Numer Algor (2018) 77:727–740
DOI 10.1007/s11075-017-0337-6

ORIGINAL PAPER

Iterative approaches to solving convex minimization problems and fixed point problems in complete CAT(0) spaces

Kritsada Lerkchaiyaphum[1] ·
Withun Phuengrattana[1,2]

Received: 17 January 2017 / Accepted: 30 April 2017 / Published online: 13 May 2017

Abstract In this paper, we propose a new modified proximal point algorithm for finding a common element of the set of common minimizers of a finite family of convex and lower semi-continuous functions and the set of common fixed points of a finite family of nonexpansive mappings in complete CAT(0) spaces, and prove some convergence theorems of the proposed algorithm under suitable conditions. A numerical example is presented to illustrate the proposed method and convergence result. Our results improve and extend the corresponding results existing in the literature.

Keywords Proximal point algorithm · Nonexpansive mappings · CAT(0) spaces

Mathematics Subject Classification (2010) 47H09 · 47H10 · 65K10 · 65K15

1 Introduction

A metric space (X, d) is said to be a *CAT(0) space* if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane (see more details in [1]). It is well known that any complete, simply

✉ Withun Phuengrattana
withun_ph@yahoo.com

Kritsada Lerkchaiyaphum
a_krit2@hotmail.com

1 Department of Mathematics, Faculty of Science and Technology, Nakhon Pathom Rajabhat University, Nakhon Pathom 73000, Thailand

2 Research Center for Pure and Applied Mathematics, Research and Development Institute, Nakhon Pathom Rajabhat University, Nakhon Pathom 73000, Thailand

connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Euclidean spaces, Hilbert spaces, the Hilbert ball [2], hyperbolic spaces [3], and $\mathbb{R}$-trees [4].

Let X be a CAT(0) space and $h : X \to (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization is to find $x \in X$ such that

$$h(x) = \min_{u \in X} h(u).$$

We denote by $\mathrm{argmin}_{u \in X} h(u)$ the set of minimizers of h. The proximal point algorithm is an important tool for solving this problem which was initiated by Martinet [5] in 1970. Recently, many convergence results concerning the proximal point algorithm for solving optimization problems have been extended from the classical linear spaces such as Hilbert spaces and Banach spaces to the setting of manifolds (see [6–10]).

In 2013, Bačák [7] considered the minimization problems in complete CAT(0) spaces to prove the following result.

Theorem 1 *Let X be a complete CAT(0) space, and $h : X \to (-\infty, \infty]$ be a convex and lower semi-continuous function. Suppose that h has a minimizer. For $x_1 \in X$, let $\{x_n\}$ be a sequence in X defined by*

$$x_{n+1} = \underset{u \in X}{argmin} \left[h(u) + \frac{1}{2\lambda_n} d(u, x_n)^2 \right], \quad \forall n \geq 1,$$

where $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then, the sequence $\{x_n\}$ Δ-converges to a minimizer of h.

Recently, Cholamjiak et al. [11] considered the problem of finding a minimizer of a convex and lower semi-continuous function and common fixed points of two nonexpansive mappings in complete CAT(0) spaces. To be more precise, they obtained the following result.

Theorem 2 *Let X be a complete CAT(0) space, and $h : X \to (-\infty, \infty]$ be a convex and lower semi-continuous function. Let T_1 and T_2 be nonexpansive mappings on X such that $\mathcal{F} = argmin_{u \in D} h(u) \cap F(T_1) \cap F(T_2)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \geq 1$ and for some a, b, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and for some λ. For $x_1 \in X$, let $\{x_n\}$ be a sequence in X defined by*

$$\begin{cases} z_n = \underset{u \in X}{\mathrm{argmin}} \left[h(u) + \frac{1}{2\lambda_n} d(u, x_n)^2 \right], \\ y_n = \beta_n x_n \oplus (1 - \beta_n) T_1 z_n, \\ x_{n+1} = \alpha_n T_1 x_n \oplus (1 - \alpha_n) T_2 y_n, \quad \forall n \geq 1. \end{cases}$$

Then, the sequence $\{x_n\}$ Δ-converges to a common element of $\mathcal{F}$.

Motivated by [7] and [11], we present the following question arises:

Question I Can we construct an iterative process for finding common minimizers of a finite family of convex and lower semi-continuous functions and common fixed points of a finite family of nonexpansive mappings in CAT(0) spaces?

The aim of this paper is to propose a new modified proximal point algorithm for finding a common element of the set of common minimizers of a finite family of convex and lower semi-continuous functions and the set of common fixed points of a finite family of nonexpansive mappings in a nonempty closed convex subset of a complete CAT(0) space and prove Δ-convergence and strong convergence theorems of the proposed algorithm under suitable conditions. A numerical example to support our main results is also given. Our results not only give an affirmative answer to the question I but also generalize the corresponding results of Bačák [7], Ariza-Ruiz et al. [6], Cholamjiak et al., and many others.

2 Preliminaries and lemmas

Let D be a nonempty subset of a CAT(0) space X. A subset D of X is said to be *convex* if D includes every geodesic segment joining any two of its points, that is, for any $x, y \in D$, we have $[x, y] \subset D$, where $[x, y] := \{\alpha x \oplus (1-\alpha)y : 0 \leq \alpha \leq 1\}$ is the unique geodesic joining x and y.

Let $T : D \to D$ be a mapping. An element $x \in D$ is called a *fixed point* of T if $x = Tx$. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in D : x = Tx\}$.

The notion of the asymptotic center can be introduced in a CAT(0) space X as follows: Let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from [12] that in a complete CAT(0) space, the asymptotic center $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

We now collect some basic properties of the Δ-convergence which will be used in the sequel.

Lemma 1 ([13]) *Every bounded sequence in a complete CAT(0) space has a Δ-convergent subsequence.*

Lemma 2 ([14]) *Let D be a nonempty closed convex subset of a complete CAT(0) space X. If $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 3 ([15]) *Let $\{x_n\}$ be a sequence in a complete CAT(0) space X with $A(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 4 ([15]) *Let D be a nonempty closed convex subset of a complete CAT(0) space X and $T : D \to D$ be a nonexpansive mapping, i.e., $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in D$. Let $\{x_n\}$ be a bounded sequence in D such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n\to\infty} x_n = x$. Then, x is a fixed point of T.*

In 2009, Kopecká and Reich [16] defined convex combinations of a finite members in the Hilbert ball. It can be extended to a CAT(0) space as follows. Let $x_1, \ldots, x_n$ be points in a CAT(0) space X and $\gamma_1, \ldots, \gamma_n \in (0, 1)$ with $\sum_{i=1}^n \gamma_i = 1$, we write

$$\gamma_1 x_1 \oplus \gamma_2 x_2 \oplus \cdots \oplus \gamma_n x_n := (1-\gamma_n)\left(\tfrac{\gamma_1}{1-\gamma_n} x_1 \oplus \tfrac{\gamma_2}{1-\gamma_n} x_2 \oplus \cdots \oplus \tfrac{\gamma_{n-1}}{1-\gamma_n} x_{n-1}\right) \oplus \gamma_n x_n. \tag{1}$$

Using (1), we obtain that

$$d\left(\gamma_1 x_1 \oplus \gamma_2 x_2 \oplus \cdots \oplus \gamma_n x_n, y\right) \leq \sum_{i=1}^n \gamma_i d(x_i, y) \ \text{ for each } \ y \in X.$$

In 2014, Chidume et al. [17] proved the following important lemma.

Lemma 5 *Let X be a CAT(0) space and $z \in X$. Let $x_1, \ldots, x_N \in X$ and $\gamma_1, \ldots, \gamma_N$ be real numbers in $[0, 1]$ such that $\sum_{i=1}^N \gamma_i = 1$. Then, the following inequality holds the following:*

$$d\left(z, \gamma_1 x_1 \oplus \gamma_2 x_2 \oplus \cdots \oplus \gamma_N x_N\right)^2 \leq \sum_{i=1}^N \gamma_i d(z, x_i)^2 - \sum_{i,j=1, i\neq j}^N \gamma_i \gamma_j d(x_i, x_j)^2.$$

Recall that a function $h : D \to (-\infty, \infty]$ is *convex* if, for any geodesic l: $[0, 1] \to D$, the composite function $h \, o \, l$ is convex. We say that a function h defined on D is *lower semi-continuous* at a point $x \in D$ if

$$h(x) \leq \liminf_{n\to\infty} h(x_n)$$

for every sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} x_n = x$. A function h is said to be *lower semi-continuous* on D if it is lower semi-continuous at any point in D. For any $\lambda > 0$, define the *Moreau-Yosida resolvent* of h as follows:

$$J_\lambda x = \operatorname*{argmin}_{u\in D}\left[h(u) + \frac{1}{2\lambda} d(u, x)^2\right]$$

and put $J_0x = x$, for all $x \in D$. This definition in metric spaces with no linear structure first appeared in [18]. The mapping J_λ is well defined for all $\lambda \geq 0$ (see [18]). If h is a proper convex and lower semi-continuous function, then the set of fixed points of the resolvent associated with h coincides with the set of minimizers of h (see [6]). Also, the resolvent J_λ of h is nonexpansive for all $\lambda > 0$ (see [19]).

The following two lemmas are useful for our main results.

Lemma 6 ([19, 20]) *Let X be a complete CAT(0) space and $h : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For each $x \in X$ and $\lambda > \mu > 0$, the following identity holds the following:*

$$J_\lambda x = J_\mu\left(\frac{\lambda-\mu}{\lambda}J_\lambda x \oplus \frac{\mu}{\lambda}x\right),$$

where J_λ is a Moreau-Yosida resolvent of h.

Lemma 7 ([21]) *Let X be a complete CAT(0) space and $h : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For each $x, y \in X$ and $\lambda > 0$, the following inequality holds the following:*

$$\frac{1}{2\lambda}d(J_\lambda x, y)^2 - \frac{1}{2\lambda}d(x, y)^2 + \frac{1}{2\lambda}d(J_\lambda x, x)^2 \leq h(y) - h(J_\lambda x),$$

where J_λ is a Moreau-Yosida resolvent of h.

3 Main results

In this section, we prove Δ-convergence and strong convergence theorems for finding a common solution of the set of a finite family of convex and lower semi-continuous functions and the set of a finite family of nonexpansive mappings in a nonempty closed convex subset of a complete CAT(0) space. In order to prove our main results, the following two lemmas are needed.

Lemma 8 *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \cap_{i=1}^N argmin_{u\in D}h_i(u) \cap \cap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by*

$$\begin{cases} y_n^{(i)} = \underset{u\in D}{\operatorname{argmin}}\left[h_i(u) + \dfrac{1}{2\lambda_n^{(i)}}d(u, x_n)^2\right], \\ z_n = \beta_n^{(0)}x_n \oplus \beta_n^{(1)}y_n^{(1)} \oplus \beta_n^{(2)}y_n^{(2)} \oplus \cdots \oplus \beta_n^{(N)}y_n^{(N)} \\ w_n = \gamma_n^{(0)}z_n \oplus \gamma_n^{(1)}T_1z_n \oplus \gamma_n^{(2)}T_2z_n \oplus \cdots \oplus \gamma_n^{(N)}T_Nz_n \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)w_n, \quad \forall n \geq 1, \end{cases} \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, *and* $\{\gamma_n^{(i)}\}$ *are sequences in* $[0, 1]$ *such that* $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^{N} \beta_n^{(i)} = 1$ *and* $\sum_{i=0}^{N} \gamma_n^{(i)} = 1$ *for all* $n \geq 1$, *and* $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} > 0$ *for all* $n \geq 1$, $i = 1, 2, \ldots, N$. *Then,* $\lim_{n\to\infty} d(x_n, p)$ *exists for* $p \in \mathcal{F}$.

Proof Let $p \in \mathcal{F}$. Then, $p = T_i p$ for all $i = 1, 2, \ldots, N$ and $h_i(p) \leq h_i(y)$ for all $y \in D$ and $i = 1, 2, \ldots, N$. It follows that

$$h_i(p) + \frac{1}{2\lambda_n^{(i)}} d(p, p)^2 \leq h_i(y) + \frac{1}{2\lambda_n^{(i)}} d(y, p)^2$$

for all $y \in D$, $i = 1, 2, \ldots, N$, and hence, $p = J_{\lambda_n^{(i)}} p$ for all $n \geq 1$. Since $y_n^{(i)} = J_{\lambda_n^{(i)}} x_n$ for all $n \geq 1$ and $i = 1, 2, \ldots, N$, it implies that

$$d(y_n^{(i)}, p) = d(J_{\lambda_n^{(i)}} x_n, J_{\lambda_n^{(i)}} p) \leq d(x_n, p). \tag{3}$$

By (2), we have that

$$\begin{aligned} d(x_{n+1}, p) &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(w_n, p) \\ &\leq \alpha_n d(x_n, p) + (1 - \alpha_n)(\gamma_n^{(0)} d(z_n, p) + \gamma_n^{(1)} d(T_1 z_n, p) \\ &\quad + \cdots + \gamma_n^{(N)} d(T_N z_n, p)) \\ &\leq \alpha_n d(x_n, p) + (1 - \alpha_n) d(z_n, p) \\ &\leq \alpha_n d(x_n, p) + (1 - \alpha_n)(\beta_n^{(0)} d(x_n, p) + \beta_n^{(1)} d(y_n^{(1)}, p) \\ &\quad + \cdots + \beta_n^{(N)} d(y_n^{(N)}, p)). \end{aligned}$$

This implies by (3) that

$$d(x_{n+1}, p) \leq d(x_n, p).$$

Hence, the sequence $\{d(x_n, p)\}$ is nonincreasing and bounded below. It follows that $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in \mathcal{F}$. □

Lemma 9 *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let* $\{h_i\}_{i=1}^{N}$ *be a finite family of proper convex and lower semi-continuous functions of D into* $(-\infty, \infty]$ *and let* $\{T_i\}_{i=1}^{N}$ *be a finite family of nonexpansive mappings of D into itself. Suppose that* $\mathcal{F} = \cap_{i=1}^{N} argmin_{u \in D} h_i(u) \cap \cap_{i=1}^{N} F(T_i)$ *is nonempty. For* $x_1 \in D$, *let* $\{x_n\}$ *be a sequence in D defined by* (2) *where* $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, *and* $\{\gamma_n^{(i)}\}$ *are sequences in* $[0, 1]$ *such that* $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^{N} \beta_n^{(i)} = 1$ *and* $\sum_{i=0}^{N} \gamma_n^{(i)} = 1$ *for all* $n \geq 1$, *and* $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ *for all* $n \geq 1$, $i = 1, 2, \ldots, N$ *and some* $\lambda^{(i)}$. *Then, we have the following:*

(i) $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ *for all* $i = 1, 2, \ldots, N$;
(ii) $\lim_{n\to\infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0$ *for all* $i = 1, 2, \ldots, N$.

Proof (i): Let $p \in \mathcal{F}$. By Lemma 8, we have $\lim_{n\to\infty} d(x_n, p)$ exists. Using (2), we have that

$$
\begin{aligned}
d(x_{n+1}, p)^2 &\le \alpha_n d(x_n, p)^2 + (1-\alpha_n) d(w_n, p)^2 - \alpha_n(1-\alpha_n) d(x_n, w_n)^2 \\
&\le \alpha_n d(x_n, p)^2 + (1-\alpha_n)\Bigg(\gamma_n^{(0)} d(z_n, p)^2 + \sum_{i=1}^{N} \gamma_n^{(i)} d(T_i z_n, p)^2 \\
&\quad -\sum_{i=1}^{N} \gamma_n^{(0)}\gamma_n^{(i)} d(z_n, T_i z_n)^2 - \sum_{i,j=1, i\neq j}^{N} \gamma_n^{(i)}\gamma_n^{(j)} d(T_i z_n, T_j z_n)^2\Bigg) \\
&\quad -\alpha_n(1-\alpha_n) d(x_n, w_n)^2 \\
&\le \alpha_n d(x_n, p)^2 + (1-\alpha_n) d(z_n, p)^2 - (1-\alpha_n)\sum_{i=1}^{N} \gamma_n^{(0)}\gamma_n^{(i)} d(z_n, T_i z_n)^2 \\
&\quad -(1-\alpha_n)\sum_{i,j=1, i\neq j}^{N} \gamma_n^{(i)}\gamma_n^{(j)} d(T_i z_n, T_j z_n)^2 - \alpha_n(1-\alpha_n) d(x_n, w_n)^2 \\
&\le \alpha_n d(x_n, p)^2 + (1-\alpha_n)\Bigg(\beta_n^{(0)} d(x_n, p)^2 + \sum_{i=1}^{N} \beta_n^{(i)} d(y_n^{(i)}, p)^2 \\
&\quad -\sum_{i=1}^{N} \beta_n^{(0)}\beta_n^{(i)} d(x_n, y_n^{(i)})^2 - \sum_{i,j=1, i\neq j}^{N} \beta_n^{(i)}\beta_n^{(j)} d(y_n^{(i)}, y_n^{(j)})^2\Bigg) \\
&\quad -(1-\alpha_n)\sum_{i=1}^{N} \gamma_n^{(0)}\gamma_n^{(i)} d(z_n, T_i z_n)^2 \\
&\quad -(1-\alpha_n)\sum_{i,j=1, i\neq j}^{N} \gamma_n^{(i)}\gamma_n^{(j)} d(T_i z_n, T_j z_n)^2 - \alpha_n(1-\alpha_n) d(x_n, w_n)^2 \\
&\le d(x_n, p)^2 - (1-\alpha_n)\sum_{i=1}^{N} \beta_n^{(0)}\beta_n^{(i)} d(x_n, y_n^{(i)})^2 \\
&\quad -(1-\alpha_n)\sum_{i,j=1, i\neq j}^{N} \beta_n^{(i)}\beta_n^{(j)} d(y_n^{(i)}, y_n^{(j)})^2 - (1-\alpha_n)\sum_{i=1}^{N} \gamma_n^{(0)}\gamma_n^{(i)} d(z_n, T_i z_n)^2 \\
&\quad -(1-\alpha_n)\sum_{i,j=1, i\neq j}^{N} \gamma_n^{(i)}\gamma_n^{(j)} d(T_i z_n, T_j z_n)^2 - \alpha_n(1-\alpha_n) d(x_n, w_n)^2.
\end{aligned}
$$

This implies by $0 < a \le \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \le b < 1$ that

$$
d(x_n, p)^2 - d(x_{n+1}, p)^2 \ge a^2(1-b)\sum_{i=1}^{N} d(x_n, y_n^{(i)})^2 + a^2(1-b)\sum_{i,j=1, i\neq j}^{N} d(y_n^{(i)}, y_n^{(j)})^2
$$

$$+a^2(1-b)\sum_{i=1}^{N} d(z_n, T_i z_n)^2 + a^2(1-b)\sum_{i,j=1, i\neq j}^{N} d(T_i z_n, T_j z_n)^2$$
$$+a(1-b)d(x_n, w_n)^2.$$

Since $\lim_{n\to\infty} d(x_n, p)$ exists and $a, b \in (0,1)$, it implies that $\lim_{n\to\infty} d(x_n, w_n) = 0$ and

$$\lim_{n\to\infty} d(x_n, y_n^{(i)}) = 0, \quad \lim_{n\to\infty} d(z_n, T_i z_n) = 0, \tag{4}$$

for all $i = 1, 2, \ldots, N$, and

$$\lim_{n\to\infty} d(y_n^{(i)}, y_n^{(j)}) = 0, \quad \lim_{n\to\infty} d(T_i z_n, T_j z_n) = 0, \tag{5}$$

for all $i, j = 1, \ldots, N$, and $i \neq j$. From nonexpansiveness of T_i, we have, for each $i = 1, 2, \ldots, N$,

$$\begin{aligned} d(x_n, T_i x_n) &\leq d(x_n, z_n) + d(z_n, T_i z_n) + d(T_i z_n, T_i x_n) \\ &\leq 2d(x_n, z_n) + d(z_n, T_i z_n) \\ &\leq 2(\beta_n^{(1)} d(x_n, y_n^{(1)}) + \cdots + \beta_n^{(N)} d(x_n, y_n^{(N)})) + d(z_n, T_i z_n). \end{aligned}$$

This implies by (4) that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, N$.

(ii): Since $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$, by Lemma 6 and nonexpansiveness of $J_{\lambda^{(i)}}$, we have

$$\begin{aligned} d(x_n, J_{\lambda^{(i)}} x_n) &\leq d(x_n, y_n^{(i)}) + d(y_n^{(i)}, J_{\lambda^{(i)}} x_n) \\ &= d(x_n, y_n^{(i)}) + d(J_{\lambda_n^{(i)}} x_n, J_{\lambda^{(i)}} x_n) \\ &= d(x_n, y_n^{(i)}) + d\left(J_{\lambda^{(i)}}\left(\frac{\lambda_n^{(i)} - \lambda^{(i)}}{\lambda_n^{(i)}} J_{\lambda_n^{(i)}} x_n \oplus \frac{\lambda^{(i)}}{\lambda_n^{(i)}} x_n\right), J_{\lambda^{(i)}} x_n\right) \\ &\leq d(x_n, y_n^{(i)}) + d\left(\left(\frac{\lambda_n^{(i)} - \lambda^{(i)}}{\lambda_n^{(i)}}\right) J_{\lambda_n^{(i)}} x_n \oplus \frac{\lambda^{(i)}}{\lambda_n^{(i)}} x_n, x_n\right) \\ &= d(x_n, y_n^{(i)}) + \left(1 - \frac{\lambda^{(i)}}{\lambda_n^{(i)}}\right) d(J_{\lambda_n^{(i)}} x_n, x_n) \\ &= d(x_n, y_n^{(i)}) + \left(1 - \frac{\lambda^{(i)}}{\lambda_n^{(i)}}\right) d(y_n^{(i)}, x_n) \\ &= \left(2 - \frac{\lambda^{(i)}}{\lambda_n^{(i)}}\right) d(y_n^{(i)}, x_n). \end{aligned}$$

Thus, by (4), we obtain that $\lim_{n\to\infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0$. □

We now get the Δ-convergence theorem in complete CAT(0) spaces.

Theorem 3 *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \cap_{i=1}^N argmin_{u\in D} h_i(u) \cap \cap_{i=1}^N F(T_i)$ is nonempty.*

For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by (2) where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \le \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \le b < 1$, $\sum_{i=0}^{N} \beta_n^{(i)} = 1$ and $\sum_{i=0}^{N} \gamma_n^{(i)} = 1$ for all $n \ge 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} \ge \lambda^{(i)} > 0$ for all $n \ge 1$, $i = 1, 2, \ldots, N$ and some $\lambda^{(i)}$. Then, $\{x_n\}$ Δ-converges to an element of $\mathcal{F}$.

Proof Lemma 8 shows that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$ and Lemma 9 also implies that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ and $\lim_{n\to\infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0$ for all $i = 1, 2, \ldots, N$.

Next, we show that $\omega_\Delta(x_n) \subset \mathcal{F}$. Let $u \in \omega_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. From Lemmas 1 and 4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_{n\to\infty} v_n = v$ for some $v \in \mathcal{F}$. So, by Lemma 3, we have $u = v$. This shows that $\omega_\Delta(x_n) \subset \mathcal{F}$. Finally, we show that the sequence $\{x_n\}$ Δ-converges to a point in $\mathcal{F}$. To this end, it suffices to show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_\Delta(x_n) \subset \mathcal{F}$ and $\{d(x_n, u)\}$ converges, by Lemma 3, we have $x = u$. Hence, $\omega_\Delta(x_n) = \{x\}$. This completes the proof. □

Next, we establish strong convergence theorems in complete CAT(0) spaces.

Theorem 4 *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{h_i\}_{i=1}^{N}$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \cap_{i=1}^{N} argmin_{u\in D} h_i(u) \cap \cap_{i=1}^{N} F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by (2) where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \le \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \le b < 1$, $\sum_{i=0}^{N} \beta_n^{(i)} = 1$ and $\sum_{i=0}^{N} \gamma_n^{(i)} = 1$ for all $n \ge 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} \ge \lambda^{(i)} > 0$ for all $n \ge 1$, $i = 1, 2, \ldots, N$ and some $\lambda^{(i)}$. Then, the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} dist(x_n, \mathcal{F}) = 0$, where $dist(x, \mathcal{F}) = \inf\{d(x, z) : z \in \mathcal{F}\}$.*

Proof The necessity is obvious and then we prove only the sufficiency. Suppose that $\liminf_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0$. Let $\varepsilon > 0$ be arbitrarily chosen. So, there exists a positive integer m_0 such that

$$\operatorname{dist}(x_n, \mathcal{F}) < \frac{\varepsilon}{4}, \quad \forall n \ge m_0.$$

In particular, $\inf\{d(x_{m_0}, p) : p \in \mathcal{F}\} < \frac{\varepsilon}{4}$. Thus, there must exist $q \in \mathcal{F}$ such that $d(x_{m_0}, q) < \frac{\varepsilon}{2}$. Then, for all $m, n \ge m_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\le d(x_{n+m}, q) + d(q, x_n) \\ &\le 2d(x_{m_0}, q) \\ &< \varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset in a complete CAT(0) space, it is complete. This implies that $\{x_n\}$ converges to some point p^* in C. Since $\mathcal{F}$ is closed and $\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0$, we have $p^* \in \mathcal{F}$. This completes the proof. □

Theorem 5 *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \cap_{i=1}^N argmin_{u\in D} h_i(u) \cap \cap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by (2) where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \beta_n^{(i)} = 1$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \geq 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ for all $n \geq 1$, $i = 1, 2, \ldots, N$ and some $\lambda^{(i)}$. If there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ and $g(r) > 0$ for all $r > 0$ such that*

$$g(dist(x, \mathcal{F})) \leq d(x, T_i x)$$

or

$$g(dist(x, \mathcal{F})) \leq d(x, J_{\lambda^{(i)}} x)$$

for some $i = 1, 2, \ldots, N$ and for all $x \in D$, then the sequence $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

Proof From Lemma 8, we know that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \mathcal{F}$. This implies that $\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F})$ exists. By Lemma 9, we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0$$

and

$$\lim_{n\to\infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0.$$

By the hypothesis, we see that

$$\lim_{n\to\infty} g\left(\operatorname{dist}(x_n, \mathcal{F})\right) \leq \lim_{n\to\infty} d(x_n, T_i x_n) = 0$$

or

$$\lim_{n\to\infty} g\left(\operatorname{dist}(x_n, \mathcal{F})\right) \leq \lim_{n\to\infty} d(x_n, J_{\lambda^{(i)}} x_n) = 0.$$

So, we have

$$\lim_{n\to\infty} g\left(\operatorname{dist}(x_n, \mathcal{F})\right) = 0.$$

Using the property of g, it follows that $\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0$. It implies by Theorem 4 that $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. This completes the proof. □

Recall that a mapping $T : D \to D$ is said to be *semi-compact* if for any sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$

such that $\{x_{n_i}\}$ converges strongly to $p \in D$. Now, we prove a strong convergence theorem in complete CAT(0) spaces.

Theorem 6 *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{h_i\}_{i=1}^N$ be a finite family of proper convex and lower semi-continuous functions of D into $(-\infty, \infty]$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Suppose that $\mathcal{F} = \cap_{i=1}^N argmin_{u\in D} h_i(u) \cap \cap_{i=1}^N F(T_i)$ is nonempty. For $x_1 \in D$, let $\{x_n\}$ be a sequence in D defined by (2) where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $0 < a \leq \alpha_n, \beta_n^{(i)}, \gamma_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \beta_n^{(i)} = 1$ and $\sum_{i=0}^N \gamma_n^{(i)} = 1$ for all $n \geq 1$, and $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ for all $n \geq 1$, $i = 1, 2, \ldots, N$ and some $\lambda^{(i)}$. If $J_{\lambda^{(i)}}$ or T_i is semi-compact for some $i = 1, 2, \ldots, N$, then the sequence $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.*

Proof Without loss of generality, we can assume T_1 is semi-compact. By Lemma 9, we have $\lim_{n\to\infty} d(x_n, T_1x_n) = 0$. So, by semi-compactness of T_1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q \in D$ as $k \to \infty$. Since $\lim_{n\to\infty} d(x_n, T_ix_n) = 0$ and $\lim_{n\to\infty} d(x_n, J_{\lambda^{(i)}}x_n) = 0$ for all $i = 1, 2, \ldots, N$, we have $d(q, T_iq) = 0$ and $d(q, J_{\lambda^{(i)}}q) = 0$ for all $i = 1, 2, \ldots, N$. This shows that $q \in \mathcal{F}$. In other cases, we can prove the strong convergence of $\{x_n\}$ to an element of $\mathcal{F}$. This completes the proof. □

Remark 1 (i) Theorems 3 and 6 generalizes the results of Cholamjiak et al. [11] from common fixed points of two nonexpansive mapping to common fixed points of a finite family of nonexpansive mappings, and from a minimize of a convex and lower semi-continuous function to common minimizers of a finite family of convex and lower semi-continuous functions.

(ii) Theorems 3 and 6 extend the main result in Bačák [7], and the corresponding results in Ariza-Ruiz et al. [6] and Cholamjiak et al. [11]. In fact, we construct a new modified proximal point algorithm for solving the constrained convex minimization problem for a finite family of convex and lower semi-continuous functions as well as the fixed point problem for a finite family of nonexpansive mappings in complete CAT(0) spaces.

(iii) In Hilbert spaces, if we set $\alpha x \oplus (1-\alpha)y := \alpha x + (1-\alpha)y$, then Theorems 3—6 can be applied to these spaces.

Finally, we give the numerical example to support our main theorem in a four-dimensional space of real numbers.

Example 1 Let $X = \mathbb{R}^4$ with the Euclidean norm and $D = \{\mathbf{x} = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}) \in \mathbb{R}^4 : -50 \leq x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)} \leq 50\}$. For each $\mathbf{x} = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}) \in D$, we define mappings T_1 and T_2 on D as follows:

$$T_1\mathbf{x} = \left(\frac{x^{(1)}+2}{3}, \frac{x^{(2)}-2}{3}, \frac{x^{(3)}-4}{3}, \frac{x^{(4)}+12}{5}\right)$$

Table 1 Numerical results of Example 1 for the algorithm (6)

Number	$\boldsymbol{x}_n = (x_n^{(1)}, x_n^{(2)}, x_n^{(3)}, x_n^{(4)})^t$	$\|\boldsymbol{x}_n - \boldsymbol{x}_{n-1}\|_2$
1	(−2.5000000, 4.6000000, 3.8000000,−5.9000000)	–
2	(0.1948474, −0.1430572, −1.2589906, −0.9740996)	8.92282e+00
3	(0.6858928, −0.8304900, −1.9042613, 1.3402575)	2.54682e+00
4	(0.8644453, −0.9556049, −1.9876400, 2.3146571)	1.00197e+00
5	(0.9421135, −0.9854949, −1.9984688, 2.7173304)	4.11326e−01
⋮	⋮	⋮
16	(0.9999964,−0.9999993, −2.0000000, 2.9999832)	2.44635e−05
17	(0.9999985, −0.9999997, −2.0000000, 2.9999931)	1.01041e−05
18	(0.9999994, −0.9999999, −2.0000000, 2.9999971)	4.17329e−06

and

$$T_2\mathbf{x} = \left(\frac{x^{(1)}+9}{10}, \frac{-x^{(2)}-5}{4}, \frac{3x^{(3)}-10}{8}, \frac{5x^{(4)}+3}{6}\right).$$

For each $\mathbf{x} \in D$, we define $h_1, h_2 : D \to (-\infty, \infty]$ by

$$h_1(\mathbf{x}) = \frac{1}{2}\|A_1\mathbf{x} - b_1\|^2, \quad h_2(\mathbf{x}) = \frac{1}{2}\|A_2\mathbf{x} - b_2\|^2,$$

where

$$A_1 = \begin{pmatrix} 1 & -3 & 6 & -1 \\ 4 & -4 & -3 & -2 \\ 3 & 5 & -4 & 1 \\ -1 & -1 & -5 & 0 \end{pmatrix} \text{ and } b_1 = \begin{pmatrix} -11 \\ 8 \\ 9 \\ 10 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & 0 & 2 & 3 \\ 4 & -5 & 1 & 3 \\ -1 & 3 & -4 & -2 \\ 3 & 0 & 1 & -4 \end{pmatrix} \text{ and } b_2 = \begin{pmatrix} 4 \\ 16 \\ -2 \\ -11 \end{pmatrix}.$$

We can check that T_1 and T_2 are nonexpansive and h_1 and h_2 are proper convex and lower semi-continuous. For $i = 1, 2$, we know by [22] that

$$\begin{aligned} J_{1^{(i)}}\mathbf{x} &= \operatorname*{argmin}_{\mathbf{u}\in D}\left[h_i(\mathbf{u}) + \frac{1}{2}\|\mathbf{u} - \mathbf{x}\|^2\right] \\ &= \operatorname{prox}_{h_i}\mathbf{x} \\ &= (I + A_i^t A_i)^{-1}(\mathbf{x} + A_i^t b_i). \end{aligned}$$

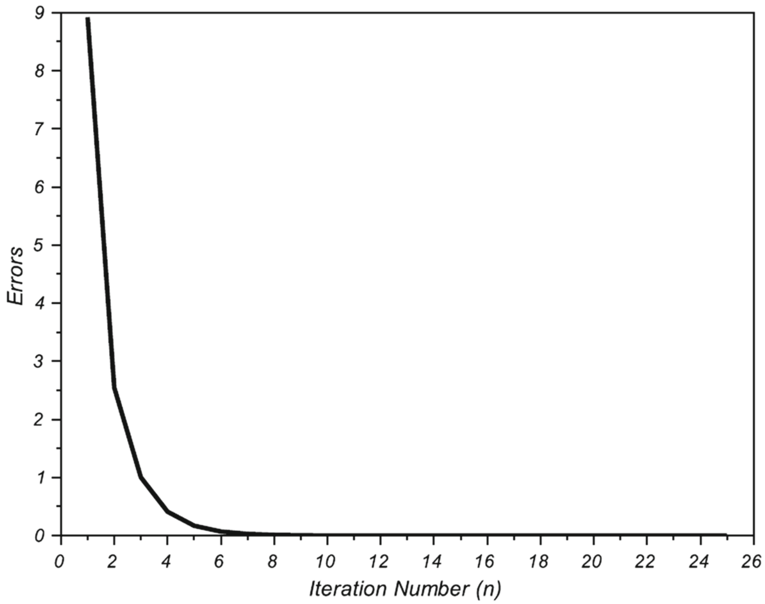

Fig. 1 The error plotting of $\|x_n - x_{n-1}\|_2$ in Table 1

Then, the algorithm (2) becomes:

$$\begin{cases} \mathbf{y}_n^{(1)} = (I + A_1^t A_1)^{-1}(\mathbf{x}_n + A_1^t b_1), \\ \mathbf{y}_n^{(2)} = (I + A_2^t A_2)^{-1}(\mathbf{x}_n + A_2^t b_2), \\ \mathbf{z}_n = \beta_n^{(0)}\mathbf{x}_n + \beta_n^{(1)}\mathbf{y}_n^{(1)} + \beta_n^{(2)}\mathbf{y}_n^{(2)}, \\ \mathbf{w}_n = \gamma_n^{(0)}\mathbf{z}_n + \gamma_n^{(1)}T_1\mathbf{z}_n + \gamma_n^{(2)}T_2\mathbf{z}_n, \\ \mathbf{x}_{n+1} = \alpha_n\mathbf{x}_n + (1-\alpha_n)\mathbf{w}_n, \quad \forall n \geq 1. \end{cases} \tag{6}$$

We choose $\beta_n^{(0)} = \frac{1}{9}$, $\beta_n^{(1)} = \frac{4}{9}$, $\beta_n^{(2)} = \frac{4}{9}$, $\gamma_n^{(0)} = \gamma_n^{(1)} = \gamma_n^{(2)} = \frac{1}{3}$, and $\alpha_n = \frac{25n-1}{625n}$. It can be observed that all the assumptions of Theorems 3 and 6 are satisfied. Using the algorithm (6) with the initial point $\mathbf{x}_1 = (-2.5, 4.6, 3.8, -5.9)^t$, we have numerical results in Table 1 and Fig. 1.

Remark 2 Table 1 and Fig. 1 show that the sequence $\{x_n\}$ converges to a unique point $(1, -1, -2, 3)^t$ which is a common element of the set of common fixed points of T_1 and T_2 and the set of common minimizers of h_1 and h_2.

Acknowledgements The authors would like to thank the referees for valuable suggestions on the research paper. This paper was supported by the Thailand Research Fund under the project RTA5780007.

References

1. Bridson, M., Haefliger, A.: Metric spaces of non-positive curvature. Springer-Verlag, Berlin, Heidelberg (1999)
2. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York (1984)
3. Kohlenbach, U.: Some logical metatheorems with applications in functional analysis. Trans. Am. Math. Soc. **357**, 89128 (2015)
4. Tits, J.: A Theorem of Lie-Kolchin for Trees, Contributions to Algebra: a Collection of Papers Dedicated to Ellis Kolchin. Academic Press, New York (1977)
5. Martinet, B.: Regularisation dinquations variationnelles par approximations successives. Rev. Fr. Inform. Rech. Oper. **4**, 154–158 (1970)
6. Ariza-Ruiz, D., Leustean, L., Lopez, G.: Firmly nonexpansive mappings in classes of geodesic spaces. Trans. Am. Math. Soc. **366**, 4299–4322 (2014)
7. Bačák, M.: The proximal point algorithm in metric spaces. Isr. J. Math. **194**, 689–701 (2013)
8. Güler, O.: On the convergence of the proximal point algorithm for convex minimization. SIAM J. Control Optim. **29**, 403–419 (1991)
9. Marino, G., Xu, H.K.: Convergence of generalized proximal point algorithm. Commun. Pure. Appl. Anal. **3**, 791–808 (2004)
10. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**, 877–898 (1976)
11. Cholamjiak, P., Abdou, A.A., Cho, Y.J.: Proximal point algorithms involving fixed points of nonexpansive mappings in CAT(0) spaces. Fixed Point Theory Appl. **2015**, 227 (2015)
12. Dhompongsa, S., Kirk, W.A., Sims, B.: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762–772 (2006)
13. Kirk, W.A., Panyanak, B.: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**, 3689–3696 (2008)
14. Dhompongsa, S., Kirk, W.A., Panyanak, B.: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. **8**, 35–45 (2007)
15. Dhompongsa, S., Panyanak, B.: On Δ-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**, 2572–2579 (2008)
16. Kopecká, E., Reich, S.: Asymptotic behavior of resolvents of coaccretive operators in the Hilbert ball. Nonlinear Anal. **70**, 3187–3194 (2009)
17. Chidume, C.E., Bello, A.U., Ndambomve, P.: Strong and Δ-convergence theorems for common fixed points of a finite family of multivalued demicontractive mappings in CAT(0) spaces, Abstr. Appl. Anal. Article ID 805168 (2014)
18. Guler, O.: On the convergence of the proximal point algorithm for convex minimization. SIAM J. Control Optim. **403419**, 29 (1991)
19. Jost, J.: Convex functionals and generalized harmonic maps into spaces of nonpositive curvature. Comment. Math Helv. **70**, 659673 (1995)
20. Mayer, U.F.: Gradient flows on nonpositively curved metric spaces and harmonic maps. Commun. Anal. Geom. **6**, 199–253 (1998)
21. Ambrosio, L., Gigli, N., Savare, G. Gradient Flows in Metric Spaces and in the Space of Probability Measures, 2nd edn. Basel, Lectures in Mathematics ETH Zrich. Birkhuser (2008)
22. Combettes, P.L., Pesquet, J.C.: Proximal Splitting Methods in Signal Processing. In: Bauschke, H.H., Burachik, R., Combettes, P.L., Elser, V., Luke, D.R., Wolkowicz, H. (eds.) Fixed-Point Algorithms for Inverse Problems in Science and Engineering, p. 185212. Springer, New York (2011)