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New error bounds for the linear complementarity problem of QN-matrices

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Abstract An error bound for the linear complementarity problem (LCP) when the involved matrices are QN-matrices with positive diagonal entries is presented by Dai et al. (Error bounds for the linear complementarity problem of QN-matrices. *Calcolo*, **53**:647-657, 2016), and there are some limitations to this bound because it involves a parameter. In this paper, for LCP with the involved matrix A being a QN-matrix with positive diagonal entries an alternative bound which depends only on the entries of A is given. Numerical examples are given to show that the new bound is better than that provided by Dai et al. in some cases.

Keywords Linear complementarity problems · Error bounds · QN-matrices · P-matrices

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1 Introduction

The linear complementarity problem (LCP) is to find a vector $x \in R^n$ such that

$$x \geq 0,\ Ax + q \geq 0,\ (Ax + q)^T x = 0 \tag{1}$$

or to show that no such vector x exists, where $A = [a_{ij}] \in R^{n\times n}$ and $q \in R^n$. We denote the problem (1) and its solutions by LCP(A, q) and x^*, respectively. For the LCP(A, q), one of the important problems is to estimate the bound of $||x - x^*||_\infty$ (i.e., error analysis of the solution), since it has widespread applications in many fields such as finding Nash equilibrium point of a bimatrix game, the contact problem and the free boundary problem for journal bearing, for details, see [1, 5, 23].

It is well-known that the LCP(A, q) has a unique solution for any $q \in R^n$ if and only if A is a P-matrix [5]. Here a real square matrix A is called a P-matrix if all its principal minors are positive. When the matrix involved is a P-matrix, Chen and Xiang gave the following error bound for the LCP(A, q) [4]:

$$||x - x^*||_\infty \leq \max_{d\in[0,1]^n} ||(I - \Lambda + \Lambda A)^{-1}||_\infty ||r(x)||_\infty, \tag{2}$$

where $r(x) = \min\{x, Ax + q\}$, $\Lambda = diag(d_i)$ and $d = [d_1, d_2, ..., d_n]^T$ with $0 \leq d_i \leq 1$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors. It should be pointed out that there exists a big challenge for (2) due to the difficulty for solving the max problem $\max_{d\in[0,1]^n} ||(I - \Lambda + \Lambda A)^{-1}||_\infty$. However, if the matrix involved belongs to a subclass of P-matrix, such as H-matrices with positive diagonals [3, 4, 11, 12, 14], B-matrices [10, 24], DB-matrices [6], SB-matrices [7, 8], B^S-matrices [13], MB-matrices [2], B-Nekrasov matrices [15, 20], weakly chained diagonally dominant B-matrices [21], then many calculable error bounds for the LCP(A, q) can be derived.

Very recently, another subclass of P-matrices: quasi-Nekrasov (QN-) matrices are introduced by Kolotilina in [17], and the corresponding error bounds for the LCP(A, q) are also achieved by Dai et al. in [9]. Here, a matrix $A = D + U + L$, where D is a diagonal matrix, L is a strictly lower triangular matrix, and U is a strictly upper triangular matrix, is called a QN-matrix [17] if its diagonal entries are nonzero and the matrix

$$G = M^{-1}\mathcal{M}(A) = I_n - M^{-1}|L||D|^{-1}|U|,$$

where

$$M = (|D| - |L|)|D|^{-1}(|D| - |U|) = \mathcal{M}(A) + |L||D|^{-1}|U|, \tag{3}$$

is strictly diagonally dominant by rows [1], where $\mathcal{M}(A) = [m_{ij}] \in R^{n,n}$ is the comparison matrix of A with the entries $m_{ii} = |a_{ii}|$ and $m_{ij} = -|a_{ij}|$, for $i \neq j$ and $i, j \in N := \{1, ..., n\}$.

Theorem 1 [9, Theorem 2.4] *Suppose that $A = [a_{ij}] \in R^{n,n}$ is a QN-matrix with positive diagonal entries such that for each $i = 1, 2, \ldots, n-1$, $a_{ij} \neq 0$ for some $j > i$ and for each $i = 2, \ldots, n$, $a_{ij} \neq 0$ for some $j < i$. Let $\xi := M^{-1}|L||D|^{-1}|U|e$, where M is given by (3), and let $W = diag(w_1, ..., w_n)$ with $w_1 = \xi_1 + \varepsilon$, $\varepsilon \in \left(0, \min\left\{1 - \xi_1, \min\limits_{2 \le i \le n} \frac{[\mathcal{M}(A)\xi]_i}{|a_{i1}|}\right\}\right)$, where $\frac{[\mathcal{M}(A)\xi]_i}{|a_{i1}|} = \infty$ when $a_{i1} = 0$, and $w_i := \xi_i$ for $i = 2, ..., n$. Then*

$$\max_{d \in [0,1]^n} ||(I - \Lambda + \Lambda A)^{-1}||_\infty \le \max\left\{\frac{\max\limits_{i\in N}\{w_i\}}{\min\limits_{i\in N}\{l_i\}}, \frac{\max\limits_{i\in N}\{w_i\}}{\min\limits_{i\in N}\{w_i\}}\right\}, \tag{4}$$

where $l_1 := \varepsilon a_{11}$ and $l_i := a_{ii}\xi_i - \sum_{j \in N\setminus\{i\}} |a_{ij}|\xi_j - \varepsilon|a_{i1}|$ for each $i \in \{2, ..., n\}$.

It is apparent from Theorem 1 that when $A = [a_{ij}] \in R^{n,n}$ is a QN-matrix such that for some $i \in \{1, 2, \ldots, n-1\}$, $a_{ij} = 0$ for any $j > i$ or for some $i \in \{2, \ldots, n\}$, $a_{ij} = 0$ for any $j < i$, Theorem 1 cannot be used to estimate $\max\limits_{d \in [0,1]^n} ||(I - \Lambda + \Lambda A)^{-1}||_\infty$, and that when $\varepsilon \to 0$,

$$l_1 := \varepsilon a_{11} \to 0, \textit{and} \min_{i\in N}\{l_i\} \to 0$$

which implies that

$$\max\left\{\frac{\max\limits_{i\in N}\{w_i\}}{\min\limits_{i\in N}\{l_i\}}, \frac{\max\limits_{i\in N}\{w_i\}}{\min\limits_{i\in N}\{w_i\}}\right\} \to +\infty.$$

These facts show that there are some limitations to the bound (4) in Theorem 1 to estimate $\max\limits_{d \in [0,1]^n} ||(I - \Lambda + \Lambda A)^{-1}||_\infty$ when A is a QN-matrix with positive diagonals. So it is interesting to find an alternative bound for LCP(A, q) to overcome these drawbacks. In this paper we address this problem, and give a new error bound which only depends on the entries of A. Numerical examples are given to show that the new bound is better than that in [9] in some cases.

2 New error bounds for LCPs of QN-matrices

We start with some preliminaries and definitions. Let $e := (1, ..., 1)^T$. A matrix is called a Z-matrix if its off-diagonal elements are nonpositive, and a Z-matrix with nonnegative inverse is a nonsingular M-matrix. It is well-known that a square matrix A is called an H-matrix if its comparison matrix $\mathcal{M}(A)$ is an M-matrix [1]. Next, six lemmas which will be used later are listed.

Lemma 1 [17] *Let $A = [a_{ij}] \in C^{n,n}, n \ge 2$, with $a_{ii} \neq 0, i \in N$. Then A is a QN-matrix if and only if*

$$e > M^{-1}|L||D|^{-1}|U|e.$$

Lemma 2 [19, Lemma 3] *Let $\gamma > 0$ and $\eta \geq 0$. Then for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + \gamma x} \leq \frac{1}{\min\{\gamma, 1\}}$$

and

$$\frac{\eta x}{1 - x + \gamma x} \leq \frac{\eta}{\gamma}.$$

Lemma 2 will be used in the proofs of the following lemma and Theorem 2.

Lemma 3 *Let $A = [a_{ij}] \in C^{n,n}$ be a QN-matrix with $a_{ii} > 0$ for all $i \in N$, and let $\tilde{A} = I - \Lambda + \Lambda A = [\tilde{a}_{ij}]$ where $\Lambda = diag(d_i)$ with $0 \leq d_i \leq 1$. Then $\tilde{A}$ is a QN-matrix.*

Proof Since $A = D + L + U$ and note that

$$\tilde{a}_{ij} = \begin{cases} 1 - d_i + d_i a_{ij}, & i = j, \\ d_i a_{ij}, & i \neq j, \end{cases}$$

it follows that $\tilde{A}$ can be split in the form of $\tilde{A} = \tilde{D} + \tilde{L} + \tilde{U}$, where $\tilde{D} = I - \Lambda + \Lambda D$, $\tilde{L} = \Lambda L$, and $\tilde{U} = \Lambda U$. Let us denote

$$\tilde{\xi} := \tilde{M}^{-1}|\tilde{L}||\tilde{D}|^{-1}|\tilde{U}|e, \tag{5}$$

where

$$\tilde{M} = (|\tilde{D}| - |\tilde{L}|)|\tilde{D}|^{-1}(|\tilde{D}| - |\tilde{U}|). \tag{6}$$

Then, by Lemma 1, we need only prove the inequality $e \geq \tilde{\xi}$ holds.

Denote

$$\tilde{v} := |\tilde{L}||\tilde{D}|^{-1}|\tilde{U}|e, \tag{7}$$

where

$$|\tilde{L}||\tilde{D}|^{-1}|\tilde{U}| = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{|\tilde{a}_{21}|}{|\tilde{a}_{11}|}|\tilde{a}_{12}| & \cdots & \frac{|\tilde{a}_{21}|}{|\tilde{a}_{11}|}|\tilde{a}_{1n}| \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{|\tilde{a}_{n1}|}{|\tilde{a}_{11}|}|\tilde{a}_{12}| & \cdots & \frac{|\tilde{a}_{n1}|}{|\tilde{a}_{11}|}|\tilde{a}_{1n}| + \frac{|\tilde{a}_{n2}|}{|\tilde{a}_{22}|}|\tilde{a}_{2n}| + \cdots + \frac{|\tilde{a}_{n,n-1}|}{|\tilde{a}_{n-1,n-1}|}|\tilde{a}_{n-1,n}| \end{bmatrix}.$$

Then, we can deduce that

$$\tilde{v}_1 = 0, and \tilde{v}_i = \sum_{j=2}^{n}\left(\sum_{k=1}^{i-1}\frac{|\tilde{a}_{ik}|}{|\tilde{a}_{kk}|}|\tilde{a}_{kj}|\right) = \sum_{k=1}^{i-1}\left(\frac{|\tilde{a}_{ik}|}{|\tilde{a}_{kk}|}\sum_{j=k+1}^{n}|\tilde{a}_{kj}|\right), i = 2, ..., n. \tag{8}$$

From (5), (6), and (7), we have

$$\tilde{\xi} := (|\tilde{D}| - |\tilde{U}|)^{-1}|\tilde{D}|(|\tilde{D}| - |\tilde{L}|)^{-1}\tilde{v}.$$

Furthermore, if we denote $\tilde{\lambda} := (|\tilde{D}| - |\tilde{L}|)^{-1}\tilde{v}$, then we can get

$$\tilde{\xi} := (|\tilde{D}| - |\tilde{U}|)^{-1}|\tilde{D}|\tilde{\lambda}, \tag{9}$$

and

$$(|\tilde{D}| - |\tilde{L}|)\tilde{\lambda} = \tilde{v}. \tag{10}$$

By (8) and (10), we can obtain the value of $\tilde{\lambda}$, recursively:

$$\tilde{\lambda}_1 = 0, and \tilde{\lambda}_i = \frac{\tilde{v}_i}{|\tilde{a}_{ii}|} + \sum_{j=1}^{i-1} \frac{|\tilde{a}_{ij}|}{|\tilde{a}_{ii}|}\tilde{\lambda}_j, i = 2, ..., n. \tag{11}$$

Moreover, it follows from the equality (9) that

$$(|\tilde{D}| - |\tilde{U}|)\tilde{\xi} = |\tilde{D}|\tilde{\lambda},$$

this implies the following recursive relations

$$\tilde{\xi}_n = \tilde{\lambda}_n, and \tilde{\xi}_i = \tilde{\lambda}_i + \sum_{j=i+1}^{n} \frac{|\tilde{a}_{ij}|}{|\tilde{a}_{ii}|}\tilde{\xi}_j, i = n-1, ..., 1. \tag{12}$$

Let

$$\lambda_1 = 0, and \lambda_i = \frac{v_i}{|a_{ii}|} + \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|}\lambda_j, i = 2, ..., n, \tag{13}$$

where

$$v_i = \sum_{k=1}^{i-1} \left(\frac{|a_{ik}|}{|a_{kk}|} \sum_{j=k+1}^{n} |a_{kj}| \right). \tag{14}$$

By Lemma 2, we next prove that for each $i = 1, 2, ..., n$,

$$\tilde{\lambda}_i \leq \lambda_i. \tag{15}$$

In fact, for $i = 1$, we have $\tilde{\lambda}_1 = 0 = \lambda_1$. For $i = 2$,

$$\begin{aligned}
\tilde{\lambda}_2 &= \frac{\tilde{v}_2}{|\tilde{a}_{22}|} + \frac{|\tilde{a}_{21}|}{|\tilde{a}_{22}|}\tilde{\lambda}_1 \\
&= \frac{1}{|\tilde{a}_{22}|}\left(\frac{|\tilde{a}_{21}|}{|\tilde{a}_{11}|}|\tilde{a}_{12}|\right) (by (8)) \\
&= \frac{d_2|a_{21}|}{1 - d_2 + d_2 a_{22}} \left(\frac{d_1|a_{12}|}{1 - d_1 + d_1 a_{11}}\right) \\
&\leq \frac{|a_{21}|}{a_{22}} \cdot \frac{|a_{12}|}{a_{11}} (by\ Lemma\ 2) \\
&= \frac{v_2}{a_{22}} \\
&= \lambda_2.
\end{aligned}$$

We now suppose that $\tilde{\lambda}_i \leq \lambda_i$ holds for $i = 3, 4, \ldots, k$ and $k < n$. Since

$$\begin{aligned}
\tilde{\lambda}_{k+1} &= \frac{\tilde{v}_{k+1}}{|\tilde{a}_{k+1,k+1}|} + \sum_{j=1}^{k} \frac{|\tilde{a}_{k+1,j}|}{|\tilde{a}_{k+1,k+1}|}\tilde{\lambda}_j \\
&= \frac{1}{|\tilde{a}_{k+1,k+1}|}\left(\sum_{l=1}^{k}\left(\frac{|\tilde{a}_{k+1,l}|}{|\tilde{a}_{ll}|}\sum_{j=l+1}^{n}|\tilde{a}_{lj}|\right)\right) + \sum_{j=1}^{k}\frac{|\tilde{a}_{k+1,j}|}{|\tilde{a}_{k+1,k+1}|}\tilde{\lambda}_j \\
&= \sum_{l=1}^{k}\left(\frac{|\tilde{a}_{k+1,l}|}{|\tilde{a}_{k+1,k+1}|}\sum_{j=l+1}^{n}\frac{|\tilde{a}_{lj}|}{|\tilde{a}_{ll}|}\right) + \sum_{j=1}^{k}\frac{|\tilde{a}_{k+1,j}|}{|\tilde{a}_{k+1,k+1}|}\tilde{\lambda}_j \\
&= \sum_{l=1}^{k}\left(\frac{d_{k+1}|a_{k+1,l}|}{1-d_{k+1}+d_{k+1}a_{k+1,k+1}}\sum_{j=l+1}^{n}\frac{d_l|a_{lj}|}{1-d_l+d_l a_{ll}}\right) \\
&\quad + \sum_{j=1}^{k}\frac{d_{k+1}|a_{k+1,j}|}{1-d_{k+1}+d_{k+1}a_{k+1,k+1}}\tilde{\lambda}_j \\
&\leq \sum_{l=1}^{k}\left(\frac{|a_{k+1,l}|}{a_{k+1,k+1}}\sum_{j=l+1}^{n}\frac{|a_{lj}|}{a_{ll}}\right) + \sum_{j=1}^{k}\frac{|a_{k+1,j}|}{a_{k+1,k+1}}\cdot\tilde{\lambda}_j \\
&= \frac{v_{k+1}}{|a_{k+1,k+1}|} + \sum_{j=1}^{k}\frac{|a_{k+1,j}|}{|a_{k+1,k+1}|}\cdot\lambda_j \\
&= \lambda_{k+1},
\end{aligned}$$

by mathematical induction we can conclude that for each $i \in N$, (15) holds.

In terms of the relation (13) and (14), $\xi = M^{-1}|L||D|^{-1}|U|e$ can be obtained from the following recursive formula as in the proof of Theorem 2.4 in [9],

$$\xi_n = \lambda_n, and \xi_i = \lambda_i + \sum_{j=i+1}^{n}\frac{|a_{ij}|}{|a_{ii}|}\xi_j, i = n-1, ..., 1. \tag{16}$$

By Lemma 2, (12), and (15), we claim that

$$\tilde{\xi} \leq \xi, \tag{17}$$

where $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n)^T$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$. In fact, for $i = n$,

$$\tilde{\xi}_n = \tilde{\lambda}_n \leq \lambda_n = \xi_n.$$

For $i = n-1$,

$$\tilde{\xi}_{n-1} = \tilde{\lambda}_{n-1} + \frac{|\tilde{a}_{n-1,n}|}{|\tilde{a}_{n-1,n-1}|}\cdot\tilde{\xi}_n \leq \lambda_{n-1} + \frac{|a_{n-1,n}|}{|a_{n-1,n-1}|}\cdot\xi_n = \xi_{n-1}.$$

Similarly, for each $i = n-2, n-3, \ldots, 1$, using the recursive relation (12), we can easily get

$$\tilde{\xi}_i = \tilde{\lambda}_i + \sum_{j=i+1}^{n} \frac{|\tilde{a}_{ij}|}{|\tilde{a}_{ii}|} \cdot \tilde{\xi}_j \leq \lambda_i + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|} \cdot \xi_j = \xi_i.$$

Therefore, we can conclude that (17) holds.

Now, it follows from the fact that A is a QN-matrix, Lemma 1, and (17) that

$$e > M^{-1}|L||D|^{-1}|U|e = \xi \geq \tilde{\xi},$$

consequently, $\tilde{A} = I - \Lambda + \Lambda D$ is a QN-matrix. The proof is completed. □

Lemma 4 [20, Lemma 3] *Let $A = [a_{ij}] \in C^{n,n}$ be a matrix with $a_{ii} > 0$ for $i \in N$ and let $\tilde{A} = I - \Lambda + \Lambda A = [\tilde{a}_{ij}]$ where $\Lambda = diag(d_i)$ with $0 \leq d_i \leq 1$. Then*

$$z_i(\tilde{A}) \leq \eta_i(A),$$

and

$$\frac{z_i(\tilde{A})}{\tilde{a}_{ii}} \leq \frac{\eta_i(A)}{\min\{a_{ii}, 1\}},$$

where $z_1(\tilde{A}) = \eta_1(A) = 1$, $z_i(\tilde{A}) = \sum_{j=1}^{i-1} \frac{|\tilde{a}_{ij}|}{|\tilde{a}_{jj}|} z_j(\tilde{A}) + 1$, and

$$\eta_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{\min\{|a_{jj}|, 1\}} \eta_j(A) + 1, i = 2, 3, \ldots, n.$$

Lemma 5 [17, Theorem 3.3] *Let $A = [a_{ij}] \in C^{n,n}, n \geq 2$, be a QN-matrix. Then*

$$||A^{-1}||_\infty \leq \max_{i \in N} \frac{\{M^{-1}e\}_i}{\{M^{-1}\mathcal{M}(A)e\}_i}. \tag{18}$$

When the matrix A is a Nekrasov matrix, Kolotilina in [17] gave the following result which shows that the bound (18) is sharper than that of Theorem 2 in [18].

Lemma 6 [17, Theorem 3.4] *Let $A = [a_{ij}] \in C^{n,n}, n \geq 2$, be a Nekrasov matrix. Then*

$$\max_{i \in N} \frac{\{M^{-1}e\}_i}{\{M^{-1}\mathcal{M}(A)e\}_i} \leq \max_{i \in N} \frac{z_i(A)}{|a_{ii}| - h_i(A)},$$

where

$$h_1(A) = \sum_{j \neq 1} |a_{1j}|, h_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j(A) + \sum_{j=i+1}^{n} |a_{ij}|, i = 2, 3, \ldots, n.$$

By Lemmas 2, 3, 4 and 5, we give the following bound for $\max_{d \in [0,1]^n} ||(I - \Lambda + \Lambda A)^{-1}||_\infty$ when A is a QN-matrix.

Theorem 2 *Let $A = [a_{ij}] \in R^{n,n}$ be a QN-matrix with $a_{ii} > 0$ for all $i \in N$, and let $\tilde{A} = I - \Lambda + \Lambda A$ where $\Lambda = diag(d_i)$ with $0 \le d_i \le 1$. Then*

$$\max_{d\in[0,1]^n} ||(I - \Lambda + \Lambda A)^{-1}||_\infty \le \max_{i\in N} \frac{\beta_i}{\{e-\xi\}_i}, \tag{19}$$

where $\xi = M^{-1}|L||D|^{-1}|U|e$, and

$$\beta_n = \alpha_n = \frac{\eta_n(A)}{min\{a_{nn}, 1\}}, \text{ and } \beta_i = \alpha_i + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|} \cdot \beta_j, i = n-1, ..., 1$$

with $\alpha_i = \frac{\eta_i(A)}{min\{a_{ii},1\}}$ for all $i \in N$ and $\eta_i(A)$ is defined in Lemma 4.

Proof Let $\tilde{A} = I - \Lambda + \Lambda A = [\tilde{a}_{ij}]$. By Lemmas 3 and 5, we have that $\tilde{A}$ is a QN-matrix, and that

$$||(I - \Lambda + \Lambda A)^{-1}||_\infty \le \max_{i\in N} \frac{\left\{\tilde{M}^{-1}e\right\}_i}{\left\{\tilde{M}^{-1}\mathcal{M}(\tilde{A})e\right\}_i}. \tag{20}$$

Denote $z(\tilde{A}) = (z_1(\tilde{A}), z_2(\tilde{A}), ..., z_n(\tilde{A}))^T$. It follows from the fact $|\tilde{D}|(|\tilde{D}| - |\tilde{L}|)^{-1}e = z(\tilde{A})$ and (6) that

$$\tilde{M}^{-1}e = (|\tilde{D}| - |\tilde{U}|)^{-1}|\tilde{D}|(|\tilde{D}| - |\tilde{L}|)^{-1}e = (|\tilde{D}| - |\tilde{U}|)^{-1}z(\tilde{A}),$$

and

$$\tilde{M}^{-1}\mathcal{M}(\tilde{A})e = (I_n - \tilde{M}^{-1}|\tilde{L}||\tilde{D}|^{-1}|\tilde{U}|)e = e - \tilde{\xi},$$

which imply that

$$||(I - \Lambda + \Lambda A)^{-1}||_\infty \le \max_{i\in N} \frac{\left\{(|\tilde{D}| - |\tilde{U}|)^{-1}z(\tilde{A})\right\}_i}{\left\{e - \tilde{\xi}\right\}_i}. \tag{21}$$

If we denote $y = (|\tilde{D}| - |\tilde{U}|)^{-1}z(\tilde{A}) = (y_1, y_2, ..., y_n)^T$, then we get that

$$(|\tilde{D}| - |\tilde{U}|)y = z(\tilde{A}),$$

i.e.,

$$\begin{bmatrix} |\tilde{a}_{11}| & -|\tilde{a}_{12}| & \cdots & -|\tilde{a}_{1n}| \\ 0 & |\tilde{a}_{22}| & \cdots & -|\tilde{a}_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\tilde{a}_{nn}| \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} z_1(\tilde{A}) \\ z_2(\tilde{A}) \\ \vdots \\ z_n(\tilde{A}) \end{bmatrix},$$

which yields the following recursive formula

$$y_n = \frac{z_n(\tilde{A})}{|\tilde{a}_{nn}|}, and y_i = \frac{z_i(\tilde{A})}{|\tilde{a}_{ii}|} + \sum_{j=i+1}^{n} \frac{|\tilde{a}_{ij}|}{|\tilde{a}_{ii}|} \cdot y_j, i = n-1, \ldots, 1. \tag{22}$$

Next, we prove that for each $i \in N$,

$$\left\{(|\tilde{D}| - |\tilde{U}|)^{-1} z(\tilde{A})\right\}_i = y_i \leq \beta_i. \tag{23}$$

In fact, for $i = n$,

$$y_n = \frac{z_n(\tilde{A})}{|\tilde{a}_{nn}|} \leq \frac{\eta_n(A)}{min\{a_{nn}, 1\}} = \beta_n.$$

For $i = n - 1$,

$$\begin{aligned} y_{n-1} &= \frac{z_{n-1}(\tilde{A})}{|\tilde{a}_{n-1,n-1}|} + \frac{|\tilde{a}_{n-1,n}|}{|\tilde{a}_{n-1,n-1}|} \cdot y_n \\ &\leq \frac{\eta_{n-1}(A)}{min\{a_{n-1,n-1}, 1\}} + \frac{|a_{n-1,n}|}{|a_{n-1,n-1}|} \cdot \beta_n (by\ Lemmas\ 2\ and\ 4) \\ &= \beta_{n-1}. \end{aligned}$$

Similarly, for each $i = n - 2, n - 3, \ldots, 1$, we have by (22) that

$$\begin{aligned} y_i &= \frac{z_i(\tilde{A})}{|\tilde{a}_{ii}|} + \sum_{j=i+1}^{n} \frac{|\tilde{a}_{ij}|}{|\tilde{a}_{ii}|} \cdot y_j \\ &\leq \frac{\eta_i(A)}{min\{a_{ii}, 1\}} + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|} \cdot \beta_j \\ &= \beta_i. \end{aligned}$$

Therefore, we can conclude that $y_i \leq \beta_i$ holds for each $i \in N$.

Now, from (17), (21), and (23), we obtain

$$\begin{aligned} ||(I - \Lambda + \Lambda A)^{-1}||_\infty &\leq \max_{i \in N} \frac{\left\{(|\tilde{D}| - |\tilde{U}|)^{-1} z(\tilde{A})\right\}_i}{\left\{e - \widetilde{\xi}\right\}_i} \\ &\leq \frac{\beta_i}{\{e - \xi\}_i}. \end{aligned} \tag{24}$$

This completes the proof. □

Remark here that the value of $\xi = M^{-1}|L||D|^{-1}|U|e$ in Theorem 2 can be easily obtained by the expression (14) and the recursive formula (13) and (16) instead of calculating M^{-1}(also see [9]), so the form of the bound (19) in Theorem 2 only involves the entries of A. Furthermore, when $0 < a_{ii} \leq 1$ for all $i \in N$, then

$$\min\{a_{ii}, 1\} = a_{ii}, and \eta_i(A) = z_i(A), \tag{25}$$

which yields the following result.

Corollary 1 *Let $A = [a_{ij}] \in R^{n,n}$ be a QN-matrix with $0 < a_{ii} \leq 1$ for all $i \in N$, and let $\tilde{A} = I - \Lambda + \Lambda A$ where $\Lambda = diag(d_i)$ with $0 \leq d_i \leq 1$. Then*

$$\max_{d \in [0,1]^n} ||(I - \Lambda + \Lambda A)^{-1}||_\infty \leq \max_{i \in N} \frac{\tilde{\beta}_i}{\{e - \xi\}_i}, \tag{26}$$

where $\xi = M^{-1}|L||D|^{-1}|U|e$, *and*

$$\tilde{\beta}_n = \frac{z_n(A)}{a_{nn}}, \text{ and } \tilde{\beta}_i = \frac{z_i(A)}{a_{ii}} + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{a_{ii}} \cdot \tilde{\beta}_j, i = n-1, ..., 1$$

with $z_i(A)$ *is defined in Lemma 4.*

Since the class of QN-matrices contains the class of Nekrasov matrices [17], the bounds (19) and (26) can also be used to estimate the bound of $\max_{d \in [0,1]^n} ||(I - \Lambda + \Lambda A)^{-1}||_\infty$ when A is a Nekrasov matrix. Here, a matrix $A = [a_{ij}] \in C^{n,n}$ is called a Nekrasov matrix [16, 22] if for each $i \in N$,

$$|a_{ii}| > h_i(A).$$

And for a Nekrasov matrix, Li et al. in [20] gave the following bound which only depends on the entries of the involved matrix.

Theorem 3 [20, Theorem 2] *Let* $A = [a_{ij}] \in R^{n,n}$ *be a Nekrasov matrix with* $a_{ii} > 0$ *for* $i \in N$, *and let* $\tilde{A} = I - \Lambda + \Lambda A$ *where* $\Lambda = diag(d_i)$ *with* $0 \le d_i \le 1$. *Then*

$$\max_{d \in [0,1]^n} ||\tilde{A}^{-1}||_\infty \le \max_{i \in N} \frac{\eta_i(A)}{\min\left\{a_{ii} - h_i(A), 1\right\}}, \tag{27}$$

where $\eta_i(A)$ *is defined in Lemma 4.*

The following theorem claims that for a Nekrasov matrix all whose diagonal entries belong to the interval (0, 1], the bound (26) in Corollary 1 is in general tighter than the bound (27) in Theorem 3.

Theorem 4 *Let* $A = [a_{ij}] \in R^{n,n}$ *be a Nekrasov matrix with* $0 < a_{ii} \le 1$ *for all* $i \in N$. *Then*

$$\max_{i \in N} \frac{\tilde{\beta}_i}{\{e - \xi\}_i} \le \max_{i \in N} \frac{\eta_i(A)}{\min\left\{a_{ii} - h_i(A), 1\right\}},$$

where ξ *and* $\tilde{\beta}_i$ *are defined in Corollary 1.*

Proof Similarly to the proof of (22) in Theorem 2, we can get that for each $i \in N$,

$$\tilde{\beta}_i = \left\{(|D| - |U|)^{-1} z(A)\right\}_i. \tag{28}$$

Then, by (25), (28) and Lemma 6, we obtain

$$\begin{aligned}\max_{i\in N}\frac{\tilde{\beta}_i}{\{e-\xi\}_i} &= \max_{i\in N}\frac{\left\{(|D|-|U|)^{-1}z(A)\right\}_i}{\{e-\xi\}_i}\\ &= \max_{i\in N}\frac{\left\{M^{-1}e\right\}_i}{\left\{M^{-1}\mathcal{M}(A)e\right\}_i}\\ &\leq \max_{i\in N}\frac{z_i(A)}{a_{ii}-h_i(A)}\\ &= \max_{i\in N}\frac{\eta_i(A)}{\min\left\{a_{ii}-h_i(A),1\right\}}.\end{aligned}$$

The conclusion follows. □

3 Numerical examples

Next examples are given to show that the bounds in Theorem 2 and Corollary 1 can improve the bounds in Theorem 1 ((2.20) of [9]) and Theorem 3 ((9) of [20]) .

Example 1 Consider the following matrix

$$A=\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\ \frac{5}{7} & 1 & \frac{3}{7} & \frac{1}{7}\\ 0 & -\frac{1}{2} & 10 & -\frac{1}{8}\\ -\frac{1}{3} & 0 & 0 & 20\end{bmatrix}.$$

It is easy to verify that A is a QN-matrix but not a Nekrasov matrix with $a_{22}=1<h_2(A)=\frac{31}{28}$. Note that A satisfies the hypothesis of Theorem 1, by (14) and the recursive formula (13) and (16) we have $\xi=(0.1573, 0.5613, 0.0555, 0.0125)^T$, and the diagonal matrix W of Theorem 1 is

$$W=diag(0.1573+\varepsilon, 0.5613, 0.0555, 0.0125)$$

with $\varepsilon\in(0, 0.5927)$. Hence, by Theorem 1 we can get the bound (4) involved with $\varepsilon\in(0, 0.5927)$ for $\max_{d\in[0,1]^4}||(I-\Lambda+\Lambda A)^{-1}||_\infty$, which is drawn in Fig. 1. Furthermore, by Theorem 2, we can obtain that the bound (19) for $\max_{d\in[0,1]^4}||(I-\Lambda+\Lambda A)^{-1}||_\infty$ is 6.1723, which is smaller than the bound (4) as shown in Fig. 1.

Example 2 Consider the following matrix

$$A=\begin{bmatrix} 1 & -\frac{3}{5} & -\frac{1}{5} & 0\\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{2}\\ -\frac{1}{5} & -\frac{2}{5} & 1 & -\frac{2}{5}\\ -1 & 0 & 0 & 1\end{bmatrix}.$$

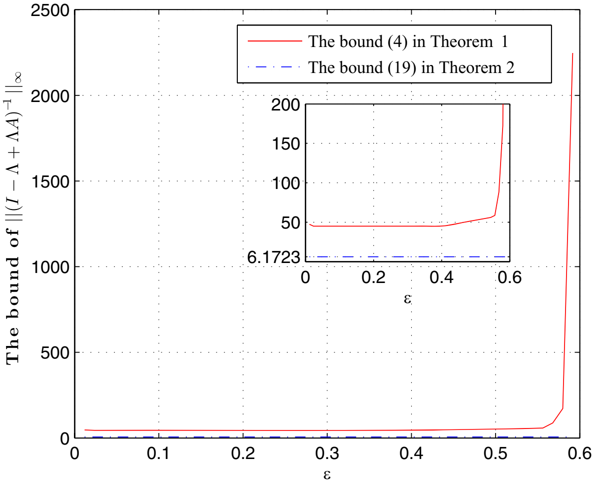

Fig. 1 The bounds in Theorems 1 and 2

Observe that A is a Nekrasov matrix and then a QN-matrix which satisfies the hypothesis of Theorem 1. By (14) and the recursive formula (13) and (16) we have $\xi = (0.6610, 0.8150, 0.8600, 0.8000)^T$. Then, by Theorem 1 we can get the bound (4) involved with $\varepsilon \in (0, 0.1390)$ and $W = diag(0.6610 + \varepsilon, 0.8150, 0.8600, 0.8000)$ for $\max\limits_{d\in[0,1]^4} ||(I - \Lambda + \Lambda A)^{-1}||_\infty$, which is drawn in Fig. 2. Moreover, the bound (26) of Corollary 1 is

$$\max_{i\in N} \frac{\tilde{\beta}_i}{\{e-\xi\}_i} = 17.8571$$

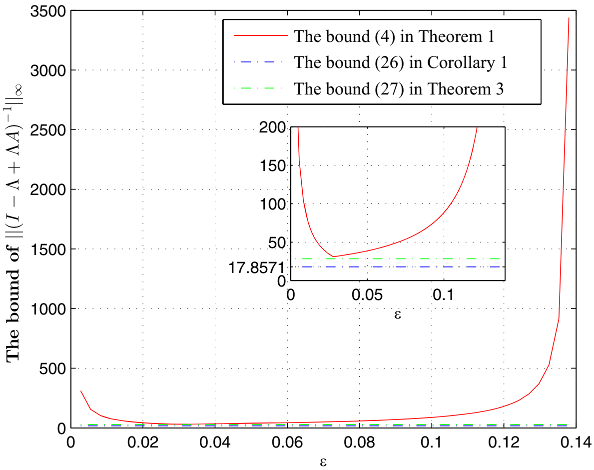

Fig. 2 The bounds in Corollary 1, Theorems 1 and 3

while the bound (27) of Theorem 3 in [20] for Nekrasov matrices is

$$\max_{i\in N} \frac{\eta_i(A)}{\min\left\{a_{ii} - h_i(A), 1\right\}} = 28.3333.$$

From Fig. 2, it is easy to see that the bound (26) in Corollary 1 is smaller than those in Theorems 1 and 3.

Example 3 Consider the following QN-matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 20 & 0 & -2 \\ -1 & 0 & 10 & -2 \\ -1 & -2 & 0 & 4 \end{bmatrix}.$$

It is easy to see that A is a QN-matrix but not a Nekrasov matrix with $h_1(A) = 3 = a_{11}$. Since $a_{21} = 0$, which does not satisfy the hypothesis of Theorem 1, we cannot use the bound (4) in Theorem 1 to estimate $\max_{d\in[0,1]^4} ||(I - \Lambda + \Lambda A)^{-1}||_\infty$. However, by Theorem 2, we have

$$\max_{d\in[0,1]^4} ||(I - \Lambda + \Lambda A)^{-1}||_\infty \leq 5.7143.$$

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