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Piecewise Chebyshevian splines: interpolation versus design

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Abstract We consider the wide class of all piecewise Chebyshevian splines with connection matrices at the knots. We prove that a spline space of this class is "good for interpolation" if and only if the spline space obtained by integration is "good for design". As a consequence, this provides us with a simple practical description of all such spline spaces which can be used for solving Hermite interpolation problems. These results strongly rely on the properties of blossoms.

Keywords Piecewise Chebyshevian splines · Connection matrices · Spline Hermite interpolation · Schoenberg-Whitney conditions · Total positivity · (Piecewise) Generalised derivatives · B-spline-type bases · Knot insertion · Blossoms

Mathematics Subject Classification (2010) 41A05 · 65D05 · 65D07 · 65D17

1 Introduction

We investigate the relations existing between Hermite interpolation and geometric design in spline spaces. Our general context to define Hermite interpolation problems will be the class of all piecewise W-spline spaces, that is, all spaces of splines with pieces taken from different W-spaces and with connection matrices at the knots. Let us recall that, on a given non-trivial interval I, a W-space is a space of sufficiently differentiable functions in which the Wronskian of any basis never vanishes on I, or, equivalently, a space in which any Taylor interpolation problem in I has a unique solution.

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This work is inspired by the non-spline case where we know how Hermite interpolation and geometric design are connected, which can be stated as follows [21, 23].

Theorem 1.1 *Given a W-space $\mathbb{E}$ on I, let $\widehat{\mathbb{E}}$ be the W-space on I composed of the primitives of all elements in $\mathbb{E}$. The following two properties are then equivalent:*

(i) *the W-space $\mathbb{E}$ is good for interpolation;*
(ii) *the W-space $\widehat{\mathbb{E}}$ is good for design.*

Moreover, when (i) is satisfied, the space $\widehat{\mathbb{E}}$ is good for interpolation in turn.

Let us comment on the latter result. By "$\mathbb{E}$ is good for interpolation", we mean that any Hermite interpolation problem has a unique solution in $\mathbb{E}$. In that case, classically, the space $\mathbb{E}$ is said to be an *Extended Chebyshev space on I* (in short, EC-space on I) [12, 39]. As for the W-space $\widehat{\mathbb{E}}$, it clearly contains the constants and it satisfies $\dim \widehat{\mathbb{E}} = \dim \mathbb{E} + 1$. The expression "$\widehat{\mathbb{E}}$ is good for geometric design" means that $\widehat{\mathbb{E}}$ possesses *blossoms*. In that case, with any $d \geqslant 1$ and any $\widehat{F} \in \widehat{\mathbb{E}}^d$, one can associate a symmetric function $\widehat{f}$ (the blossom of $\widehat{F}$), defined on $I^{\dim \mathbb{E}}$ in a geometrical way by means of intersections of osculating flats to a given mother function in $\widehat{\mathbb{E}}$. The properties of blossoms are strongly involved in the proof of Theorem 1.1. Finally, the last claim in Theorem 1.1 recalls the well-known fact that the class of all EC-spaces on a given interval is closed under integration, which readily follows from Rolle's theorem.

Proving the spline version of Theorem 1.1 is the main purpose of the present work. As a matter of fact, it can be stated similarly to Theorem 1.1, simply assuming I to be a closed bounded interval $[a, b]$ and replacing the W-space $\mathbb{E}$ by a piecewise W-spline space (for short, PW-spline space) on $([a, b]; \mathbb{T})$, where $\mathbb{T}$ is a finite sequence of interior knots. This yields:

Theorem 1.2 *Let $\mathbb{S}$ be a PW-spline space on $([a, b]; \mathbb{T})$, and let $\widehat{\mathbb{S}}$ be the PW-spline space obtained from $\mathbb{S}$ by continuous integration on $[a, b]$. The following two properties are then equivalent:*

(i) *the PW-spline space $\mathbb{S}$ is good for interpolation;*
(ii) *the PW-spline space $\widehat{\mathbb{S}}$ is good for design.*

Moreover, when (i) is satisfied, the space $\widehat{\mathbb{S}}$ is good for interpolation in turn.

This calls for some important preliminary observations. The expression "$\widehat{\mathbb{S}}$ is good for design" is now well established: it means that blossoms exist in the PW-spline space $\widehat{\mathbb{S}}$, given that spline blossoms only have to be defined on a symmetric restricted set of tuples depending on the knot-vector, by means of (possibly left/right) osculating flats. In contrast, we will have to give the precise definition of the expression "$\mathbb{S}$ is good for interpolation". This definition (Section 4) will take into account some features specific to the spline context. Firstly, to be of interest, a property is expected to be refinable, that is, to be preserved under knot insertion. Secondly, it is well known and easily seen that, as soon as a spline space possesses a basis of the B-spline type, for a given Hermite interpolation problem to be unisolvent, it is necessary that the interpolation sites and the knots satisfy some interlacing property often referred to as *the Schoenberg-Whitney conditions*. To conclude this preliminary presentation, let us mention that we could as well state Theorem 1.2 within the more restricted framework of piecewise Chebyshevian splines, that is, splines with pieces taken from different EC-spaces. Indeed, the property (ii) of Theorem 1.2 implies that each section-space of $\mathbb{S}$ is an EC-space on its own interval (see Theorem 1.1).

Let us now describe the organisation of the article. We present the W-spline context in the next section. It is the largest refinable context to define splines with linear connections between a number of left/right derivatives at each interior knot. It is also the largest context in which we can consider the most general spline Hermite interpolation problems. We explore in all details such problems with special attention to the difficulty inherent in the presence of connection matrices. These preliminary investigations are intended to facilitate the work in the subsequent sections. Unlike most papers dealing with spline interpolation, at each interior knot, we allow interpolation conditions beyond the number of derivatives involved in the corresponding connection conditions. Although bases of the B-spline type are a priori not expected to exist, we establish necessary conditions for a given interpolation problem to be unisolvent (Schoenberg-Whitney conditions [4, 37]), and we show how to possibly split such a problem into several simpler ones.

Theorem 1.2 is mainly based on many crucial results on blossoms which have been achieved during the last two decades. These results are succinctly reminded in Section 3, in particular their fundamental link with B-spline bases, from which we could eventually derive the description of all piecewise W-spline spaces good for design. Since blossoms cannot exist in $\widehat{\mathbb{S}}$ without the spline space $\mathbb{S}$ being a piecewise Chebyshevian spline space, in each section-space, we can replace the ordinary derivatives by *generalised derivatives* which can alternatively be used to write the connection conditions. The presence of blossoms in $\widehat{\mathbb{S}}$ is actually characterised by the existence of convenient generalised derivatives relative to which the connections are expressed by identity matrices [27]. This strong result is a key point in the proof of Theorem 1.2 given in Section 4. Besides, it inherently contains a constructive way to obtain practical necessary and sufficient conditions for $\widehat{\mathbb{S}}$ to be good for design, as was illustrated in [7, 29], for instance. Once Theorem 1.2 is established, these conditions are necessary and sufficient conditions for $\mathbb{S}$ to be good for interpolation as well. The examples treated in [7, 29] will thus enable us to illustrate unisolvence of spline interpolation in two different frameworks: firstly, a class of L-splines producing surprisingly powerful tension effects; secondly, geometrically continuous cubic splines, with special emphasis on *interpolation beyond design*. Final remarks are given in Section 6, including comparison with the note [13] in which a different proof of Theorem 1.2 had been announced, modelled on [5].

2 Preliminaries

In this section, we introduce the piecewise W-spline (PW-spline) context and we describe how to state the most general Hermite interpolation problems taking account of the possible presence of connection matrices. We also analyse such problems with a view to reduce the difficulties.

2.1 Piecewise W-splines

From now on, we consider a fixed interval $[a, b]$, $a < b$, and a sequence $\mathbb{T}$ of *interior knots*

$$\mathbb{T} := (t_1, \dots, t_q), \quad \text{with } t_0 := a < t_1 < t_2 < \cdots < t_q < t_{q+1} := b.$$

We will a priori not deal with functions on $[a, b]$, but with *piecewise functions F on* $([a, b]; \mathbb{T})$, in the sense that F is defined separately on each interval $[t_k^+, t_{k+1}^-]$, implying in particular that, for each $k = 1, \dots, q$, both $F(t_k{}^-)$ and $F(t_k{}^+)$ are defined, with possibly $F(t_k{}^-) \neq F(t_k{}^+)$. In such a case, unless explicitly mentioned, F is not a function on $[a, b]$. We shall deliberately use the somewhat abusive notation $F : \cup_{k=0}^{q}[t_k^+, t_{k+1}^-] \to \mathbb{R}$ to stress this fact. All properties of piecewise functions on $([a, b]; \mathbb{T})$ will be introduced separately on each $[t_k^+, t_{k+1}^-]$. For instance, given two piecewise functions F and G on $([a, b]; \mathbb{T})$, the equality $F = G$ (resp. the positivity of F on $([a, b]; \mathbb{T})$) means that $F(x) = G(x)$ (resp. $F(x) > 0$) for all $x \in [t_k^+, t_{k+1}^-]$, and all $k = 0, \dots, q$. We denote by $PC^n([a, b]; \mathbb{T})$ the set of all piecewise functions on $([a, b]; \mathbb{T})$ which are C^n on each interval $[t_k^+, t_{k+1}^-]$, and by $PC^n_+([a, b]; \mathbb{T})$ the set of all elements of $PC^n([a, b]; \mathbb{T})$ which are positive on $([a, b]; \mathbb{T})$.

Throughout the paper, for any $x \in [a, b]$ and any non-negative integer μ, the notation $x^{[\mu]}$ stands for x repeated μ times. Along with $\mathbb{T}$, we also consider

– a given sequence $m_1, \dots, m_q$ of *interior multiplicities*: for each k, m_k is the multiplicity of the knot t_k, with $0 \leqslant m_k \leqslant n + 1$; this yields the associated knot-vector

$$K := \left(t_0{}^{[m_0]}, t_1{}^{[m_1]}, \dots, t_q{}^{[m_q]}, t_{q+1}{}^{[m_{q+1}]}\right), \qquad \text{where } m_0 := m_{q+1} := n + 1;$$

– a given sequence $(R_1, \dots, R_q)$ of *connection matrices*: for each $k = 1, \dots, q$, R_k is a lower triangular matrix of order $(n + 1 - m_k)$ with positive diagonal entries;
– a given sequence $(\mathbb{E}_0, \mathbb{E}_1, \dots, \mathbb{E}_q)$ of *section-spaces*: for each $k = 0, \dots, q$, $\mathbb{E}_k \subset C^n([t_k, t_{k+1}])$ is an $(n + 1)$-dimensional W-space on $[t_k, t_{k+1}]$.

Definition 2.1 Based on the latter data, we define the *Piecewise W-spline (for short, PW-spline) space on* $([a, b]; \mathbb{T})$ as the linear space $\mathbb{S}$ composed of all piecewise functions S on $([a, b]; \mathbb{T})$ which satisfy

1) for each $k = 0, \dots, q$, there exists a function $F_k \in \mathbb{E}_k$ such that S coincides with F_k on $[t_k^+, t_{k+1}^-]$;
2) for $k = 1, \dots, q$, S satisfies the connection conditions

$$\left(S(t_k^+), S'(t_k^+), \dots, S^{(n-m_k)}(t_k^+)\right)^T = R_k \left(S(t_k^-), S'(t_k^-), \dots, S^{(n-m_k)}(t_k^-)\right)^T, \ 1 \leqslant k \leqslant q. \tag{1}$$

It is well known that

$$\dim \mathbb{S} = n + 1 + m, \quad \text{with } m := \sum_{i=1}^{q} m_i.$$

A spline S in the PW-spline space $\mathbb{S}$ is a priori not a function on $[a, b]$, but a piecewise function on $([a, b]; \mathbb{T})$. At a given interior knot t_k, $k = 1, \ldots, q$, the possible discontinuities have two different origins: either $m_k = n + 1$, and there is no connection condition at t_k, or $m_k \leqslant n$ and $S(t_k^+) = a_k S(t_k^-)$, with a positive $a_k \neq 1$. Nevertheless, in the latter case, the structure of the connection matrix guarantees that a spline $S \in \mathbb{S}$ vanishes $p \leqslant n + 1 - m_k$ at t_k^- if and only if it vanishes p times at t_k^+, and that it vanishes exactly $p < n + 1 - m_k$ at t_k^- if and only if it vanishes exactly p times at t_k^+. In the latter case, the positivity of the diagonal entries in R_k ensures sign consistency from t_k^- to t_k^+.

The lower triangular nature of the connection matrices is also essential to permit *refinability*, that is, to preserve the structure under *knot insertion*. A PW-spline space $\mathbb{S}^\star$ with $(n + 1)$-dimensional section-spaces $\mathbb{S}$ is said to be obtained from $\mathbb{S}$ by knot insertion if $\mathbb{S} \subset \mathbb{S}^\star$. It is based on a knot-vector

$$K^\star := \left(t_0{}^{[m_0]}, t_1^{\star [m_1^\star]}, \ldots, t_{q^\star}^{\star}{}^{[m_{q^\star}^\star]}, t_{q+1}{}^{[m_{q+1}]}\right),$$

also said to be obtained from $\mathbb{K}$ by knot insertion. The inclusion $\mathbb{S} \subset \mathbb{S}^\star$ means that

- the new knot-vector $\mathbb{K}^\star$ is a refinement of $\mathbb{K}$ in the sense that each knot t_k is a knot $t_{k^\star}^\star$ in $\mathbb{K}^\star$, with new multiplicity $m_{k^\star}^\star \geqslant m_k$; the connection matrix $R_{k^\star}^\star$ at t_k in the new space $\mathbb{S}^\star$ is the lower triangular matrix obtained by deleting the last $(m_{k^\star}^\star - m_k)$ rows and columns of R_k;
- at each knot $t_k^\star$ in $\mathbb{K}^\star$ with multiplicity $m_k^\star$ which is not a knot in $\mathbb{K}$, the connection matrix in the PW-spline space $\mathbb{S}^\star$ is the identity matrix of order $(n + 1 - m_k^\star)$.

We conclude this presentation recalling that piecewise multiplication by any $\omega \in PC_+^n([a, b]; \mathbb{T})$ transforms the PW-spline space $\mathbb{S}$ into another PW-spline space on $([a, b]; \mathbb{T})$, in which the diagonal of the new (lower triangular) connection matrix at t_k is obtained by multiplying the one of R_k by the positive number $\omega(t_k^+)/\omega(t_k^-)$. This will implicitly be involved throughout the article.

2.2 Hermite interpolation in PW-spline spaces

In this difficult and large context, the first delicate task consists in specifying what exactly is meant by a Hermite interpolation problem in the PW-spline space $\mathbb{S}$. With this in view, we start with

- interpolation sites $x_1 < \cdots < x_r$ in $\cup_{k=0}^{q}[t_k^+, t_{k+1}^-]$ (with the convention that $t_k^- < t_k^+$ for $k = 1, \ldots, q$): which we refer to as *nodes*,
- interpolation multiplicities: positive integers $\mu_1, \ldots, \mu_r \leqslant n + 1$.

Extending the notation $x^{[\mu]}$ for any $x \in \cup_{k=0}^{q}[t_k^+, t_{k+1}^-]$, the associated *node-vector* is defined by $\mathbb{Y} := (x_1{}^{[\mu_1]}, \dots, x_r{}^{[\mu_r]})$. We can then consider problems of the form:

$$\text{find } S \in \mathbb{S} \text{ such that } S^{(p)}(x_i) = a_{i,p}, \quad 0 \leqslant p \leqslant \mu_i - 1,\ 1 \leqslant i \leqslant r, \tag{2}$$

where $a_{i,j}$, $i = 1, \dots, r$, $j = 0, \dots, \mu_i - 1$, are any real numbers referred to as *the interpolation data*. Before saying more on the total number of conditions in (2), we have to discuss what happens at interior knots. Clearly, for the previous interpolation problem to have at least one solution, it is necessary that the interpolation data satisfy the following conditions:

$$\begin{aligned}&\text{for any integer } \ k \in \{1, \dots, q\} \text{ such that } x_i = t_k^- \text{ and } x_{i+1} = t_k^+, \text{ for some } i,\ 1 \leqslant i \leqslant r-1,\\ &\qquad (a_{i+1,0}, a_{i+1,1}, \dots, a_{i+1,\nu(k)-1})^T = R_{k,\nu(k)} \,.\, (a_{i,0}, a_{i,1}, \dots, a_{i,\nu(k)-1})^T,\end{aligned} \tag{3}$$

where

$$\nu(k) := \mathrm{Min}(\mu_i, \mu_{i+1}, n + 1 - m_k).$$

and where $R_{k,\nu(k)}$ denotes the square matrix of order $\nu(k)$ obtained by restricting the connection matrix R_k to the entries in its first $\nu(k)$ rows and columns. Then, $\nu(k)$ among the first $2\nu(k)$ interpolation conditions at the two nodes x_i and x_{i+1} being redundant, we can thus keep only those at one of them. Furthermore, we then do not change the problem if we assume that

$$\text{for } k = 1, \dots, q : \nu(k) < n + 1 - m_k \quad \Rightarrow \mu_i = \mu_{i+1}. \tag{4}$$

Taking the latter considerations into account, it is more convenient to present interpolation problems in the PW-spline space $\mathbb{S}$ differently. Each interior knot t_k is allocated three integers $\nu(k), \alpha_k^-, \alpha_k^+$, with

$$0 \leqslant \nu(k) \leqslant n + 1 - m_k \text{ and } 0 \leqslant \alpha_k^-, \alpha_k^+ \leqslant m_k \text{ for } 1 \leqslant k \leqslant q, \tag{5}$$

along with the following *consistency conditions*

$$\text{for each } k = 1, \dots, q, \quad \nu(k) < n - m_k + 1 \quad \Rightarrow \quad \alpha_k^- = \alpha_k^+ = 0. \tag{6}$$

For $1 \leqslant k \leqslant q$, choose $\varepsilon_k \in \{-, +\}$. We define the node-vector as a sequence $\mathbb{Y} = (y_{-n}, \dots, y_m)$ in $\cup_{k=0}^{q}[t_k^+, t_{k+1}^-]$, with $y_{-n} \leqslant y_{-n+1} \leqslant \dots \leqslant y_{m-1} \leqslant y_m$, and with, up to permutation,

$$(y_{-n}, \dots, y_m) = \left(x_1{}^{[\mu_1]}, \dots, x_r{}^{[\mu_r]}, t_1^{\varepsilon_1}{}^{[\alpha_1^{\varepsilon_1}+\nu(1)]}, t_1^{-\varepsilon_1}{}^{[\alpha_1^{-\varepsilon_1}]}, \dots, t_q^{\varepsilon_q}{}^{[\alpha_q^{\varepsilon_q}+\nu(q)]}, t_q^{-\varepsilon_q}{}^{[\alpha_q^{-\varepsilon_q}]}\right), \tag{7}$$

where

$$\begin{aligned}&x_1, \dots, x_r \in [a, b] \setminus \{t_1, \dots, t_q\}, \quad 0 < \mu_i \leqslant n + 1 \text{ for } 1 \leqslant i \leqslant r,\\ &\sum_{i=1}^{r} \mu_i + \sum_{k=1}^{q} \left[\alpha_k^- + \nu(k) + \alpha_k^+\right] = n + 1 + m.\end{aligned} \tag{8}$$

The latter node-vector $\mathbb{Y}$ leads to Hermite interpolation problems associated with any given interpolation data

$$\gamma_{i,p}, \quad 0 \leqslant p \leqslant \mu_i - 1,\ 1 \leqslant i \leqslant r,$$
$$\eta_{k,p}^{\varepsilon_k}, \quad 0 \leqslant p \leqslant \nu(k) + \alpha_k^{\varepsilon_k} - 1, \text{ and } \eta_{k,p}^{-\varepsilon_k}, \quad \nu(k) \leqslant p \leqslant \nu(k) + \alpha_k^{-\varepsilon_k} - 1, \text{ for } 1 \leqslant k \leqslant q.$$

In a systematic way, we start by defining all missing $\eta_{k,p}^{-\varepsilon_k}$, $0 \leqslant p \leqslant \nu(k) - 1$, via the equalities

$$(\eta_{k,1}^+, \ldots, \eta_{k,\nu(k)-1}^+) = R_{k,\nu(k)}(\eta_{k,1}^-, \ldots, \eta_{k,\nu(k)}^-), \quad 1 \leqslant k \leqslant q.$$

This enables us to exchange the rôles of t_k^- and t_k^+ in the node-vector $\mathbb{Y}$. The associated *Hermite interpolation problem* $(\mathcal{H})$ in the PW-spline space $\mathbb{S}$ then consists in searching for an element $S \in \mathbb{S}$ which satisfies the total amount of $(n+1+m)$ conditions (9), (10), (11) below

$$S^{(p)}(x_i) = \gamma_{i,p}, \quad 0 \leqslant p \leqslant \mu_i - 1, \quad 1 \leqslant i \leqslant r, \tag{9}$$

and, for $k = 1, \ldots, q$,

$$S^{(p)}(t_k^\varepsilon) = \eta_{k,p}^\varepsilon, \quad 0 \leqslant p \leqslant \nu(k) - 1 \text{ for } \textit{either } \varepsilon = -, \textit{ or } \varepsilon = +; \tag{10}$$
$$S^{(p)}(t_k^\varepsilon) = \eta_{k,p}^\varepsilon, \quad \nu(k) \leqslant p \leqslant \nu(k) + \alpha_k^\varepsilon - 1 \text{ for } \textit{both } \varepsilon = -, \textit{ and } \varepsilon = +. \tag{11}$$

Now that we can exchange the rôles of t_k^- and t_k^+, for the sake of symmetry, it is even preferable not to allocate each integer $\nu(k)$, $1 \leqslant k \leqslant q$, to $t_k^{\varepsilon_k}$, but directly to t_k, therefore writing the node-vector $\mathbb{Y}$ symmetrically as (up to permutation)

$$\mathbb{Y} = \left(x_1{}^{[\mu_1]}, \ldots, x_r{}^{[\mu_r]}, t_1^{-[\alpha_1^-]}, t_1{}^{[\nu(1)]}, t_1^{+[\alpha_1^+]}, \ldots, t_q^{-[\alpha_q^-]}, t_q{}^{[\nu(q)]}, t_q^{+[\alpha_q^+]}\right). \tag{12}$$

Let us select any fixed k_0, $1 \leqslant k_0 \leqslant q$. The interpolation conditions can then be separated into three disjoint categories:

- those concerning only $\cup_{i=0}^{k_0-1}[t_i^+, t_{i+1}^-]$, the total number of which will be denoted by $\lambda(k_0)$;
- those concerning only $\cup_{i=k_0}^{q}[t_i^+, t_{i+1}^-]$, the total number of which will be denoted by $\varrho(k_0)$;
- those concerning both $\cup_{i=0}^{k_0-1}[t_i^+, t_{i+1}^-]$ and $\cup_{i=k_0}^{q}[t_i^+, t_{i+1}^-]$, that is, the $\nu(k_0)$ conditions

$$S^{(p)}(t_{k_0}^\varepsilon) = \eta_{k_0,p}^\varepsilon, \quad 0 \leqslant p \leqslant \nu(k_0) - 1 \text{ for either } \varepsilon = -, \text{ or } \varepsilon = +. \tag{13}$$

Observe that

$$\lambda(k) + \nu(k) + \varrho(k) = n + 1 + m, \quad 1 \leqslant k \leqslant q.$$

Let us denote by $\mathbb{S}_{k_0}^-$ the restriction of $\mathbb{S}$ to $\cup_{i=0}^{k_0-1}[t_i^+, t_{i+1}^-]$ and by $\mathbb{S}_{k_0}^+$ its restriction to $\cup_{i=k_0}^{q}[t_i^+, t_{i+1}^-]$. It is natural to consider the following two problems

- $(\mathcal{H}_{k_0}^-)$: find a spline S^- in the PW-spline space $\mathbb{S}_{k_0}^-$ satisfying the $\lambda(k_0) + \nu(k_0)$ interpolation conditions which concern $\cup_{i=0}^{k_0-1}[t_i^+, t_{i+1}^-]$, among which the $\nu(k_0)$ conditions (13) obtained with $\varepsilon = -$;

– $(\mathcal{H}_{k_0}^+)$: find a spline S^+ in the PW-spline space $\mathbb{S}_{k_0}^+$ satisfying the $\varrho(k_0)+\nu(k_0)$ interpolation conditions which concern $\cup_{i=k_0}^{q}[t_i^+, t_{i+1}^-]$, among which the $\nu(k_0)$ conditions (13) obtained with $\varepsilon = +$.

It should be observed that $(\mathcal{H}_{k_0}^-)$ cannot be considered a Hermite interpolation problem in the PW-spline space $\mathbb{S}_{k_0}^-$ stricto sensu because, without additional assumptions, we do not know whether or not the equality

$$\lambda(k_0) + \nu(k_0) = \dim(\mathbb{S}_{k_0}^-) = n + 1 + \sum_{i=1}^{k_0-1} m_i \tag{14}$$

is satisfied. Similarly, $(\mathcal{H}_{k_0}^+)$ will be a Hermite interpolation problem in the PW-spline space $\mathbb{S}_{k_0}^-$ only if we can make sure that

$$\varrho(k_0) + \nu(k_0) = \dim(\mathbb{S}_{k_0}^+) = n + 1 + \sum_{i=k_0+1}^{q} m_i. \tag{15}$$

Still, whether or not (14) and (15) hold, any solution $S \in \mathbb{S}$ to the initial problem $(\mathcal{H})$ yields a solution $S^- \in \mathbb{S}_{k_0}^-$ to $(\mathcal{H}_{k_0}^-)$ and a solution $S^+ \in \mathbb{S}_{k_0}^+$ to $(\mathcal{H}_{k_0}^+)$ by restriction to $\cup_{i=0}^{k_0-1}[t_i^+, t_{i+1}^-]$ and $\cup_{i=k_0}^{q}[t_i^+, t_{i+1}^-]$, respectively. Conversely, let us start from a solution $S^- \in \mathbb{S}_{k_0}^-$ to $(\mathcal{H}_{k_0}^-)$ and a solution $S^+ \in \mathbb{S}_{k_0}^+$ to $(\mathcal{H}_{k_0}^+)$. Let S be the piecewise function on $([a,b];\mathbb{T})$ built from S^- and S^+. Though S obviously satisfies the $(n+1+m)$ interpolation conditions required by $(\mathcal{H})$, it is not necessarily a solution to $(\mathcal{H})$ because it is not necessarily an element of the PW-spline space $\mathbb{S}$. Note that, for S to belong to $\mathbb{S}$, it is sufficient that

$$\nu(k_0) = n + 1 - m_{k_0}. \tag{16}$$

When condition (16) holds, solving any Hermite interpolation problem $(\mathcal{H})$ associated with $\mathbb{Y}$ is thus equivalent to solving separately the corresponding two problems $(\mathcal{H}_{k_0}^-)$ and $(\mathcal{H}_{k_0}^+)$, even though these may still fail to be Hermite interpolation problems in $\mathbb{S}_{k_0}^-$, $\mathbb{S}_{k_0}^+$.

2.3 Schoenberg-Whitney conditions

After the previous discussion, we can establish necessary conditions for any problem $(\mathcal{H})$ to have a unique solution in $\mathbb{S}$.

Theorem 2.2 *Consider any given Hermite interpolation problem $(\mathcal{H})$ based on the node-vector $\mathbb{Y}$ defined by (12). If $(\mathcal{H})$ possesses a unique solution in $\mathbb{S}$, then the node vector $\mathbb{Y}$ satisfies the following conditions*

$$\textit{for each } k = 1, \ldots, q, \quad \lambda(k) \geqslant \sum_{i=1}^{k} m_i \textit{ and } \varrho(k) \geqslant \sum_{i=k}^{q} m_i, \tag{17}$$

which we refer to as the Schoenberg-Whitney conditions (for short, SW-conditions).

Proof With no loss of generality, we can assume that all interpolation data are 0. Given an integer k_0, $1 \leqslant k_0 \leqslant q$, assume that $\lambda(k_0) < \sum_{i=1}^{k_0} m_i$. We shall then prove the existence of a non-zero $S \in \mathbb{S}$ which is a solution to $(\mathcal{H})$. Let us first build S^- the restriction of S to $\cup_{k=0}^{k_0-1}[t_k^+, t_{k+1}^-]$. It must satisfy the interpolation conditions concerned, that is, a total amount of conditions equal $\lambda(k_0) + \nu(k_0)$. In case $\nu(k_0) < n + 1 - m_{k_0}$, we even add $(n + 1 - m_{k_0} - \nu(k_0))$ additional conditions, requiring S^- to have all derivatives up to order $(n - m_{k_0})$ to be zero at $t_{k_0}^-$. This way, the total amount of interpolation requirements on S^- is

$$\lambda(k_0) + n + 1 - m_{k_0} < n + 1 + \sum_{i=1}^{k_0} m_i - m_{k_0} = n + 1 + \sum_{i=1}^{k_0-1} m_i = \dim \mathbb{S}_{k_0}^-.$$

We can thus choose a non-zero $S^- \in \mathbb{S}_{k_0}^-$ satisfying all previous interpolation conditions. For the restriction S^+ of S to $\cup_{k=k_0}^{q}[t_k^+, t_{k+1}^-]$, we take 0. Since S^- vanishes at least $n - m_{k_0} + 1$ times at $t_{k_0}^-$, the function S built from S^- and S^+ is indeed a non-zero element of $\mathbb{S}$ which satisfies the interpolation problem $(\mathcal{H})$.

Similar arguments can be used to prove the right part of (17). □

When the node-vector $\mathbb{Y}$ satisfies the SW-conditions (17), we will say as well that any Hermite interpolation problem $(\mathcal{H})$ based on $\mathbb{Y}$ satisfies the SW-conditions. We can now complete the discussion above as follows.

Theorem 2.3 *Given a PW-spline space based on the knot-vector $\mathbb{K}$, we assume that condition (16) holds for some integer k_0, $1 \leqslant k_0 \leqslant q$. The following properties are then equivalent*

- (i) *the node-vector $\mathbb{Y}$ satisfies the SW-conditions (17);*
- (ii) *for any Hermite interpolation problem $(\mathcal{H})$ in $\mathbb{S}$, based on $\mathbb{Y}$, the two problems $(\mathcal{H}_{k_0}^-)$ and $(\mathcal{H}_{k_0}^+)$ presented above are Hermite interpolation problems in the PW-spline spaces $\mathbb{S}_{k_0}^-$, $\mathbb{S}_{k_0}^+$, respectively, and each of them satisfies the SW-conditions.*

Furthermore, if (i) holds, then, solving $(\mathcal{H})$ is equivalent to solving separately the $(\mathcal{H}_{k_0}^-)$ and $(\mathcal{H}_{k_0}^+)$.

Proof First, observe that the last statement results from the discussion above.

•(i) ⇒ (ii): Suppose that $\mathbb{Y}$ satisfies (17). Since $\lambda(k_0)+\nu(k_0)+\varrho(k_0) = n+1+m$, we first note that (16) holds if and only if

$$\lambda(k_0) + \varrho(k_0) = m + m_{k_0} = \sum_{i=1}^{k_0} m_i + \sum_{i=k_0}^{q} m_i.$$

Given that $\lambda(k_0) \geqslant \sum_{i=1}^{k_0} m_i$ and $\varrho(k_0) \geqslant \sum_{i=k_0}^{q} m_i$, the previous equality implies

$$\lambda(k_0) = \sum_{i=1}^{k_0} m_i, \quad \varrho(k_0) = \sum_{i=k_0}^{q} m_i. \tag{18}$$

From (18) and (16), one can easily deduce that both (14) and (15) hold. In other words, $(\mathcal{H}_{k_0}^-)$ and $(\mathcal{H}_{k_0}^+)$ are indeed Hermite interpolation problems in the PW-spline spaces $\mathbb{S}_{k_0}^-$, $\mathbb{S}_{k_0}^+$, respectively.

It remains to check that they both satisfy the SW-conditions (17). With each interior knot t_k, $k \in \{1, \ldots, t_{k_0-1}\}$ of the PW-spline space $\mathbb{S}_{k_0}^-$, we associate two numbers $\lambda^-(k)$, $\varrho^-(k)$ defined similarly to $\lambda(k)$, $\varrho(k)$ in $\mathbb{S}$. Clearly,

$$\lambda^-(k) = \lambda(k), \quad \varrho^-(k) = \varrho(k) - \varrho(k_0), \quad 1 \leqslant k \leqslant k_0 - 1. \tag{19}$$

Taking account of (18) and of $(\mathcal{H})$ satisfying (17) this yields

$$\lambda^-(k) \geqslant \sum_{i=1}^{k} m_i, \quad \varrho^-(k) \geqslant \sum_{i=k}^{q} m_i - \sum_{i=k_0}^{q} m_i = \sum_{i=k}^{k_0-1} m_i,$$

which proves that $(\mathcal{H}_{k_0}^-)$ satisfies (17). Symmetric arguments can be used for $\mathbb{S}_{k_0}^+$.

•(ii) $\Rightarrow$ (i) : Suppose that (ii) holds. This means in particular that the two conditions (14) and (15) are fulfilled. Taking account of (16), one can deduce from them that the two equalites in (18) are satisfied. Using (19), the symmetric equalities for $\mathbb{S}_{k_0}^+$, and the fact that both $(\mathcal{H}_{k_0}^-)$ and $(\mathcal{H}_{k_0}^+)$ satisfy the SW-conditions (17), it is easy to derive that $(\mathcal{H})$ itself satisfies (17). □

Remark 2.4 Let us comment on Theorem 2.3.

1- Iterated application of Theorem 2.3 ensures that solving any given Hermite interpolation problem satisfying the SW-conditions can be replaced by solving a number of Hermite interpolation problems (in PW-spline spaces obtained by restriction of $\mathbb{S}$) which all satisfy the SW-conditions plus the condition

$$\nu(k) < n + 1 - m_k \text{ at all their interior knots } t_k. \tag{20}$$

2- Condition (16) holds in particular whenever the interior knot t_{k_0} has multiplicity $m_{k_0} = n+1$, in which case $\nu(k_0) = 0$, that is, there is no interpolation condition of the form (10) at t_{k_0}. Nevertheless, there may be $\alpha_{k_0}^- \leqslant n+1$ interpolation conditions at $t_{k_0}^-$ and $\alpha_{k_0}^+ \leqslant n+1$ interpolation conditions at $t_{k_0}^+$. Solving a given Hermite interpolation problem $(\mathcal{H})$ satisfying the SW-conditions (17) in a W-spline space with interior multiplicities less than or equal to $(n+1)$ can thus equivalently be replaced by solving a number of Hermite interpolation problems all satisfying (17) in PW-spline spaces with interior multiplicities less than or equal to n. This important observation will enable us to assume all interior multiplicities bounded by n if needed.

Suppose there exists $k_0 \in \{1, \ldots, q\}$ such that at least one of the two integers $\alpha_{k_0}^-$, $\alpha_{k_0}^+$ is not zero. Then the consistency condition (6) implies that $\nu(k_0) = n + 1 - m_{k_0}$, and according to Theorem 2.3, we can split the problem $(\mathcal{H})$. It is therefore interesting to focus on the situation where

$$\alpha_k^- = \alpha_k^+ = 0 \quad \text{for } k = 1, \ldots, q. \tag{21}$$

Throughout the rest of the article, the knot-vector $\mathbb{K}$ will also be denoted as

$$\mathbb{K} = (\xi_{-n}, \xi_{-n+1}, \ldots, \xi_{m+n+1}), \quad \text{with } \xi_k \leqslant \xi_{k+1} \text{ for } -n \leqslant k \leqslant m+n. \tag{22}$$

This notation will be useful to state the SW-conditions in a more usual way under the restricted assumption (21). For each integer, $k = 0, \dots, q+1$, let j_k be the greatest integer j such that $\xi_j \leqslant t_k$. We thus have

$$j_0 = 0, \qquad j_k = \sum_{i=1}^{k} m_i, \quad 1 \leqslant k \leqslant q.$$

Moreover, each knot t_k, $k = 0, \dots, q+1$, with positive multiplicity satisfies

$$\xi_j = t_k \quad \Leftrightarrow \quad j_k - m_k + 1 \leqslant j \leqslant j_k. \tag{23}$$

Theorem 2.5 *Consider a node-vector $\mathbb{Y}$ for which* (21) *holds. The following properties are then equivalent:*

1. *$\mathbb{Y}$ satisfies the SW-conditions* (17)*;*
2. *the interior knots interlace the nodes as follows:*

$$y_{\ell-n-1} < \xi_\ell < y_\ell, \quad 1 \leqslant \ell \leqslant m. \tag{24}$$

3. *for all integers $0 \leqslant k < k' \leqslant q+1$, such that $m_k, m_{k'} > 0$, the relative interior of $[t_k, t_{k'}]$ (in $[a, b]$) contains at least $\left(-n - 1 + \sum_{i=k}^{k'} m_i\right)$ nodes (counted with their multiplicities).*

Proof When (21) holds, for each $k = 1, \dots, q$, t_k^-, t_k^+ do not appear in the node-vector $\mathbb{Y}$ written as (12) up to permutation. Therefore, the two numbers $\lambda(k)$ et $\varrho(k)$ can be defined as follows:

$$\lambda(k) := \#\{j, -n \leqslant j \leqslant m \mid y_j < t_k\}, \quad \varrho(k) := \#\{j, -n \leqslant j \leqslant m \mid y_j > t_k\}.$$

•(i) $\Rightarrow$ (ii): Suppose the existence of some ℓ_0, $1 \leqslant \ell_0 \leqslant m$, such that $y_{\ell_0-n-1} \geqslant \xi_{\ell_0}$. Then, $\xi_{\ell_0} = t_{k_0}$ for some integer $k_0 \in \{1, \dots, q\}$, implying that

$$m_{k_0} > 0, \quad j_{k_0} - m_{k_0} + 1 \leqslant \ell_0 \leqslant j_{k_0}.$$

Then, the indices j for which $y_j < t_{k_0}$ all belong to the set $\{-n, \dots, \ell_0 - n - 2\}$. Therefore,

$$\lambda(k_0) \leqslant \ell_0 - 1 \leqslant j_{k_0} - 1 < j_{k_0} = \sum_{i=1}^{k_0} m_i,$$

which contradicts (17). Similar arguments show that, if there exists an integer $\ell_0 \in \{1, \dots, m\}$ such that $y_{\ell_0} \leqslant \xi_{\ell_0}$, the SW-conditions (17) cannot be satisfied either.

•(ii) $\Rightarrow$ (iii): Let an integer k, $1 \leqslant k \leqslant q$, satisfy $m_k > 0$. Then, combining (24) and (23) shows that

$$-n \leqslant j \leqslant j_k - n - 1 \quad \Rightarrow \quad y_j < t_k, \qquad j_k - m_k + 1 \leqslant j \leqslant m \quad \Rightarrow \quad y_j > t_k.$$

It is therefore straightforward to obtain (iii).

•(iii) $\Rightarrow$ (i): That $\lambda(k) \geqslant \sum_{i=1}^{k} m_i$ and $\mu(k) \geqslant \sum_{i=k}^{q} m_i$ for each $k = 1, \dots, q$ such that $m_k > 0$, is contained in (iii). The case $m_k = 0$ readily follows. □

3 The background

In this section, we briefly review the background necessary for Theorem 1.2. We will mainly list various ways to characterise the fact that a PW-spline space is good for design. Beforehand, it is necessary to remind the reader of the construction of Extended Chebyshev (piecewise) spaces by means of (piecewise) generalised derivatives.

3.1 PEC-splines and (piecewise) weight functions

Given a non-trivial real interval I, an $(n+1)$-dimensional space $\mathbb{E} \subset C^n(I)$ is an EC-space on I if and only if any non-zero $F \in \mathbb{E}$ vanishes at most n times on I, counting multiplicities up to $(n+1)$. Let us recall the classical procedure to build EC-spaces via systems of weight functions on I. A *system of weight functions on* I is a sequence $(w_0, \ldots, w_n)$ of functions defined on I, with the requirement that, for $i = 0, \ldots, n$, w_i is positive and C^{n-i} on I. Classically, such a system generates associated *generalised derivatives* $L_0, \ldots, L_n$, recursively defined on $C^n(I)$ as follows:

$$L_0 F := \frac{F}{w_0}, \quad L_i F := \frac{D L_{i-1} F}{w_i} \quad \text{for } 1 \leqslant i \leqslant n, \tag{25}$$

where D stands for the ordinary differentiation. As is well known, the set of all functions $F \in C^n(I)$ such that $L_n F$ is constant on I is then an $(n+1)$-dimensional EC-space on I. We denote it by $EC(w_0, \ldots, w_n)$. Conversely, in case the interval I is closed and bounded, any given $(n+1)$-dimensional EC-space $\mathbb{E}$ on I is of the form $\mathbb{E} = EC(w_0, \ldots, w_n)$. In other words, on the closed bounded interval $I = [a, b]$, $a < b$, (i) of Theorem 1.1 is equivalent to the existence of a system $(w_0, \ldots, w_n)$ of weight functions on I such that $\mathbb{E} = EC(w_0, \ldots, w_n)$, or such that $\widehat{E} = EC(\mathbb{I}, w_0, \ldots, w_n)$. Here, and throughout the article, we use the notation $\mathbb{I}$ for the constant $\mathbb{I}(x) = 1$ for all x in any given interval. The infinitely many different such systems of weight functions such that $\mathbb{E} = EC(w_0, \ldots, w_n)$ were described in [26].

Let us come back to the piecewise W-spline space $\mathbb{S}$ presented in Definition 2.1. We now assume that it is a PEC-spline space, i.e., the section-spaces are EC-spaces on their intervals. For each $k = 0, \ldots, q$, select any system $(w_0^k, \ldots, w_n^k)$ of weight functions on $[t_k, t_{k+1}]$ such that

$$\mathbb{E}_k = EC(w_0^k, \ldots, w_n^k), \quad k = 0, \ldots, q,$$

For each $k = 0, \ldots, q$, denote by $L_0^k, \ldots, L_n^k$ the associated generalised derivatives on $C^n([t_k, t_{k+1}])$. Then, instead of (1), we can alternatively express the connection conditions at the interior knot t_k, $k = 1, \ldots, q$, as follows:

$$\left(L_0^k(t_k^+), L_1^k(t_k^+), \ldots, L_{n-m_k}^k(t_k^+)\right)^T = \overline{R}_k \left(L_0^{k-1}(t_k^-), L_1^{k-1}(t_k^-), \ldots, L_{n-m_k}^{k-1}(t_k^-)\right)^T, \tag{26}$$

the new matrix $\overline{R}_k$ being lower triangular with positive diagonal entries due to the positivity of all weight functions. For convenience, for each $i = 0, \dots, n$, denote by w_i the piecewise function on $([a, b]; \mathbb{T})$ whose restriction to each $[t_k^+, t_{k+1}^-]$ coincides with w_i^k. According to our notations and terminology, each w_i is positive on $([a, b]; \mathbb{T})$, and therefore, $w_i \in PC_+^{n-i}([a, b]; \mathbb{T})$. We say that such a sequence is a *system of piecewise weight functions on* $([a, b]; \mathbb{T})$. We can now associate with it *piecewise generalised derivatives*, defined on $PC^n([a, b]; \mathbb{T})$ by formulæ similar to (25), which now have the meaning of piecewise equalities. Then, we can as well describe the spline space $\mathbb{S}$ as the set of all piecewise functions $S \in PC^n([a, b]; \mathbb{T})$ such that $L_n S$ is piecewise constant on $([a, b]; \mathbb{T})$ and which satisfy the connection conditions

$$\left(L_0(t_k^+), L_1(t_k^+), \dots, L_{n-m_k}(t_k^+)\right)^T = \overline{R}_k \left(L_0(t_k^-), L_1(t_k^-), \dots, L_{n-m_k}(t_k^-)\right)^T, \; k = 1, \dots, q. \quad (27)$$

To conclude this section, consider again the PW-spline space $\mathbb{S}$ presented in Definition 2.1, but now in the special case where $m_k = 0$ for all $k = 1, \dots, q$. Then, $\mathbb{S}$ is $(n + 1)$-dimensional. In that case, we will rather name it $\mathbb{E}$ instead of $\mathbb{S}$. According to our reminder in the previous section, we can now count the zeroes in $\mathbb{E}$ with multiplicities up to $(n + 1)$. When any non-zero element of $\mathbb{E}$ vanishes at most n times, we say that $\mathbb{E}$ *is an Extended Chebyshev Piecewise space* (for short, ECP-space) *on* $([a, b]; \mathbb{T})$ [22–24, 27]. In particular, a piecewise version of Rolle's theorem [23] shows that it is so when the ($(n+1)$-th order) connection matrices in (27) are identity matrices, as recalled in the theorem below.

Theorem 3.1 *Let* $(w_0, \dots, w_n)$ *be a system of piecewise weight functions on* $([a, b]; \mathbb{T})$*, with associated piecewise generalised derivatives* $L_0, \dots, L_n$*. Denote by* $ECP(w_0, \dots, w_n)$ *the set of all piecewise functions* $F \in PC^n([a, b]; \mathbb{T})$ *such that:*

1. $L_n F$ *is constant on* $[a, b]$*;*
2. *for each* $i = 1, \dots, n - 1$*,* $L_i F$ *is continuous on* $[a, b]$.

The space $ECP(w_0, \dots, w_n)$ *is an* $(n + 1)$*-dimensional ECP-space on* $([a, b]; \mathbb{T})$. *Conversely, any* $(n + 1)$*-dimensional ECP-space on* $([a, b]; \mathbb{T})$ *is of the form* $ECP(w_0, \dots, w_n)$ *for some system* $(w_0, \dots, w_n)$ *of piecewise weight functions on* $([a, b]; \mathbb{T})$.

3.2 Design with PW-splines

Our background will now concern the class of all PW-splines which can be used to design, according to the definition below. Though blossoms and their properties are inherently connected with design, we will say the least possible about them to facilitate the reading. Readers interested in this elegant and powerful tools can refer to [16–23, 25, 27, 35], for instance, and additional references therein.

Definition 3.2 In short, we say that the PW-spline space $\mathbb{S}$ introduced in Definition 2.1 is *good for design* when the following two properties are satisfied:

1- $\mathbb{S}$ contains the constants;
2- blossoms exist in $\mathbb{S}$.

In case one interior knot t_{k_0}, $1 \leqslant k_0 \leqslant q$, is of multiplicity $m_{k_0} = n+1$, all tools involved in the present section can be introduced separately in each of the PW-spline spaces $\mathbb{S}_{k_0}^-$ and $\mathbb{S}_{k_0}^+$. Without loss of generality, throughout the section, we therefore assume that

$$0 \leqslant m_k \leqslant n, \quad 1 \leqslant k \leqslant q. \tag{28}$$

Definition 3.3 In the PW-spline $\mathbb{S}$, a sequence $(B_{-n}, B_{-n+1}, \ldots, B_m)$, is said to be a B-spline-like basis of $\mathbb{S}$ if, for each ℓ, B_ℓ satisfies the following properties:

(BSLB)$_1$ *support property*: $B_\ell(x) = 0$ for $x < \xi_\ell$ or $x > \xi_{\ell+n+1}$;
(BSLB)$_2$ *positivity property*: $B_\ell(x^\varepsilon) > 0$ for $\xi_\ell < x < \xi_{\ell+n+1}$ and $\varepsilon = \pm$;
(BSLB)$_1$ *endpoint property*: B_ℓ vanishes exactly $(n-s+1)$ times at ξ_ℓ^+ and exactly $(n-s'+1)$ at $\xi_{\ell+n+1}^-$, where $s := \#\{j \geqslant \ell \mid \xi_j = \xi_\ell\}$ and $s' := \#\{j \leqslant \ell+n+1 \mid \xi_j = \xi_{\ell+n+1}\}$.

Definition 3.4 In the PW-spline $\mathbb{S}$, a B-spline basis is a B-spline-like basis $(N_{-n}, N_{-n+1}, \ldots, N_m)$ which is normalised, in the sense that $\sum_{\ell=-n}^m N_\ell = \mathbb{1}$.

Let us now comment on each requirement in Definition 3.2. For $\mathbb{S}$ to contain the constants it is necessary and sufficient to assume that

– firstly, each section-space $\mathbb{E}_k$, $k = 0, \ldots, q$, contains the constants;
– secondly, the first column of each connection matrix R_k is equal to $(1, \underbrace{0, \ldots, 0}_{(n-m_k)\text{ times}})^T$.

For the remainder of the section, the PW-spline space $\mathbb{S}$ is assumed to contain the constants. Then, it can as well be defined as the set of all continuous functions $S : [a,b] \to \mathbb{R}$ which satisfy the connection conditions

$$\big(S'(t_k^+), \ldots, S^{(n-m_k)}(t_k^+)\big)^T = M_k\big(S'(t_k^-), \ldots, S^{(n-m_k)}(t_k^-)\big)^T, \quad 1 \leqslant k \leqslant q, \tag{29}$$

where M_k is the lower triangular matrix of order $(n - m_k)$, with positive diagonal entries, obtained after deleting the first row and column of R_k. Note that the splines in $\mathbb{S}$ are then *geometrically continuous* at t_k in the weak sense of continuity of the Frenet frames of order $(n - m_k)$. Besides, when the PW-spline space $\mathbb{S}$ contains constants, if D denotes the piecewise differentiation on $([a,b]; \mathbb{T})$, the space $D\mathbb{S}$ is in turn a PW-spline space on $([a,b]; \mathbb{T})$, with n-dimensional section-spaces.

Without giving the precise definition of blossoms, recall that, when they exist, i.e., when $\mathbb{S}$ is good for design, blossoms are symmetric functions of n variables defined by means of intersections of (possibly left/right) osculating flats on the symmetric set $\mathbb{A}_n(\mathbb{K})$ of all n-tuples in $[a,b]^n$ which are *admissible* (with respect to the knot vector $\mathbb{K}$). An n-tuple $(x_1, \ldots, x_n) \in [a, b]^n$ is admissible when each interior knot t_k, $k = 1, \ldots, q$, which satisfies

$$\min(x_1, \ldots, x_n) < t_k < \max(x_1, \ldots, x_n),$$

appears in the sequence $(x_1, \ldots, x_n)$ a number of times at least equal to its multiplicity m_k. Blossoms satisfy a crucial (and difficult to prove) property on $\mathbb{A}_n(\mathbb{K})$: they are *pseudoaffine* in each variable. This property extends the simple affinity in each variable of polynomial spline blossoms [36]. Roughly speaking, for fixed convenient $x_1, \ldots, x_{n-1}, c, d$, $c < d$, and $x \in [c, d]$, the value $s(x_1, \ldots, x_{n-1}, x)$ of the blossom s of a given $S \in \mathbb{S}^d$, $d \geqslant 1$, is as convex combination of $s(x_1, \ldots, x_{n-1}, c)$ and $s(x_1, \ldots, x_{n-1}, d)$, with coefficients independent of S. Along with symmetry, this pseudoaffinity property permits the evaluation of all values of the blossom s of a given $S \in \mathbb{S}^d$, $d \geqslant 1$, as convex combinations of the poles or control points of $S \in \mathbb{S}^d$ defined as

$$P_\ell := s(\xi_{\ell+1}, \xi_{\ell+2}, \ldots, \xi_{\ell+n}), \quad \ell = -n, \ldots, m,$$

through an n-step de Boor-type algorithm. This yields non-negative functions ν_ℓ on $\mathbb{A}_n(\mathbb{K})$ (independent of the spline S) such that

$$s(x_1, \ldots, x_n) = \sum_{\ell=-n}^{m} \nu_\ell(x_1, \ldots, x_n) P_\ell, \text{ with } \sum_{\ell=-n}^{m} \nu_\ell(x_1, \ldots, x_n) = 1, \ (x_1, \ldots, x_n) \in \mathbb{A}_n(\mathbb{K}). \tag{30}$$

The most important design algorithms (knot insertion, passage from the poles to the Bézier points of each section, ...) are contained in the previous description, and they are corner-cutting algorithms. Besides, the geometrical definition of blossoms makes it obvious that s coincides with S on the diagonal of $[a, b]^n$ (*diagonal property* of blossoms). From (30), we can thus derive:

$$S(x) = s(x^{[n]}) = \sum_{\ell=-n}^{m} N_\ell(x) P_\ell, \quad \sum_{\ell=-n}^{m} N_\ell(x) = 1, \quad x \in [a, b]. \tag{31}$$

The functions N_ℓ, $-n \leqslant \ell \leqslant m$, form the B-spline basis of the spline space $\mathbb{S}$ and their blossoms are the functions ν_ℓ involved in (30). That knot insertion is corner-cutting explains why the B-spline basis is *totally positive on* $[a, b]$, i.e., given any sequence $y_{-n} < y_{-n+1} < \cdots < y_m$ in $[a, b]$, the matrix with entries $N_\ell(y_j)$, $-n \leqslant \ell, j \leqslant m$, is totally positive (i.e., all its minors are non-negative), see [19]. Concerning the importance of total positivity, see for instance [8, 11].

The previous description justifies the terminology employed in Definition 3.2. The pseudoaffinity of blossoms is more generally the main underlying tool in the theorem below which gathers the crucial results on which the present work is based. In particular, it gives a complete description of the class of all PW-spline spaces good for design. For the equivalence (i) $\Leftrightarrow$ (ii), see [17, 18, 20, 25]. For all other points, see [27].

Theorem 3.5 *Assume the PW-spline* $\mathbb{S}$ *to contain the constants. Then, all properties listed below are equivalent:*

(i) $\mathbb{S}$ *is good for design;*

(ii) *$\mathbb{S}$ possesses a B-spline basis and so does any spline space derived from $\mathbb{S}$ by knot insertion;*

(iii) *the PW-spline space $D\mathbb{S}$ possesses a B-spline-like basis and so does any spline space derived from $D\mathbb{S}$ by knot insertion;*

(iv) *there exists a system $(w_1, \dots, w_n)$ of piecewise weight functions on $([a, b]; \mathbb{T})$ such that*

$$ECP(\mathbb{1}, w_1, \dots, w_n) \subset \mathbb{S}; \tag{32}$$

(v) *there exists a system $(w_1, \dots, w_n)$ of piecewise weight functions on $([a, b]; \mathbb{T})$ such that*

$$ECP(w_1, \dots, w_n) \subset D\mathbb{S}. \tag{33}$$

(vi) *there exists a positive piecewise function $\Omega \in PC_+^{n-1}([a, b]; \mathbb{T})$ such the PW-spline space $\mathbb{S}^\diamond$ obtained by piecewise division of all elements of $D\mathbb{S}$ by Ω is good for design.*

Furthermore, when (i) holds, the B-spline basis $(N_{-n}, \dots, N_m)$ is the optimal normalised totally positive basis in the spline space $\mathbb{S}$.

Remark 3.6 Theorem 3.5 is valid when $q = 0$ (that is, when there is no interior knot), or when $q > 0$ with $m_k = 0$ for $k = 1, \dots, q$ — in which cases it provides us with various characterisations of EC-spaces good for design on $[a, b]$ [21, 23, 26], or ECP-spaces good for design on $([a, b]; \mathbb{T})$ [22–24], respectively. Note that, in both cases, (ii) and (iii) can equivalently be replaced by the existence of a Bernstein basis in $\mathbb{S}$ (resp., a Bernstein-like basis in $D\mathbb{S}$) relative to each couple $(c, d) \in [a, b]^2$, $c < d$.

Remark 3.7 As observed in the introduction, without loss of generality, in addition to $\mathbb{S}$ containing the constants, we could directly have assumed the PW-spline space $D\mathbb{S}$ to be a PEC-spline space. This amounts to assuming each section-space to be of the form $\mathbb{E}_k = EC(\mathbb{1}, w_1^k, \dots, w_n^k)$ for some system $(w_1^k, \dots, w_n^k)$ of weight functions on $[t_k, t_{k+1}]$. With L_0^k equal to the identity on $C^n([t_k, t_{k+1}])$, we could then replace the connection relations (29) by

$$\left(L_1^k(t_k^+), \dots, L_{n-m_k}^k(t_k^+)\right)^T = \overline{M}_k \left(L_1^{k-1}(t_k^-), \dots, L_{n-m_k}^{k-1}(t_k^-)\right)^T, \; k = 1, \dots, q, \tag{34}$$

the matrix $\overline{M}_k$ being obtained by deleting the first row and column in $\overline{R}_k$. All other approaches of piecewise Chebyshevian splines start with given systems of weight functions, given *totally positive matrices* $\overline{M}_k$, and connection conditions (34). P. J. Barry initiated them in [1], with additional assumptions on the differentiability of the weight functions to make it possible to consider the dual EC-spaces (see also [33, 34]). This total positivity assumption is only a sufficient condition for obtaining "good" properties for the spline space $\mathbb{S}$, such as existence of B-spline bases. By comparison, we clearly established the advantages of the blossoming approach in [17], with four-dimensional section-spaces. The results gathered in Theorem 3.5 (see [27]) definitively highlighted the powerfulness of blossoms: in a good for design PW-spline space, we can always find appropriate systems of weight functions so that all connection matrices $\overline{M}_k$ involved in (34) are identity matrices, that is, the simplest possible totally positive matrices.

4 Interpolation versus design

The crucial results recalled in Theorem 3.5 will enable us not only to demonstrate Theorem 1.2, but also to describe and characterise the class of all PW-spline spaces which are "good for interpolation", as defined in Definition 4.1 below.

4.1 The results

Definition 4.1 Given a PW-spline space $\mathbb{S}$ as introduced in Definition 2.1, for short, we say that $\mathbb{S}$ is *good for interpolation* when

1- any Hermite interpolation problem in $\mathbb{S}$ which satisfies the SW-conditions has a unique solution;
2- the same holds in any spline space obtained from $\mathbb{S}$ under knot insertion.

This terminology is justified by the fact that, when dealing with spline Hermite interpolation problems, it is interesting to refine the knot-vector. On the other hand, when the PW-spline space $\mathbb{S}^\star$ is obtained from $\mathbb{S}$ by knot insertion, a priori there is no obvious correlation between the unisolvence of all Hermite interpolation problems based on node-vectors satisfying the SW-conditions in $\mathbb{S}$ and in $\mathbb{S}^\star$.

Example 4.2 Let us illustrate Definition 4.1 with a trivial example. Assume that $\mathbb{S}$ is a PW-spline space with all interior multiplicities given by $m_k = n+1$, $k = 1, \ldots, q$. In $\mathbb{S}$, solving a Hermite interpolation problem satisfying the SW-conditions consists in solving separately in each section-space a Hermite interpolation problem in $(n+1)$ data. It clearly results that $\mathbb{S}$ is good for interpolation in the sense of Definition 4.1 if and only for each $k = 0, \ldots, q$, the section-space $\mathbb{E}_k$ is an EC-space on $[t_k, t_{k+1}]$, i.e., if and only if $\mathbb{S}$ is a PEC-spline space. By contrast, if we no longer assume that $m_k = n+1$, $k = 1, \ldots, q$, then a PEC-spline space is not necessarily good for interpolation.

The object of the present section is to prove the following theorem.

Theorem 4.3 *Let $\mathbb{S}$ be the PW-spline space described in Definition 2.1. The following properties are equivalent:*

(i) *the PW-spline space $\mathbb{S}$ is good for interpolation;*
(ii) *$\mathbb{S}$ possesses a B-spline-like basis and so does any PW-spline space obtained from $\mathbb{S}$ by knot insertion;*
(iii) *there exists a system $(w_0, \ldots, w_n)$ of piecewise weight functions on $([a, b]; \mathbb{T})$ such that*

$$ECP(w_0, \ldots, w_n) \subset \mathbb{S}; \tag{35}$$

The equivalence between (ii) and (iii) is already contained in Theorem 3.5. That (i) $\Rightarrow$ (ii) will be proved in Section 4.2 and that (iii) $\Rightarrow$ (i) in Section 4.4, while Section 4.3 will establish intermediate results on bounds of zeroes. Once Theorem 4.3 is proven, additional characterisations of being good for interpolation will follow from Theorem 3.5, among which the equivalence between (i) and (ii) of Theorem 1.2. The last claim in Theorem 1.2—i.e., "good for design" implies "good for interpolation"—is then trivially obtained with $w_0 = \mathbb{I}$ in (35). As a matter of fact, we can even state more, because we know how to select all $w_0 = \Omega \in PC_+^{n-1}([a, b]; \mathbb{T})$ involved in the property (vi) of Theorem 3.5 (see [27], Theorem 5.2). Indeed, the terminology introduced in Definition 4.1 enables us to restate Corollary 5.4 of [27] as follows (to be compared with the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 3.5).

Theorem 4.4 *A PW-spline space $\mathbb{S}$ being given, the following two properties are equivalent:*

(i) $\mathbb{S}$ *is good for design:*
(ii) $\mathbb{S}$ *is good for interpolation and* $\mathbb{S}$ *possesses a B-spline basis.*

4.2 Proof of Theorem 4.3: (i) implies (ii)

Proposition 4.5 *Assume that any Hermite interpolation satisfying the SW conditions is unisolvent in $\mathbb{S}$. Then, the same property holds true in any PW-spline space obtained by restriction of $\mathbb{S}$.*

Proof Let $\mathbb{S}^*$ be the restriction of $\mathbb{S}$ to a subinterval $[a^*, b^*]$, $a \leqslant a^* < b^* \leqslant b$. Out of symmetry and iteration arguments, it is sufficient to consider the following two cases:

1- $a^* = a, t_q < b^* < t_{q+1} = b$;
2- $a^* = a, b^* = t_q$.

Let $\mathbb{Y}^*$ be a node-vector of a given Hermite interpolation $(\mathcal{H}^*)$ in $\mathbb{S}^*$, assumed to satisfy the SW-conditions. In case 1, $\dim \mathbb{S}^* = \dim \mathbb{S}$, the same node-vector and the same data define a Hermite interpolation problem $(\mathcal{H})$ in $\mathbb{S}$ which satisfies the SW-conditions. By restriction to $[a^*, b^*]$, the unique solution to $(\mathcal{H})$ in $\mathbb{S}$ provides us with a unique solution to $(\mathcal{H}^*)$ in $\mathbb{S}^*$. In case 2, $\dim \mathbb{S} = n + 1 + m^*$, with $m^* := \sum_{k=1}^{q-1} m_k = m - m_q$. Write the node-vector $\mathbb{Y}^*$ as

$$\mathbb{Y}^* = (y^*_{-n}, \ldots, y^*_{m^*}), \quad \text{with } y^*_j \leqslant y^*_{j+1} \text{ for } -n \leqslant j \leqslant m^* - 1.$$

Let the node-vector $\mathbb{Y} = (y_{-n}, \ldots, y_m)$ be defined by

$$y_j := y^*_j \text{ for } j = -n, \ldots, m^*, \quad y_{m^*+1} = \cdots = y_{m-1} = y_m = b.$$

It is easily checked that the node-vector $\mathbb{Y}$ satisfies the SW-conditions relative to $\mathbb{S}$. Selecting any m_q additional data at b, we transform $(\mathcal{H}^*)$ into a Hermite interpolation problem $(\mathcal{H})$ in $\mathbb{S}$ which has a unique solution. By restriction to $\cup_{k=0}^{q-1}[t_k^+, t_{k+1}^-]$, we obtain a unique solution to $(\mathcal{H}^*)$ in $\mathbb{S}^*$. □

As a special case, for $k = 0, \ldots, q$, the section-space $\mathbb{E}_k$ can be viewed as the PW-spline space obtained from $\mathbb{S}$ by restriction to the subinterval $[t_k, t_{k+1}]$. For this restriction, being good for interpolation simply means that $\mathbb{E}_k$ is an EC-space on $[t_k, t_{k+1}]$. We can therefore state the following:

Corollary 4.6 *If any Hermite interpolation satisfying the SW-conditions is unisolvent in a PW-spline space $\mathbb{S}$, then $\mathbb{S}$ is a PEC-spline space.*

On account of Definition 4.1, that (i) $\Rightarrow$ (ii) in Theorem 4.3 readily follows from the following proposition.

Proposition 4.7 *Assume that any Hermite interpolation satisfying the SW-conditions is unisolvent in $\mathbb{S}$. Then, $\mathbb{S}$ possesses B-spline-like bases.*

Proof Suppose that an interior knot t_k, $1 \leqslant k \leqslant q$, has multiplicity $m_k = n + 1$. Then, we split the spline space $\mathbb{S}$ into $\mathbb{S}_k^-$ and $\mathbb{S}_k^+$. From Proposition 4.5, we know that, in both spline spaces, any Hermite interpolation problem satisfying the corresponding SW-conditions is unisolvent. On the other hand, searching for B-spline-like bases in $\mathbb{S}$ amounts to searching for B-spline-like bases separately in $\mathbb{S}_k^-$ and $\mathbb{S}_k^+$. Accordingly, without loss of generality, we can assume that we are working under the assumption

$$m_k \leqslant n \quad \text{for } k = 1, \dots, q.$$

Under this assumption, for any $k = 1, \dots, q$, a spline $S \in \mathbb{S}$ vanishes at t_k^- if and only if it vanishes at t_k^+. Consider an integer j, $-n \leqslant j \leqslant m$. Let $k < k'$ be the two integers such that

$$\xi_j = t_k, \quad \xi_{j+n+1} = t_{k'},$$

and let $\mathbb{S}_{k,k'}$ denote the restriction of $\mathbb{S}$ to $\cup_{i=k}^{k'-1}[t_i^+, t_{i+1}^-]$. We thus have

$$j = j_k - m_k + r \text{ with } 1 \leqslant r \leqslant m_k, \quad j + n + 1 = j_{k'} - m_{k'} + r' \text{ with } 1 \leqslant r' \leqslant m_{k'},$$

so that

$$n + 1 = m_k - r + r' + \sum_{i=k+1}^{k'-1} m_i.$$

Accordingly,

$$\dim \mathbb{S}_{k,k'} = n + 1 + \sum_{i=k+1}^{k'-1} m_i = 2(n+1) - m_k + r - r'.$$

According to Proposition 4.5, in the PW-spline space $\mathbb{S}_{k,k'}$, any Hermite interpolation problem satisfying the SW-conditions has a unique solution. Consider the problem of finding an element $S \in \mathbb{S}_{k,k'}$ such that

$$S^{(q)}(t_k) = 0, \quad 0 \leqslant q \leqslant n - m_k + r - 1, \tag{36}$$
$$S^{(n-m_k+r)}(t_k^+) = 1, \tag{37}$$
$$S^{(q)}(t_{k'}) = 0, \quad 0 \leqslant q \leqslant n - r'. \tag{38}$$

This Hermite interpolation problem in $(2n + 2 - m_k + r - r')$ data satisfies the SW-conditions relative to $\mathbb{S}_{k,k'}$ since the numbers of interpolation conditions at the end points respectively satisfy

$$n + 1 - m_k + r = \sum_{i=k+1}^{k'-1} m_i + r' > \sum_{i=k+1}^{p} m_i, \quad n + 1 - r' = m_k - r$$
$$+ \sum_{i=k+1}^{k'-1} m_i \geqslant \sum_{i=p}^{k'-1} m_i, \quad k + 1 \leqslant p \leqslant k' - 1. \tag{39}$$

Let $\overline{B}_j \in \mathbb{S}_{k,k'}$ denote the unique element satisfying (36)–(38).

Suppose the existence of an x_0 in the open interval $]t_k, t_{k'}[$ such that $\overline{B}_j(x_0) = 0$ (in the sense that both $\overline{B}_j(x_0^-) = 0$ and $\overline{B}_j(x_0^+) = 0$ in case x_0 is an interior knot in $\mathbb{S}_{k,k'}$). Then $\overline{B}_j$ would be a solution to the Hermite interpolation in $(2n+2-m_k+r-r')$ data obtained by replacing in the previous one the condition (37) by $S(x_0) = 0$. This interpolation problem too satisfy the SW-conditions relative to $\mathbb{S}_{k,k'}$ (see (39)). Its interpolation data being all 0, this would imply $\overline{B}_j = 0$, which would contradict the fact that $\overline{B}_j^{(n-m_k+r-1)}(t_k^+) = 1$. The interpolating conditions at t_k implying the positivity of S on a sufficiently small interval contained in $]t_k, t_{k+1}[$, it follows that $\overline{B}_j(x^\varepsilon) > 0$ for all $x \in]t_k, t_{k'}[$, and $\varepsilon = \pm$.

Denote by B_j the element of $\mathbb{S}$ obtained by extending $\overline{B}_j$ by 0 on $[t_0, t_k^-]$ (if $k \geqslant 1$) and $[t_{k'}^+, t_{q+1}]$ (if $k' \leqslant q$). The sequence $(B_{-n}, \ldots, B_m)$ so obtained is a Bernstein-like basis in $\mathbb{S}$. □

4.3 Interpolation and zeroes

In this subsection, we characterise the unisolvence of convenient interpolation problems in terms of upper bounds on the numbers of zeroes of splines in $\mathbb{S}$.

First recall that, in a W-space on a non-trivial interval I, a function cannot be zero on a non-trivial subinterval of I without it being zero on the whole of I. Accordingly, a PW-spline $S \in \mathbb{S}$ being given, for each $k = 0, \ldots, q$, either S has only isolated zeroes on $[t_k^+, t_{k+1}^-]$ or S is zero on the whole of $[t_k^+, t_{k+1}^-]$. In the latter case, we say that $[t_k^+, t_{k+1}^-]$ is a zero interval for S.

Second, we have to specify the definition of the multiplicity of an interior knot t_k, $k = 1, \ldots, q$, as a zero of S. Supposing that neither $[t_{k-1}^+, t_k^-]$ nor $[t_k^+, t_{k+1}^-]$ are zero intervals for S, with no possible ambiguity the exact left/right multiplicities $\mu_k^-(S), \mu_k^+(S) \leqslant n$ of t_k as a left/right zero of S are defined by

$$S^{(i)}(t_k^-) = 0 \text{ for } 0 \leqslant i \leqslant \mu_k^-(S) - 1, \quad S^{(i)}(t_k^-) \neq 0 \text{ for } i = \mu_k^-(S),$$
$$S^{(i)}(t_k^+) = 0 \text{ for } 0 \leqslant i \leqslant \mu_k^+(S) - 1, \quad S^{(i)}(t_k^+) \neq 0 \text{ for } i = \mu_k^+(S).$$

Definition 4.8 Given $k = 1, \ldots, q$, assume that neither $[t_{k-1}^+, t_k^-]$ nor $[t_k^+, t_{k+1}^-[$ are zero intervals of a given spline $S \in \mathbb{S}$.

1- If $\mu_k^-(S) < n + 1 - m_k$, then $\mu_k^+(S) = \mu_k^-(S)$. The multiplicity $\mu_k(S)$ of t_k as a zero of S is defined as $\mu_k(S) := \mu_k^+(S) = \mu_k^-(S)$.
2- If $\mu_k^-(S) \geqslant n + 1 - m_k$, then $\mu_k^+(S) \geqslant n + 1 - m_k$. The multiplicity $\mu_k(S)$ of t_k as a zero of S is understood as

$$\mu_k(S) := n + 1 - m_k + \beta_k^-(S) + \beta_k^+(S), \tag{40}$$

the non-negative integers $\beta_k^-(S)$, $\beta_k^+(S)$ being defined by

$$\mu_k^-(S) = n + 1 - m_k + \beta_k^-(S), \quad \mu_k^+(S) = n + 1 - m_k + \beta_k^+(S). \tag{41}$$

In other words, we add the left and right multiplicities (as would be natural for piecewise functions on $([a, b]; \mathbb{T})$) given that, below $(n + 1 - m_k)$, successive zero conditions at t_k^+ follow from those at t_k^- and they should not be counted twice. The previous two cases can be summarized by the common formula

$$\mu_k(S) := \mu_k^-(S) + \mu_k^+(S) - \min\left(n + 1 - m_k, \mu_k^-(S), \mu_k^+(S)\right). \tag{42}$$

It should be observed that, with Definition 4.8, the count of zeroes of a given spline $S \in \mathbb{S}$ is related to the space $\mathbb{S}$. If S is an element of another spline space, the count will be different. Subsequently, we will not allocate numbers of zeroes to zero intervals. Given $S \in \mathbb{S}$, and given any two integers k, k', $0 \leqslant k < k' \leqslant q + 1$, such that none of the intervals $[t_i^+, t_{i+1}^-]$, $i = k, \ldots, k' - 1$ is a zero interval for S, Definition 4.8 makes it possible to define with no ambiguity the total number $Z(S; [t_k, t_{k'}])$ of zeroes of S on the interval $[t_k, t_{k'}]$ as follows:

$$Z(S; [t_k, t_{k'}]) := \mu_k^+(S) + \mu_{k'}^-(S) + \sum_{i=k+1}^{k'-1} \mu_i(S) + \sum_{i=k}^{k'-1} Z(S;]t_i, t_{i+1}[).$$

Theorem 4.9 *For a given PW-spline space $\mathbb{S}$, the following two properties are equivalent:*

(i) *any Hermite interpolation satisfying the SW-conditions is unisolvent in $\mathbb{S}$;*
(ii) *given any $S \in \mathbb{S}$, and any $0 \leqslant k < k' \leqslant q + 1$, such that none of the intervals $[t_i^+, t_{i+1}^-]$, $i = k, \ldots, k' - 1$, is a zero interval for S, we have*

$$Z(S; [t_k, t_{k'}]) \leqslant n + \sum_{k<i<k'} m_i = \dim \mathbb{S}_{k,k'} - 1. \tag{43}$$

Proof •(i) $\Rightarrow$ (ii): Suppose that any Hermite interpolation satisfying the SW conditions is unisolvent in $\mathbb{S}$. Then, for $0 \leqslant k < k' \leqslant q + 1$, the same property is true in the PW-spline space $\mathbb{S}_{k,k'}$ (Proposition 4.5). It is therefore sufficient to prove that

$$Z(S; [a, b]) \leqslant n + m = n + \sum_{i=1}^{q} m_i, \tag{44}$$

for any given spline $S \in \mathbb{S}$ with no zero interval. For $s = 0, \dots, q$, denote by $\mathcal{P}(s)$ the following property

$$\text{for } k = 0, \dots, q - s,\ \ Z(S; [t_k, t_{k+s+1}]) \leqslant n + \sum_{i=k+1}^{k+s} m_i.$$

We know that the property $\mathcal{P}(0)$ holds true (Corollary 4.6) and (44) is the property $\mathcal{P}(q)$. Assume that $q \geqslant 1$ and that $\mathcal{P}(s)$ is satisfied for each $s \leqslant q - 1$, while $\mathcal{P}(q)$ fails to be true. We thus have $Z(S; [a, b]) \geqslant n + 1 + m$. On account of Definition 4.8, we can select $(n + 1 + m)$ (multiple) zeroes of S so as to form a Hermite interpolation problem $(\mathcal{H})$ with all interpolation data equal to 0, to which S yields a non-zero solution in $\mathbb{S}$. Let us allocate to $(\mathcal{H})$ the notations introduced in Section 2. For $k = 1, \dots, q$, we thus have

$$n + 1 + \sum_{i=1}^{q} m_i = \lambda(k) + Z(S; [t_k; t_{q+1}]) \leqslant \lambda(k) + n + \sum_{i=k+1}^{q} m_i,$$

the inequality resulting from our assumption that $\mathcal{P}(q - k)$ holds true. This implies $\lambda(k) \geqslant \sum_{i=1}^{k} m_i$. Symmetric arguments work for $\varrho(k)$. Therefore, $(\mathcal{H})$ satisfies the SW-conditions, which contradicts the fact that $(\mathcal{H})$ has a non-zero solution. Therefore, if $\mathcal{P}(s)$ is satisfied for each $s \leqslant q - 1$, then (44) is satisfied too. The induction will be complete by applying (44) in a convenient space $\mathbb{S}_{k,k'}$, replacing q by some s, $1 \leqslant s < q$.

•(ii) $\Rightarrow$ (i): Assume that (ii) holds. Given a node-vector $\mathbb{Y}$ satisfying the SW-conditions relative to $\mathbb{S}$, let us prove that the zero spline is the only solution to the Hermite interpolation problem $(\mathcal{H})$ based on $\mathbb{Y}$ when all interpolation data are equal to zero. Since (ii) holds in any $\mathbb{S}_{k,k'}$, from Theorem 2.3 and Remark 2.4, with no loss of generality, we can assume that $\nu(k) \leqslant n - m_k$ for all $k = 1, \dots, q$, implying both that $\alpha_k^- = \alpha_k^+ = 0$ and that $m_k \leqslant n$ for all $k = 1, \dots, q$. Let $S \in \mathbb{S}$ be a non-zero solution to $(\mathcal{H})$. For each $k = 1, \dots, q$ such that neither $[t_{k-1}^+, t_k^-]$ nor $[t_k, t_{k+1}^-]$ are zero intervals for S, we can say that $\mu_k(S) \geqslant \nu(k)$. If S had no zero interval, we would therefore have $Z(S; [a, b]) \geqslant \dim \mathbb{S}$, which would contradict (43) for $k = 0$, $k' = q + 1$. Accordingly, at least one interval $[t_k^+, t_{k+1}^-]$, $k = 0, \dots, q$, is a zero interval for S. We can therefore find either an integer $k \in \{1, \dots, q\}$ such that none of the intervals $[t_i^+, t_{i+1}^-]$, $i = 0, \dots, k - 1$ (resp., $i = k, \dots, q$) is a zero interval for S while S is zero on $[t_k^+, t_{k+1}^-]$ (resp. on $[t_{k-1}^+, t_k^-]$), or two integers $k, k' \in \{1, \dots, q\}$, $k < k'$, such that S is zero on both $[t_{k-1}^+, t_k^-]$ and $[t_{k'}^+, t_{k'+1}^-]$, and none of the interval $[t_i^+, t_{i+1}^-]$, $i = k, \dots, k' - 1$, is a zero interval for S. In the first case, we know that $Z(S; [a, t_k[) \geqslant \lambda(k)$ and S vanishes at least $(n + 1 - m_k)$ times at t_k^- since it is zero on $[t_k^+, t_{k+1}^-]$. Taking account of the SW-conditions (17), we thus have

$$Z(S; [a, t_k]) \geqslant \lambda(k) + (n + 1 - m_k) \geqslant n + 1 + \sum_{i=1}^{k-1} m_i = \dim \mathbb{S}_k^-.$$

This contradicts (43). Out of symmetry, the second case leads to a similar contradiction. Consider the third case. Then, from (iii) of Theorem 2.5, we know that at least

$\left(-n-1+\sum_{i=k}^{k'} m_i\right)$ nodes are contained in the interval $]t_k, t_{k'}[$. The spline S being zero on both $[t_{k-1}^+, t_k^-]$ and $[t_{k'}^+, t_{k'+1}^-]$, we have

$$Z(S;[t_k,t_{k'}]) \geqslant \left(-n-1+\sum_{i=k}^{k'} m_i\right)+(n+1-m_k)+(n+1-m_{k'})$$
$$= n+1+\sum_{k<i<k'} m_i = \dim \mathbb{S}_{k,k'}.$$

Again, this contradicts (43). Therefore, S must be zero everywhere on $[a,b]$. □

4.4 Proof of Theorem 4.3: (iii) ⇒ (i)

In this subsection, we assume the existence of a system $(w_0,\ldots,w_n)$ of piecewise weight functions on $([a,b];\mathbb{T})$ satisfying (35), and we denote by $L_0, L_1\ldots, L_n$ the associated piecewise generalised derivatives. The count of zeroes can now be done using the piecewise generalised derivatives $L_0,\ldots,L_n$ rather than the ordinary (left/right) derivatives. For instance, for $1\leqslant k\leqslant q$, when neither $[t_{k-1}^+, t_k^-]$ nor $[t_k^+, t_{k+1}^-]$ are zero intervals for $S\in\mathbb{S}$, the exact left/right multiplicities $\mu_k^-(S), \mu_k^+(S)\leqslant n$ of t_k as a left/right zero of S are as well defined by

$$L_iS(t_k^-) = 0 \text{ for } 0\leqslant i\leqslant \mu_k^-(S)-1, \quad L_iS(t_k^-)\neq 0 \text{ for } i=\mu_k^-(S),$$
$$L_iS(t_k^+) = 0 \text{ for } 0\leqslant i\leqslant \mu_k^+(S)-1, \quad L_iS(t_k^+)\neq 0 \text{ for } i=\mu_k^+(S).$$

Moreover, from (35), we know that the connection conditions satisfied by any spline $S\in\mathbb{S}$ can be expressed as follows:

$$L_jS(t_k^+) = L_jS(t_k^-) \quad \text{for } j=0,\ldots,n-m_k, \quad k=1,\ldots,q. \tag{45}$$

This will be a key point in the proof of Theorem 4.10 below.

Any PW-spline space $\mathbb{S}^\star$ obtained from $\mathbb{S}$ by insertion of knots satisfies $ECP(w_0,\ldots,w_n)\subset\mathbb{S}\subset\mathbb{S}^\star$. Moreover, the inclusion $ECP(w_0,\ldots,w_n)\subset\mathbb{S}$ induces a similar inclusion for any $\mathbb{S}_{k,k'}$, $0\leqslant k<k'\leqslant q+1$. Therefore, on account of Theorem 4.9, that (iii) ⇒ (i) in Theorem 4.3 will clearly result from the bounds obtained below.

Theorem 4.10 *Let the PW-spline space $\mathbb{S}$ satisfy (35). Given $S\in\mathbb{S}$, we assume that S has no zero interval. Then the total number $Z(S;[a,b])$ of zeroes of S on $[a,b]$ is bounded above as follows:*

$$Z(S;[a,b])\leqslant n+m-z(S;]a,b[) = \dim\mathbb{S}-1-z(S;]a,b[), \tag{46}$$

where $z(S;]a,b[)$ is the number of interior knots t_k, $1\leqslant k\leqslant q$, such that $\mu_k(S)\geqslant n+1-m_k$, or equivalently, such that $L_0S(t_k)=L_1S(t_k)=\cdots=L_{n-m_k}S(t_k)=0$.

The proof of Theorem 4.9 relies on a number of preliminary results. The successive steps are rather classical when bounding numbers of zeroes in spline spaces (e.g., [9, 10, 34, 38]). Nevertheless, we will give all details, for the context is not exactly the same, and also we do not use the same definition for the multiplicity of an interior knot as a zero of a spline. Subsequently, the notation $\mathcal{S}^-$ will count the number of strict changes of sign in any sequence of real numbers, that is, the number of sign changes after deletion of all zeroes in the sequence. The notation $\mathcal{S}^+$ will count the weak sign changes, that is, the maximum number of possible sign changes after replacing each zero in the sequence by either 1 or -1. In particular, given a piecewise function $F \in PC^n([a,b];\mathbb{T})$, which vanishes $\mu \leqslant n$ times at $x^\varepsilon \in \cup_{k=0}^q [t_k^+, t_{k+1}^-]$, we have

$$\mathcal{S}^-\big(F(x^\varepsilon), F'(x^\varepsilon), \dots, F^{(n)}(x^\varepsilon)\big) \leqslant n-\mu \quad \text{and} \quad \mathcal{S}^+\big(F(x^\varepsilon), F'(x^\varepsilon), \dots, F^{(n)}(x^\varepsilon)\big) \geqslant \mu,$$

or, equivalently,

$$\mathcal{S}^-\big(L_0F(x^\varepsilon), L_1F(x^\varepsilon), \dots, L_nF(x^\varepsilon)\big) \leqslant n-\mu \text{ and } \mathcal{S}^+\big(L_0F(x^\varepsilon), L_1F(x^\varepsilon), \dots, L_nF(x^\varepsilon)\big) \geqslant \mu. \tag{47}$$

Subsequently, S denotes a given spline in $\mathbb{S}$.

Lemma 4.11 *For each integer $i = 0, \dots, q$ such that $[t_i^+, t_{i+1}^-]$ is not a zero interval for S, denote by $p_i \leqslant n$ the smallest integer such that L_{p_i} is constant on $[t_i, t_{i+1}]$. Then, the total number $Z(S;]t_i, t_{i+1}[) \leqslant n$ of S on $]t_i, t_{i+1}[$ satisfies*

$$Z(S;]t_i, t_{i+1}[) \leqslant \mathcal{S}_{p_i}^-(t_i^+) - \mathcal{S}_{p_i}^+(t_{i+1}^-), \quad i = 0, \dots, q, \tag{48}$$

where, for each $p \leqslant n$, the condensed notations $\mathcal{S}_p^-(t_i^\varepsilon)$ and $\mathcal{S}_p^-(t_i^\varepsilon)$ respectively stand for

$$\mathcal{S}^-\big(L_0S(t_i^\varepsilon), L_1S(t_i^\varepsilon), \dots, L_pS(t_i^\varepsilon)\big), \quad \mathcal{S}^+\big(L_0S(t_i^\varepsilon), L_1S(t_i^\varepsilon), \dots, L_pS(t_i^\varepsilon)\big).$$

Proof This classical result [39] is known as the Budan-Fourier theorem in the EC-space $\mathbb{E}_i = EC(w_0^i, \dots, w_n^i)$, where $w_0^i, \dots, w_n^i$ are the restrictions of $w_0, \dots, w_n$ to $[t_i^+, t_{i+1}^-]$. □

Lemma 4.12 *Given an integer $i \in \{1, \dots, q\}$, assume that neither $[t_{i-1}^+, t_i^-]$ nor $[t_i^+, t_{i+1}^-]$ are zero intervals for S. Then we have, with the notation introduced in Lemma 4.11,*

$$\begin{cases} \mu_i(S) \leqslant m_i - \mathcal{S}_{p_i}^-(t_i^+) + \mathcal{S}_{p_{i-1}}^+(t_i^-) - 1 & \textit{if } \mu_i(S) \geqslant n+1-m_i, \\ \mu_i(S) \leqslant m_i - \mathcal{S}_{p_i}^-(t_i^+) + \mathcal{S}_{p_{i-1}}^+(t_i^-) & \textit{if } \mu_i(S) < n+1-m_i. \end{cases} \tag{49}$$

Proof We know that

$$\mathcal{S}_{p_{i-1}}^+(t_i^-) \geqslant \mu_i^-(S), \quad \mathcal{S}_{p_i}^-(t_i^+) = \mathcal{S}_n^-(t_i^+) \leqslant n - \mu_i^+(S).$$

Accordingly, we can state that

$$m_i - \mathcal{S}_{p_i}^-(t_i^+) + \mathcal{S}_{p_i-1}^+(t_i^-) \geqslant m_i - n + \mu_i^-(S) + \mu_i^+(S)$$

- Let us first assume that $\mu_i(S) \geqslant n+1-m_i$. Then, the inequality claimed in the first line of (49) is trivially satisfied since; according to (40) and (41), we have

$$\mu_i^-(S) + \mu_i^+(S) + (m_i - n) = \mu_i(S) + 1.$$

- Let us now assume that $\mu_i(S) \leqslant n - m_i$. In that case, we have $m_i \leqslant n$. From the connection conditions (45), we can derive that

$$\mathcal{S}^-_{n-m_i}(t_i^-) = \mathcal{S}^-_{n-m_i}(t_i^+). \tag{50}$$

Given that $\mu_i(S) = \mu_i^+(S) = \mu_i^-(S)$, we can sharpen the bounds in (47) as follows:

$$\begin{aligned} \mathcal{S}^+_{p_{i-1}}(t_i^-) &\geqslant \mathcal{S}^-_{p_{i-1}}(t_i^-) + \mu_i \geqslant \mathcal{S}^-_{n-m_i}(t_i^-) + \mu_i, \\ \mathcal{S}^-_{p_i}(t_i^+) &= \mathcal{S}^-_n(t_i^+) \leqslant \mathcal{S}^-_{n-m_i}(t_i^+) + m_i. \end{aligned} \tag{51}$$

Combining (51) and (50), it readily follows that

$$\mathcal{S}^+_{p_{i-1}}(t_i^-) - \mathcal{S}^-_{p_i}(t_i^+) \geqslant \mu_i - m_i,$$

which is the inequality claimed in the second line of (49). □

Lemma 4.13 *If none of the intervals $[t_i^+, t_{i+1}^-]$, $i = k, \ldots, k+s$, is a zero interval for S, then*

$$Z(S;]t_k, t_{k+s+1}[) \leqslant \mathcal{S}^-_{p_k}(t_k^+) - \mathcal{S}^+_{p_{k+s}}(t^-_{k+s+1}) + \sum_{i=k+1}^{k+s} m_i - z(S;]t_k, t_{k+s+1}[), \tag{52}$$

with z defined as in Theorem 4.10.

Proof With the notations introduced above, we have

$$Z(S;]t_k, t_{k+s+1}[) = \sum_{i=k}^{k+s} Z(S;]t_i, t_{i+1}[) + \sum_{i=k+1}^{k+s} \mu_i(S).$$

The claimed inequality (52) is simply obtained by application of (51) for $i = k, \ldots, k+s$ and of (49) for $i = k+1, \ldots, k+s$. □

Proof of Theorem 4.9 From Lemma 4.13 we know that

$$Z_{]a,b[}(S) \leqslant \mathcal{S}^-_{p_0}(a) - \mathcal{S}^+_{p_q}(b) + m - z(S;]a, b[) = \mathcal{S}^-_n(a) - \mathcal{S}^+_{p_q}(b) + m.$$

Accordingly, if μ_0 (resp., μ_{q+1}) denotes the multiplicity of $a^+ = a = t_0$ (resp. $b^- = b = t_{q+1}$) as a zero of S, we have

$$Z_{[a,b]}(S) = \mu_0 + Z_{]a,b[}(S) + \mu_{q+1} \leqslant \big(\mathcal{S}^-_n(a) + \mu_0\big) + \big(\mu_{q+1} - \mathcal{S}^+_{p_q}(b)\big) + m - z(S;]a, b[).$$

Since $\mathcal{S}^-_n(a) + \mu_0 \leqslant n$ and $\mu_{q+1} - \mathcal{S}^+_{p_q}(b) \leqslant 0$ (see (47)), the proof is complete. □

4.5 A few remarks

We would like to conclude this section with a few observations.

1– We have presented what we consider the natural way to count the zeroes in a given PW-spline S, limiting this count to sequences of consecutive non-zero intervals for S. As already mentioned, this approach differs from what is generally done (e.g., [9, 10, 34, 38]). We believe that our definition is also the most efficient, as pointed out below. In addition to (35), assume that $m_k = n + 1$, $k = 1, \ldots, q$. Then, for any $S \in \mathbb{S}$ with no zero interval, $z(S;]a, b[) = q$, and formula (46) yields:

$$Z(S; [a, b]) \leqslant n + (n + 1)q - q = \dim \mathbb{S} - 1 - q = (q + 1)n, \qquad (53)$$

which is consistent with the bounds $Z(S; [t_k, t_{k+1}]) \leqslant n$, $k = 0, \ldots, q$, resulting from the section-spaces being EC-spaces on their intervals. Instead of (53), the general bound provided in the articles cited above would be $Z(S; [a, b]) \leqslant \dim \mathbb{S} - 1 = n + (n + 1)q$.

Take the following trivial example, obtained under the additional assumption that $q = 1$ (i.e., there is only one interior knot), and that S vanishes exactly n times at t_1^- and exactly n times at t_1^+, that is, $\beta_1^-(S) = \beta_1^+(S) = n$. Then, according to (40), our multiplicity of the interior knot t_1 as a zero of S is $\mu_1(S) = 2n$. The bound $Z(S; [a, b]) \leqslant 2n$ provided in (53) is thus the best possible: it tells us that S vanishes neither on $[a, t_1[$ nor on $]t_1, b]$, which again is consistent with the fact that $\mathbb{E}_0$ and $\mathbb{E}_1$ are EC-spaces on their intervals. For comparison, in the various papers mentioned above, the multiplicity of t_1 as a zero of S would be defined either by n, or by $n + 1$, depending on the signs of S close to t_k^- and t_k^+. Accordingly, in all cases, the best bound for the zeroes of S on $[a, t_1[\cup]t_1, b]$ would be $2n + 1 - (n + 1) = n$.

2– Two different bounds of zeroes have been encountered: (43) and (46). As a matter of fact, in any given PW-spline space, a spline S being given, with no zero interval, the inequality (44) holds true if and only if so does (46). Indeed, one can easily check that our definition of the multiplicities of interior knots as zeroes implies that each of the two inequalities (44) and (46) is equivalent to the fact that (43) is satisfied for any $0 \leqslant k < k' \leqslant q + 1$ such that $\mu_i(S) \leqslant n - m_i$ for $k < i < k'$.

3– Let us work within the property (iii) of Theorem 4.3. Under the requirement that $\nu(k) \leqslant n + 1 - m_k$ at all interior knots, any Hermite interpolate problem in $\mathbb{S}$, based on a node-vector $\mathbb{Y} = \left(x_1{}^{[\mu_1]}, \ldots, x_r{}^{[\mu_r]}\right)$, with $a \leqslant x_1 < \cdots < x_r \leqslant b$, and positive $\mu_1, \ldots, \mu_r$ can as well be presented as the search for an $S \in \mathbb{S}$ satisfying

$$L_j S(x_i) = c_{i,j}, \quad j = 0, \ldots, \mu_i - 1, \quad i = 1, \ldots, r,$$

for convenient real numbers $c_{i,j}$. The unisolvence of any Hermite interpolation problem satisfying the SW-conditions in the spline space $L_0\mathbb{S}$ could have been deduced from either [38] or from [34]. Then we could have derived it in $\mathbb{S}$ afterwards. Nevertheless, we preferred to give a complete proof of the implication (iii) $\Rightarrow$ (i) with reference to our Definition 4.1 to stress the difference with other counts of zeroes.

5 Consequences

Theorem 4.3 provides us with a description of the whole class of all PEC-spline spaces which are good for interpolation. A PEC-spline space $\mathbb{S}$ being given; it also provides us with a constructive way to answer the question: is $\mathbb{S}$ good for interpolation or not? Given that, in each section-space $\mathbb{E}_k$, $k = 0, \ldots, q$, we know an explicit construction of all possible systems $(w_0^k, \ldots, w_n^k)$ of weight functions on $[t_k, t_{k+1}]$ such that $\mathbb{E}_k = EC(w_0^k, \ldots, w_n^k)$, in theory the question we have to answer is: can we find such systems so that, in terms of the associated generalised derivatives, the connection matrices will be the identity matrices? Answering this question is all the more difficult as n increases and as the multiplicities decrease. As a matter of fact, this is also the constructive way to answer the question: is the PEC-spline space $\widehat{\mathbb{S}}$ obtained from $\mathbb{S}$ by continuous integration good for design or not? We have been able to answer this latter question in some specific cases [7, 29]. Subsequently, we take advantage from these cases to illustrate the unisolvence of Hermite interpolation problems in $\mathbb{S}$ via Theorem 4.3.

5.1 Example 1: Hermite interpolation with W-splines

In this section, we assume that $\mathbb{E} \subset C^n([a, b])$ is an $(n + 1)$-dimensional W-space on $[a, b]$. We also assume that it is not an EC-space on the whole of $[a, b]$. As usual, we consider the sequence $\mathbb{T}$ of interior knots and the knot-vector $\mathbb{K}$ in (22). Here, for $k = 0, \ldots, q$, the section-space $\mathbb{E}_k$ is the restriction of $\mathbb{E}$ to $[t_k, t_{k+1}]$, and at each interior knot t_k, $k = 1, \ldots, q$, the splines in $\mathbb{S}$ are C^{n-m_k}, i.e., the connection matrix R_k is the identity matrix of order $(n+1-m_k)$. We can therefore assume that $m_k > 0$ for $k = 1, \ldots, q$.

It is known that one can find a positive number ℓ such that $\mathbb{E}$ is an EC-space on each interval $[c, d] \subset [a, b]$, if and only if $0 < d - c < \ell \leqslant b - a$ [39]. This positive number is called *the critical length of* $\mathbb{E}$ *on* $[a, b]$, see [28]. We can then state the following:

Proposition 5.1 *Let $\ell > 0$ be the critical length of $\mathbb{E}$ on $[a, b]$. For the W-spline space $\mathbb{S}$ to be good for interpolation it is necessary that*

$$t_{k+1} - t_k < \ell, \quad k = 0, \ldots, q, \tag{54}$$

while it is sufficient that

$$\xi_{k+n} - \xi_k < \ell, \quad k = 0, \ldots, m + 1 - n. \tag{55}$$

Proof Condition (54) means that the section-spaces are EC-spaces on their intervals, which is necessary for $\mathbb{S}$ to be good for interpolation. The second claimed statement corresponds to the obvious inclusion

$$\mathbb{A}_{n+1}(\mathbb{K}) \subset \bigcup_{k=0}^{m+1-n} [\xi_k, \xi_{k+n}]^{n+1}. \tag{56}$$

To make this clear, let $\widehat{\mathbb{E}} \subset C^{n+1}([a, b])$ satisfy $D\widehat{\mathbb{E}} = \mathbb{E}$. Take $(n + 1)$ functions $\widehat{\Phi}_0, \ldots, \widehat{\Phi}_n \in \widehat{\mathbb{E}}$ whose first derivatives span the space $\mathbb{E}$. Via convenient intersections of osculating flats, we can define the blossom $\widehat{\varphi}$ of $\widehat{\Phi} := (\widehat{\Phi}_0, \ldots, \widehat{\Phi}_n)^T$. It is well defined on some subset of $[a, b]^{n+1}$ containing the diagonal. We denote this subset by $\Delta(\widehat{\Phi})$. From (56), we can say that

$$\Delta(\widehat{\varphi}) \supset \bigcup_{k=0}^{m+1-n} [\xi_k, \xi_{k+n}]^{n+1} \quad \Rightarrow \quad \Delta(\widehat{\varphi}) \supset \mathbb{A}_{n+1}(\mathbb{K}). \tag{57}$$

The left inclusion in (57) means that $\widehat{\mathbb{E}}$ is an EC-space good for design on each $[\xi_k, \xi_{k+n}]$, that is, $\mathbb{E}$ is an EC-space on each $[\xi_k, \xi_{k+n}]$. This is satisfied when (55) holds. The right inclusion in (57) means that the spline space $\widehat{\mathbb{S}}$ is good for design, that is, the spline space $\mathbb{S}$ is good for interpolation. This concludes the proof. □

Remark 5.2 Within this remark, we work under the stronger requirement that $\mathbb{E} \subset C^{n+1}([a, b])$. Then, the $(n+1)$-dimensional W-space on $[a, b]$ is the null space of a linear differential operator L on $C^{n+1}([a, b])$, of the form

$$L = D^{n+1} + \sum_{i=0}^{n} a_i D^i,$$

where $a_0, \dots, a_n$ are continuous functions on $[a, b]$. The class of all W-spline space then coincides with the class of all L-spline spaces [39]. Under the additional assumption that, for $i = 0, \dots, n$, $a_i \in C^{n+1-i}([a, b])$ (which makes it possible to consider the adjoint of L), the study of minors of LB-spline collocation matrices carried out in [15] (see also [31, 32]) tells us that our W-spline space is good for interpolation when $\xi_{k+n+1} - \xi_k < \ell$, for all convenient k. By comparison, our conditions (55) thus represent an improvement.

Example 5.3 A positive number a being given, consider the space $\mathbb{E}$ spanned on $\mathbb{R}$ by four functions

$$\begin{aligned} U_1(x) &:= \cosh(ax)\cos x, \quad U_2(x) := \cosh(ax)\sin x, \\ U_3(x) &:= \sinh(ax)\cos x, \quad U_4(x) := \sinh(ax)\sin x. \end{aligned} \tag{58}$$

One can check that $\mathbb{E}$ is the null space of the linear differential operator L with constant coefficients

$$L := D^4 + 2(1 - a^2)D^2 + (a^2 + 1)^2 D^0.$$

The critical length ℓ on $\mathbb{R}$ of the W-space $\mathbb{E}$ is located in the open interval $]\pi, \frac{3\pi}{2}[$, see [6]. Here, all interior knots are simple, and we assume that $t_{k+1} - t_k = h$ for each $k = 0, \dots, q$, with $0 < h < \ell$. We proved in [7] that, if $q \geqslant 2$, the spline space $\widehat{\mathbb{S}}$ is good for design if and only if $h < \pi$. This is therefore the necessary and sufficient condition for the spline space $\mathbb{S}$ to be good for interpolation. Observe that our general sufficient condition (55) is much more restrictive since it implies that $h < \frac{\pi}{2}$. For comparison, the sufficient conditions in [15] would imply $h < \frac{3\pi}{8}$.

Let $\widehat{\widehat{\mathbb{S}}}$ be obtained by one additional step of continuous integration, with therefore pieces taken from the space $\widehat{\widehat{\mathbb{E}}}$ spanned by the six functions $1, x, U_1(x), U_2(x), U_3(x), U_4(x)$. The condition $h < \pi$ is sufficient for each of the spline spaces $\widehat{\mathbb{S}}$, $\widehat{\widehat{\mathbb{S}}}$ to be good for interpolation. We conjecture that it is also necessary, but this is not

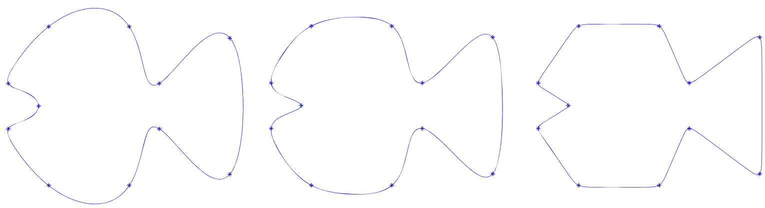

Fig. 1 Lagrange interpolation by C^4 splines: equispaced knots with $t_{k+1} - t_k = 3$, section-spaces spanned by $1, x, \cosh(ax)\cos x, \cosh(ax)\sin x, \sinh(ax)\cos x, \sinh(ax)\sin x$. From left to right: $a = 0.1$; $a = 1$; $a = 8$

proved so far. In [7], we showed the very strong shape effects resulting from the two parameters a, h when designing with the space $\widehat{\mathbb{S}}$. In Fig. 1, we consider Lagrange interpolating parametric spline curves in $\widehat{\widehat{\mathbb{S}}}$. For the sake of simplicity, we assume that we are dealing with periodic splines and that the nodes coincide with the knots. We therefore have symmetric cardinal B-spline bases with knot spacing h. As the parameter a tends to infinity, the splines behave as C^4 piecewise affine splines (!), all the more efficiently as h is closer to the limit π^-. This is why in all pictures we take $h = 3$. Then, visually speaking, we already obtain nearly perfect shape preservation with rather small values of a ($a = 8$).

5.2 Example 2: geometrically continuous cubic splines

Throughout the present section, we are dealing with simple equispaced knots $t_k = k$, $k = 0, \ldots, q+1$. We denote by $\widehat{\mathbb{S}}$ the associated space of geometrically continuous quadric splines, described by (29), the entries of the connection matrices at the interior knots being denoted as follows:

$$M_k = \begin{bmatrix} a_k & 0 & 0 \\ b_k & c_k & 0 \\ d_k & e_k & f_k \end{bmatrix}, \qquad \text{with } a_k, c_k, f_k > 0 \text{ for all } k = 1, \ldots, q. \tag{59}$$

In [29], we obtained necessary and sufficient conditions for $\widehat{\mathbb{S}}$ to be good for design. However, in [29], the integer k was ranging over the whole of $\mathbb{Z}$. Adapting the results to our present context, we can state, with

$$\begin{aligned} B_k &:= b_k + 3a_k + 3c_k, \quad E_k := e_k + 4c_k + 4f_k, \\ D_k &:= d_k + 3e_k + 4b_k + 6(a_k + 2c_k + f_k), \quad k = 1, \ldots, q. \end{aligned} \tag{60}$$

Theorem 5.4 *If $q \geqslant 2$, the spline space $\widehat{\mathbb{S}}$ of geometrically continuous quadric splines with connection matrices (59) is good for design if and only if the quantities introduced in (60) satisfy the following conditions:*

$$\begin{aligned} &B_k > 0, \quad D_k > 0, \quad E_k B_k - c_k D_k \quad \textit{for } k = 1, \ldots q, \\ &4B_k B_{k+1} f_{k+1} < D_k(E_{k+1}B_{k+1} - c_{k+1}D_{k+1}) \quad \textit{for } k = 1, \ldots, q-1. \end{aligned} \tag{61}$$

We now consider the space $\mathbb{S}$ of all geometrically continuous cubic splines with connection conditions

$$\big(S'(t_k^+), S''(t_k^+)\big)^T = \begin{bmatrix} c_k & 0 \\ e_k & f_k \end{bmatrix} \big(S'(t_k^-), S''(t_k^-)\big)^T, \text{ with } c_k, f_k > 0 \text{ for } k = 1, \ldots, q. \tag{62}$$

Such connections for cubic splines were first considered by B. Barsky in [2], in the special case $f_k = {c_k}^2$ and $e_k \geqslant 0$, i.e., G^2 continuity with total positivity of the connection matrices (see also [3] and [9, 10] for extensions to higher degrees). The following result was first obtained in [17], see also [29].

Theorem 5.5 *The space $\mathbb{S}$ of geometrically continuous cubic splines with connection conditions (62) is good for design if and only*

$$X_k := e_k + 2(c_k + f_k) > 0 \quad \textit{for } k = 1, \ldots, q. \tag{63}$$

Our purpose is now to obtain necessary and sufficient conditions for $\mathbb{S}$ to be good for interpolation and to compare them with (63). We obtain the following:

Theorem 5.6 *Assume that $q \geqslant 2$. The space $\mathbb{S}$ of all geometrically continuous cubic splines with connection conditions (62) is good for interpolation if and only if, with the notation introduced in (63)*

$$\begin{aligned} &X_k + 2(c_k + 1)\,\min(f_k, 1) > 0 \textit{ for } k = 1, \ldots, q, \textit{ and} \\ &\big[X_k + 2(c_k + 1)\big]\big[X_{k+1} + 2f_{k+1}(c_{k+1} + 1)\big] > 4(c_k + 1)(c_{k+1} + 1)f_{k+1} \textit{ for } k = 1, \ldots, q-1. \end{aligned} \tag{64}$$

Proof Consider the space $\widehat{\mathbb{S}}$ of geometrically continuous quadric splines defined by the connection matrices (59) in which we take

$$a_k = 1, \quad b_k = d_k = 0, \quad \text{for } k = 1, \ldots, q.$$

With these data, Theorem 5.4 shows that the conditions (64) are necessary and sufficient for $\widehat{\mathbb{S}}$ to be good for design, that is, for $\mathbb{S}$ to be good for interpolation. □

Remark 5.7 Comparison of (63) and (64) makes it obvious that "$\mathbb{S}$ good for design" implies "$\mathbb{S}$ good for interpolation" as stated in Theorem 1.2 (see also Theorem 4.4). When $q = 1$, one can check the two properties are equivalent.

Subsequently, we illustrate conditions (64) in several situations, all of which guarantee preservation of the symmetry of the data. Without real loss of generality, we assume that we are dealing with periodic data. Non-negative values of the parameters e_k correspond to totally positive connection matrices. In that case, some specific interpolation problems by geometrically polynomial splines were investigated in [9] (see also [14]). We will only mention this case incidentally, for comparison and for the sake of completeness. We are more specifically interested in illustrating unisolvence of interpolation problems, with special focus on *"interpolation beyond design"*.

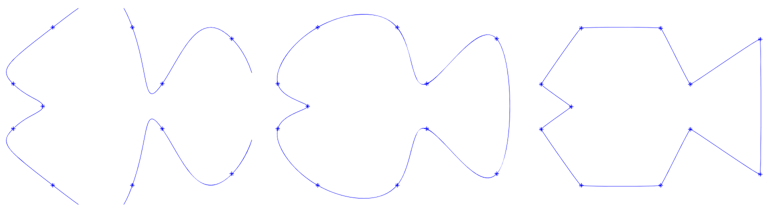

Fig. 2 Lagrange interpolation by C^1, G^2, cubic splines with connection conditions (62) and (65), see Example 5.8. From left to right: $e = -3$ (design/interpolation limit: -4); $e = 0$ (ordinary cubic splines); $e = 100$

Example 5.8 Assume that all the connection matrices in (62) are the same, that is,

$$c_k = c, \quad f_k = f, \quad e_k = e, \quad k = 1, \dots, q. \tag{65}$$

Then, it is easily seen that the conditions (64) reduce to $X > 0$, with $X := e + 2(c + f)$. In other words, in that case, $\mathbb{S}$ is good for interpolation if and only if it is good for design. In order to preserve the symmetry of the data (see [30]), we must additionally assume that $c = f = 1$. The spline space $\mathbb{S}$ depends on the sole parameter e which ranges over the interval $]-4, +\infty[$. It is a good opportunity to show how efficient this parameter is for tension properties when it tends to infinity, while when $e \to -4^+$, the interpolating curve becomes looser and looser. See Fig. 2. On purpose, for better comparison, we have kept the same frame in all figures in spite of the left curve extending off the edges.

Example 5.9 In our second example, we investigate examples of local shape effects. Consider the case where each connection matrix is the identity matrix, except at one selected control point, indicated with a circle, where it is equal $M = \begin{bmatrix} c & 0 \\ e & f \end{bmatrix}$, given that, in order to preserve symmetry, we have to take the connection matrix $\begin{bmatrix} 1/c & 0 \\ e/(cf) & 1/f \end{bmatrix}$ at the symmetric control point, indicated with a square. Set $X := e + 2(c + f)$. With our interpolating data, two different situations can be encountered:

1- The control points marked with a circle and a square are consecutive. Then conditions (64) reduce to $X > 0$. In that case, the spline space $\mathbb{S}$ is good for design if and only if it is good for interpolation. This case occurs in the sixth situation of Figs. 3 and 4.

2- Otherwise, the matrix M is preceded and followed by the identity matrix. In that case, the spline space is good for interpolation if and only

$$X + (c + 1)\min(f, 1) > 0, \tag{66}$$

that is,

$$e + 3c + 2f + 1 > 0 \quad \text{if } f \geqslant 1, \qquad e + 2c + 3f + cf > 0 \quad \text{if } f \leqslant 1.$$

We can see that the condition “$\mathbb{S}$ is good for interpolation” is strictly weaker than “$\mathbb{S}$ is good for design”. As an instance, when $c = f = 1$, we have $X = e + 4$

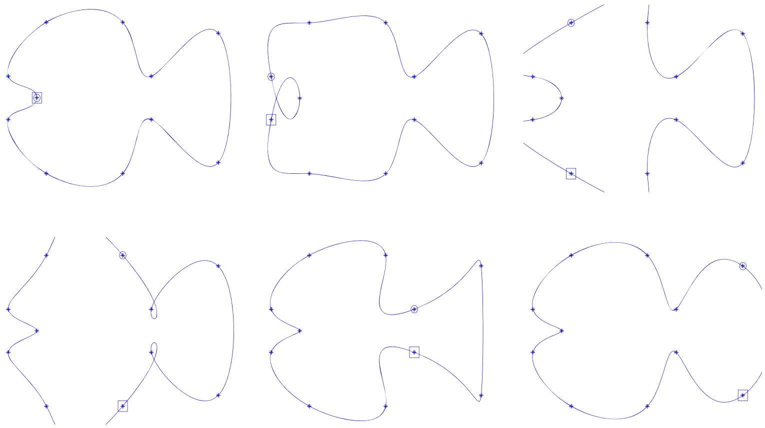

Fig. 3 Lagrange interpolation at the knots with C^1, G^2 cubic splines which are C^2 everywhere except at a circle/square according to Example 5.9 with $c = f = 1$, $e = -5$ (beyond the design limit -4) in each case, except the rightmost bottom one where $c = f = 1$, $e = -3$. See Example 5.9

and (66) means that $e + 6 > 0$. Note that taking $c = f = 1$ is compulsory for symmetry preservation when the circled point is its own symmetric, as occurs in the first situation of Figs. 3 and 4.

Figure 3 illustrates the limit values for "being good for interpolation" and more specifically the comparison with the corresponding limit for design, according to the discussion above. Though not specially concerned with the quality of the interpolating curves, we briefly comment on this point. Figure 4 illustrates the efficiency of the sole parameter e for local shape preservation when its tends to infinity. Oppositely, when e tends to the negative limit provided by (66), we may obtain oscillations or loops (e.g., Fig. 3, up, middle). This drawback can easily be overcome by increasing the value of e so as to get closer to the corresponding picture in Fig. 4. However,

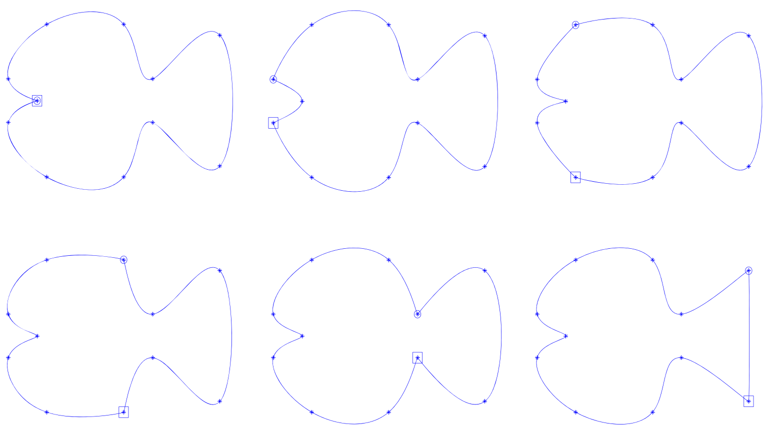

Fig. 4 Lagrange interpolation at the knots with C^1, G^2 cubic splines which are C^2 everywhere except at a circle/square according to Example 5.9 with $c = f = 1$, $e = 100$ in each case

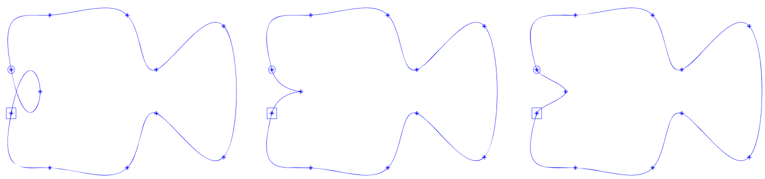

Fig. 5 Interpolation beyond design: examples of G^1 cubic splines which are C^2 everywhere except the circle/square (see Example 5.9). In all examples $f = 1$. From left to right: $c = 1$, $e = -5$ (design limit -4); $c = 3$, $e = -9$ (design limit -8); $c = 9$, $e = -21$ (design limit -20)

if ever we are interested in keeping the general shape, it is alternatively possible to modify the three parameters c, e, f at the same time. We can indeed change the curvature in a predictable way by increasing/decreasing the value of c, while remaining in the "beyond design" context. Examples are given in Fig. 5: at a circle/square the splines are no longer C^1, but G^1.

Remark 5.10 Let $\mathbb{S}$ be any space of geometrically continuous cubic splines in a situation "interpolation beyond design", such as illustrated in Figs. 3 and 5. Then, $\mathbb{S}$ being good for interpolation, it possesses B-spline-like bases, and so does any $\mathbb{S}^\star$ obtained from $\mathbb{S}$ by knot insertion. By contrast, since $\mathbb{S}$ is not good for design, Theorem 4.4 ensures that $\mathbb{S}$ does not possess a B-spline basis. Since $\mathbb{S}$ contains the constants, this means that the expansion of the function $\mathbb{1}$ in any given B-spline-like basis of $\mathbb{S}$ possesses at least one non-positive coefficient.

6 Concluding remarks

1– This work has enabled us to identify the whole class of spline spaces which can be used for solving appropriate Hermite interpolation problems within the largest possible context of PW-spline spaces on a given $([a, b]; \mathbb{T})$. It is the class of all spline spaces based on ECP-spaces on $([a, b]; \mathbb{T})$. The foundations of this work was the previously obtained description of the whole class of spline spaces which can be used for design within the same context [27].

Our main concern was to establish the crucial links existing between spline Hermite interpolation and spline design. As announced in Theorem 1.2, differentiation/integration transforms "good for design" into "good for interpolation" and vice versa, while decreasing/increasing the dimension of the section-spaces by one. From the more complete Theorems 4.3 and 3.5, we can additionally say that multiplication by a piecewise weight function also transform "good for design" into the weaker property "good for interpolation" and vice versa. Denoting by $\mathcal{D}_n([a, b]; \mathbb{T})$ into $\mathcal{I}_n([a, b]; \mathbb{T})$ the classes of all PW- (or PEC-) spline spaces which are good for respectively design and interpolation, we can symbolically summarize this as follows:

$$\mathcal{D}_{n+1}([a, b]; \mathbb{T}) = \int \mathcal{I}_n([a, b]; \mathbb{T}), \quad \mathcal{I}_n([a, b]; \mathbb{T}) = \bigcup_{\Omega \in PC_+^n([a,b];\mathbb{T})} \Omega\, \mathcal{D}_n([a, b]; \mathbb{T}).$$

In other words, piecewise generalised derivatives constitute the fundamental tools ensuring the passage from spline interpolation to spline design and conversely.

2– Theorem 1.2 was already announced in the note [13]. As briefly indicated in [13], the proof of its implication (i) $\Rightarrow$ (ii) which was initially planned consisted in the two main points listed below:

– Firstly, in all given PW-spline spaces good for design, the expansions of the B-spline bases in the new B-spline bases obtained after insertion of knots all have exactly the same structure. This structure is a clear consequence of the properties of blossoms, mainly of their pseudoaffinity.
– Secondly, the geometric proof of the positivity of all minors of polynomial collocation B-spline matrices under SW-conditions provided in [5], with a view to spline interpolation, based on knot insertion, depends only on the previous structure.

Applied to a PW-spline space $\mathbb{S}$ assumed to be good for design, the proof carried out in [5] should have enabled us to demonstrate that, if, after knot insertion, we obtain a PW-spline space $D\mathbb{S}^\star$ good for interpolation, then the initial PW-spline space $D\mathbb{S}$ itself is good for interpolation. The conclusion would then have been achieved via Example 4.2. Unfortunately, a very careful checking showed that the proof in question in [5] contained a very small mistake in the set of values over which an index is ranging, difficult or even impossible to rectify. Fortunately, the characterisation of all spline spaces good for design obtained in [27] enabled us to replace this incorrect expected proof of Theorem 1.2 by a correct one, based on totally different arguments.

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