Numer Algor (2018) 77:313–326
DOI 10.1007/s11075-017-0317-x

ORIGINAL PAPER

A new construction of Szász-Mirakyan operators

Ali Aral[1] · **Gulsum Ulusoy**[2] · **Emre Deniz**[1]

Received: 13 May 2016 / Accepted: 17 March 2017 / Published online: 30 March 2017

Abstract The paper aims to study a generalization of Szász-Mirakyan-type operators such that their construction depends on a function ρ by using two sequences of functions. To show how the function ρ play a crucial role in the design of the operator, we reconstruct the mentioned operators which preserve exactly two test functions from the set $\{1, \rho, \rho^2\}$. We show that these operators provide weighted uniform approximation over unbounded interval. We establish the degree of approximation in terms of a weighted moduli of smoothness associated with the function ρ. Also a Voronovskaya type result is presented. Finally some graphical examples of the mentioned operators are given. Our results show that mentioned operators are sensitive or flexible to point of wive of the rate of convergence to f, depending on our selection of ρ.

Keywords Szász-Mirakyan-type operators · King-type operators · Voronovskaya type theorem · Korovkin-type theorem · Weighted modulus of continuity · Positive linear operators

✉ Emre Deniz
emredeniz--@hotmail.com

Ali Aral
aliaral73@yahoo.com

Gulsum Ulusoy
ulusoygulsum@hotmail.com

[1] Kırıkkale University Department of Mathematics, Kırıkkale, Turkey

[2] Department of Mathematics, Faculty of Science, Cankırı Karatekin University, Cankırı, Turkey

1 Introduction

Approximation theory has great potential with wide variety of application of different branch of mathematics. The basis of the theory of approximation is the theorem discovered by Weierstrass [15] in 1885, which asserts that each continuous function defined on closed interval can be approximated uniformly on this interval by a polynomial with any degree. Deviating from Korovkin theorem, we can test the statement of Weierstrass' approximation theorem for three test functions $e_0(x) = 1$, $e_1(x) = x$, $e_2(x) = x^2$, see [2]. For this reason, this test functions set is the most useful tool in approximation theory. Most of the approximating operators preserve e_0. Besides, a part of these operators also preserves the second test function e_1. On this point, King [11] introduced a sequence of positive linear operators which modify the Bernstein operators and preserve the test functions e_0 and e_2 on $[0, 1]$ to obtain better error estimation. King's approach was further investigated by several authors. In this direction, in [5] a King-type operator, which reproduce e_1 and e_2 of the Bernstein type was constructed. Then in [7], the authors obtained an anologue of King's results for Szász-Mirakyan operators. Following this, P. Braica [10] approached the same problem introducing the Szász-Mirakjan operators of the King type, reproducing e_0 and e_2 and also reproducing e_1 and e_2 . Moreover, three different-type operators which preserve exactly two test functions from the set $\{e_0, e_1, e_2\}$ were considered by [12] Pop et al.

On the other hand, Bernstein type operators defined by $B_n\left(f \circ \tau^{-1}\right) \circ \tau$, under general assumptions on τ, were considered in [6], and they showed that its Korovkin set is $\left\{e_0, \tau, \tau^2\right\}$ instead of $\{e_0, e_1, e_2\}$. Durrmeyer-type generalization of mentioned operators were also studied in [1]. Very recently, Aral et al. [3] introduced a similar modification of the Szász-Mirakjan operators to investigate approximation properties of the announced operators acting on functions defined on unbounded intervals. Further, pointwise convergence of the operators introduced in [3] was extensively studied in [14]. Let us recall that construction.

Let

(p_1) ρ is a continuously differentiable function on $\mathbb{R}^+$,
(p_2) $\rho(0) = 0$, $\inf_{x\in[0,\infty)} \rho'(x) \geq 1$.

The generalized Szász-Mirakjan operators are defined by

$$S_n^{\rho}(f; x) = S_n\left(\left(f \circ \rho^{-1}\right) \circ \rho\right)(x) = \sum_{k=0}^{\infty} \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right) \mathscr{P}_{n,\rho,k}(x), \qquad (1)$$

where $\mathscr{P}_{n,\rho,k}(x) := \exp(-n\rho(x))(n\rho(x))^k / k!$, S_n are the classical Szász-Mirakjan operators and can be obtained from S_n^{ρ} as a particular case $\rho(x) = x$.

The main purpose of our paper is to construct sequences of operators depending on two sequences of functions $\alpha_n(x)$ and $\beta_n(x)$. For this purpose, we use sequences of Szász-Mirakjan-type operators which are based on a function ρ. By means of a suitable selection of sequences of functions, we can determine the operators of the general class which preserve exactly two test functions $\{1, \rho\}$, $\{1, \rho^2\}$ or $\{\rho, \rho^2\}$. We emphasize that our operators extend known results for the aforementioned particular cases. We aim to work in weighted spaces, the weight function being related to ρ to ensure the mentioned operator would be an approximation process on the entire semi-axis.

In the following section, the construction of the announced class of operators is also presented. Then, we give sufficient conditions which ensure weighted uniform convergence on the semi-axis and quantitative estimate in weighted spaces. The main tool is a weighted modulus, associated with the function ρ. Also, a Voronovskaya-type result is given. Some special cases of new operators are presented as examples. Finally some graphical examples are given. Our results show that our operators are sensitive or flexible to the point of view of the rate of convergence to f, depending on our selection ρ. Moreover, our generalization allows us to take general results to special classes of operators rediscovering known operators.

2 The construction of Szász-Mirakjan-type operators

Let $I \subset [0, \infty)$ be an interval, f is a continuous function on $\mathbb{R}^+$, $n_0 \in \mathbb{N}$ given, $\mathbb{N}_1 = \{n \in \mathbb{N}\, | n \geq n_0\}$, $\alpha_n, \beta_n : I \to \mathbb{R}$ be positive functions on I, such that $\beta_n - \alpha_n \geq 0$, for any $x \in I$, any $n \in \mathbb{N}_1$. We consider operators of the following form

$$\widetilde{S}_n^{\rho}(f; x) =: \widetilde{S}_n^{\rho} f(x) = e^{-\alpha_n(x)} \sum_{k=0}^{\infty} \frac{(\beta_n(x))^k}{k!} \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right), \tag{2}$$

where the function ρ satisfies the conditions (ρ_1) and (ρ_2).

To show the new class of operators is an approximation process, we impose several assumptions on the operators (2). In this direction, we impose several assumptions that can satisfy our sequence of operators.

First, we impose the condition

$$\widetilde{S}_n^{\rho}(1; x) = 1 + u_n(x), \tag{3}$$

where $u_n : I \to \mathbb{R}$ is a function. Under the assumption that α_n and β_n are positive functions on I, such that $\beta_n - \alpha_n \geq 0$, $n \in \mathbb{N}_1$, From (2) we have

$$\begin{aligned}\widetilde{S}_n^{\rho}(1; x) &= \sum_{k=0}^{\infty} e^{-\alpha_n(x)} \frac{(\beta_n(x))^k}{k!} \\ &= e^{\beta_n(x)-\alpha_n(x)}, \ x \in I.\end{aligned}$$

Thus, we immediately obtain

$$e^{\beta_n(x)-\alpha_n(x)} = 1 + u_n(x). \tag{4}$$

We also impose that $\widetilde{S}_n^{\rho}$ maps ρ to the same function, more precisely,

$$\widetilde{S}_n^{\rho}(\rho; x) = \rho(x) + v_n(x), \quad x \in I \tag{5}$$

where $v_n : I \to \mathbb{R}$ is a function. Considering (2), we get

$$\widetilde{S}_n^{\rho}(\rho; x) = \frac{\beta_n(x)}{n} e^{\beta_n(x)-\alpha_n(x)}$$

Based on the above equality, we get

$$\frac{\beta_n(x)}{n} e^{\beta_n(x)-\alpha_n(x)} = \rho(x) + v_n(x) \tag{6}$$

for any $n \in \mathbb{N}_1$ and any $x \in I$. From (4) and (6) it follows

$$\beta_n(x) = n\frac{\rho(x) + v_n(x)}{1 + u_n(x)}, \tag{7}$$

for any $n \in \mathbb{N}_1$ and any $x \in I$, and relation (4) ensure that

$$\beta_n(x) - \alpha_n(x) = \ln(1 + u_n(x))$$

and

$$\alpha_n(x) = n\frac{\rho(x) + v_n(x)}{1 + u_n(x)} - \ln(1 + u_n(x)), \tag{8}$$

where $u_n(x) > -1$, for any $n \in \mathbb{N}_1$ and any $x \in I$.

As a consequence of these requirements, the operators (2) become:

$$\begin{aligned} \widetilde{S}_n^{\rho}(f;x) &= \sum_{k=0}^{\infty} e^{-\alpha_n(x)} \frac{(\beta_n(x))^k}{k!} \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^{\infty} e^{-n\frac{\rho(x)+v_n(x)}{1+u_n(x)}} (1 + u_n(x)) \frac{1}{k!} \left(n\frac{\rho(x) + v_n(x)}{1 + u_n(x)}\right)^k \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right) \end{aligned} \tag{9}$$

for any $n \in \mathbb{N}_1$ and any $x \in I$.

In order to obtain a weighted approximation process from this sequence, we suppose that the following equalities:

$$|u_n(x)| \leq u_n \text{ and } |v_n(x)| \leq v_n, x \in I \tag{10}$$

such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = 0$ are fulfilled. At this point, we emphasized that, based on weighted Korovkin theorem, the values of these limits and above relations on α_n and β_n guarantee that $(\widetilde{S}_n^{\rho})_{n\geq n_0}$ is an approximation process on I.

From the generalized operators (9), we can derive some sequences of linear positive operators investigated in literature by making an adequate selection of u_n, v_n, and ρ as we give the following :

(i) if we choose $u_n(x) = 0$, $v_n(x) = 0$, the operators (9) reduce to operators which given by (1),

(ii) if we choose $u_n(x) = 0$, $v_n(x) = 0$, $\rho(x) = x$, the operators (9) reduce to Szász- Mirakjan operators given in [13] by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

(iii) if we choose $u_n(x) = 0$, $v_n(x) = -\frac{1}{2n} + \frac{\sqrt{4n^2\rho^2(x)+1}}{2n} - \rho(x)$, $\rho(x) = x$, the operators (9) reduce to the operators given in [7] by

$$D_n^*(f;x) = e^{\frac{1-\sqrt{4n^2\rho^2(x)+1}}{2}} \sum_{k=0}^{\infty} \frac{\left(\sqrt{4n^2\rho^2(x)+1}-1\right)^k}{2^k k!} f\left(\frac{k}{n}\right),$$

(iv) if we choose $u_n(x) = \frac{1}{nx-1}$, $v_n(x) = 0$, $\rho(x) = x$, the operators (9) reduce to operators given in [4] by

$$S_n^*(f;x) = \frac{nx}{nx-1} e^{1-nx} \sum_{k=0}^{\infty} \frac{(nx-1)^k}{k!} f\left(\frac{k}{n}\right).$$

We mention some obvious properties of the new class of operators in the following lemmas.

Lemma 1 *The operators defined by (9) verify, for each $x \in I$, the following identities*

$$\widetilde{S}_n^{\rho}(1;x) = 1 + u_n(x),\ \widetilde{S}_n^{\rho}(\rho;x) = \rho(x) + v_n(x), \tag{11}$$

$$\widetilde{S}_n^{\rho}(\rho^2;x) = \frac{(\rho(x)+v_n(x))^2}{1+u_n(x)} + \frac{\rho(x)+v_n(x)}{n}, \tag{12}$$

$$\widetilde{S}_n^{\rho}(\rho^3;x) = \frac{(\rho(x)+v_n(x))^3}{(1+u_n(x))^2} + 3\frac{(\rho(x)+v_n(x))^2}{n(1+u_n(x))} + \frac{\rho(x)+v_n(x)}{n^2}. \tag{13}$$

Lemma 2 *If we define the central moments of operator of degree m*

$$\mu_{n,m}^{\rho}(x) := \widetilde{S}_n^{\rho}((\rho(t)-\rho(x))^m;x),$$

then we have

$$\mu_{n,1}^{\rho}(x) = v_n(x) - \rho(x)u_n(x), \tag{14}$$

and

$$\mu_{n,2}^{\rho}(x) = \frac{(\rho(x)+v_n(x))^2}{1+u_n(x)} + \frac{(\rho(x)+v_n(x))(1-2n\rho(x))}{n} + (1+u_n(x))\rho^2(x). \tag{15}$$

Since the identities are easily obtained by direct computation, we omit the proof.

3 Direct result

Let a real valued function ρ defined on $\mathbb{R}$ satisfies the conditions (ρ_1) and (ρ_2). We note that the function ρ characterizes not only the growth of functions which are approximated but also defines the test function set $\{e_0, \rho, \rho^2\}$ in a Korovkin-type theorem.

Let $\varphi(x) = 1 + \rho^2(x)$ be a weight function and let $B_{\varphi}(\mathbb{R}^+)$ be the space defined by

$$B_{\varphi}(\mathbb{R}^+) = \left\{f : \mathbb{R}^+ \to \mathbb{R},\ |f(x)| \le M_f\varphi(x),\ x \ge 0\right\},$$

where M_f is a constant depend only on f. $B_\varphi\left(\mathbb{R}^+\right)$ is a normed space with the norm

$$\|f\|_\varphi = \sup_{x\in\mathbb{R}^+} \frac{f(x)}{\varphi(x)}.$$

We define also the spaces

$$\begin{aligned} C_\varphi\left(\mathbb{R}^+\right) &= \left\{f \in B_\varphi\left(\mathbb{R}^+\right), f \text{ is continuous on} \mathbb{R}^+\right\}, \\ C_\varphi^*\left(\mathbb{R}^+\right) &= \left\{f \in C_\varphi\left(\mathbb{R}^+\right), \lim_{x\to\infty} f(x)/\varphi(x) = \text{const.}\right\}, \\ U_\varphi\left(\mathbb{R}^+\right) &= \left\{f \in C_\varphi\left(\mathbb{R}^+\right), f(x)/\varphi(x) \text{ is uniformly continuous on} \mathbb{R}^+\right\}. \end{aligned}$$

It is obvious that $C_\varphi^*\left(\mathbb{R}^+\right) \subset U_\varphi\left(\mathbb{R}^+\right) \subset C_\varphi\left(\mathbb{R}^+\right) \subset B_\varphi\left(\mathbb{R}^+\right)$.

In [8], the author proved the following weighted Korovkin-type theorems.

We consider $(L_n)_{n\geq 1}$ a sequence of positive linear operators acting from $C_\varphi\left(\mathbb{R}^+\right)$ to $B_\varphi\left(\mathbb{R}^+\right)$.

Lemma 3 [8] *The positive linear operators L_n, $n \geq 1$, act from $C_\varphi\left(\mathbb{R}^+\right)$ to $B_\varphi\left(\mathbb{R}^+\right)$ if and only if inequality*

$$|L_n(\varphi; x)| \leq K_n \varphi(x),$$

holds, where K_n is a positive constant depending on n.

Theorem 1 [8] *Let the sequence of linear positive operators (L_n), $n \geq 1$,acting from $C_\varphi\left(\mathbb{R}^+\right)$ to $B_\varphi\left(\mathbb{R}^+\right)$ satisfy the three conditions*

$$\lim_{n\to\infty} \left\|L_n\rho^\nu - \rho^\nu\right\|_\varphi = 0, \nu = 0, 1, 2.$$

Then for any function $f \in C_\varphi^*\left(\mathbb{R}^+\right)$,

$$\lim_{n\to\infty} \|L_n f - f\|_\varphi = 0.$$

Therefore, we can present the following result.

Theorem 2 *For each function $f \in C_\varphi^*\left(\mathbb{R}^+\right)$ following relation*

$$\lim_{n\to\infty} \sup_{x\in I} \frac{\left|\widetilde{S}_n^\rho(f; x) - f(x)\right|}{\varphi(x)} = 0$$

provided that (10) holds.

Proof If f is $C_\varphi^*\left(\mathbb{R}^+\right)$ then $|f(x)| \leq M_f \varphi(x)$, $x \geq 0$. $\widetilde{S}_n^\rho$ being linear and positive is monotone. The relation

$$\widetilde{S}_n^\rho(\varphi; x) = 1 + u_n(x) + \frac{(\rho(x) + v_n(x))^2}{1 + u_n(x)} + \frac{\rho(x) + v_n(x)}{n}$$

implies that the operator $\widetilde{S}_n^\rho$ maps the space $C_\varphi\left(\mathbb{R}^+\right)$ into $B_\varphi\left(\mathbb{R}^+\right)$.

In view of the relations (11), (12) and the assumption (10), one can write

$$\lim_{n\to\infty}\sup_{x\in I}\frac{\left|\widetilde{S}_n^{\rho}\left(\rho^{\nu}\right)-\rho^{\nu}\right|}{\varphi(x)}=0,\ \nu=0,1,2. \tag{16}$$

Each function $\widetilde{S}_n^{\rho}(f)$ is defined on I. To extend it on $\mathbb{R}^+$, consider the sequence of operators as follows

$$G_n(f;x)=\begin{cases}\widetilde{S}_n^{\rho}(f;x) & \text{if} x\in I\\ f(x) & \text{if} x\in\mathbb{R}^+\backslash I.\end{cases}$$

Then obviously, we can write

$$\|G_n(f)-f\|_{\varphi}=\sup_{x\in I}\frac{\widetilde{S}_n^{\rho}(f;x)-f(x)}{\varphi(x)}. \tag{17}$$

Applying Theorem 1 to operators $L_n \equiv G_n$ our theorem will be finished. Therefore, it is sufficient to prove

$$\lim_{n\to\infty}\left\|G_n\left(\rho^{\nu}\right)-\rho^{\nu}\right\|_{\varphi}=0,\ \nu=0,1,2.$$

Since

$$\left\|G_n\left(\rho^{\nu}\right)-\rho^{\nu}\right\|_{\varphi}=\sup_{x\in I}\frac{\left|\widetilde{S}_n^{\rho}\left(\rho^{\nu}\right)-\rho^{\nu}\right|}{\varphi(x)}$$

using (16), we obtain

$$\lim_{n\to\infty}\|G_n(f)-f\|_{\varphi}=0.$$

Considering (17) we have desired result. □

4 Order of approximation

In order to obtain a quantitative type theorem, we use following weighted modulus of continuity:

$$\omega_{\rho}(f;\delta)=\sup_{\substack{x,t\in\mathbb{R}^+\\|\rho(t)-\rho(x)|\le\delta}}\frac{|f(t)-f(x)|}{\varphi(t)+\varphi(x)},\ \delta>0$$

where $f\in C_{\varphi}\left(\mathbb{R}^+\right)$. This modulus was recently considered by A. Holhoş [9]. We observe that $\omega_{\rho}(f;0)=0$ for every $f\in C_{\varphi}\left(\mathbb{R}^+\right)$ and the function $\omega_{\rho}(f;\delta)$ is nonnegative and nondecreasing with respect to δ for $f\in C_{\varphi}\left(\mathbb{R}^+\right)$ and also $\lim_{\delta\to 0}\omega_{\rho}(f;\delta)=0$ for every $f\in U_{\varphi}\left(\mathbb{R}^+\right)$.

For our purposes we recall the following result.

Theorem 3 *([9]) Let $L_n : C_\varphi(\mathbb{R}^+) \to B_\varphi(\mathbb{R}^+)$ be a sequence of positive linear operators with*

$$\left\| L_n\left(\rho^0\right) - \rho^0 \right\|_{\varphi^0} = a_n, \tag{18}$$

$$\left\| L_n(\rho) - \rho \right\|_{\varphi^{\frac{1}{2}}} = b_n, \tag{19}$$

$$\left\| L_n\left(\rho^2\right) - \rho^2 \right\|_{\varphi} = c_n, \tag{20}$$

$$\left\| L_n\left(\rho^3\right) - \rho^3 \right\|_{\varphi^{\frac{3}{2}}} = d_n, \tag{21}$$

where a_n, b_n, c_n and d_n tend to zero as $n \to \infty$. Then

$$\left\| L_n(f) - f \right\|_{\varphi^{\frac{3}{2}}} \leq (7 + 4a_n + 2c_n)\, \omega_\rho(f; \delta_n) + \|f\|_\varphi a_n \tag{22}$$

for all $f \in C_\varphi(\mathbb{R}^+)$, where

$$\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n.$$

Theorem 4 *For all $f \in C_\varphi(\mathbb{R}^+)$,we have*

$$\left\| \widetilde{S}_n^\rho(f) - f \right\|_{\varphi^{\frac{3}{2}}} \leq \left(7 + 4u_n + 2\left(2v_n + v_n^2 + \frac{2}{n} + \frac{2v_n}{n}\right)\right) \omega_\rho(f; \delta_n),$$

where

$$\delta_n = \frac{16}{n} + \frac{4}{n^2} + 3u_n + 20v_n + \frac{22v_n}{n} + \frac{4v_n}{n^2} + 8v_n^2 + \frac{6v_n^2}{n} + v_n^3 + 2\sqrt{(1 + u_n)\left(\frac{2}{n} + u_n + 4v_n + \frac{2v_n}{n} + v_n^2\right)}.$$

Proof To apply Theorem 3, we should calculate the sequences a_n, b_n, c_n and d_n. It is obvious that from (11),

$$\left\| \widetilde{S}_n^\rho\left(\rho^0\right) - \rho^0 \right\|_{\varphi^0} = \sup_{x \in I} u_n(x) \leq u_n = a_n$$

and

$$\left\| \widetilde{S}_n^\rho(\rho) - \rho \right\|_{\varphi^{\frac{1}{2}}} = \sup_{x \in I} \frac{v_n(x)}{\sqrt{1 + \rho^2(x)}} \leq v_n = b_n.$$

Also by (12) we have

$$\left\| \widetilde{S}_n^\rho\left(\rho^2\right) - \rho^2 \right\|_{\varphi} \leq 2v_n + v_n^2 + \frac{2}{n} + \frac{2v_n}{n} = c_n.$$

Finally using (13), we get

$$\left\| \widetilde{S}_n^{\rho}\left(\rho^3\right) - \rho^3 \right\|_{\varphi^{\frac{3}{2}}} \leq \frac{4}{n^2} + \frac{6}{n} + \frac{4v_n}{n^2} + \frac{12v_n}{n} + 3v_n + \frac{6v_n^2}{n} + 3v_n^2 + v_n^3 = d_n.$$

Thus the equalities (18)–(21) are calculated.
Using the statement (22), we obtain desired result. □

Remark 1 Using $\lim_{\delta\to 0} \omega_{\rho}(f;\delta) = 0$ and Theorem 4, we have

$$\lim_{n\to\infty} \left\| \widetilde{S}_n^{\rho}(f) - f \right\|_{\varphi^{\frac{3}{2}}} = 0$$

for $f \in U_{\varphi}\left(\mathbb{R}^+\right)$.

5 A Voronovskaya-type theorem

Here, we shall focus on a pointwise convergence of $\widetilde{S}_n^{\rho}$ by obtaining Voronovskaya-type theorem. We need the following lemma given in [9].

Lemma 4 *For every $f \in C_{\varphi}\left(\mathbb{R}^+\right)$,for $\delta > 0$ and for all $u, x \geq 0$,*

$$|f(u) - f(x)| \leq (\varphi(u) + \varphi(x))\left(2 + \frac{|\rho(u) - \rho(x)|}{\delta}\right)\omega_{\rho}(f,\delta) \qquad (23)$$

holds.

Theorem 5 *Let $f \in C_{\varphi}\left(\mathbb{R}^+\right)$, $x \in I$ and suppose that the first and second derivatives of $f \circ \rho^{-1}$ exist at $\rho(x)$. If the second derivative of $f \circ \rho^{-1}$ is bounded on $\mathbb{R}^+$ and*

$$\lim_{n\to\infty} nu_n(x) = l_1, \lim_{n\to\infty} nv_n(x) = l_2,$$

then we have

$$\lim_{n\to\infty} n\left[\widetilde{S}_n^{\rho}(f;x) - f(x)\right] = f(x)l_1 + (l_2 - \rho(x)l_1)\left(f\circ\rho^{-1}\right)'(\rho(x)) + \frac{1}{2}\rho(x)\left(f\circ\rho^{-1}\right)''(\rho(x)).$$

Proof By the Taylor expansion of $f \circ \rho^{-1}$ at the point $\rho(x) \in \mathbb{R}^+$, there exists ξ lying between x and t such that

$$f(t) = \left(f\circ\rho^{-1}\right)(\rho(t)) = \left(f\circ\rho^{-1}\right)(\rho(x)) + \left(f\circ\rho^{-1}\right)'(\rho(x))(\rho(t) - \rho(x)) + \frac{\left(f\circ\rho^{-1}\right)''(\rho(x))(\rho(t) - \rho(x))^2}{2} + \lambda_x(t)(\rho(t) - \rho(x))^2,$$

where

$$\lambda_x(t) := \frac{\left(f \circ \rho^{-1}\right)''(\rho(\xi)) - \left(f \circ \rho^{-1}\right)''(\rho(x))}{2}. \tag{24}$$

Note that the assumptions on f together with definition (24) ensure that $|\lambda_x(t)| \leq M$ for all t and converges to zero as $t \to x$. Applying the operator (9) to the above equality, we get

$$n\left[\widetilde{S}_n^{\rho}(f;x) - f(x)\right] = nf(x)u_n(x) + \left(f \circ \rho^{-1}\right)'(\rho(x))\, n\widetilde{S}_n^{\rho}(\rho(t) - \rho(x))$$
$$+\frac{\left(f \circ \rho^{-1}\right)''(\rho(x))\, n\widetilde{S}_n^{\rho}(\rho(t) - \rho(x))^2}{2} + n\widetilde{S}_n^{\rho}\left(\lambda_x(t)(\rho(t) - \rho(x))^2; x\right).$$

By (11), (14), and (15), we have

$$\lim_{n\to\infty} n\widetilde{S}_n^{\rho}(\rho(t) - \rho(x); x) = l_2 - \rho(x)l_1,$$

$$\lim_{n\to\infty} n\widetilde{S}_n^{\rho}(((\rho(t) - \rho(x))^2; x) = \rho(x).$$

Therefore, we can write

$$\lim_{n\to\infty} n\left[\widetilde{S}_n^{\rho}(f;x) - f(x)\right] = f(x)l_1 + (l_1 - \rho(x)l_2)\left(f \circ \rho^{-1}\right)'(\rho(x))$$
$$+\frac{\rho(x)}{2}\left(f \circ \rho^{-1}\right)''(\rho(x))$$
$$+\lim_{n\to\infty} nS_n^{\rho}\left(|\lambda_x(t)|(\rho(t) - \rho(x))^2; x\right).$$

To complete the proof, we must estimate the last term $\left|nS_n^{\rho}\left(|\lambda_x(t)|(\rho(t) - \rho(x))^2; x\right)\right|$. Since $\lim_{t\to x}\lambda_x(t) = 0$ for every $\varepsilon > 0$, let $\delta > 0$ such that $|\lambda_x(t)| < \varepsilon$ for every $t \geq 0$. Applying the Cauchy-Schwarz inequality we infer

$$\lim_{n\to\infty} n\widetilde{S}_n^{\rho}\left(|\lambda_x(t)|(\rho(t) - \rho(x))^2; x\right) \leq \varepsilon \lim_{n\to\infty} n\widetilde{S}_n^{\rho}\left((\rho(t) - \rho(x))^2; x\right)$$
$$+\frac{M}{\delta^2}\lim_{n\to\infty} n\widetilde{S}_n^{\rho}\left((\rho(t) - \rho(x))^4; x\right).$$

Since

$$\lim_{n\to\infty} n\widetilde{S}_n^{\rho}\left((\rho(t) - \rho(x))^4; x\right) = 0,$$

we have

$$\lim_{n\to\infty} n\widetilde{S}_n^{\rho}\left(|\lambda_x(t)|(\rho(t) - \rho(x))^2; x\right) = 0.$$

This proves the theorem. □

6 Some special case of $\widetilde{S}_n^\rho$

(1) $\widetilde{S}_n^\rho$ operators preserving the functions 1 and ρ:

If we choose $u_n(x) = v_n(x) = 0$, the operators (9) reduce to S_n^ρ operators defined in [3]. As consequences of Theorem 4 and Theorem 5, we refined the following results.

Theorem 6 *For all* $f \in C_\varphi\left(\mathbb{R}^+\right)$ *we have*

$$\left\| S_n^\rho(f) - f \right\|_{\varphi^{\frac{3}{2}}} \leq \left(7 + \frac{4}{n}\right) \omega_\rho \left(f; \frac{20}{n} + \sqrt{\frac{8}{n}}\right).$$

Theorem 7 *Let* $f \in C_\varphi\left(\mathbb{R}^+\right)$, $x \in I$ *and suppose that the first and second derivatives of* $f \circ \rho^{-1}$ *exist at* $\rho(x)$. *If the second derivative of* $f \circ \rho^{-1}$ *is bounded on* $\mathbb{R}^+$, *then we have*

$$\lim_{n\to\infty} n\left[S_n^\rho(f;x) - f(x)\right] = \frac{1}{2}\rho(x)\left(f \circ \rho^{-1}\right)''(\rho(x)).$$

Remark 2 If we choose $u_n(x) = v_n(x) = 0$, $\rho(x) = x$ the operators (9) reduce to classical Szász operators.

(2) $\widetilde{S}_n^\rho$ operators preserving the functions 1 and ρ^2 :

If we choose $u_n(x) = 0$, $v_n(x) = -\frac{1}{2n} + \frac{\sqrt{4n^2x^2+1}}{2n} - x$ the operators (9) reduce to D_n^* operators which were defined in [7]. As consequences of Theorem 5, we refined the following results.

Theorem 8 *Let* $f \in C_\varphi\left(\mathbb{R}^+\right)$, $x \in I$ *and suppose that the first and second derivatives of* f *exist at* x. *If the second derivative of* f *is bounded on* $\mathbb{R}^+$, *then we have*

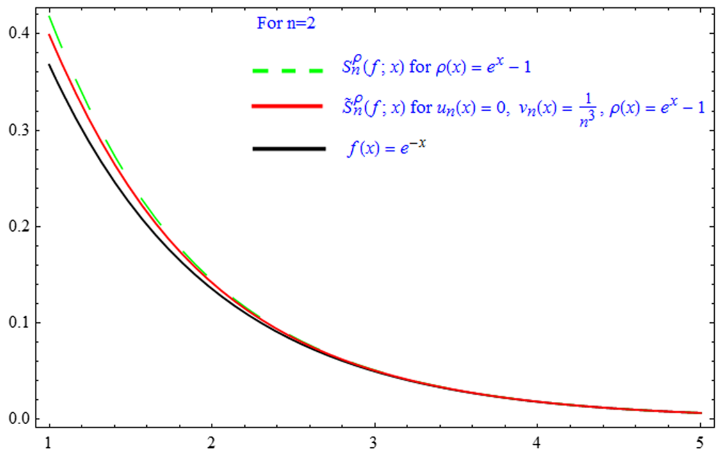
$$\lim_{n\to\infty} n\left[D_n^*(f;x) - f(x)\right] = -\frac{1}{2}f'(x) + \frac{x}{2}f''(x).$$

(3) $\widetilde{S}_n^\rho$ operators preserving the functions ρ and ρ^2 :

If we choose $u_n(x) = \frac{1}{nx-1}$, $v_n(x) = 0$, the operators (9) reduce to S_n^* operators which defined in [4]. As consequences of Theorem 4 and Theorem 5, we refined the following results.

Theorem 9 *Let* $f \in C_\varphi\left(\mathbb{R}^+\right)$, $x \in \mathbb{R}$ *and suppose that the first and second derivatives of* f *exist at* x. *If the second derivative of* f *is bounded on* $\mathbb{R}^+$, *then we have*

$$\lim_{n\to\infty} n\left[S_n^*(f;x) - f(x)\right] = \frac{f(x)}{x} - f'(x) + \frac{x}{2}f''(x).$$

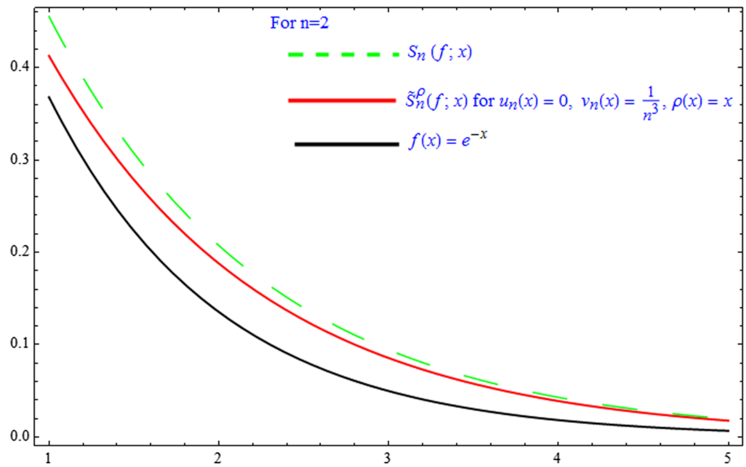


7 Graphics

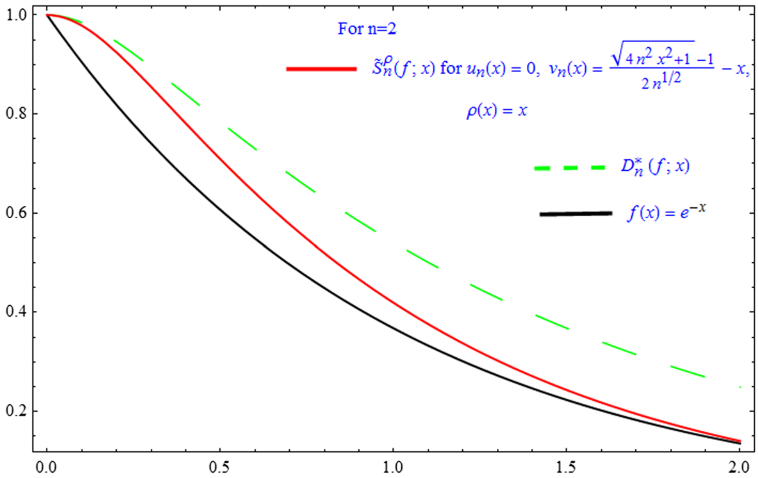
In this section, we give some illustrative graphs which compare $\widetilde{S}_n^\rho$ operator with S_n^ρ, S_n, D_n^* and S_n^*, respectively.

(i) Here, the standard line, the red line and the dashed line one represent, respectively, the function f, $\widetilde{S}_n^\rho f$ and $S_n^\rho f$. The operators comparisons are given in the following graph;

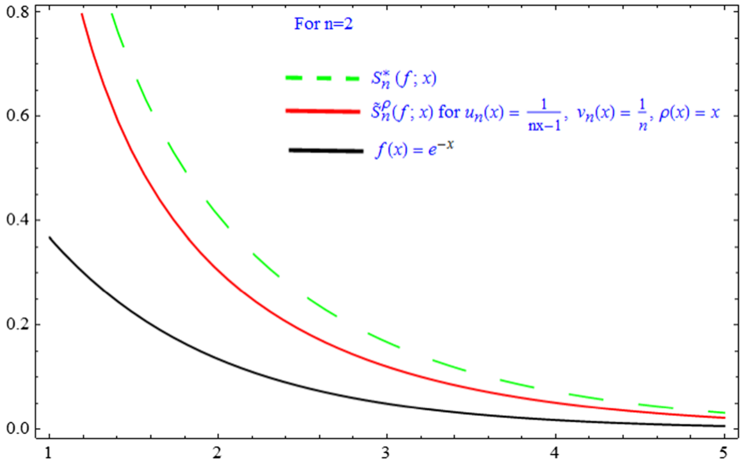
(ii) Here, the standard line, the red line, and the dashed line one represent, respectively, the function f, $\widetilde{S}_n^\rho f$ and $S_n f$. The operators comparisons are given in the following graph;



(iii) Here, the standard line, the red line, and the dashed line one represent, respectively, the function f, $\widetilde{S}_n^\rho f$ and $D_n^* f$. The operators comparisons are given in the following graph;



(iv) Here, the standard line, the red line, and the dashed line one represent, respectively, the function f, $\widetilde{S}_n^{\rho} f$ and $S_n^* f$. The operators comparisons are given in the following graph;



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