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Modified Newton-NSS method for solving systems of nonlinear equations

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Abstract By making use of the normal and skew-Hermitian splitting (NSS) method as the inner solver for the modified Newton method, we establish a class of modified Newton-NSS method for solving large sparse systems of nonlinear equations with positive definite Jacobian matrices at the solution points. Under proper conditions, the local convergence theorem is proved. Furthermore, the successive-overrelaxation (SOR) technique has been proved quite successfully in accelerating the convergence rate of the NSS or the Hermitian and skew-Hermitian splitting (HSS) iteration method, so we employ the SOR method in the NSS iteration, and we get a new method, which is called modified Newton SNSS method. Numerical results are given to examine its feasibility and effectiveness.

Keywords Normal and skew-Hermitian splitting · Modified Newton-NSS method · Successive overrelaxation · Modified Newton-SNSS method

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1 Introduction

Considering the large sparse systems of nonlinear equation:

$$F(x) = 0, \tag{1}$$

where $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a continuously differentiable nonlinear mapping defined on the open convex domain $\mathbb{D}$ in the n-dimensional complex linear space $\mathbb{C}^n$. This kind of nonlinear equation in many scientific computing and engineering applications has a wide range of applications, see [1–3]. The effective method to solve the nonlinear (1) is the inexact Newton method, see [4–6]:

$$x_{k+1} = x_k + s_k,\ F'(x_k)s_k = -F(x_k) + r_k,\ k \geq 0,$$

where $F'(x_k)$ is Jacobian matrix of $F(x)$ in each step of x_k and r_k is a residual yielded by the inner iteration. Every step of iteration for the above method, we can use classic splitting methods, see [3] or Krylov subspace method, see [7, 8] to solve Newton equation $F'(x_k)s_k = -F(x_k)$. Recently, based on the use of Hermitian and skew-Hermitian splitting iteration method [9], Wu et al. [10] have proposed the modified Newton-HSS method to solve non-Hermitian positive definite systems of nonlinear equations, and established the local and semilocal convergence theorems. On the other hand, Bai et al. [11] generalized the Hermitian/skew-Hermitian splitting iteration method to the normal/skew-Hermitian splitting (NSS) iteration method, putting forward the NSS method, and gave out the local convergence theorem. We remark that, in actual applications, there are situations that a matrix may be more naturally splitted into its normal and skew-Hermitian parts rather than its Hermitian and skew-Hermitian parts. Particularly, in some special conditions, the NSS iteration outperforms the Hermitian and skew-Hermitian splitting (HSS) iteration. See [12, 13], the NSS iteration method succeeds in solving continuous Sylvester equations, and the NSS iteration method considerably outperforms the HSS iteration method in both iteration steps and CPU time. Unlike the HSS iteration method, the NSS splitting is not unique for a given matrix A, we consider a NS splitting, where

$$N = H + icI,\ S = S_0 - icI,$$

and c is real number, see [14]. In the text, the symbol $\|\cdot\|$ denotes the 2-norm of vector or matrix.

In this paper, we utilize the NSS method as the inner solver of the modified Newton method, and we construct the modified Newton-NSS method. Under proper conditions, the local convergence theorem is proved. Moreover, we use successive-overrelaxation (SOR) method in NSS iteration to accelerate the convergence rate of the NSS iteration method, then we get a new method, that is the modified Newton SNSS method, and numerical results are given to examine their feasibility and effectiveness.

The paper is organized as follows. In Section 2, we introduce the modified Newton-NSS method. In Section 3, we display the convergence property of the modified Newton-NSS method. In Section 4, we employ the method of SOR to accelerate the modified Newton-NSS method. In Section 5, numerical results are presented to confirm the effectiveness of our method. Finally, in Section 6, some brief conclusions are given.

2 The modified Newton-NSS method

In this section, firstly, we introduce the Normal and skew-Hermitian splitting (NSS) method. Bai et al. [11] have generalized the Hermitian and skew-Hermitian (HS) splitting to the Normal/skew-Hermitian (NS) splitting $A = N + S$, where $N \in \mathbb{C}^{n\times n}$ is a normal matrix and $S \in \mathbb{C}^{n\times n}$ is a skew-Hermitian matrix, called as the NSS iteration method.

Given an initial guess $x_0 \in \mathbb{C}^n$, compute x_{k+1} for $k = 0, 1, 2, \ldots$ using the following iteration scheme until $\{x_k\}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + N)x_{k+\frac{1}{2}} = (\alpha I - S)x_k + b, \\ (\alpha I + S)x_{k+1} = (\alpha I - N)x_{k+\frac{1}{2}} + b, \end{cases} \tag{2}$$

where α is a given positive constant and I denotes the identity matrix. Combining the two equations of (2) into the form

$$x_{k+1} = T(\alpha)x_k + G(\alpha)b, \tag{3}$$

leads to

$$x_{k+1} = T(\alpha)^{k+1}x_0 + \sum_{j=0}^{k} T(\alpha)^j G(\alpha)b, \text{ k=0,1,2,}\ldots, \tag{4}$$

where

$$T(\alpha) = (\alpha I + S)^{-1}(\alpha I - N)(\alpha I + N)^{-1}(\alpha I - S),$$

and

$$G(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + N)^{-1}.$$

Here, $T(\alpha)$ is the iteration matrix of the NSS method. In fact, splitting A into the form

$$A(\alpha) = B(\alpha) - C(\alpha),$$

with

$$B(\alpha) = \frac{1}{2\alpha}(\alpha I + N)(\alpha I + S),$$
$$C(\alpha) = \frac{1}{2\alpha}(\alpha I - N)(\alpha I - S),$$

also results in (4), and

$$T(\alpha) = B(\alpha)^{-1}C(\alpha),$$
$$G(\alpha) = B(\alpha)^{-1}.$$

Now, we employ the modified Newton method

$$\begin{cases} y_k = x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} = y_k - F'(x_k)^{-1}F(y_k). \end{cases} \tag{5}$$

as the outer iteration and we employ the NSS method as the inner iteration. In other words, we apply the NSS iteration method to the following linear systems:

$$F'(x_k)d_k = -F(x_k), \tag{6}$$

$$F'(x_k)h_k = -F(y_k). \tag{7}$$

Then, the modified Newton-NS S method for solving nonlinear system $F(x) = 0$ is obtained.

The modified Newton-NSS iteration method Let $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be a continuously differentiable function with the positive-definite Jacobian matrix $F'(x)$ at any point $x \in \mathbb{D}$. Given an initial guess $x_0 \in \mathbb{D}$, a positive constant α and two sequence $\{l_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$ of positive integers, compute x_{k+1} for $k = 0, 1, 2, \ldots$ until x_k converge (see Algorithm 1).

Algorithm 1 MN–NSS (the modified Newton–NSS) algorithm

```
Given an initial guess x_0, positive constants α and tol, and two positive integer
sequences {l_k}_{k=0}^∞, {m_k}_{k=0}^∞.
for k = 0, 1, 2, ... ‖F(x_k)‖ ≥tol‖F(x_0)‖ do
    Set d_{k,0} = h_{k,0} := 0.
    for l = 0, 1, ..., l_k − 1, apply Algorithm NSS to the linear system (6):
```

$$\begin{cases} (\alpha I + N(x_k))d_{k,l+\frac{1}{2}} = (\alpha I - S(x_k))d_{k,l} - F(x_k), \\ (\alpha I + S(x_k))d_{k,l+1} = (\alpha I - N(x_k))d_{k,l+\frac{1}{2}} - F(x_k), \end{cases}$$

and obtain d_{k,l_k} such that $\left\| F(x_k) + F^{'}(x_k)d_{k,l_k} \right\| \le \eta_k \|F(x_k)\|$ for some $\eta_k \in [0, 1)$.

```
    Set y_k = x_k + d_{k,l_k}.
    Compute F(y_k).
    for m = 0, 1, ..., m_k − 1, apply Algorithm NSS to the linear system (7):
```

$$\begin{cases} (\alpha I + N(x_k))h_{k,m+\frac{1}{2}} = (\alpha I - S(x_k))h_{k,m} - F(y_k), \\ (\alpha I + S(x_k))h_{k,m+1} = (\alpha I - N(x_k))h_{k,m+\frac{1}{2}} - F(y_k), \end{cases}$$

and obtain h_{k,m_k} such that $\left\| F(y_k) + F^{'}(x_k)h_{k,m_k} \right\| \le \tilde{\eta}_k \|F(y_k)\|$ for some $\tilde{\eta}_k \in [0, 1)$.

```
    x_{k+1} = y_k + h_{k,m_k}.
end for
```

Based on the use of the (4), after straightforward operations, we can obtain the following uniform expressions for d_{k,l_k} and h_{k,m_k},

$$\begin{aligned} d_{k,l_k} &= -\sum_{j=0}^{l_k-1} T(\alpha; x_k)^j G(\alpha; x_k) F(x_k), \\ h_{k,m_k} &= -\sum_{j=0}^{m_k-1} T(\alpha; x_k)^j G(\alpha; x_k) F(y_k), \end{aligned}$$

where

$$T(\alpha; x) = (\alpha I + S(x))^{-1}(\alpha I - N(x))(\alpha I + N(x))^{-1}(\alpha I - S(x)),$$

and

$$G(\alpha; x) = 2\alpha(\alpha I + S(x))^{-1}(\alpha I + N(x))^{-1}.$$

Then, the modified Newton-NSS method can be rewritten as

$$\begin{cases} y_k = x_k - \sum_{j=0}^{l_k-1} T(\alpha; x_k)^j G(\alpha; x_k) F(x_k), \\ x_{k+1} = y_k - \sum_{j=0}^{m_k-1} T(\alpha; x_k)^j G(\alpha; x_k) F(y_k), k = 0, 1, 2 \ldots, \end{cases} \quad (8)$$

define matrics

$$\begin{aligned} B(\alpha; x) &= \frac{1}{2\alpha}(\alpha I + N(x))(\alpha I + S(x)), \\ C(\alpha; x) &= \frac{1}{2\alpha}(\alpha I - N(x))(\alpha I - S(x)). \end{aligned}$$

Then the Jacobian matrix $F'(x)$ can be hold that

$$F' = B(\alpha; x) - C(\alpha; x),$$

and

$$T(\alpha; x) = B(\alpha; x)^{-1} C(\alpha; x), B(\alpha; x) = G(\alpha; x)^{-1}, \quad (9)$$

$$F'(x)^{-1} = (I - T(\alpha; x)^{-1}) G(\alpha; x). \quad (10)$$

Hence, from the (7), we can equivalently express the modified Newton-NSS method as the following form

$$\begin{cases} y_k = x_k - (I - T(\alpha; x_k)^{l_k}) F'(x_k)^{-1} F(x_k), \\ x_{k+1} = y_k - (I - T(\alpha; x_k)^{m_k}) F'(x_k)^{-1} F(y_k), k = 0, 1, 2 \ldots \end{cases}$$

3 The local convergence theorem

A nonlinear mapping $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is Gateaux-(or G-) differentiable at an interior point x of $\mathbb{D}$ if there exists a linear operator $J \in \mathbb{C}^{n\times n}$ such that, for any $h \in \mathbb{C}^n$,

$$\lim_{t\to 0} \frac{1}{t} \|F(x+th) - F(x) - tJh\| = 0.$$

$F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is said to be G-differentiable on an open set $\mathbb{D}_0 \subset \mathbb{D}$ if it is G-differentiable at any point in $\mathbb{D}_0$.

The perturbation lemma plays a fundamental role in the subsequent discussion; see Lemma 2.3.2 in [3].

Lemma 3.1 *Let $M, N \in \mathbb{C}^{n\times n}$ and assume that M is nonsingular, with $\left\|M^{-1}\right\| \leq$ æ. If $\|M - N\| \leq \delta$ and δæ < 1, then N is also nonsingular, and*

$$\left\|N^{-1}\right\| \leq \frac{\text{æ}}{1 - \delta\text{æ}}.$$

In the following text, we prove that the modified Newton-NSS method has the similar local convergence properties as the modified Newton-HSS under similar conditions.

Let $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}_0 \subset \mathbb{D}$ of a point $x_* \in \mathbb{D}$ at which $F'(x)$ is continuous, positive definite, and $F(x_*) = 0$. Suppose $F'(x) = N(x) + S(x)$. Denote with $N(x_*, r)$ an open ball centered at x_* with radius $r > 0$.

Lemma 3.2 *If $r \in (0, \frac{1}{\gamma L})$ and for all $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$, assume the following conditions hold:*

Assumption A1 (The Bounded Condition) there exist positive constants β and γ such that

$$\max\{\|N(x_*)\| \ \|S(x_*)\|\} \leq \beta \text{ and } \left\|F'(x_*)^{-1}\right\| \leq \gamma.$$

Assumption A2 (The Lipschitz Condition) there exist nonnegative constants L_h and L_s such that

$$\begin{aligned} \|N(x) - N(x_*)\| &\leq L_h \|x - x_*\| \ , \\ \|S(x) - S(x_*)\| &\leq L_s \|x - x_*\| \ . \end{aligned}$$

Then $F'(x)^{-1}$ exists for any $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$. And the following inequalities hold with $L := L_h + L_s$ for all $x, y \in \mathbb{N}(x_*, r)$:

$$\left\| F'(x) - F'(x_*) \right\| \le L \|x - x_*\| \tag{11}$$

$$\left\| F'(x)^{-1} \right\| \le \frac{\gamma}{1 - \gamma L \|x - x_*\|}, \tag{12}$$

$$\|F(y)\| \le \frac{L}{2}(\|y - x_*\|)^2 + 2\beta \|y - x_*\| , \tag{13}$$

$$\left\| y - x_* - F'(x)^{-1} F(y) \right\| \le \frac{\gamma}{1 - \gamma L \|x - x_*\|} \left(\frac{L}{2} \|y - x_*\| + L \|x - x_*\| \right) \|y - x_*\| . \tag{14}$$

Proof The Lipschitz condition directly implies

$$\begin{aligned} \left\| F'(x) - F'(x_*) \right\| &= \|N(x) + S(x) - N(x_*) - S(x_*)\| \le \|N(x) - N(x_*)\| \\ &\quad + \|S(x) - S(x_*)\| \\ &\le (L_h + L_s) \|x - x_*\| = L \|x - x_*\| . \end{aligned}$$

Hence

$$\left\| F'(x_*)^{-1}(F'(x_*) - F'(x)) \right\| \le \left\| F'(x_*)^{-1} \right\| \left\| F'(x_*) - F'(x) \right\| \le \gamma L \|x - x_*\| < 1.$$

By making use of Banach Lemma, $F'(x)^{-1}$ exists, and

$$\left\| F'(x)^{-1} \right\| \le \frac{\gamma}{1 - \gamma L \|x - x_*\|}.$$

Since

$$\begin{aligned} F(y) &= F(y) - F(x_*) - F'(x)(y - x_*) + F'(x_*)(y - x_*) \\ &= \int_0^1 (F'(x_* + t(y - x_*)) - F'(x_*))dt(y - x_*) + F'(x_*)(y - x_*), \end{aligned}$$

the bounded condition leads to

$$\left\| F'(x_*) \right\| = \|N(x_*) + S(x_*)\| \le \|N(x_*)\| + \|S(x_*)\| \le 2\beta,$$

and

$$\begin{aligned} \|F(y)\| &\le \left\| \int_0^1 (F'(x_* + t(y-x_*)) - F'(x_*))dt(y-x_*) \right\| + \left\| F'(x_*)(y-x_*) \right\| \\ &\le \frac{L}{2}(\|y-x_*\|)^2 + 2\beta \|y-x_*\| . \end{aligned}$$

Clearly, it holds that

$$\begin{aligned} y - x_* - F'(x)^{-1}F(y) &= -F'(x)^{-1}(F(y) - F(x_*) - F'(x)(y-x_*)) \\ &= -F'(x)^{-1}(F(y) - F(x_*) - F'(x_*)(y-x_*)) \\ &\quad + F'(x)^{-1}(F'(x) - F'(x_*))(y-x_*) \\ &= -F'(x)^{-1}\int_0^1 (F'(x_* + t(y-x_*)) - F'(x_*))dt(y-x_*) \\ &\quad + F'(x)^{-1}(F'(x) - F'(x_*))(y-x_*). \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| y - x_* - F'(x_*)^{-1}F(y) \right\| &\le \left\| -F'(x)^{-1} \right\| \left(\int_0^1 \left\| F'(x_* + t(y-x_*)) - F'(x_*) \right\| dt \right. \\ &\quad \left. + \left\| F'(x) - F'(x_*) \right\| \right) \|y - x_*\| \\ &\le \frac{\gamma}{1 - \gamma L \|x - x_*\|} \left(\frac{L}{2}\|y - x_*\| + L\|x - x_*\| \right) \|y - x_*\|. \end{aligned}$$

□

Lemma 3.3 *Under the assumptions A1 and A2, suppose* $r \in (0, r_0)$ *and define* $r_0 := \min_{1 \le j \le 2}\{r_+^j\}$*, where*

$$\begin{aligned} r_+^{(1)} &= \frac{\alpha + \beta}{L}\left(\sqrt{\frac{2\tau\alpha\theta}{\gamma(2 + \tau\theta)(\alpha + \beta)^2} + 1} - 1 \right), \\ r_+^{(2)} &= \frac{1 - 2\beta\gamma[(\tau + 1)\theta]^u}{3\gamma L}, \end{aligned}$$

with $u = \min\{l_*, m_*\}$*,* $l_* = \liminf_{k \to \infty} l_k$*,* $m_* = \liminf_{k \to \infty} m_k$*, and the constant u satisfies*

$$u > \left\lfloor -\frac{\ln(2\beta\gamma)}{\ln((\tau + 1)\theta)} \right\rfloor,$$

where the symbol $\lfloor \bullet \rfloor$ *is used to denote the smallest integer no less than the corresponding real number,* $\tau \in (0, \frac{1-\theta}{\theta})$ *a prescribed positive constant and*

$$\theta \equiv \theta(\alpha; x_*) = \|T(\alpha; x_*)\| \le \max_{\lambda \in \sigma(N(x_*))} \frac{|\alpha - \lambda|}{|\alpha - \lambda|} \equiv \sigma(\alpha; x_*).$$

Then, for any $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$, $t \in (0, r)$ *and* $v > u$, *it holds that*

$$\begin{aligned} \|T(\alpha; x)\| &\le (\tau+1)\theta < 1, \\ g(t; v) &= \frac{2\gamma}{1-\gamma L t}(Lt + \beta[(\tau+1)\theta]^{v}) < g(r_0; u) < 1. \end{aligned}$$

Proof Based on the assumption A1, it holds

$$\begin{aligned} \left\|B(\alpha; x_*)^{-1}\right\| &= \left\|(I - T(\alpha; x_*))F'(x_*)^{-1}\right\| \\ &\le \|I - T(\alpha; x_*)\| \left\|F'(x_*)^{-1}\right\| \\ &\le (1 + \|T(\alpha; x_*)\|) \left\|F'(x_*)^{-1}\right\| \le 2\gamma. \end{aligned} \tag{15}$$

Here, we use the equality (9) and the fact

$$\|T(\alpha; x_*)\| \le \sigma(\alpha; x_*) < 1.$$

Because

$$\begin{aligned} N(x)S(x) - N(x_*)S(x_*) &= [N(x) - N(x_*)]S(x) + N(x_*)[S(x) - S(x_*)] \\ &= [N(x) - N(x_*)][S(x) - S(x_*)] + [N(x) - N(x_*)]S(x_*) \\ &\quad + N(x_*)[S(x) - S(x_*)], \end{aligned}$$

it follows from both assumptions A1 and A2 that for all $x \in \mathbb{N}(x_*, r)$ we have

$$\begin{aligned} \|N(x)S(x) - N(x_*)S(x_*)\| &\le \|N(x) - N(x_*)\| \, \|S(x) - S(x_*)\| \\ &\quad + \|N(x) - N(x_*)\| \, \|S(x_*)\| \\ &\quad + \|N(x_*)\| \, \|S(x) - S(x_*)\| \\ &\le L_s L_h (\|x - x_*\|)^2 + \beta(L_s + L_h)\|x - x_*\| \\ &\le \frac{1}{2}(L_s + L_h)^2(\|x - x_*\|)^2 + \beta(L_s + L_h)\|x - x_*\| \\ &= \frac{1}{2}L^2(\|x - x_*\|)^2 + \beta L\|x - x_*\| . \end{aligned} \tag{16}$$

Noticing that the equivalent expression

$$B(\alpha; x) = \frac{1}{2\alpha}(\alpha^2 I + \alpha F'(x) + N(x)S(x)),$$

and

$$C(\alpha; x) = \frac{1}{2\alpha}(\alpha^2 I - \alpha F'(x) + N(x)S(x)),$$

straightforwardly lead to the equalities

$$B(\alpha; x) - B(\alpha; x_*) = \frac{1}{2}(F'(x) - F'(x_*)) + \frac{1}{2\alpha}(N(x)S(x) - N(x_*)S(x_*)),$$

and

$$C(\alpha; x) - C(\alpha; x_*) = -\frac{1}{2}(F'(x) - F'(x_*)) + \frac{1}{2\alpha}(N(x)S(x) - N(x_*)S(x_*)).$$

From (10) and (16), we can further obtain the estimates

$$\begin{aligned}\|B(\alpha;x)-B(\alpha;x_*)\| &\le \tfrac{1}{2}\left\|F'(x)-F'(x_*)\right\|+\tfrac{1}{2\alpha}\left\|N(x)S(x)-N(x_*)S(x_*)\right\| \\ &\le \tfrac{L^2}{4\alpha}(\|x-x_*\|)^2+\tfrac{L(\alpha+\beta)}{2\alpha}\|x-x_*\|\ ,\end{aligned} \tag{17}$$

and

$$\|C(\alpha;x)-C(\alpha;x_*)\| \le \frac{L^2}{4\alpha}(\|x-x_*\|)^2+\frac{L(\alpha+\beta)}{2\alpha}\|x-x_*\|\ . \tag{18}$$

Hence, by making use of the perturbation lemma, (15) and (17), it follows

$$\left\|B(\alpha;x)^{-1}\right\| \le \frac{4\alpha\gamma}{2\alpha-\gamma(L^2(\|x-x_*\|)^2+2(\alpha+\beta)L\|x-x_*\|)}, \tag{19}$$

hold for all $x\in\mathbb{N}(x_*,r)$, provided r is small enough such that $L\gamma\|x-x_*\|<2\alpha$ and

$$\gamma(L^2(\|x-x_*\|)^2+2(\alpha+\beta)L\|x-x_*\|)<2\alpha.$$

Using (8), we immediately give the equality

$$\begin{aligned}T(\alpha;x)-T(\alpha;x_*) &= B(\alpha;x)^{-1}C(\alpha;x)-B(\alpha;x_*)^{-1}C(\alpha;x_*) \\ &= B(\alpha;x)^{-1}((C(\alpha;x)-C(\alpha;x_*))-(B(\alpha;x) \\ &\qquad -B(\alpha;x_*))T(\alpha;x_*)).\end{aligned}$$

Based on (17) and (18), we can obtain that

$$\begin{aligned}\|T(\alpha;x)-T(\alpha;x_*)\| &\le \left\|B(\alpha;x)^{-1}\right\|\,[\|C(\alpha;x)-C(\alpha;x_*)\| \\ &\quad +\|B(\alpha;x)-B(\alpha;x_*)\|\ \|T(\alpha;x_*)\|\,] \\ &\le \frac{2\gamma(L^2(\|x-x_*\|)^2+2(\alpha+\beta)L\|x-x_*\|)}{2\alpha-\gamma(L^2(\|x-x_*\|)^2+2(\alpha+\beta)L\|x-x_*\|)}.\end{aligned}$$

Let us further restrict r so small that $L\gamma\|x-x_*\|<1$ and

$$\gamma(L^2(\|x-x_*\|)^2+2(\alpha+\beta)L\|x-x_*\|)<\frac{2\tau\alpha\theta}{2+\tau\theta}.$$

Then it holds that

$$\frac{2\gamma(L^2(\|x-x_*\|)^2+2(\alpha+\beta)L\|x-x_*\|)}{2\alpha-\gamma(L^2(\|x-x_*\|)^2+2(\alpha+\beta)L\|x-x_*\|)}<\tau\theta,$$

and hence,

$$\begin{aligned}\|T(\alpha;x)\| &\le \|T(\alpha;x)-T(\alpha;x_*)\|+\|T(\alpha;x_*)\| \\ &\le \tfrac{2\gamma(L^2(\|x-x_*\|)^2+2(\alpha+\beta)L\|x-x_*\|)}{2\alpha-\gamma(L^2(\|x-x_*\|)^2+2(\alpha+\beta)L\|x-x_*\|)}+\theta \\ &\le (\tau+1)\theta.\end{aligned} \tag{20}$$

□

Theorem 3.1 *Under the assumptions A1 and A2, then for any $x_0\in\mathbb{N}(x_*,r)$ and any sequences $\{l_k\}_{k=0}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$ of positive integers, the iteration sequence $\{x_k\}_{k=0}^{\infty}$*

generated by the modified Newton-NSS method is well-defined and converges to x_*. Moreover, it holds that

$$\limsup_{k\to\infty}(\|x_k - x_*\|\,)^{\frac{1}{k}} \leq g(r_0; u)^2,$$

with $u = \min\{l_*, m_*\}$, $l_* = \liminf_{k\to\infty} l_k$, $m_* = \liminf_{k\to\infty} m_k$.

Proof From lemmas 3.2 and 3.3, we have

$$\begin{aligned}
\|y_k - x_*\| &= \left\| x_k - x_* - (I - T(\alpha; x_k)^{l_k})F'(x_k)^{-1}F'(x_k) \right\| \\
&\leq \left\| x_k - x_* - F'(x_k)^{-1}F'(x_k) \right\| + \left\| T(\alpha; x_k)^{l_k} \right\| + \left\| F'(x_k)^{-1}F'(x_k) \right\| \\
&\leq \frac{\gamma}{1 - \gamma L \|x_k - x_*\|} \frac{3L}{2}(\|x_k - x_*\|\,)^2 + [(\tau + 1)\theta]^{l_k} \\
&\quad \times \frac{\gamma}{1 - \gamma L \|x_k - x_*\|}(\frac{L}{2}(\|x_k - x_*\|\,)^2 + 2\beta \|x_k - x_*\|\,) \\
&= \frac{(3 + [(\tau + 1)\theta]^{l_k})\gamma L}{2(1 - \gamma L \|x_k - x_*\|)}(\|x_k - x_*\|\,)^2 + \frac{2\beta\gamma[(\tau + 1)\theta]^{l_k}}{1 - \gamma L \|x_k - x_*\|} \|x_k - x_*\| \\
&\leq \frac{2\gamma}{1 - \gamma L \|x_k - x_*\|}(L \|x_k - x_*\| + \beta[(\tau + 1)\theta]^{l_k}) \|x_k - x_*\| \\
&= g(\|x_k - x_*\|\,; l_k) \|x_k - x_*\| < g(r_0; u) \|x_k - x_*\| < \|x_k - x_*\|\,,
\end{aligned}$$

and

$$\begin{aligned}
\|x_{k+1} - x_*\| &= \left\| y_k - x_* - (I - T(\alpha; x_k)^{m_k})F'(x_k)^{-1}F'(y_k) \right\| \\
&\leq \left\| x_k - x_* - F'(x_k)^{-1}F'(y_k) \right\| + \left\| T(\alpha; x_k)^{m_k} \right\| + \left\| F'(x_k)^{-1}F'(y_k) \right\| \\
&\leq \frac{\gamma}{1 - \gamma L \|x_k - x_*\|}(\frac{L}{2} \|y_k - x_*\| + L \|x_k - x_*\|\,) \|y_k - x_*\| \\
&\quad + \frac{\gamma[(\tau + 1)\theta]^{m^k}}{1 - \gamma L \|x_k - x_*\|}(\frac{L}{2}(\|y_k - x_*\|\,)^2 + 2\beta \|y_k - x_*\|\,) \\
&= (\frac{\gamma}{1 - \gamma L \|x_k - x_*\|}(\frac{1 + [(\tau + 1)\theta]^{m^k}}{2} \|y_k - x_*\| + \|x_k - x_*\|\,) \\
&\quad + \frac{2\beta\gamma[(\tau + 1)\theta]^{m^k}}{1 - \gamma L \|x_k - x_*\|}) \|y_k - x_*\| \\
&\leq \frac{2\gamma g(\|x_k - x_*\|\,; l_k)}{1 - \gamma L \|x_k - x_*\|}(\frac{1 + g(\|x_k - x_*\|\,; l_k)}{2} \\
&\quad \times L \|x_k - x_*\| + \beta[(\tau + 1)\theta]^{m^k}) \|x_k - x_*\| \\
&< \frac{2\gamma g(\|x_k - x_*\|\,; l_k)}{1 - \gamma L \|x_k - x_*\|}(L \|x_k - x_*\| + \beta[(\tau + 1)\theta]^{m^k}) \|x_k - x_*\| \\
&= g(\|x_k - x_*\|\,; l_k) g(\|x_k - x_*\|\,; m_k) \|x_k - x_*\| \\
&\leq g(\|x_k - x_*\|\,; u)^2 \|x_k - x_*\| < g(r_0; u)^2 \|x_k - x_*\| < \|x_k - x_*\|\ .
\end{aligned}$$

We can further prove that $\{x_k\}_{k=0}^{\infty} \subset \mathbb{N}(x_*, r)$ convergences to x_* by induction. When $k = 0$, we can get $\|x_0 - x_*\| < r < r_0$ and

$$\|x_1 - x_*\| < g(\|x_0 - x_*\|\,; u)^2\, \|x_0 - x_*\| < \|x_0 - x_*\| < r,$$

since $x_0 \in \mathbb{N}(x_*, r)$. Hence $x_1 \in \mathbb{N}(x_*, r)$. Now, when $k = n$,suppose that $x_n \in \mathbb{N}(x_*, r)$ and then we can straightforwardly deduce the estimate

$$\begin{aligned}\|x_{n+1} - x_*\| &< g(\|x_n - x_*\|\,; u)^2\, \|x_n - x_*\| \\ &< g(r_0; u)^2\, \|x_n - x_*\| < g(r_0; u)^{2(n+1)}\, \|x_0 - x_*\| < r,\end{aligned}$$

which shows that $x_{n+1} \in \mathbb{N}(x_*, r)$ for $k = n + 1$. Moreover, as $n \to \infty$, $x_{n+1} \to x_*$. This completes the proof of Theorem. □

4 The SOR acceleration

From the definition of the NSS iteration, we can obtain the fixed-point equations

$$\begin{aligned}(\alpha I + N)x &= (\alpha I + S)y + b, \\ (\alpha I - S)y &= (\alpha I - N)x + b.\end{aligned} \tag{21}$$

These two fixed-point equations have the following relationships with the large sparse non-Hermitian and positive definite system of linear equations, A is nonsingular :

$$Ax = b,\ A \in \mathbb{C}^{n\times n},\ b \in \mathbb{C}^n. \tag{22}$$

Lemma 4.1 *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $N \in \mathbb{C}^{n\times n}$ be a normal matrix and $S \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix such that $A = N + S$. If $x^* \in \mathbb{C}^n$ is the exact solution of (22), then $z^* = \begin{bmatrix} x^* \\ x^* \end{bmatrix} \in \mathbb{C}^{2n}$ is the exact solution of the linear system*

$$A_0(\alpha)z \equiv \begin{bmatrix} \alpha I + N & -(\alpha I - S) \\ -(\alpha I - N) & \alpha I + S \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} \equiv c. \tag{23}$$

Conversely, it is also established.

Proof Details see Theorem 3.3 in [11]. □

The block Jacobi iteration for the fixed-point (21), or for the block linear system (7), is

$$\begin{aligned}x_{k+1} &= (\alpha I + N)^{-1}[(\alpha I - S)y_k + b], \\ y_{k+1} &= (\alpha I + S)^{-1}[(\alpha I - N)x_k + b],\end{aligned}$$

or equivalently,

$$z_{(k+1)} = \mathcal{B}(\alpha)z_k + c(\alpha),$$

where $z_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$ and

$$\mathcal{B} = \begin{bmatrix} 0 & (\alpha I + N)^{-1}(\alpha I - S) \\ (\alpha I + S)^{-1}(\alpha I - N) & 0 \end{bmatrix},$$

and the block SOR iteration for the fixed-point (21), is

$$\begin{aligned} x_{k+1} &= (1-\omega)x_k + \omega(\alpha I + N)^{-1}[(\alpha I - S)y_k + b], \\ y_{k+1} &= (1-\omega)y_k + \omega(\alpha I + S)^{-1}[(\alpha I - N)x_k + b], \end{aligned}$$

with ω the relaxation parameter, or equivalently,

$$z_{k+1} = \mathcal{C}_\omega(\alpha)z_k + c_\omega(\alpha),$$

where

$$\mathcal{C}_\omega(\alpha) = \begin{bmatrix} (1-\omega)I & \omega(\alpha I + N)^{-1}(\alpha I - S) \\ \omega(1-\omega)(\alpha I + S)^{-1}(\alpha I - N) & (1-\omega)I + \omega^2 T(\alpha) \end{bmatrix}, \tag{24}$$

and $T(\alpha)$ is the NSS iteration matrix. There are the convergence theorem of the block SOR given in [11], as follows:

Theorem 4.1 *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $N \in \mathbb{C}^{n\times n}$ be a normal matrix and $S \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix such that $A = N + S$.*

(a) *When all of the eigenvalues of the block Jacobi matrix $\mathcal{B}$ are real, the block SOR matrix $\mathcal{C}_\omega(\alpha)$ is convergent if and only if $0 < \omega < 2$.*
(b) *When some of the eigenvalues of the block Jacobi matrix $\mathcal{B}(\alpha)$ are complex, the block SOR matrix $\mathcal{C}_\omega(\alpha)$ is convergent if for some positive number $\tau \in (0, 1)$ and each eigenvalue $\mu = \delta + \imath\beta$ of $\mathcal{B}(\alpha)$, the point (δ, β) lies in the interior of the ellipse*

$$\Phi(1, \tau) = \{(\delta, \beta) : \delta^2 + \frac{\beta^2}{\tau^2} = 1\},$$

and ω satisfies

$$0 < \omega < \frac{2}{1+\tau}. \tag{25}$$

Conversely, if the block SOR matrix converges, then all eigenvalues of $\mathcal{B}(\alpha)$ lie inside $\Phi(1, \tau)$ for some $\tau \in (0, 1)$. Moreover, if some μ lies on $\Phi(1, \tau)$ and if the block SOR matrix converges, then the (25) holds.

Proof Details see Theorem 3.6 in [11]. □

Based on the above preparations, we can now establish the SOR acceleration of the modified Newton-NSS method for solving the system of nonlinear equations $F(x) = 0$, which uses the modified Newton iteration as the outer iteration and the block SOR iteration as the inner iteration.

Algorithm 2 MN–SNSS (the modified Newton–SOR NSS) algorithm

Given an initial guess x_0, positive constants α and tol, and two positive integer sequences $\{l_k\}_{k=0}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$.
for $k = 0, 1, 2, \ldots \|F(x_k)\| \geq \text{tol}\|F(x_0)\|$ **do**
Set $d_{k,0} = h_{k,0} := 0$.
for $l = 0, 1, \ldots, l_k - 1$, apply Algorithm NSS to the linear system(5):

$$\begin{cases} (\alpha I + N(x_k))d_{k,l+1} = (1-\omega)(\alpha I + N(x_k))d_{k,l} + \omega[(\alpha I - S(x_k))D_{k,j} - F(x_k)], \\ (\alpha I + S(x_k))D_{k,l+1} = (1-\omega)(\alpha I + S(x_k))D_{k,l} + \omega[(\alpha I - N(x_k))d_{k,j} - F(x_k)], \end{cases}$$

and obtain D_{k,l_k} such that $\left\| F(x_k) + F^{'}(x_k)d_{k,l_k} \right\| \leq \eta_k \|F(x_k)\|$ for some $\eta_k \in [0, 1)$.
Set $y_k = x_k + D_{k,l_k}$.
Compute $F(y_k)$.
for $m = 0, 1, \ldots, m_k - 1$, apply Algorithm NSS to the linear system(6):

$$\begin{cases} (\alpha I + N(x_k))h_{k,l+1} = (1-\omega)(\alpha I + N(x_k))h_{k,l} + \omega[(\alpha I - S(x_k))H_{k,j} - F(y_k)], \\ (\alpha I + S(x_k))H_{k,l+1} = (1-\omega)(\alpha I + S(x_k))H_{k,l} + \omega[(\alpha I - N(x_k))h_{k,j} - F(y_k)], \end{cases}$$

and obtain H_{k,m_k} such that $\left\| F(y_k) + F^{'}(x_k)H_{k,m_k} \right\| \leq \tilde{\eta_k} \|F(y_k)\|$ for some $\tilde{\eta_k} \in [0, 1)$.
$x_{k+1} = y_k + H_{k,m_k}$.
end for

5 Numerical examples

In this section, we compare our methods with the modified Newton-HSS (MN-HSS) by the example given in [10], and the numerical results show that the spectral radius of the MN-NSS iteration matrices are always greater than those of the MN-HSS iteration matrices and the modified Newton-SNSS method is more competitive than the MN-HSS method. We consider the two-dimensional nonlinear convection-diffusion equations

$$\begin{cases} -(u_{xx} + u_{yy}) + q_1 u_x + q_2 u_y = -e^u, \text{ for } (x, y) \in \Omega, \\ u(x, y) = 0, \text{ for } (x, y) \in \partial\Omega, \end{cases}$$

where $\Omega = (0, 1) \times (0, 1)$, with $\partial\Omega$ its boundary, and q_1, q_2 are positive constants used to measure magnitudes of the convective terms. By applying the centered finite difference scheme on the equidistant discretization grid with the stepsize $h = \frac{1}{N+1}$, the system of nonlinear equations $F(x) = 0$ is obtained with following form

$$F(x) = Mx + h^2\Phi(x) = 0,$$

Table 1 The optimal values α for modified Newton-NSS1 method

N	$q = 600$			$q = 800$			$q = 1000$		
	0.1	0.2	0.4	0.1	0.2	0.4	0.1	0.2	0.4
30	8.0	8.0	8.0	8.1	8.3	8.3	8.1	8.4	8.0
40	8.3	8.5	8.3	8.4	8.1	8.0	8.2	8.3	8.4
50	3.9	3.8	4.2	7.0	7.2	7.3	7.2	7.0	7.0

where N is a prescribed positive integer,

$$\begin{aligned} M &= T_x \otimes I + I \otimes T_y, \\ \Phi(x) &= (e^{x_1}, e^{x_2}, \ldots, e^{x_n})^T, \end{aligned}$$

with T_x and T_y being tridiagonal matrices given by

$$\begin{aligned} T_x &= \text{tridiag}(-1, -Re_1, 2, -1 + Re_1), \\ T_y &= \text{tridiag}(-1, -Re_2, 2, -1 + Re_2), \end{aligned}$$

here,$Re_j = \frac{1}{2} q_j h$, $j = 1, 2$, $Re = \max\{Re_1, Re_2\}$ is the mesh Reynolds number, $\otimes$ the Kronecker product symbol, and $n = N \times N$. Here, we choose the same parameters as those given in [16]. The positive constant $q_2 = \frac{1}{h}$, the initial guess $x_0 = 0$, the stopping criterion for the outer Newton iteration is set to be

$$\frac{\|F(x_k)\|_2}{\|F(x_0)\|_2} \le 10^{-6}$$

and the prescribed tolerances η_k and $\tilde{\eta}_k$ for controlling the accuracy of the block SOR iteration are both set to be η.

In the implementations, we set $c = 0.01$ for the MN-NSS method denoted as the MN-NSS1 method and set $c = 1.0$ for the MN-NSS method denoted as the MN-NSS2 method. We find that the CPU times of the MN-NSS1 method are less than these of the MN-NSS2 method. We use the optimal parameters α for the modified Newton-NSS1 method listed in Table 1, the optimal parameters α for the modified

Table 2 The optimal values α for modified Newton-NSS2 method

N	$q = 600$			$q = 800$			$q = 1000$		
	0.1	0.2	0.4	0.1	0.2	0.4	0.1	0.2	0.4
30	8.2	8.0	8.0	8.0	8.0	8.0	8.1	8.3	8.0
40	8.1	8.4	8.5	8.3	8.2	8.0	8.4	8.2	8.1
50	4.1	4.3	4.2	7.0	7.0	7.3	7.0	7.0	7.0

Table 3 The optimal values α for modified Newton-HSS method

N	$q = 600$			$q = 800$			$q = 1000$		
	0.1	0.2	0.4	0.1	0.2	0.4	0.1	0.2	0.4
30	3.1	2.9	3.6	3.5	3.3	3.5	3.7	3.8	4.4
40	3.1	2.8	3.4	3.4	3.3	3.5	4.1	3.6	3.7
50	2.0	2.1	2.8	3.0	2.6	2.9	3.2	3.0	2.8

Table 4 The optimal values α for modified Newton-SNSS method

N	$q = 600$			$q = 800$			$q = 1000$		
	0.1	0.2	0.4	0.1	0.2	0.4	0.1	0.2	0.4
30	3.4	3.4	4.1	3.4	3.5	4.0	3.4	3.4	4.1
40	3.2	3.1	4.4	3.2	3.2	4.3	3.2	3.1	4.4
50	3.5	3.1	3.0	3.5	3.3	3.1	3.5	3.0	3.1

Table 5 The optimal values ω for modified Newton-SNSS method

N	$q = 600$			$q = 800$			$q = 1000$		
	0.1	0.2	0.4	0.1	0.2	0.4	0.1	0.2	0.4
30	0.84	0.85	0.90	0.84	0.85	0.90	0.84	0.85	0.90
40	0.88	0.85	0.91	0.88	0.85	0.91	0.88	0.85	0.91
50	0.90	0.89	0.89	0.90	0.89	0.89	0.90	0.89	0.89

Table 6 $\eta = 0.1$, $N = 30$

q	Method	Error estimates	CPU times (s)	Outer IT	Inner IT
600	MN-NSS1	4.5366×10^{-7}	7.962517	3	49
	MN-NSS2	4.3356×10^{-7}	8.477883	3	52
	MN-HSS	3.6278×10^{-7}	4.614020	3	49
	MN-SNSS	1.18761×10^{-7}	1.64416	3	20
800	MN-NSS1	3.6964×10^{-7}	9.867488	3	56
	MN-NSS2	4.2752×10^{-7}	10.301429	3	58
	MN-HSS	4.9047×10^{-7}	5.23838	3	53
	MN-SNSS	6.3317×10^{-7}	1.682617	3	20
1000	MN-NSS1	4.9425×10^{-7}	11.831364	3	62
	MN-NSS2	5.4973×10^{-7}	12.184616	3	63
	MN-HSS	5.7774×10^{-7}	5.866053	3	59
	MN-SNSS	2.6900×10^{-8}	1.662240	3	20

Table 7 $\eta = 0.1, N = 40$

q	Method	Error estimates	CPU times (s)	Outer IT	Inner IT
600	MN-NSS1	4.2093×10^{-7}	25.663544	3	60
	MN-NSS2	4.1633×10^{-7}	27.860602	3	65
	MN-HSS	3.3513×10^{-7}	10.016936	3	45
	MN-SNSS	1.4016×10^{-7}	6.537703	3	24
800	MN-NSS1	3.6975×10^{-7}	26.982290	3	59
	MN-NSS2	5.8142×10^{-7}	27.381107	3	59
	MN-HSS	5.1506×10^{-7}	12.579769	3	53
	MN-SNSS	6.4947×10^{-8}	6.327009	3	23
1000	MN-NSS1	6.3443×10^{-7}	28.999633	3	58
	MN-NSS2	4.3558×10^{-7}	31.478180	3	64
	MN-HSS	7.1399×10^{-7}	13.614789	3	59
	MN-SNSS	8.6677×10^{-8}	5.834393	3	22

Newton-NSS2 method listed in Table 2, the optimal parameters α for the modified Newton-HSS method listed in Table 3, and the optimal parameters α for the modified Newton-SNSS method listed in Table 4, which yield the smallest value of CPU time. Here, we set Hermitian matrix H as normal matrix N and we adopt the experimentally optimal parameters ω for the Newton-SNSS method, listed in Table 5.

The results which are shown in Tables 6, 7, 8, 9, 10, 11, and 12 indicate that the modified Newton-SNSS methods outperforms the modified Newton-HSS in the sense of number of inner iterations and CPU time. From the numerical results, we observe that the CPU time for the MN-HSS method is about 2.5 times in average

Table 8 $\eta = 0.1, N = 50$

q	Method	Error estimates	CPU times (s)	Outer IT	Inner IT
600	MN-NSS1	4.9212×10^{-7}	59.413742	3	49
	MN-NSS2	4.8010×10^{-7}	70.931500	3	57
	MN-HSS	4.1182×10^{-7}	27.491654	3	49
	MN-SNSS	1.2404×10^{-7}	19.592829	3	30
800	MN-NSS1	3.3001×10^{-7}	65.153644	3	61
	MN-NSS2	4.8254×10^{-7}	75.445771	3	67
	MN-HSS	3.9851×10^{-7}	28.32209	3	53
	MN-SNSS	1.4468×10^{-7}	17.939034	3	27
1000	MN-NSS1	4.2091×10^{-7}	74.805796	3	67
	MN-NSS2	5.0247×10^{-7}	76.746795	3	67
	MN-HSS	3.6745×10^{-7}	33.731306	3	59
	MN-SNSS	8.5513×10^{-8}	17.487255	3	26

Table 9 $\eta = 0.2$, $N = 30$

q	Method	Error estimates	CPU times (s)	Outer IT	Inner IT
600	MN-NSS1	8.2935×10^{-7}	8.309378	4	52
	MN-NSS2	5.9674×10^{-7}	8.869997	4	53
	MN-HSS	4.7251×10^{-7}	4.375032	4	44
	MN-SNSS	4.2602×10^{-7}	1.756669	4	19
800	MN-NSS1	7.0054×10^{-7}	9.394007	4	54
	MN-NSS2	8.4335×10^{-7}	10.011003	4	55
	MN-HSS	8.8260×10^{-7}	4.912147	4	50
	MN-SNSS	6.7665×10^{-7}	1.965945	4	20
1000	MN-NSS1	9.5585×10^{-7}	11.148087	4	60
	MN-NSS2	9.7813×10^{-7}	11.510977	4	61
	MN-HSS	5.713169×10^{-7}	5.713169	4	58
	MN-SNSS	2.24780×10^{-7}	1.660845	4	18

of that for the MN-SNSS method. In the tables, the Outer IT and Inner IT mean the number of outer iterations and inner iterations. From the tables ,we can observe that the error estimates and CPU time in MN-SNSS method are smaller. Also, in Fig. 1, the results show that the spectral radius of the MN-NSS1 iteration matrices are always greater than those of the MN-HSS iteration matrices, therefore, for the MN-NSS iteration method, an initial point has a wider scope.

Table 10 $\eta = 0.2$, $N = 40$

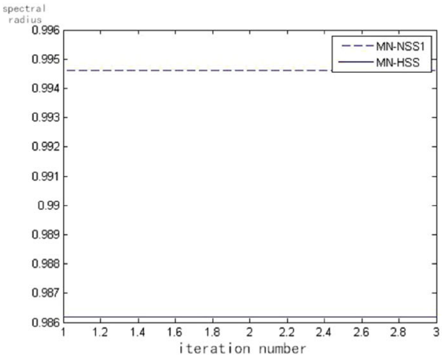
q	Method	Error estimates	CPU times (s)	Outer IT	Inner IT
600	MN-NSS1	3.9359×10^{-7}	22.023706	4	53
	MN-NSS2	6.1254×10^{-7}	25.924621	4	57
	MN-HSS	6.7542×10^{-7}	10.687251	4	45
	MN-SNSS	3.5661×10^{-7}	6.585178	4	22
800	MN-NSS1	6.3431×10^{-7}	27.390020	4	59
	MN-NSS2	8.1484×10^{-7}	28.807108	4	59
	MN-HSS	9.1259×10^{-7}	12.417761	4	51
	MN-SNSS	5.1312×10^{-7}	6.203906	4	20
1000	MN-NSS1	9.2658×10^{-7}	29.955763	4	61
	MN-NSS2	9.9294×10^{-7}	33.518957	4	65
	MN-HSS	3.5042×10^{-8}	17.221869	5	67
	MN-SNSS	2.5770×10^{-7}	6.174436	4	20

Table 11 $\eta = 0.2$, $N = 50$

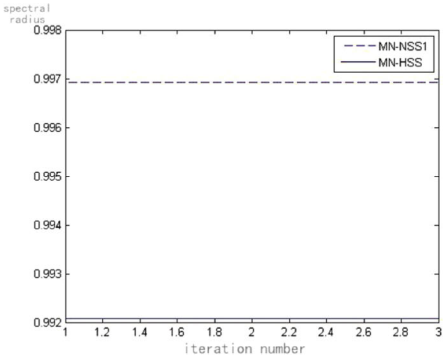
q	Method	Error estimates	CPU times (s)	Outer IT	Inner IT
600	MN-NSS1	3.9898×10^{-7}	58.13681	4	48
	MN-NSS2	8.3841×10^{-7}	64.960205	4	55
	MN-HSS	6.9440×10^{-7}	26.0821781	4	46
	MN-SNSS	9.4305×10^{-8}	20.738478	4	29
800	MN-NSS1	6.5147×10^{-7}	65.140163	4	61
	MN-NSS2	9.7837×10^{-7}	67.908378	4	62
	MN-HSS	8.6441×10^{-7}	28.675378	4	50
	MN-SNSS	4.0200×10^{-8}	20.407093	4	28
1000	MN-NSS1	9.5523×10^{-7}	69.602890	4	61
	MN-NSS2	9.3257×10^{-7}	72.911546	4	63
	MN-HSS	8.4289×10^{-7}	32.395389	4	56
	MN-SNSS	3.4427×10^{-8}	6.174436	4	27

Table 12 $\eta = 0.4$, $N = 30$

q	Method	Error estimates	CPU times (s)	Outer IT	Inner IT
600	MN-NSS1	7.2568×10^{-7}	8.598955	7	51
	MN-NSS2	4.1958×10^{-7}	9.536475	7	55
	MN-HSS	3.8186×10^{-7}	4.155098	7	47
	MN-SNSS	9.2850×10^{-8}	2.474202	6	25
800	MN-NSS1	3.0281×10^{-7}	10.837617	7	60
	MN-NSS2	3.8171×10^{-7}	10.611237	7	58
	MN-HSS	4.9190×10^{-7}	5.284114	7	52
	MN-SNSS	1.4485×10^{-7}	2.326768	6	24
1000	MN-NSS1	4.8887×10^{-7}	11.779672	7	60
	MN-NSS2	4.9682×10^{-7}	12.476918	7	63
	MN-HSS	6.7426×10^{-7}	5.435648	7	56
	MN-SNSS	3.0810×10^{-8}	2.136836	6	23



(a) $N = 30, \eta = 0.1$, q=600



(b) $N = 40, \eta = 0.1$, q=600

Fig. 1 Spectral radius

6 Conclusions

We employ normal/skew-Hermitian splitting iteration method as inner iteration and the modified Newton method as outer iteration, we propose the modified Newton-NSS method, and Newton-NSS method is a competitive method for solving large sparse nonlinear systems with non-Hermitian positive definite Jacobian matrices. We have proved our method has local convergence property. In fact, we consider SOR method to accelerate the Newton-NSS method, and our numerical results have demonstrated the feasibility and effectiveness.

However, the NS splitting is not unique for a given matrix, it is an interesting topic in our future study. And according to the practical choice of the relaxion parameter ω in the SOR acceleration scheme, the technique proposed in [15] may be adopted in actual applications. Furthermore, we could develope a class of new inexact preconditioners for solving the block two-by-two linear systems to (24), see [17–25].

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References

1. Bai, Z.-Z.: A class of two-stage iterative methods for systems of weakly nonlinear equations. Numer. Algor. **14**, 295–319 (1997)
2. Rheinboldt, W.C.: Methods for Solving Systems of Nonliear Equations. SIAM, Philadephia, PA (1998)
3. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York (1970)
4. Dembo, R., Eisenstat, S., Steihaug, T.: Inexact newton method. SIAM J. Numer. Anal. **19**, 400–408 (1982)
5. Benzi, M., Szyld, D.B.: Existence and uniqueness of splittings for stationary iterative methods with applications to alternating methods. Numer. Math. **76**, 309–321 (1997)
6. Deuflhard, P.: Newton Methods for Nonlinear Problems. Springer, Berlin (2004)
7. An, H.-B., Bai, Z.-Z.: A globally convergent Newton-GMRES method for large sparse systems of nonlinear equations. Appl. Numer. Math. **57**, 235–252 (2007)
8. Saad, Y.: Iterative Methods for Sparse Linear Systems, 2nd edn. SIAM, PA, USA (2003)
9. Bai, Z.-Z., Golub, G.H., Ng, M.K.: Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. SIAM J. Matrix Anal. Appl. **24**, 603–626 (2002)
10. Wu, Q.-B., Chen, M.-H.: Convergence analysis of modified Newton-HSS method for solving systems of nonlinear equations. Numer. Algor. **64**, 659–683 (2013)
11. Bai, Z.-Z., Golub, G.H., Ng, M.: On successive-overrelaxation acceleration of the Hermitian and skew-Hermitian splitting iterations. Numer. Linear Algebra Appl. **14**, 319–335 (2007)
12. Zheng, Q.-Q., Ma, C.: On normal and skew-Hermitian splitting iteration methods for large sparse continuous Sylvester equations. J. Comput. Appl. Math. **268**, 145–154 (2014)
13. Wang, X., Li, W.-W., Mao, L.: On positive-definite and skew-Hermitian splitting iteration methods for continuous Sylvester equation AX+XB=C. Comput. Math. Appl. **66**, 2352–2361 (2013)
14. Toutounian, F., Hezari, D.: Accelerated normal and skew-Hermitian splitting methods for positive definite linear systems. IJNAO **3**, 31–34 (2013)
15. Bai, Z.-Z., Chi, X.-B.: Asymptotically optimal successive overrelaxation methods for systems of linear equations. J. Comput. Math. **21**, 603–612 (2003)
16. Bai, Z.-Z., Guo, X.: The Newton-HSS methods for systems of nonlinear equations with positive-definite Jacobian matrices. J. Comput. Math. **28**, 235–260 (2010)

17. Zhang, J.-H., Dai, H.: Inexact splitting-based block preconditioners for block two-by-two linear systems. Appl. Math. Lett. **60**, 89–95 (2016)
18. Lang, C., Ren, Z.-R.: Inexact rotated block triangular preconditioners for a class of block two-by-two matrices. J. Engrg. Math. **93**, 87–98 (2017)
19. Bai, Z.-Z.: Rotated block triangular preconditioning based on PMHSS. Sci. China. Math. **56**, 2523–2538 (2013)
20. Bai, Z.-Z.: On preconditioned iteration methods for complex linear systems. J. Eng. Math. **93**, 41–60 (2017)
21. Bai, Z.-Z.: Structured preconditioners for nonsingular matrices of block two-by-two structures. Math. Comput. **75**, 791–815 (2006)
22. Benzi, M., Bertaccini, D.: Block preconditioning of real-valued iterative algorithms for complex linear systems. IMA J. Numer. Anal. **28**, 598–618 (2008)
23. Guo, X.-X., Wang, S.: Modified HSS iteration methods for a class of non-Hermitian positive-definite linear systems. Appl. Math. Comput. **218**, 10122–10128 (2012)
24. Bai, Z.-Z., Benzi, M., Chen, F.: Modified HSS iteration methods for a class of complex symmetric linear systems. Computing **87**, 93–111 (2010)
25. Bai, Z.-Z., Benzi, M., Chen, F.: On preconditioned MHSS iteration methods for complex symmetric linear systems. Numer. Algor. **56**, 297–317 (2011)