies $index(I - \mathcal{T}(\alpha, \beta)) = 1$, or equivalently,*

$$rank(I - \mathcal{T}(\alpha, \beta)) = rank((I - \mathcal{T}(\alpha, \beta))^2). \tag{14}$$

Proof Inasmuch as $\mathcal{T}(\alpha, \beta) = \mathcal{P}_{MGSS}^{-1}\mathcal{Q}_{MGSS} = I - \mathcal{P}_{MGSS}^{-1}\mathcal{A}$, (14) holds if

$$null\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right) = null\left(\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right)^2\right).$$

It is obvious that $null\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right) \subseteq null\left(\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right)^2\right)$. Thus, we only need to prove

$$null\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right) \supseteq null\left(\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right)^2\right).$$

Let $x = (x_1^*, x_2^*)^* \in \mathbb{C}^{m+n} \in null\left(\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right)^2\right)$, then it has $\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right)^2 x = 0$. Denote by $y = \mathcal{P}_{MGSS}^{-1}\mathcal{A}x$. Careful calculation gives

$$\begin{aligned} y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ \frac{2}{\beta}B^T & I \end{pmatrix} \begin{pmatrix} \alpha I + 2A + \frac{4}{\beta}BB^T & 0 \\ 0 & \beta I \end{pmatrix}^{-1} \begin{pmatrix} I & -\frac{2}{\beta}B \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} \left(\alpha I + 2A + \frac{4}{\beta}BB^T\right)^{-1}\left(Ax_1 + Bx_2 + \frac{2}{\beta}BB^Tx_1\right) \\ \frac{2}{\beta}B^T\left(\alpha I + 2A + \frac{4}{\beta}BB^T\right)^{-1}\left(Ax_1 + Bx_2 + \frac{2}{\beta}BB^Tx_1\right) - \frac{1}{\beta}B^Tx_1 \end{pmatrix}, \end{aligned}$$

i.e.,

$$\begin{cases} y_1 = \left(\alpha I + 2A + \dfrac{4}{\beta}BB^T\right)^{-1}\left(Ax_1 + Bx_2 + \dfrac{2}{\beta}BB^Tx_1\right), \\ y_2 = \dfrac{2}{\beta}B^T\left(\alpha I + 2A + \dfrac{4}{\beta}BB^T\right)^{-1}\left(Ax_1 + Bx_2 + \dfrac{2}{\beta}BB^Tx_1\right) - \dfrac{1}{\beta}B^Tx_1. \end{cases} \tag{15}$$

Since $\mathcal{P}_{MGSS}^{-1}\mathcal{A}y = \left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right)^2 x = 0$, it holds that $\mathcal{A}y = 0$, i.e.,

$$Ay_1 + By_2 = 0, \ -B^Ty_1 = 0. \tag{16}$$

Since A is positive definite, from the first equation of (16) we can easily get $y_1 = -A^{-1}By_2$. Then, substituting y_1 into the second equation of (16), we obtain $B^TA^{-1}By_2 = 0$, which leads to $By_2 = 0$. Taking $By_2 = 0$ into $y_1 = -A^{-1}By_2$, we obtain $y_1 = 0$. Hence, the first equation of (15) becomes

$$y_1 = \left(\alpha I + 2A + \frac{4}{\beta}BB^T\right)^{-1}\left(Ax_1 + Bx_2 + \frac{2}{\beta}BB^Tx_1\right) = 0.$$

Substituting $y_1 = 0$ into y_2 yields $y_2 = -\frac{1}{\beta} B^T x_1$. Since $By_2 = 0$, $-\frac{1}{\beta} BB^T x_1 = 0$, it has $x_1^* BB^T x_1 = 0$. This results in $B^T x_1 = 0$, then we get $y_2 = -\frac{1}{\beta} B^T x_1 = 0$. Hence, we obtain $y = 0$, which means that

$$null\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right) \supseteq null\left(\left(\mathcal{P}_{MGSS}^{-1}\mathcal{A}\right)^2\right). \tag{17}$$

(17) implies the conclusion in Lemma 4.2. □

In the sequel, we show that the iteration scheme (3) satisfies the condition (ii) in Lemma 4.1. Without loss of generality, we assume that $rank(B) = r < n$. Let $B = U(B_r, 0)V^*$ be the singular decomposition of the matrix B, where

$$B_r = \begin{pmatrix} \Sigma_r \\ 0 \end{pmatrix} \in \mathbb{C}^{m\times r}, \quad \Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r) \in \mathbb{C}^{r\times r}$$

with $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$ being two unitary matrices and σ_i $(i = 1, 2, \cdots, r)$ being a singular value of B.

We introduce a block diagonal matrix

$$P = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

which is an $(m + n) \times (m + n)$ unitary matrix, and the iteration matrix $\mathcal{T}(\alpha, \beta)$ is unitarily similar to the matrix $\hat{\mathcal{T}}(\alpha, \beta) = P^*\mathcal{T}(\alpha, \beta)P$. Hence, the matrix $\mathcal{T}(\alpha, \beta)$ has the same spectrum with the matrix $\hat{\mathcal{T}}(\alpha, \beta)$. Thus, we only need to analyze the pseudo-spectral radius of the matrix $\hat{\mathcal{T}}(\alpha, \beta)$ now.

Denoting $\hat{A} = U^*AU$, then it holds that

$$\begin{aligned}
\hat{\mathcal{T}}(\alpha, \beta) &= P^* \begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I + A & B \\ -B^T & \beta I \end{pmatrix} P \\
&= \begin{pmatrix} \alpha I + 2U^*AU & 2U^*BV \\ -2V^*B^TU & \beta I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I + U^*AU & U^*BV \\ -V^*B^TU & \beta I \end{pmatrix} \\
&= \begin{pmatrix} \alpha I + 2\hat{A} & 2B_r & 0 \\ -2B_r^T & \beta I & 0 \\ 0 & 0 & \beta I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I + \hat{A} & B_r & 0 \\ -B_r^T & \beta I & 0 \\ 0 & 0 & \beta I \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \alpha I + 2\hat{A} & 2B_r \\ -2B_r^T & \beta I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I + \hat{A} & B_r \\ -B_r & \beta I \end{pmatrix} & 0 \\ 0 & I_{n-r} \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\mathcal{T}}(\alpha, \beta) & 0 \\ 0 & I_{n-r} \end{pmatrix}.
\end{aligned} \tag{18}$$

Then, from (18), $\gamma(\hat{\mathcal{T}}(\alpha, \beta)) < 1$ holds if and only if $\rho(\tilde{\mathcal{T}}(\alpha, \beta)) < 1$.

Note that $\tilde{\mathcal{T}}(\alpha, \beta)$ can be viewed as the iteration matrix of the MGSS iteration method applied to the nonsingular saddle point problem

$$\begin{pmatrix} \hat{A} & B_r \\ -B_r^T & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \hat{f} \\ -\hat{g} \end{pmatrix},$$

where $\hat{A} = U^*AU$ and $\hat{y}, \hat{g} \in \mathbb{R}^r$.

$\rho(\tilde{\mathcal{T}}(\alpha, \beta)) < 1$ implies $\gamma(\mathcal{T}(\alpha, \beta)) = \gamma(\hat{\mathcal{T}}(\alpha, \beta)) < 1$. By making use of the proof of Theorem 3.1, we derive the following result.

Lemma 4.3 *Let A be nonsymmetric positive definite, B be rank deficient and $\alpha \geq 0, \beta > 0$ be two given constants. Then, the pseudo-spectral radius of the matrix $\mathcal{T}(\alpha, \beta)$ is less than 1, i.e., $\gamma(\mathcal{T}(\alpha, \beta)) < 1$ for all $\alpha \geq 0$ and $\beta > 0$.*

It follows from Lemmas 4.2 and 4.3 that two conditions in Lemma 4.1 are satisfied naturally. Thus, the following theorem readily follows from Lemmas 4.1–4.3.

Theorem 4.1 *Let A be nonsymmetric positive definite, B be rank deficient and $\alpha \geq 0, \beta > 0$ be two given constants. Then the MGSS iteration method is semi-convergent for solving the singular saddle point problem (1) for all $\alpha \geq 0$ and $\beta > 0$.*

5 Spectral analysis of the MGSS preconditioned matrix

In this section, we will analyze the spectral properties of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$, since the convergence behavior relates closely to the eigenvalue distribution of the preconditioned matrix. The following theorem is given to describe the eigenvalue distribution of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$.

Theorem 5.1 *Let the MGSS preconditioner be defined as in (4) and $(\lambda, (u^*, v^*)^*)$ be an eigenpair of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$. Then if B is of full column rank and $B^T u = 0$, then*

$$\frac{\lambda_{\min}(H)(\alpha + 2\lambda_{\min}(H))}{(\alpha + 2\rho(H))^2 + 4\rho(S)^2} \leq Re(\lambda) \leq \frac{\rho(H)(\alpha + 2\rho(H)) + 2\rho(S)^2}{(\alpha + 2\lambda_{\min}(H))^2},$$
$$|Im(\lambda)| \leq \frac{\alpha\rho(S)}{(\alpha + 2\lambda_{\min}(H))^2}, \tag{19}$$

where $Re(\lambda)$ and $Im(\lambda)$ denote the real part and the imaginary part of λ, respectively. If B is rank deficient and $u = 0$, then $\lambda = 0$. Besides, if B is rank deficient and $B^T u = 0$, then $\lambda = 0$ or λ satisfies the Inequalities (19). If $B^T u \neq 0$, then the eigenvalues of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ satisfy

$$\lambda_+ = \frac{1}{2} + \frac{(z_1 - \alpha\beta - \beta a_1) + i(z_2 - \beta b_1)}{2(\alpha\beta + 2\beta a_1 + 4c_1 + 2i\beta b_1)},\ \lambda_- = \frac{1}{2} - \frac{(z_1 + \alpha\beta + \beta a_1) + i(z_2 + \beta b_1)}{2(\alpha\beta + 2\beta a_1 + 4c_1 + 2i\beta b_1)}, \tag{20}$$

where

$$\frac{u^* A u}{u^* u} = a_1 + i b_1,\ \frac{u^* B B^T u}{u^* u} = c_1 \tag{21}$$

and z_1, z_2 are real numbers and $z_1 + i z_2$ is one of the square roots of $a_2 + b_2 i$, with

$$a_2 = \beta^2(a_1^2 - b_1^2) - 4\alpha\beta c_1,\ b_2 = 2\beta^2 a_1 b_1$$

and

$$z_1 = \sqrt{\frac{\sqrt{\left[\beta^2\left(a_1^2 - b_1^2\right) - 4\alpha\beta c_1\right]^2 + 4\beta^4 a_1^2 b_1^2} + \beta^2\left(a_1^2 - b_1^2\right) - 4\alpha\beta c_1}{2}},$$

$$\begin{aligned} z_2 &= sign(b_1)\sqrt{\frac{\sqrt{\left[\beta^2\left(a_1^2 - b_1^2\right) - 4\alpha\beta c_1\right]^2 + 4\beta^4 a_1^2 b_1^2} - \beta^2\left(a_1^2 - b_1^2\right) + 4\alpha\beta c_1}{2}} \\ &:= sign(b_1) z_3, \end{aligned} \tag{22}$$

and the second root of $a_2+b_2 i$ is $-(z_1+iz_2)$. The eigenvalues $\lambda_\pm$ satisfy the following inequality:

$$\left|\lambda_\pm - \frac{1}{2}\right|^2 \leq \frac{(\alpha\beta + 2\beta\rho(H))^2 + (\beta\rho(S) + \sqrt{\beta^2\rho(S)^2 + 4\alpha\beta\rho(BB^T)})^2}{4(\alpha\beta + 2\beta\lambda_{\min}(H) + 4\lambda_{\min}(BB^T))^2}. \tag{23}$$

When $\beta \to 0_+$, it holds that

$$\begin{cases} \lambda_+ = \dfrac{1}{2} + \dfrac{(z_1 - \alpha\beta - \beta a_1) + i(z_2 - \beta b_1)}{2(\alpha\beta + 2\beta a_1 + 4c_1 + 2i\beta b_1)} \to \dfrac{1}{2}, \\ \lambda_- = \dfrac{1}{2} - \dfrac{(z_1 + \alpha\beta + \beta a_1) + i(z_2 + \beta b_1)}{2(\alpha\beta + 2\beta a_1 + 4c_1 + 2i\beta b_1)} \to \dfrac{1}{2}, \end{cases}$$

i.e., for $\alpha > 0$, the eigenvalues of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ tend to cluster near the point $(\frac{1}{2}, 0)$ as $\beta \to 0_+$; and when $\alpha \to 0_+$, it has

$$\begin{cases} \lambda_+ = \dfrac{1}{2} + \dfrac{(z_1 - \alpha\beta - \beta a_1) + i(z_2 - \beta b_1)}{2(\alpha\beta + 2\beta a_1 + 4c_1 + 2i\beta b_1)} \to \dfrac{1}{2} + \dfrac{(z_1 - \beta a_1) + i(z_2 - \beta b_1)}{2(2\beta a_1 + 4c_1 + 2i\beta b_1)} = \dfrac{1}{2}, \\ \lambda_- = \dfrac{1}{2} - \dfrac{(z_1 + \alpha\beta + \beta a_1) + i(z_2 + \beta b_1)}{2(\alpha\beta + 2\beta a_1 + 4c_1 + 2i\beta b_1)} \to \dfrac{1}{2} - \dfrac{(z_1 + \beta a_1) + i(z_2 + \beta b_1)}{2(2\beta a_1 + 4c_1 + 2i\beta b_1)} = \dfrac{1}{2} - \dfrac{\beta a_1 + i\beta b_1}{2\beta a_1 + 4c_1 + 2i\beta b_1}. \end{cases}$$

That is, for $\beta > 0$, the eigenvalues of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ tend to cluster near the points $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{\beta a_1 c_1 + 2c_1^2}{\left(\beta a_1^2 + 2c_1\right)^2 + \beta^2 b_1^2}, -\frac{\beta b_1 c_1}{\left(\beta a_1^2 + 2c_1\right)^2 + \beta^2 b_1^2}\right)$ as $\alpha \to 0_+$. In addition, the eigenvalues of $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ tend to cluster near the points $\left(\frac{\alpha_0\beta_0^2 a_1 + 2\beta_0^2(a_1^2+b_1^2) + 12\beta_0 a_1 c_1 + (\alpha_0\beta_0 + 4c_1)(4c_1 + z_1) + 2\beta_0(a_1 z_1 + |b_1 z_2|)}{2\left[(\alpha_0\beta_0 + 2\beta_0 a_1 + 4c_1)^2 + 4\beta_0^2 b_1^2\right]}, \frac{(\alpha_0\beta_0 + 2\beta_0 a_1 + 4c_1)(z_2 + \beta_0 b_1) - 2\beta_0 b_1(\beta_0 a_1 + 4c_1 + z_1)}{2\left[(\alpha_0\beta_0 + 2\beta_0 a_1 + 4c_1)^2 + 4\beta_0^2 b_1^2\right]}\right)$ and $\left(\frac{\alpha_0\beta_0^2 a_1 + 2\beta_0^2(a_1^2+b_1^2) + 12\beta_0 a_1 c_1 + (\alpha_0\beta_0 + 4c_1)(4c_1 - z_1) - 2\beta_0(a_1 z_1 + |b_1 z_2|)}{2\left[(\alpha_0\beta_0 + 2\beta_0 a_1 + 4c_1)^2 + 4\beta_0^2 b_1^2\right]}, \frac{(\alpha_0\beta_0 + 2\beta_0 a_1 + 4c_1)(\beta_0 b_1 - z_2) - 2\beta_0 b_1(\beta_0 a_1 + 4c_1 - z_1)}{2\left[(\alpha_0\beta_0 + 2\beta_0 a_1 + 4c_1)^2 + 4\beta_0^2 b_1^2\right]}\right)$ as $\alpha \to \alpha_0$ and $\beta \to \beta_0$ $(0 \leq \alpha_0 < +\infty, 0 < \beta_0 < +\infty)$.

Proof Let $(\lambda, (u^*, v^*)^*)$ be an eigenpair of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$, we consider the eigenvalue problem $\mathcal{P}_{MGSS}^{-1}\mathcal{A}\eta = \lambda\eta$, where $\eta = (u^*, v^*)^*$, then it holds that

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} \alpha I + 2A & 2B \\ -2B^T & \beta I \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

which can be equivalently rewritten as

$$\begin{cases} Au = \lambda(\alpha I + 2A)u + (2\lambda - 1)Bv, \\ (2\lambda - 1)B^T u = \lambda\beta v. \end{cases} \tag{24}$$

If B has full column rank and $u = 0$, then it follows from the second equation of (24) that $\lambda v = 0$ and therefore $v = 0$, which contradicts with the assumption that $(u^*, v^*)^*$ is an eigenvector. Hence, $u \neq 0$. If B is of full column rank and $B^T u = 0$, then from the second equation of (24), we have $v = 0$ and

$$Au = \lambda(\alpha I + 2A)u. \tag{25}$$

Owing to $u \neq 0$, it holds that the definition $\frac{u^*}{u^* u}$ does make sense. Premultiplying (25) with $\frac{u^*}{u^* u}$ and utilizing the symbols defined as in (21) give

$$\lambda = \frac{a_1 + ib_1}{\alpha + 2a_1 + 2ib_1} = \frac{a_1(\alpha + 2a_1) + 2b_1^2 + i\alpha b_1}{(\alpha + 2a_1)^2 + 4b_1^2}. \tag{26}$$

It is easy to verify that $\lambda \to (\frac{1}{2}, 0)$ as $\alpha \to 0_+$. Besides, (26) implies that

$$Re(\lambda) = \frac{a_1(\alpha + 2a_1) + 2b_1^2}{(\alpha + 2a_1)^2 + 4b_1^2}, \quad Im(\lambda) = \frac{\alpha b_1}{(\alpha + 2a_1)^2 + 4b_1^2}.$$

Since

$$\lambda_{\min}(H) \le a_1 = \frac{1}{2}\left(\frac{u^* A u}{u^* u} + \frac{u^* A^T u}{u^* u}\right) = \frac{u^* H u}{u^* u} \le \rho(H),$$
$$0 \le |b_1| = \frac{1}{2}\left|\frac{1}{i}\left(\frac{u^* A u}{u^* u} - \frac{u^* A^T u}{u^* u}\right)\right| = \left|\frac{u^* i S u}{u^* u}\right| \le \rho(S),$$

it is not difficult to derive (19).

If B is rank deficient and $u = 0$, then from the second equation of (24), we derive $\lambda = 0$. Additionally, if B is rank deficient and $B^T u = 0$, then it holds that $\lambda = 0$ or $v = 0$, $\lambda \neq 0$ by virtue of the second equation of (24). Similar to the derivation of (19), we also deduce (19) as B is rank deficient, $v = 0$ and $\lambda \neq 0$.

Subsequently, we assume that $B^T u \neq 0$. Then, $\lambda \neq 0$ and $u \neq 0$. Otherwise, it follows from the second equation of (24) that $B^T u = 0$, a contradiction. By making use of the second equation of (24), we have $v = \frac{(2\lambda-1)B^T u}{\lambda\beta}$. Then, substituting v into the first equation of (24) gives

$$\lambda^2(\alpha\beta I + 2\beta A + 4BB^T)u - \lambda(4BB^T + \beta A)u + BB^T u = 0. \tag{27}$$

By multiplying $\frac{u^*}{u^* u}$ on (27) from the left and using the symbols defined as in (21), it follows that

$$\lambda^2(\alpha\beta + 2\beta a_1 + 2i\beta b_1 + 4c_1) - \lambda(4c_1 + \beta a_1 + i\beta b_1) + c_1 = 0,$$

which can be equivalently transformed into the following equation

$$\lambda^2 - \lambda\frac{4c_1 + \beta a_1 + i\beta b_1}{\alpha\beta + 2\beta a_1 + 2i\beta b_1 + 4c_1} + \frac{c_1}{\alpha\beta + 2\beta a_1 + 2i\beta b_1 + 4c_1} = 0. \tag{28}$$

By solving (28), we obtain its two roots as follows:

$$\lambda_+ = \frac{1}{2} + \frac{(z_1 - \alpha\beta - \beta a_1) + i(z_2 - \beta b_1)}{2(\alpha\beta + 2\beta a_1 + 4c_1 + 2i\beta b_1)},\ \lambda_- = \frac{1}{2} - \frac{(z_1 + \alpha\beta + \beta a_1) + i(z_2 + \beta b_1)}{2(\alpha\beta + 2\beta a_1 + 4c_1 + 2i\beta b_1)}, \tag{29}$$

where z_1 and z_2 are given by (22). Applying (22) leads to

$$\begin{aligned} z_1 &= \sqrt{\frac{\sqrt{\left[\beta^2\left(a_1^2 - b_1^2\right) - 4\alpha\beta c_1\right]^2 + 4\beta^4 a_1^2 b_1^2} + \beta^2\left(a_1^2 - b_1^2\right) - 4\alpha\beta c_1}{2}} \\ &= \sqrt{\frac{\sqrt{\beta^4\left(a_1^2 + b_1^2\right)^2 - 8\alpha c_1\beta^3\left(a_1^2 - b_1^2\right) + 16\alpha^2\beta^2 c_1^2} + \beta^2\left(a_1^2 - b_1^2\right) - 4\alpha\beta c_1}{2}} \\ &\le \sqrt{\frac{\sqrt{\left[\beta^2\left(a_1^2 + b_1^2\right) + 4\alpha\beta c_1\right]^2} + \beta^2\left(a_1^2 - b_1^2\right) - 4\alpha\beta c_1}{2}} = \beta a_1, \end{aligned} \tag{30}$$

$$\begin{aligned} |z_2| = z_3 &= \sqrt{\frac{\sqrt{\left[\beta^2\left(a_1^2 - b_1^2\right) - 4\alpha\beta c_1\right]^2 + 4\beta^4 a_1^2 b_1^2} - \beta^2\left(a_1^2 - b_1^2\right) + 4\alpha\beta c_1}{2}} \\ &\le \sqrt{\frac{\sqrt{\left[\beta^2\left(a_1^2 + b_1^2\right) + 4\alpha\beta c_1\right]^2} - \beta^2\left(a_1^2 - b_1^2\right) + 4\alpha\beta c_1}{2}} = \sqrt{\beta^2 b_1^2 + 4\alpha\beta c_1}, \end{aligned} \tag{31}$$

which yield that

$$\begin{aligned} \left|\lambda_\pm - \frac{1}{2}\right|^2 &= \frac{(\alpha\beta + \beta a_1 \pm z_1)^2 + (\beta b_1 \pm z_2)^2}{4[(\alpha\beta + 2\beta a_1 + 4c_1)^2 + 4\beta^2 b_1^2]} \\ &\le \frac{(\alpha\beta + 2\beta a_1)^2 + (\beta|b_1| + \sqrt{\beta^2 b_1^2 + 4\alpha\beta c_1})^2}{4[(\alpha\beta + 2\beta a_1 + 4c_1)^2 + 4\beta^2 b_1^2]} := f(a_1, b_1, c_1). \end{aligned} \tag{32}$$

It is evident that an upper bound of $\left|\lambda_\pm - \frac{1}{2}\right|^2$ is $f(a_1, b_1, c_1)$, with a_1, b_1, c_1 being bounded as follows:

$$\lambda_{\min}(H) \le a_1 \le \rho(H),\ 0 \le |b_1| \le \rho(S),\ 0 \le b_1^2 \le \rho(S)^2,\ \lambda_{\min}(BB^T) \le c_1 \le \rho(BB^T),$$

from which one may deduce the following result

$$\left|\lambda_{\pm}-\frac{1}{2}\right|^2 \le f(a_1,b_1,c_1) \le \frac{(\alpha\beta+2\beta\rho(H))^2+(\beta\rho(S)+\sqrt{\beta^2\rho(S)^2+4\alpha\beta\rho(BB^T)})^2}{4(\alpha\beta+2\beta\lambda_{\min}(H)+4\lambda_{\min}(BB^T))^2}.$$

Furthermore, it is not difficult to verify that $z_1, z_2 \to 0$ as $\beta \to 0_+$, and therefore for $\alpha > 0$, $\lambda_+, \lambda_- \to (\frac{1}{2}, 0)$ as $\beta \to 0_+$. Moreover, if $\alpha \to 0_+$, then it follows from (22) that $z_1 \to \beta a_1$ and $z_2 \to \beta b_1$, thus

$$\begin{cases} \lambda_+ \to \dfrac{1}{2} + \dfrac{(z_1-\beta a_1)+i(z_2-\beta b_1)}{2(2\beta a_1+4c_1+2i\beta b_1)} = \dfrac{1}{2}, \\ \lambda_- \to \dfrac{1}{2} - \dfrac{(z_1+\beta a_1)+i(z_2+\beta b_1)}{2(2\beta a_1+4c_1+2i\beta b_1)} = \dfrac{1}{2} - \dfrac{\beta a_1+i\beta b_1}{2\beta a_1+4c_1+2i\beta b_1}, \end{cases}$$

which means that for $\beta > 0$, the eigenvalues of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ tend to cluster near the points $(\frac{1}{2}, 0)$ and $\left(\frac{\beta a_1 c_1+2c_1^2}{(\beta a_1^2+2c_1)^2+\beta^2 b_1^2}, -\frac{\beta b_1 c_1}{(\beta a_1^2+2c_1)^2+\beta^2 b_1^2}\right)$ as $\alpha \to 0_+$. Additionally, it is easily seen that the eigenvalues of $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ tend to cluster near the points $\left(\frac{\alpha_0\beta_0^2 a_1+2\beta_0^2(a_1^2+b_1^2)+12\beta_0 a_1 c_1+(\alpha_0\beta_0+4c_1)(4c_1+z_1)+2\beta_0(a_1 z_1+|b_1 z_2|)}{2[(\alpha_0\beta_0+2\beta_0 a_1+4c_1)^2+4\beta_0^2 b_1^2]}, \frac{(\alpha_0\beta_0+2\beta_0 a_1+4c_1)(z_2+\beta_0 b_1)-2\beta_0 b_1(\beta_0 a_1+4c_1+z_1)}{2[(\alpha_0\beta_0+2\beta_0 a_1+4c_1)^2+4\beta_0^2 b_1^2]}\right)$ and $\left(\frac{\alpha_0\beta_0^2 a_1+2\beta_0^2(a_1^2+b_1^2)+12\beta_0 a_1 c_1+(\alpha_0\beta_0+4c_1)(4c_1-z_1)-2\beta_0(a_1 z_1+|b_1 z_2|)}{2[(\alpha_0\beta_0+2\beta_0 a_1+4c_1)^2+4\beta_0^2 b_1^2]}, \frac{(\alpha_0\beta_0+2\beta_0 a_1+4c_1)(\beta_0 b_1-z_2)-2\beta_0 b_1(\beta_0 a_1+4c_1-z_1)}{2[(\alpha_0\beta_0+2\beta_0 a_1+4c_1)^2+4\beta_0^2 b_1^2]}\right)$ as $\alpha \to \alpha_0$ and $\beta \to \beta_0$ $(0 \le \alpha_0 < +\infty, 0 < \beta_0 < +\infty)$. □

Remark 5.1 It follows from Theorem 5.1 that

$$\begin{aligned} Re(\lambda_+) &= \frac{\alpha\beta^2 a_1+2\beta^2\left(a_1^2+b_1^2\right)+12\beta a_1 c_1+(\alpha\beta+4c_1)(4c_1+z_1)+2\beta(a_1 z_1+|b_1 z_2|)}{2\left[(\alpha\beta+2\beta a_1+4c_1)^2+4\beta^2 b_1^2\right]} > 0, \\ Re(\lambda_-) &= \frac{\alpha\beta^2 a_1+2\beta^2\left(a_1^2+b_1^2\right)+12\beta a_1 c_1+(\alpha\beta+4c_1)(4c_1-z_1)-2\beta(a_1 z_1+|b_1 z_2|)}{2\left[(\alpha\beta+2\beta a_1+4c_1)^2+4\beta^2 b_1^2\right]} \\ &\ge \frac{8c_1(\beta a_1+2c_1)}{2\left[(\alpha\beta+2\beta a_1+4c_1)^2+4\beta^2 b_1^2\right]} > 0 \end{aligned}$$

as $\alpha \ge 0$, $\beta > 0$ and $B^T u \ne 0$, and if B is of full column rank and $B^T u = 0$, then from (19), we infer that $Re(\lambda) > 0$, where $(\lambda, (u^*, v^*)^*)$ is an eigenpair of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$. Thus, all eigenvalues of $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ have positive real parts and lie in a positive box as B is of full column rank, which may result in fast convergence of Krylov subspace acceleration. Besides, from the proof of Theorem 5.1, it can be seen that when $B^T u = 0$ and $\alpha \to 0_+$, it holds that $\lambda \to \left(\frac{1}{2}, 0\right)$ or $\lambda = 0$; when $B^T u \ne 0$, $\lambda \to \left(\frac{1}{2}, 0\right)$ as $\beta \to 0_+$ for $\alpha \ge 0$. This implies that the MGSS preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ with proper parameters α and β has much denser spectrum distribution compared with the saddle point matrix $\mathcal{A}$. Consequently, when the MGSS preconditioner is applied for the GMRES method, the rate of convergence (semi-convergence) can be improved considerably. This fact is further confirmed by the numerical results presented in Tables 2, 3, 7 and 8 of Section 6. What is more, if $B^T u \neq 0$, then $c_1 > 0$ and

$$\begin{aligned}(\alpha\beta + 2\beta a_1)^2 + (\beta|b_1| + \sqrt{\beta^2 b_1^2 + 4\alpha\beta c_1})^2 &= (\alpha\beta + 2\beta a_1)^2 + 2\beta^2 b_1^2 + 4\alpha\beta c_1 \\ &\quad + 2\beta|b_1|\sqrt{\beta^2 b_1^2 + 4\alpha\beta c_1} \\ &\leq (\alpha\beta + 2\beta a_1)^2 + 2\beta^2 b_1^2 + 4\alpha\beta c_1 + 2\beta|b_1|\sqrt{\beta^2|b_1|^2 + 4\alpha\beta c_1 + \left(\frac{2\alpha c_1}{|b_1|}\right)^2} \\ &= (\alpha\beta + 2\beta a_1)^2 + 4\beta^2 b_1^2 + 8\alpha\beta c_1 < (\alpha\beta + 2\beta a_1 + 4c_1)^2 + 4\beta^2 b_1^2,\end{aligned}$$

then it follows from (32) that

$$\left|\lambda_\pm - \frac{1}{2}\right|^2 \leq \frac{(\alpha\beta + 2\beta a_1)^2 + (\beta|b_1| + \sqrt{\beta^2 b_1^2 + 4\alpha\beta c_1})^2}{4[(\alpha\beta + 2\beta a_1 + 4c_1)^2 + 4\beta^2 b_1^2]} < \frac{1}{4},$$

which implies that $\left|\lambda_\pm - \frac{1}{2}\right| < \frac{1}{2}$ when $B^T u \neq 0$. When $B^T u = 0$, $\lambda = 0$ or λ satisfies (26). From (26), it has

$$\left|\lambda - \frac{1}{2}\right|^2 = \frac{\alpha^2}{4[(\alpha + 2a_1)^2 + 4b_1^2]} < \frac{1}{4}.$$

We summarize the above discussions and obtain that all eigenvalues of $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ are located in a circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$.

Owing to the fact that the convergence of Krylov subspace methods is not only dependent on the eigenvalue distribution of the preconditioned matrix, but also on the corresponding eigenvectors of the preconditioned matrix [1, 4] except for the case that the preconditioned matrix is symmetric, we next discuss the eigenvector distribution of $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ in the following theorem.

Theorem 5.2 *Let the MGSS preconditioner $\mathcal{P}_{MGSS}$ be defined as in (4). If B is of full column rank and $\alpha = 0$, then the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ has $m + t$ $(0 \leq t \leq m)$ linearly independent eigenvectors, and if B is of full column rank and $\alpha > 0$, then the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ has t $(0 \leq t \leq m)$ linearly independent eigenvectors. If B is rank deficient and $\alpha = 0$, then the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ has $m + i + j$ $(0 \leq i \leq m, 1 \leq j \leq n)$ linearly independent eigenvectors, and if B is rank deficient and $\alpha > 0$, then the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ has $i + j$ $(0 \leq i \leq m, 1 \leq j \leq n)$ linearly independent eigenvectors. There are*

1) m eigenvectors of the form $\begin{pmatrix} u_l \\ 0 \end{pmatrix}$ $(1 \leq l \leq m)$ that correspond to the eigenvalue $\frac{1}{2}$ as $\alpha = 0$, where $u_l \neq 0$ $(1 \leq l \leq m)$ are arbitrary linearly independent vectors;

2) If B is of full column rank, t $(0 \leq t \leq m)$ eigenvectors of the form $\begin{pmatrix} u_l^1 \\ \frac{(2\lambda-1)B^T u_l^1}{\lambda\beta} \end{pmatrix}$

$(1 \le l \le t)$ *that correspond to the eigenvalues* $\lambda \neq \frac{1}{2}$, *where* u_l^1 $(1 \le l \le t)$ *satisfy* $\lambda\beta A u_l^1 = \beta\lambda^2(\alpha I + 2A)u_l^1 + (2\lambda - 1)^2 BB^T u_l^1$.

3) If B *is rank deficient,* i $(0 \le i \le m)$ *eigenvectors of the form* $\begin{pmatrix} u_l^1 \\ \frac{(2\lambda-1)B^T u_l^1}{\lambda\beta} \end{pmatrix}$ $(1 \le l \le i)$ *that correspond to the eigenvalues* $\lambda \neq \frac{1}{2}, 0$, *where* u_l^1 $(1 \le l \le i)$ *satisfy* $\lambda\beta A u_l^1 = \beta\lambda^2(\alpha I + 2A)u_l^1 + (2\lambda - 1)^2 BB^T u_l^1$*; and* j $(1 \le j \le n)$ *eigenvectors of the form* $\begin{pmatrix} 0 \\ v_l^2 \end{pmatrix}$ $(1 \le l \le j)$ *that correspond to the eigenvalue 0, where* $v_l^2 \neq 0$ $(1 \le l \le j)$ *satisfy* $Bv_l^2 = 0$.

Proof Let λ be an eigenvalue of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ be the corresponding eigenvector. To investigate the eigenvector distribution of $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$, we consider (24) as follows:

$$\begin{cases} Au = \lambda(\alpha I + 2A)u + (2\lambda - 1)Bv, \\ (2\lambda - 1)B^T u = \lambda\beta v. \end{cases} \tag{33}$$

We first consider the case that B has full column rank. If $u = 0$, then it follows from the second equation of (33) that $\lambda v = 0$ and therefore $v = 0$, which contradicts with the assumption that $(u^*, v^*)^*$ is an eigenvector. Hence $u \neq 0$. If $\lambda = \frac{1}{2}$, then from (33) we can easily get $\alpha u = 0$ and $v = 0$. If $\alpha = 0$, then (33) satisfies naturally for the case of $\lambda = \frac{1}{2}$. Hence, there are m linearly independent eigenvectors of the form $\begin{pmatrix} u_l \\ 0 \end{pmatrix}$ $(l = 1, 2, \cdots, m)$ that correspond to the eigenvalue $\frac{1}{2}$ as $\alpha = 0$, where u_l $(l = 1, 2, \cdots, m)$ are arbitrary linearly independent vectors. If $\alpha > 0$, then $u = 0$ and $v = 0$, a contradiction. If $\lambda \neq \frac{1}{2}$, then it follows from the second equation of (33) that $v = \frac{(2\lambda-1)B^T u}{\lambda\beta}$. Substituting v into the first equation of (33) results in

$$\lambda\beta A u = \beta\lambda^2(\alpha I + 2A)u + (2\lambda - 1)^2 BB^T u. \tag{34}$$

If there exists $u \neq 0$ which satisfies (34), there will be t $(1 \le t \le m)$ linearly independent eigenvectors of the form $\begin{pmatrix} u_l^1 \\ v_l^1 \end{pmatrix}$ $(1 \le l \le t)$ that correspond to the eigenvalues $\lambda \neq \frac{1}{2}$. Here, $u_l^1 \neq 0$ $(1 \le l \le k)$ satisfy $\lambda\beta A u_l^1 = \beta\lambda^2(\alpha I + 2A)u_l^1 + (2\lambda - 1)^2 BB^T u_l^1$ and the forms of v_l^1 $(1 \le l \le t)$ are

$$v_l^1 = \frac{(2\lambda - 1)B^T u_l^1}{\lambda\beta}.$$

Next, we consider the case that B is rank deficient. In this case, $\lambda = 0$ is an eigenvalue of $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$. If $\lambda = 0$, then from (33), it holds that $B^T u = 0$ and $Au = -Bv$, which lead to $B^T A^{-1} Bv = 0$, and therefore $Bv = 0$ is due to the fact that A^{-1} is positive definite, thus $u = 0$. Recalling that B is rank deficient, then there exists $v \neq 0$ which satisfies $Bv = 0$, hence there will be j $(1 \le j \le n)$ linearly independent eigenvectors of the form $\begin{pmatrix} 0 \\ v_l^2 \end{pmatrix}$ $(1 \leq l \leq j)$ that correspond to the eigenvalue 0, where $v_l^2 \neq 0$ $(1 \leq l \leq j)$ satisfy $Bv_l^2 = 0$. With a quite similar strategy utilized in the case that B has full column rank, we also can obtain the forms of the eigenvectors that correspond to $\lambda = \frac{1}{2}$ and $\lambda \neq 0, \frac{1}{2}$ for the case that B is rank deficient.

Now, we show the linear independence of the $m + t$ eigenvectors when B is of full column rank and $\alpha = 0$. Let $c^{(1)} = [c_1^{(1)}, c_2^{(1)}, \cdots, c_m^{(1)}]^T$ and $c^{(2)} = [c_1^{(2)}, c_2^{(2)}, \cdots, c_t^{(2)}]^T$ be two vectors with $0 \leq t \leq m$. Then, we need to show that

$$\begin{pmatrix} u_1 & \cdots & u_m \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_m^{(1)} \end{pmatrix} + \begin{pmatrix} u_1^1 & \cdots & u_t^1 \\ v_1^1 & \cdots & v_t^1 \end{pmatrix} \begin{pmatrix} c_1^{(2)} \\ \vdots \\ c_t^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{35}$$

holds if and only if the vectors $c^{(1)}$ and $c^{(2)}$ both are zero vectors. Recalling that the first matrix in (35) arises from the case $\lambda_l = \frac{1}{2}$ $(l = 1, 2, \cdots, m)$ in 1), and the second matrix from the case $\lambda_l \neq \frac{1}{2}$ $(l = 1, 2, \cdots, t)$ in 2). Multiplying both sides of (35) from left with $2\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ leads to

$$\begin{pmatrix} u_1 & \cdots & u_m \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_m^{(1)} \end{pmatrix} + \begin{pmatrix} u_1^1 & \cdots & u_t^1 \\ v_1^1 & \cdots & v_t^1 \end{pmatrix} \begin{pmatrix} 2\lambda_1 c_1^{(2)} \\ \vdots \\ 2\lambda_t c_t^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{36}$$

Then, by subtracting (35) from (36), it holds that

$$\begin{pmatrix} u_1^1 & \cdots & u_t^1 \\ v_1^1 & \cdots & v_t^1 \end{pmatrix} \begin{pmatrix} (2\lambda_1 - 1)c_1^{(2)} \\ \vdots \\ (2\lambda_t - 1)c_t^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the eigenvalues $\lambda_l \neq \frac{1}{2}$ and $\begin{pmatrix} u_l^1 \\ v_l^1 \end{pmatrix}$ $(1 \leq l \leq t)$ are linearly independent, we infer that $c_l^{(2)} = 0$ $(l = 1, 2, \cdots, t)$. Because of the linear independence of u_l $(l = 1, 2, \cdots, m)$, it follows that $c_l^{(1)} = 0$ $(l = 1, 2, \cdots, m)$. Therefore, the $m + t$ eigenvectors are linearly independent.

In the sequel, we verify that the $m + i + j$ eigenvectors are linearly independent when B is rank deficient and $\alpha = 0$. Let $c^{(1)} = [c_1^{(1)}, c_2^{(1)}, \cdots, c_m^{(1)}]^T$, $c^{(2)} = [c_1^{(2)}, c_2^{(2)}, \cdots, c_i^{(2)}]^T$ and $c^{(3)} = [c_1^{(3)}, c_2^{(3)}, \cdots, c_j^{(3)}]^T$ be three vectors with $0 \leq i \leq m$ and $1 \leq j \leq n$, and

$$\begin{pmatrix} u_1 & \cdots & u_m \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_m^{(1)} \end{pmatrix} + \begin{pmatrix} u_1^1 & \cdots & u_i^1 \\ v_1^1 & \cdots & v_i^1 \end{pmatrix} \begin{pmatrix} c_1^{(2)} \\ \vdots \\ c_i^{(2)} \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ v_1^2 & \cdots & v_j^2 \end{pmatrix} \begin{pmatrix} c_1^{(3)} \\ \vdots \\ c_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{37}$$

It is necessary for us to prove that (37) holds if and only if the vectors $c^{(1)}$, $c^{(2)}$ and $c^{(3)}$ are all zero vectors, where the first matrix consists of the eigenvectors that correspond to the eigenvalue $\frac{1}{2}$ for the case 1), and the second and the third matrices consist of those for the case 3). Premultiplying (37) with $2\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ and going through the same algebraic operations as before, we also obtain

$$\begin{pmatrix} u_1^1 & \cdots & u_i^1 \\ v_1^1 & \cdots & v_i^1 \end{pmatrix} \begin{pmatrix} (2\lambda_1 - 1)c_1^{(2)} \\ \vdots \\ (2\lambda_i - 1)c_i^{(2)} \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 \\ v_1^2 & \cdots & v_j^2 \end{pmatrix} \begin{pmatrix} c_1^{(3)} \\ \vdots \\ c_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Inasmuch as $\lambda_l \neq \frac{1}{2}$ and u_l^1 $(1 \leq l \leq i)$ are linearly independent, it holds that $c_l^{(2)} = 0$ $(l = 1, 2, \cdots, i)$. Then, it has

$$\begin{pmatrix} 0 & \cdots & 0 \\ v_1^2 & \cdots & v_j^2 \end{pmatrix} \begin{pmatrix} c_1^{(3)} \\ \vdots \\ c_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

As the vectors v_l^2 $(l = 1, 2, \cdots, j)$ are also linearly independent, we have $c_l^{(3)} = 0$ $(l = 1, 2, \cdots, j)$. Thus, (37) reduces to

$$\begin{pmatrix} u_1 & \cdots & u_m \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_m^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since u_l $(l = 1, 2, \cdots, m)$ are linearly independent, we have $c_l^{(1)} = 0$ $(l = 1, 2, \cdots, m)$. As a result, it holds that the $m + i + j$ eigenvectors are linearly independent.

Finally, we prove that the $i + j$ eigenvectors are linearly independent when B is rank deficient and $\alpha > 0$. Let $c^{(1)} = [c_1^{(1)}, c_2^{(1)}, \cdots, c_i^{(1)}]^T$ and $c^{(2)} = [c_1^{(2)}, c_2^{(2)}, \cdots, c_j^{(2)}]^T$ be two vectors with $0 \leq i \leq m$, $1 \leq j \leq n$. It is left to show that

$$\begin{pmatrix} u_1^1 & \cdots & u_i^1 \\ v_1^1 & \cdots & v_i^1 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_i^{(1)} \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ v_1^2 & \cdots & v_j^2 \end{pmatrix} \begin{pmatrix} c_1^{(2)} \\ \vdots \\ c_j^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

holds if and only if the vectors $c^{(1)}$ and $c^{(2)}$ both are zero vectors. Since u_l^1 $(1 \leq l \leq i)$ are linearly independent, it follows that $c_l^{(1)} = 0$ $(l = 1, 2, \cdots, i)$. Because of the linear independence of v_l^2 $(l = 1, 2, \cdots, j)$, it holds that $c_l^{(2)} = 0$ $(l = 1, 2, \cdots, j)$. Consequently, the above $i + j$ eigenvectors are linearly independent. □

6 Numerical experiments

In this section, we carry out two numerical examples to validate the effectiveness of the MGSS iteration method and the MGSS preconditioned GMRES method. In the meanwhile, we compare the MGSS iteration method with the GSS and the GMSS iteration methods, and also compare the MGSS preconditioner with the SS, GSS, M-SS, GMSS, and MSS ones for the GMRES method from aspects of the number of iterations (denoted by "IT") and the elapsed CPU times (denoted by "CPU"). All codes are run in MATLAB R2016a and all experiments are performed on an Intel(R) Pentium(R) CPU G3240T 2.70 GHz, 4.0GB memory and XP operating system. In our implementations, the linear systems $(\alpha I + A + \frac{1}{\alpha}BB^T)x = b$, $(\alpha I + A + \frac{1}{\beta}BB^T)x = b$ and $(\alpha I + 2A + \frac{4}{\beta}BB^T)x = b$ involved in the SS, GSS, and MGSS iterations, respectively, are solved inexactly by the GMRES method. In addition, the linear systems with the coefficient matrices $\alpha I + 2H + \frac{1}{\alpha}BB^T$ and $\alpha I + 2H + \frac{1}{\beta}BB^T$ are solved inexactly by the conjugate gradient (CG) method. The inner GMRES and the inner CG methods are terminated if the current residuals of the inner iterations satisfy $\|r^{(k)}\| < 10^{-7} \times \|r^{(0)}\|$, where $r^{(k)}$ denotes the residual of the kth GMRES iteration or the kth CG iteration.

In all the tests, the initial vector $\mathbf{x}^{(0)}$ is set to be a zero vector and the right-hand side vector b is chosen such that the exact solution of the saddle point problem (1) is a vector of all ones. The iterations are terminated as soon as the current iterate $\mathbf{x}^{(k)}$ satisfies

$$\text{RES} = \frac{\sqrt{\|f - Ax^{(k)} - By^{(k)}\|_2^2 + \|g - B^T x^{(k)}\|_2^2}}{\sqrt{\|f\|_2^2 + \|g\|_2^2}} < 10^{-6},$$

and we use "–" to indicate that the corresponding iteration method does not satisfy the prescribed stopping criterion until 500 iteration steps.

In our numerical experiments, the parameters adopted in the iteration methods are the experimentally found optimal ones that minimize the total number of iteration steps for those methods. In addition, to implement the tested preconditioners efficiently and obtain fast convergence rates of the corresponding preconditioned GMRES methods, the parameters involved in these preconditioners should be chosen appropriately. Here, we adopt two ways to compare the involved preconditioners' numerical efficiencies. First, by making use of the methods applied in [18], the parameters chosen for the tested preconditioners in Tables 2, 3, 7 and 8 of our numerical experiments are as follows:

$$\alpha_{SS} = \frac{\|B\|_2^2}{\|A\|_2}; \quad \alpha_{GSS} = \upsilon,\ \beta_{GSS} = \frac{\|B\|_2^2}{\|A\|_2}; \quad \alpha_{M-SS} = \frac{\|B\|_2^2}{2\|H\|_2};$$
$$\alpha_{GMSS} = \upsilon,\ \beta_{GMSS} = \frac{\|B\|_2^2}{2\|H\|_2}; \quad \alpha_{MSS} = \frac{2\|B\|_2^2}{\|A\|_2}; \quad \alpha_{MGSS} = \upsilon,$$
$$\beta_{MGSS} = \frac{2\|B\|_2^2}{\|A\|_2},$$

where υ and $\|A\|_2$ denote the viscosity value and the Euclidean norm of the matrix A, respectively. On the other hand, we list the numerical results of the tested preconditioned GMRES methods for different values of parameters α and β for each value of υ in Tables 4, 5, 9 and 10.

Table 1 Numerical results for the three iteration methods with $v = 0.1$

Method		p		
		16	32	64
	α_{exp}	20	51	125
	β_{exp}	2.7	5	1.5
GSS	IT	58	72	102
	CPU	0.2428	1.0425	15.3155
	RES	8.79e-07	8.68e-07	9.80e-07
	α_{exp}	22	36	38
	β_{exp}	16	8.3	5.9
GMSS	IT	66	73	89
	CPU	0.4360	1.2271	16.4183
	RES	8.45e-07	9.09e-07	9.50e-07
	α_{exp}	0.2	0.5	0.2
	β_{exp}	0.1	0.1	0.1
MGSS	IT	21	21	21
	CPU	0.1438	0.6311	7.0078
	RES	9.88e-07	9.85e-07	9.57e-07

Example 6.1 Consider the nonsymmetric nonsingular saddle point problem structured as (1) with the following coefficient sub-matrices [37]:

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2},$$

$$B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times p^2},$$

$$T = \frac{v}{h^2} \cdot \mathrm{tridiag}(-1, 2, -1) + \frac{1}{2h} \cdot \mathrm{tridiag}(-1, 0, 1) \in \mathbb{R}^{p \times p},$$

$$F = \frac{1}{h} \cdot \mathrm{tridiag}(-1, 1, 0) \in \mathbb{R}^{p \times p}.$$

The symbol $\otimes$ denotes the Kronecker product and $h = \frac{1}{p+1}$ is the discretization mesh size.

In Table 1, we list the parameters involved in the tested methods which are chosen to be the experimentally found optimal ones that minimize the total number of iteration steps for those methods, as well as the numerical results of the GSS, GMSS, and MGSS iteration methods when $v = 0.1$ with respect to different grids 16×16, 32×32, and 64×64. Moreover, numerical results of the GMRES method and the preconditioned GMRES methods incorporated with the SS, GSS, M-SS, GMSS, MSS and MGSS preconditioners are listed in Tables 2 and 3 for

Table 2 Numerical results for the seven preconditioned GMRES methods with $\upsilon = 1$

Preconditioner		p			
		16	32	48	64
I	IT	121	264	429	–
	CPU	0.1550	3.8574	24.7021	–
	RES	7.21e-07	9.74e-07	9.95e-07	–
	α	0.9995	0.9999	1.0000	1.0000
$\mathcal{P}_{SS}$	IT	10	12	13	14
	CPU	0.1130	0.5992	2.1863	9.9617
	RES	9.16e-07	7.41e-07	5.21e-07	2.10e-07
	α	1	1	1	1
	β	0.9995	0.9999	1.0000	1.0000
$\mathcal{P}_{GSS}$	IT	10	12	13	14
	CPU	0.0515	0.3958	2.2862	10.0452
	RES	9.12e-07	7.41e-07	5.21e-07	2.10e-07
	α	0.4997	0.5000	0.5000	0.5000
$\mathcal{P}_{M-SS}$	IT	15	15	16	16
	CPU	0.0803	0.4858	2.6539	10.6907
	RES	3.29e-07	7.63e-07	6.33e-07	8.29e-07
	α	1	1	1	1
	β	0.4997	0.5000	0.5000	0.5000
$\mathcal{P}_{GMSS}$	IT	15	15	16	16
	CPU	0.0907	0.4491	2.5381	10.8585
	RES	3.31e-07	7.80e-07	6.49e-07	8.50e-07
	α	1.9989	1.9999	2.0000	2.0000
$\mathcal{P}_{MSS}$	IT	10	12	13	14
	CPU	0.0848	0.5783	2.2968	10.0103
	RES	9.16e-07	7.41e-07	5.21e-07	2.10e-07
	α	1	1	1	1
	β	1.9989	1.9999	2.0000	2.0000
$\mathcal{P}_{MGSS}$	IT	10	11	12	12
	CPU	0.0494	0.3832	2.0440	8.8767
	RES	3.06e-07	6.16e-07	4.01e-07	3.01e-07

$\upsilon = 1$ and 0.1 on different uniform grids, respectively. To further show the advantages of the MGSS preconditioner over the GSS and the GMSS ones, numerical results of the GSS, GMSS, and MGSS preconditioned GMRES methods with different values of α and β for $\upsilon = 1$ and $\upsilon = 0.1$ are listed in Tables 4 and 5, respectively.

From numerical results listed in Tables 1, 2, 3, 4 and 5, we can conclude some observations as follows.

Table 3 Numerical results for the seven preconditioned GMRES methods with $v = 0.1$

Preconditioner		p			
		16	32	48	64
I	IT	115	240	367	495
	CPU	0.1326	3.4868	20.4798	81.8770
	RES	9.50e-07	9.34e-07	9.80e-07	9.73e-07
	α	9.9931	9.9992	9.9998	9.9999
$\mathcal{P}_{SS}$	IT	24	26	27	28
	CPU	0.1283	0.8748	4.6786	18.9283
	RES	5.93e-07	4.11e-07	6.79e-07	6.89e-07
	α	0.1	0.1	0.1	0.1
	β	9.9931	9.9992	9.9998	9.9999
$\mathcal{P}_{GSS}$	IT	14	15	15	16
	CPU	0.0747	0.4911	2.5376	11.2892
	RES	7.09e-07	6.49e-07	9.17e-07	6.30e-07
	α	4.9974	4.9996	4.9999	5.0000
$\mathcal{P}_{M-SS}$	IT	25	26	27	27
	CPU	0.1012	0.7269	4.0839	16.4194
	RES	4.91e-07	5.27e-07	4.37e-07	4.46e-07
	α	0.1	0.1	0.1	0.1
	β	4.9974	4.9996	4.9999	5.0000
$\mathcal{P}_{GMSS}$	IT	23	24	25	25
	CPU	0.3316	0.7313	3.6292	15.8545
	RES	6.78e-07	8.45e-07	5.48e-07	6.72e-07
	α	19.9861	19.9983	19.9995	19.9998
$\mathcal{P}_{MSS}$	IT	24	26	27	28
	CPU	0.2198	0.8668	4.5341	19.8549
	RES	5.94e-07	4.11e-07	6.79e-07	6.89e-07
	α	0.1	0.1	0.1	0.1
	β	19.9861	19.9983	19.9995	19.9998
$\mathcal{P}_{MGSS}$	IT	14	15	15	15
	CPU	0.0740	0.4912	2.4140	10.5017
	RES	4.46e-07	3.12e-07	4.38e-07	5.86e-07

- From Table 1, it can be observed that the IT of the GSS and the GMSS iteration methods increase as the increasing of the problem size, but that of the MGSS iteration method keeps constant. Among these methods, the MGSS iteration method requires the least IT and CPU times, which implies that the MGSS iteration method is superior to the other two methods in terms of computing efficiency.

Table 4 Numerical results for the three preconditioned GMRES methods with $\upsilon = 1$

p	(α, β)	$\mathcal{P}_{GSS}$		$\mathcal{P}_{GMSS}$		$\mathcal{P}_{MGSS}$	
		IT	CPU	IT	CPU	IT	CPU
16	(0.6, 0.8)	10	0.0748	15	0.0658	8	0.0472
	(0.2, 0.5)	8	0.0407	15	0.0619	7	0.0373
	(0.25, 0.15)	6	0.0406	12	0.0535	5	0.0288
	(0.05, 0.08)	5	0.0334	11	0.0488	4	0.0323
	(1, 0.8)	10	0.0503	15	0.0669	8	0.0436
	(1.2, 1.5)	12	0.0616	17	0.0722	10	0.0524
32	(0.6, 0.8)	11	0.3878	17	0.4845	8	0.2902
	(0.2, 0.5)	9	0.3235	15	0.4300	7	0.2417
	(0.25, 0.15)	7	0.2386	13	0.4227	6	0.2274
	(0.05, 0.08)	6	0.2549	11	0.3548	5	0.1962
	(1, 0.8)	11	0.4006	17	0.4913	9	0.2952
	(1.2, 1.5)	13	0.4570	18	0.5579	11	0.3939
64	(0.6, 0.8)	12	8.5137	18	11.9339	10	7.0529
	(0.2, 0.5)	10	7.2786	16	10.7576	8	5.7586
	(0.25, 0.15)	8	5.8886	13	8.7388	6	4.5626
	(0.05, 0.08)	6	4.6207	12	8.1109	5	3.8777
	(1, 0.8)	13	9.2250	18	12.1414	10	7.0628
	(1.2, 1.5)	15	10.6303	20	12.4562	12	8.4906

- By comparing the results in Tables 2 and 3, we see that that the GMRES method with no preconditioner converges very slowly and it is even not convergent within 500 iteration steps when $p = 64$ and $\upsilon = 1$. All aforementioned preconditioners can accelerate the convergence rate of the GMRES method, and the MGSS preconditioner is more efficient than other five preconditioners according to IT and CPU times. When $p \leq 48$, the IT of the GSS preconditioned GMRES method is almost the same as that of the MGSS preconditioned GMRES method. However, for $p = 64$, the IT of the MGSS preconditioned GMRES method is less than that of the GSS preconditioned GMRES method.
- Tables 4 and 5 show that for different parameters, the MGSS preconditioned GMRES method requires less IT and CPU times than the other two preconditioned GMRES methods, which means that the MGSS preconditioner outperforms the GSS and the GMSS preconditioners in accelerating the convergence of the GMRES method for solving the saddle point problem in Example 6.1.

To better show the convergence behavior of the tested iteration methods with the experimentally found optimal parameters in Table 1, we plot the residual curves of the tested iteration methods in Fig. 1. Figure 1 clearly shows that among these iteration methods, the MGSS iteration one is the most effective method as its residual reduces the fastest.

Table 5 Numerical results for the three preconditioned GMRES methods with $\upsilon = 0.1$

p	(α, β)	$\mathcal{P}_{GSS}$		$\mathcal{P}_{GMSS}$		$\mathcal{P}_{MGSS}$	
		IT	CPU	IT	CPU	IT	CPU
16	(0.6, 0.8)	7	0.0418	17	0.1916	6	0.0326
	(0.2, 0.5)	6	0.0313	16	0.0743	5	0.0313
	(0.25, 0.15)	5	0.0350	15	0.0629	4	0.0316
	(0.05, 0.08)	4	0.0456	14	0.0687	4	0.0425
	(1, 0.8)	8	0.0446	17	0.0774	6	0.0551
	(1.2, 1.5)	9	0.0554	18	0.0747	7	0.0408
32	(0.6, 0.8)	8	0.3138	16	0.4968	6	0.2598
	(0.2, 0.5)	6	0.2848	16	0.5158	5	0.2113
	(0.25, 0.15)	5	0.2108	14	0.4361	5	0.1955
	(0.05, 0.08)	4	0.1632	14	0.4282	4	0.1543
	(1, 0.8)	8	0.2757	17	0.6801	7	0.2483
	(1.2, 1.5)	10	0.3776	19	0.5779	8	0.2670
64	(0.6, 0.8)	8	5.9215	17	10.8365	7	5.1948
	(0.2, 0.5)	7	5.2904	16	10.4351	6	4.5064
	(0.25, 0.15)	6	4.5613	14	9.2630	5	3.9649
	(0.05, 0.08)	5	3.9050	14	9.2313	4	3.2229
	(1, 0.8)	9	6.4687	17	11.2160	7	5.1702
	(1.2, 1.5)	11	7.7398	19	12.3902	8	5.8822

To further confirm the effectiveness of the MGSS preconditioned GMRES method compared with the GSS and the GMSS ones, we illustrate the changing of their IT with parameters $\alpha = \beta$ from 0.1 to 10 with step size 0.1 in Fig. 2. From Fig. 2, we can observe that the MGSS preconditioned GMRES method needs less IT than the other two preconditioned GMRES ones with the changing of α. What is more, the MGSS preconditioner is more insensitive to the parameter α than the other two

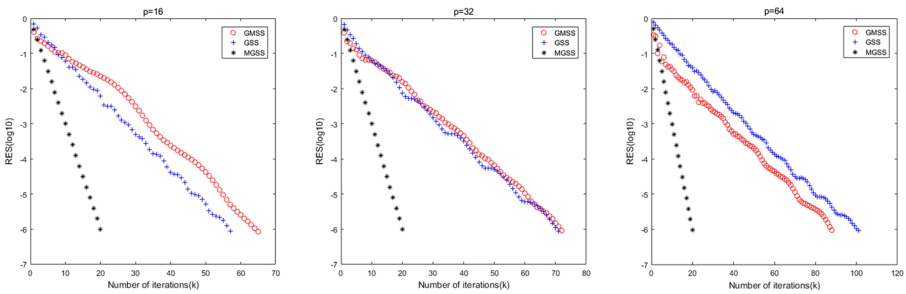

Fig. 1 Convergence curves of algorithms with $\upsilon = 0.1$ for $p = 16$, $p = 32$, and $p = 64$, respectively

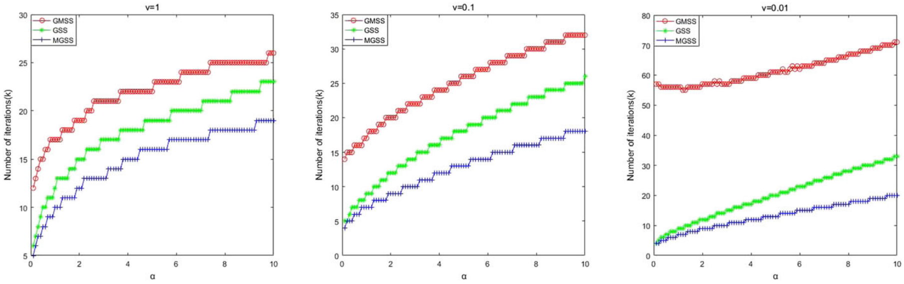

Fig. 2 IT of three preconditioned GMRES methods with varying $\alpha = \beta$ for $p = 32$

preconditioners. From these two points, our proposed preconditioner is more effective and practical for solving the nonsymmetric nonsingular saddle point problems, in comparison with the GSS and the GMSS preconditioners.

Figure 3 demonstrates the eigenvalue distributions of the six preconditioned matrices with experimentally found optimal parameters for $\upsilon = 1$ and $p = 32$. As seen from Fig. 3, the eigenvalue distributions of the preconditioned matrix $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ are clustered more closely than those of the other ones. This further confirms that the MGSS preconditioner outperforms the other five preconditioners for the GMRES method.

Example 6.2 Consider the nonsymmetric singular saddle point problem structured as (1) with the following coefficient sub-matrices [42]:

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2},\ B = \begin{pmatrix} \hat{B} & b_1 & b_2 \end{pmatrix} \in \mathbb{R}^{2p^2 \times (p^2+2)},$$

where

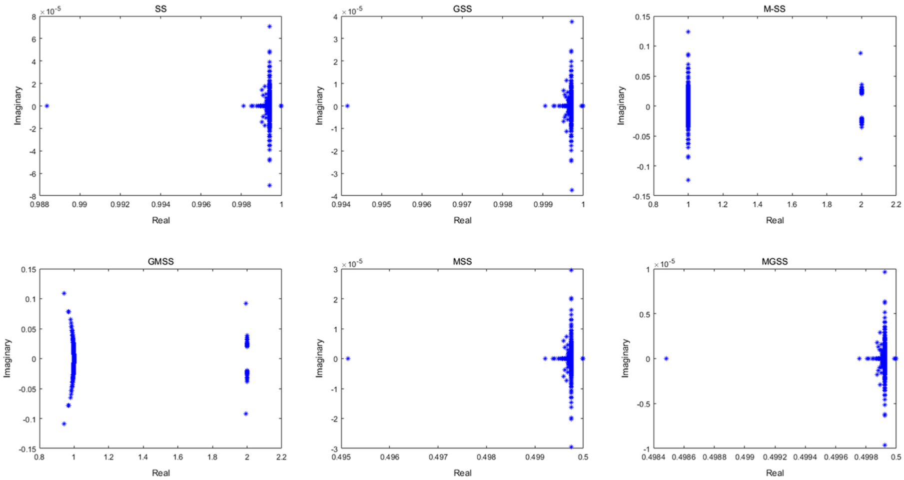

Fig. 3 The eigenvalue distributions of the six preconditioned matrices for $p = 32$ and $\upsilon = 1$

Table 6 Numerical results for the three iteration methods with $v = 0.1$

Method		p		
		16	32	64
	α_{exp}	13	29	66
	β_{exp}	39	53	60
GSS	IT	85	136	230
	CPU	0.2729	1.7084	32.7261
	RES	9.48e-07	9.83e-07	9.73e-07
	α_{exp}	16	18	24
	β_{exp}	75	134.4	240
GMSS	IT	143	213	337
	CPU	0.6913	3.5548	62.2509
	RES	9.86e-07	9.90e-07	9.98e-07
	α_{exp}	0.02	0.01	0.05
	β_{exp}	0.1	0.05	0.1
MGSS	IT	21	21	21
	CPU	0.0842	0.5365	7.9819
	RES	9.53e-07	9.54e-07	9.54e-07

$$T = \frac{v}{h^2} \cdot \text{tridiag}(-1, 2, -1) + \frac{1}{2h} \cdot \text{tridiag}(-1, 0, 1) \in \mathbb{R}^{p\times p},\ \hat{B} = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2\times p^2},$$
$$b_1 = \hat{B}\begin{pmatrix} e \\ 0 \end{pmatrix},\ b_2 = \hat{B}\begin{pmatrix} 0 \\ e \end{pmatrix},\ e = (1, 1, \cdots, 1) \in \mathbb{R}^{p^2/2},$$
$$F = \frac{1}{h} \cdot \text{tridiag}(-1, 1, 0) \in \mathbb{R}^{p\times p},\ h = \frac{1}{p+1}.$$

Here, $\otimes$ denotes the Kronecker product symbol and $h = \frac{1}{p+1}$ is the discretization meshsize.

Table 6 reports the IT, CPU times and relative residual (RES) of the tested iteration methods with respect to different values of the problem size p for $v = 0.1$. We adopt the parameters of the tested methods to be the experimentally found optimal ones. From Table 6, we observe that the MGSS iteration method outperforms the GSS and the GMSS iteration methods in terms of the IT and CPU times, and the advantage of the MGSS iteration method becomes more pronounced as the system size increases.

With respect to different sizes of the coefficient matrix, we list the numerical results of the SS, GSS, M-SS, GMSS, MSS, and MGSS preconditioned GMRES methods with two different values of v ($v = 1$ and $v = 0.1$) in Tables 7 and 8, respectively. From Tables 7 and 8, we can conclude some observations as follows. Firstly, without preconditioning, the GMRES method converges very slowly. Secondly, all the discussed preconditioners can improve the convergence behavior of the

Table 7 Numerical results for the seven preconditioned GMRES methods with $v = 1$

Preconditioner		p			
		16	32	48	64
I	IT	145	278	366	465
	CPU	0.2146	4.1297	20.2558	76.1434
	RES	7.95e-07	9.79e-07	9.71e-07	9.71e-07
	α	6.5105	12.7108	18.9346	25.1644
$\mathcal{P}_{SS}$	IT	16	21	24	26
	CPU	0.0941	0.6623	3.8408	17.4955
	RES	9.05e-07	5.84e-07	8.19e-07	9.49e-07
	α	1	1	1	1
	β	6.5105	12.7108	18.9346	25.1644
$\mathcal{P}_{GSS}$	IT	12	13	13	12
	CPU	0.0879	0.4463	2.2246	8.4683
	RES	9.71e-07	7.57e-07	5.29e-07	9.66e-07
	α	3.2553	6.3554	9.4673	12.5822
$\mathcal{P}_{M-SS}$	IT	18	21	23	24
	CPU	0.1031	0.5961	3.2619	14.5996
	RES	8.29e-07	9.82e-07	7.33e-07	8.64e-07
	α	1	1	1	1
	β	3.2553	6.3554	9.4673	12.5822
$\mathcal{P}_{GMSS}$	IT	18	22	23	23
	CPU	0.1018	0.6227	3.2238	14.5045
	RES	9.17e-07	3.20e-07	5.93e-07	9.22e-07
	α	13.0210	25.4217	37.8619	50.3288
$\mathcal{P}_{MSS}$	IT	16	21	24	26
	CPU	0.0799	0.7197	3.9288	17.6278
	RES	9.06e-07	5.84e-07	8.19e-07	9.49e-07
	α	1	1	1	1
	β	13.0210	25.4217	37.8619	50.3288
$\mathcal{P}_{MGSS}$	IT	12	12	11	11
	CPU	0.0801	0.4196	1.9405	7.8001
	RES	5.34e-07	4.87e-07	8.65e-07	5.35e-07

GMRES method efficiently, but the MGSS preconditioned GMRES method returns better numerical results than the other preconditioned GMRES methods in terms of IT and CPU times. Thirdly, the IT of the GSS and the MGSS preconditioned GMRES methods are almost constant for $v = 1$ and even reduce with size grows for $v = 0.1$. Lastly, the M-SS and the GMSS preconditioned GMRES methods have worse convergence behaviors as v becomes small.

Table 8 Numerical results for the seven preconditioned GMRES methods with $\upsilon = 0.1$

Preconditioner		p			
		16	32	48	64
I	IT	122	237	350	461
	CPU	0.1422	3.3291	19.3841	76.1620
	RES	8.71e-07	9.87e-07	9.99e-07	9.82e-07
	α	65.0943	127.1068	189.3452	251.6436
$\mathcal{P}_{SS}$	IT	58	87	109	128
	CPU	0.4285	2.9045	16.9213	83.8172
	RES	9.24e-07	9.20e-07	9.27e-07	9.16e-07
	α	0.1	0.1	0.1	0.1
	β	65.0943	127.1068	189.3452	251.6436
$\mathcal{P}_{GSS}$	IT	17	17	14	13
	CPU	0.1532	0.5840	2.4159	9.7652
	RES	9.33e-07	4.86e-07	9.50e-07	7.68e-07
	α	32.5526	63.5542	94.6729	125.8219
$\mathcal{P}_{M-SS}$	IT	44	61	73	83
	CPU	0.2846	1.6266	9.9969	51.6923
	RES	9.26e-07	8.07e-07	8.43e-07	8.50e-07
	α	0.1	0.1	0.1	0.1
	β	32.5526	63.5542	94.6729	125.8219
$\mathcal{P}_{GMSS}$	IT	38	46	53	56
	CPU	0.1518	1.2584	7.3594	33.5253
	RES	7.07e-07	9.84e-07	8.04e-07	8.29e-07
	α	130.1886	254.2136	378.6905	503.2872
$\mathcal{P}_{MSS}$	IT	58	87	109	128
	CPU	0.2778	2.8788	17.1494	82.8753
	RES	9.24e-07	9.20e-07	9.27e-07	9.16e-07
	α	0.1	0.1	0.1	0.1
	β	130.1886	254.2136	378.6905	503.2872
$\mathcal{P}_{MGSS}$	IT	17	15	14	12
	CPU	0.1381	0.5395	2.4073	9.1421
	RES	4.65e-07	9.91e-07	7.75e-07	7.05e-07

Furthermore, the performances of the GSS, GMSS, and MGSS preconditioned GMRES methods for different choices of α and β with $\upsilon = 1$ and $\upsilon = 0.1$ are exhibited in Tables 9 and 10, respectively. From the numerical results shown in Tables 9 and 10, we see that the MGSS preconditioner is superior to the GSS and the GMSS preconditioners in terms of the IT and CPU times.

The graphs of RES (log10) against number of iterations of in Table 6 for three different sizes are displayed in Fig. 4. As observed in Fig. 4, the MGSS iteration

Table 9 Numerical results for the three preconditioned GMRES methods with $\upsilon = 1$

p	(α, β)	$\mathcal{P}_{GSS}$		$\mathcal{P}_{GMSS}$		$\mathcal{P}_{MGSS}$	
		IT	CPU	IT	CPU	IT	CPU
16	(0.5, 0.8)	9	0.0484	15	0.0650	7	0.0495
	(0.3, 0.6)	8	0.0410	15	0.0707	6	0.0348
	(0.45, 0.25)	7	0.0394	13	0.0682	5	0.0291
	(0.1, 0.05)	4	0.0243	10	0.0725	4	0.0326
	(1.2, 0.8)	9	0.0460	15	0.1082	7	0.0424
	(1.8, 1.5)	11	0.0577	16	0.1700	9	0.0464
32	(0.5, 0.8)	9	0.3467	16	0.4792	7	0.2328
	(0.3, 0.6)	8	0.2720	15	0.4482	7	0.2773
	(0.45, 0.25)	7	0.2625	13	0.3908	6	0.2675
	(0.1, 0.05)	5	0.1826	10	0.3738	4	0.1601
	(1.2, 0.8)	10	0.3154	16	0.6572	8	0.2900
	(1.8, 1.5)	12	0.4139	17	0.6794	9	0.3033
64	(0.5, 0.8)	10	7.1345	17	10.5319	8	5.8608
	(0.3, 0.6)	8	5.7044	16	10.5535	7	5.2030
	(0.45, 0.25)	7	5.1791	14	9.3277	6	4.5605
	(0.1, 0.05)	5	3.9282	10	7.1549	4	3.3178
	(1.2, 0.8)	10	7.2011	17	10.6561	8	5.8994
	(1.8, 1.5)	12	8.3663	18	11.5827	10	7.2121

Table 10 Numerical results for the three preconditioned GMRES methods with $\upsilon = 0.1$

p	(α, β)	$\mathcal{P}_{GSS}$		$\mathcal{P}_{GMSS}$		$\mathcal{P}_{MGSS}$	
		IT	CPU	IT	CPU	IT	CPU
16	(0.5, 0.8)	6	0.0345	18	0.0976	5	0.0296
	(0.3, 0.6)	6	0.0365	17	0.0717	5	0.0300
	(0.45, 0.25)	5	0.0302	17	0.0698	4	0.0277
	(0.1, 0.05)	4	0.0256	15	0.0663	3	0.0197
	(1.2, 0.8)	7	0.0367	18	0.0780	6	0.0357
	(1.8, 1.5)	9	0.0492	19	0.0817	7	0.0404
32	(0.5, 0.8)	7	0.2727	17	0.5033	5	0.1827
	(0.3, 0.6)	6	0.2186	17	0.4947	5	0.2233
	(0.45, 0.25)	5	0.2199	16	0.4751	4	0.1468
	(0.1, 0.05)	4	0.1533	15	0.4473	3	0.1582
	(1.2, 0.8)	7	0.2456	18	0.5349	6	0.2173
	(1.8, 1.5)	9	0.3269	19	0.6920	7	0.2928
64	(0.5, 0.8)	7	5.3262	17	11.1527	5	4.0357
	(0.3, 0.6)	6	4.7161	17	11.0136	5	3.9860
	(0.45, 0.25)	5	4.0828	16	10.5630	5	3.9198
	(0.1, 0.05)	4	3.3399	15	10.0314	3	2.6391
	(1.2, 0.8)	8	6.0988	17	11.2515	6	4.6330
	(1.8, 1.5)	9	6.5748	19	11.8322	7	5.3267

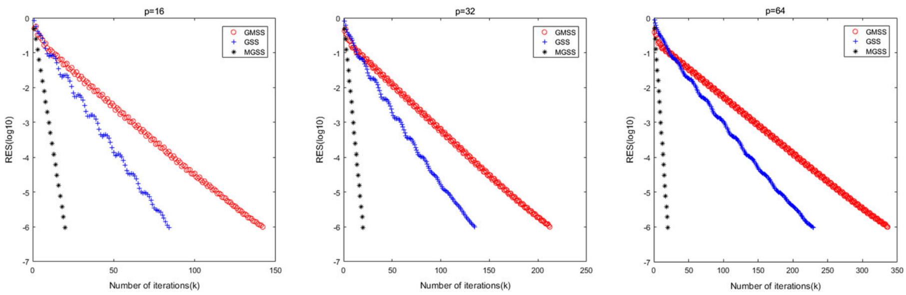

Fig. 4 Convergence curves of algorithms with $v = 0.1$ for $p = 16$, $p = 32$, and $p = 64$, respectively

method leads to much better performance than the GSS and the GMSS iteration methods. It is worthy noting that the IT of the GSS and the GMSS iteration methods increase when p becomes large, but this is not true for the MGSS iteration method.

In order to compare the effects of the GSS, GMSS, and the MGSS preconditioned GMRES methods with respect to the parameters α and β, we test these methods with $\alpha = \beta$ and plot the IT of the three preconditioned GMRES methods with α from 0.1 to 10 with step size 0.1 in Fig. 5. The conclusions obtained from Fig. 5 are similar to those of Fig. 2.

In order to better investigate the performances of the tested preconditioned GMRES methods, Fig. 6 depicts the eigenvalue distributions of the SS, GSS, M-SS, GMSS, MSS, and MGSS preconditioned matrices with experimentally found optimal parameters for $v = 0.1$ and $p = 32$. These subfigures clearly show that the eigenvalue distribution of the MGSS preconditioned matrix is more clustered compared with those of the other ones. From the view point of clustering properties of spectrum, the MGSS preconditioner established in this paper is better than the SS, GSS, M-SS, GMSS, and MSS preconditioners and it can act as an efficient preconditioner for solving the singular saddle point problems by the preconditioned GMRES method. In the meanwhile, as in accordance with the results of Remark 5.1, we find that the all eigenvalues of $\mathcal{P}_{MGSS}^{-1}\mathcal{A}$ are located in a circle centered at $(0.5, 0)$ with radius 0.5 in Fig. 6.

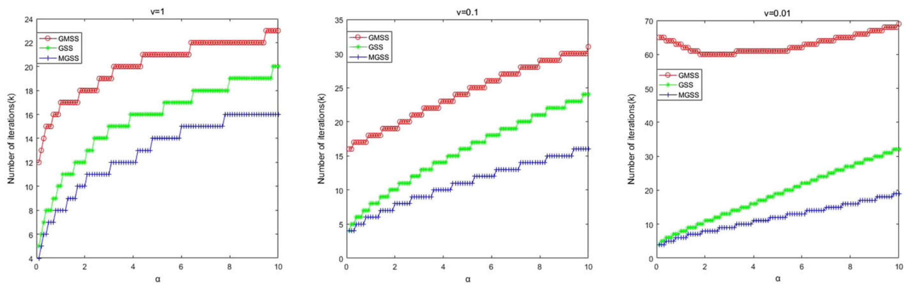

Fig. 5 IT of three preconditioned GMRES methods with varying $\alpha = \beta$ for $p = 32$

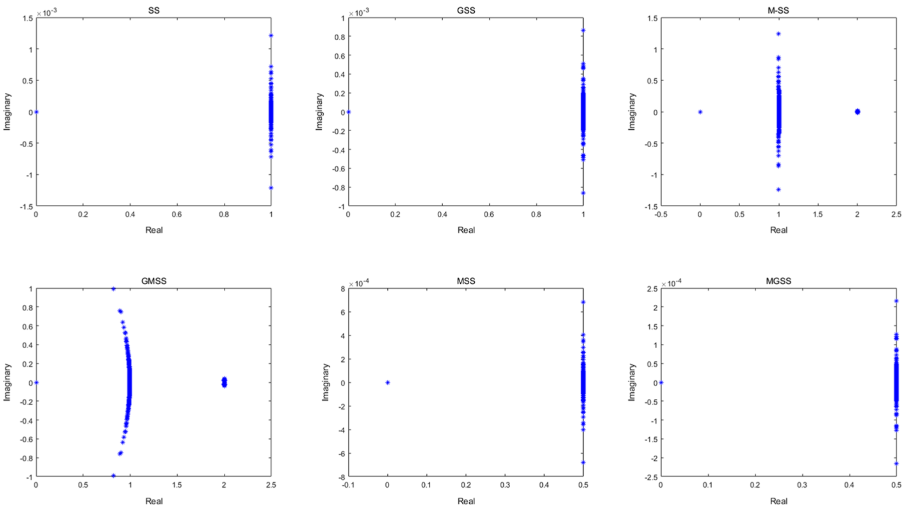

Fig. 6 The eigenvalue distributions of the six preconditioned matrices for $p = 32$ and $\upsilon = 0.1$

7 Conclusions

To solve the nonsymmetric saddle point problems, by combining the GSS and MSS of a matrix, we establish a modified generalized shift-splitting (MGSS) iteration method and the corresponding preconditioner called the MGSS preconditioner in this paper. The unconditional convergence and semi-convergence of the MGSS iteration method for solving nonsingular and singular saddle point problems, respectively, are discussed in detail. Moreover, eigenproperties of the preconditioned matrix are described. Numerical results given in Section 6 illustrate that the efficiency of the MGSS iteration method and the MGSS preconditioner for the saddle point problems with nonsymmetric positive definite (1,1) parts, and confirm that they outperform some existing ones. We should point out that the MGSS preconditioner may not have the optimality property, i.e., the iteration counts depend on the parameters α and β (see Figs. 2 and 5). Besides, admittedly, the choice of the optimal parameters of the MGSS iteration method and the MGSS preconditioned GMRES method is a challenging problem that deserves further study. For most iterative methods, this work is very complicated. Nevertheless, by adopting certain approximation strategies, there have been practically useful formulas for obtaining nearly optimal iteration parameters; see [22, 32, 39]. To further investigations, we would like to study how to further improve the MGSS preconditioner and choose the optimal parameters for the MGSS iteration method.

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