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Hermite interpolation with symmetric polynomials

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Abstract We study the Hermite interpolation problem on the spaces of symmetric bivariate polynomials. We show that the multipoint Berzolari-Radon sets solve the problem. We also give a Newton formula for the interpolation polynomial and use it to prove a continuity property of the interpolation polynomial with respect to the interpolation points.

Keywords Polynomial interpolation · Hermite interpolation · Multipoint berzolari-radon set

1 Introduction

Let $\mathscr{P}(\mathbb{R}^2)$ be the vector space of all polynomials (of real coefficients) in $\mathbb{R}^2$ and $\mathscr{P}_n(\mathbb{R}^2)$ the subspace consisting of all polynomials of degree at most n. The vector space $\mathscr{P}(\mathbb{R}^2)$ is endowed with the norm

$$\|p\|_\infty = \max_{j+k\le d} |c_{jk}| \quad \text{with} \quad p(x, y) = \sum_{j+k\le d} c_{jk} x^j y^k.$$

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For $p \in \mathscr{P}(\mathbb{R}^2)$, we shall also denote by p the associated algebraic curve $\{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}$. Let $\mathscr{S}$ and $\mathscr{A}$ be the subspaces of $\mathscr{P}(\mathbb{R}^2)$ that consists of all symmetric polynomials and antisymmetric polynomials respectively,

$$\mathscr{S} = \{p \in \mathscr{P}(\mathbb{R}^2) : p(x, y) = p(x, -y)\},$$
$$\mathscr{A} = \{p \in \mathscr{P}(\mathbb{R}^2) : p(x, y) = -p(x, -y)\}.$$

Let us define $\mathscr{S}_n = \mathscr{S} \cap \mathscr{P}_n(\mathbb{R}^2)$ and $\mathscr{A}_n = \mathscr{A} \cap \mathscr{P}_n(\mathbb{R}^2)$. In [7], it is shown that

$$\dim \mathscr{S}_n = \lfloor \frac{n+3}{2} \rfloor \lfloor \frac{n+2}{2} \rfloor \quad \text{and} \quad \mathscr{A}_n = y\mathscr{S}_{n-1}.$$

Also in [7], Carnicer and Godés studied the Lagrange interpolation problem for symmetric polynomials. They constructed a $\mathscr{S}_n$-Berzolari-Radon set (BR set for short) X in the upper half plane that solves the interpolation problem. More precisely, X consists of $\dim \mathscr{S}_n$ distinct points distributed on lines. The authors also proposed a Newton formula for the symmetric interpolation polynomial. In [7], it is shown that the Lagrange interpolation problems for $\mathscr{A}_n$ and $\mathscr{P}_n(\mathbb{R}^2)$ are direct consequences from the corresponding problems for $\mathscr{S}_n$.

In this paper, we consider a problem of Hermite interpolation for $\mathscr{S}_n$. More precisely, the problem means to find a polynomial in $\mathscr{S}_n$ which matches, on a set of distinct points in $\mathbb{R} \times [0, \infty)$, values of a function, and its partial derivatives. When no partial derivative appears, our problem reduces to that studied in [7]. Here, we also deal with the case in which the number of interpolation conditions is equal to $\dim \mathscr{S}_n$. Roughly speaking, a univariate Hermite interpolation is the result of the collapsing of points in a univariate Lagrange interpolation. When n real points coalesce to a single point, the derivative up to order $n-1$ will arise. Similarly, if we let some points of a Berzolari-Radon set coalesce along the lines containing them, we will get directional derivatives with respect to the vectors that are parallel with the lines. Based on this observation, we introduce in this paper the multipoint Berzolari-Radon sets (MBR sets for short) whose points are not necessarily distinct. It is proved that the MBR set solves the Hermite interpolation problem. A Newton formula for interpolation polynomial is also given in this paper. Remark that our method to prove these results is different from [7]. Indeed, Carnicer an Godés showed that the interpolation operator corresponding to the Berzolari-Radon set is a bijective map onto the set of symmetric polynomials $\mathscr{S}_n$. They constructed a symmetric polynomial of the Newton form that matches the value of the interpolated function at the BR set. Here, we prove that the interpolation operator corresponding to the MBR set is injective. Our Newton formula for the interpolation polynomial at the MBR set is slightly different from [7]. Our method is as follows. We first construct a precise bivariate polynomial that interpolates (in a Hermite type) a function at points in the MBR set lying on a line. We then collect interpolation conditions along with these polynomials to obtain the formula. Moreover, the formula enables us to prove the continuity property of Hermite interpolation at MBR sets with respect to the interpolation points. It is worth pointing out that detemining the limit of multivariate Lagrange and Hermite interpolants is not an easy problem (see [5, 9]). In a recent work, based on a results of Bos and Calvi, Calvi and Phung [6] proved that the limit of Lagrange projectors at Bos configurations on irreducible algebraic curves in $\mathbb{C}^2$ are the Hermite projectors introduced by Bos and Calvi [3, 4]. Finally, it can be said that the BR sets are similar to the Bos configurations [2]. The paper [8] deals with Hermite systems of points lying on lines and allowing coalescences of points and lines for Hermite problems and is an interesting precedent of the multipoint Berzolari-Radon sets introduced in the paper. The Hermite problem for $\mathscr{P}_d(\mathbb{R}^2)$ was studied in [3, 4].

The paper is organized as follows. In Section 2, we recall properties of univariate Hermite interpolation. Section 3 is devoted to Hermite interpolation on straight lines. In Section 4, we give a divisibility criterion for $\mathscr{S}_n$ which is use to prove the poisedness of the interpolation problem. In Section 5, we study Hermite interpolation with $\mathscr{S}_n$ at the MBR sets. Finally, Section 6 contains some examples.

2 Univariate hermite interpolation

Let $t_1, \dots, t_\lambda$ be λ distinct real numbers. Let $\mu_1, \dots, \mu_\lambda$ be λ positive integers and $d = \mu_1 + \cdots + \mu_\lambda$. The following result is well known.

Theorem 1 *Given a function f for which $D^{\mu_i - 1} f(t_i)$ exists for $i = 1, \dots, \lambda$. Then there exists a unique $p \in \mathscr{P}_{d-1}(\mathbb{R})$ such that*

$$p^{(j)}(t_i) = f^{(j)}(t_i), \quad 1 \le i \le \lambda, \ 0 \le j \le \mu_i - 1.$$

The polynomial p in Theorem 1 is denoted by $\mathbf{H}[\{(t_1; \mu_1), \dots, (t_\lambda; \mu_\lambda)\}; f]$ and called the Hermite interpolation polynomial.

In studying Hermite interpolation, it is convenient to use interpolation sets in which elements may be repeated. For example, if $A = \{1, 1, 3, -2, 1, -7, 1, 3, 3\}$, then we can write $A = \{(1; 4), (-2; 1), (3; 3), (-7; 1)\}$. More generally, any set $A = \{s_1, \dots, s_d\} \subset \mathbb{R}$ can be identified with $\{(t_1; \mu_1), \dots, (t_\lambda; \mu_\lambda)\}$. Here, the t_is are pairwise distinct and $(t_i; \mu_i)$ means that t_i is repeated μ_i-times. Hence, we can write $\mathbf{H}[A; f]$ instead of $\mathbf{H}[\{(t_1; \mu_1), \dots, (t_\lambda; \mu_\lambda)\}; f]$. In the case where the s_is are pairwise distinct, the interpolation polynomial becomes the ordinary Lagrange interpolation polynomial. The Hermite interpolation polynomial can be written into the Newton form in which the coefficients are the divided differences. Using the continuity property of the divided difference which follows from the Hermite-Genocchi's formula (see [1, Theorem 1.9]), one can prove that the univariate Hermite interpolation is continuous with respect to the interpolation points and the interpolated function (see for instance [1, Theorem 1.4]).

Theorem 2 *Let $I \subset \mathbb{R}$ be an interval and $f \in C^{d-1}(I)$. Then the map*

$$(t_1, \dots, t_d) \in I^d \mapsto \mathbf{H}[\{t_1, \dots, t_d\}; f]$$

is continuous. Moreover, if $\{t_i^k\}$ is a sequence in I that tends to t_i as $k \to \infty$ for $i = 1, \dots, d$, and f_k converges to f in the standard topology of $C^{d-1}(I)$, then

$$\lim_{k \to \infty} \mathbf{H}[\{t_1^k, \dots, t_d^k\}; f_k] = \mathbf{H}[\{t_1, \dots, t_d\}; f].$$

The following result is stated in [10]. For the sake of completeness, we give the proof.

Proposition 1 *Let $t_1, \ldots, t_\lambda$ be distinct real numbers in $(0, a]$ and $\mu_1, \ldots, \mu_\lambda$ positive integers. Let f be a function defined on $(0, a]$ such that there exists $f^{(\mu_i-1)}(t_i)$ for $i = 1, \ldots, \lambda$. Let $f^*(t) = f(\sqrt{t})$ and $\hat{f}$ the even extension of f, i.e., $f(t) = \hat{f}(t) = \hat{f}(-t)$ for $0 < t \le a$. Then*

$$\mathbf{H}[\{(t_1;\mu_1),\ldots,(t_\lambda;\mu_\lambda),(-t_1;\mu_1),\ldots,(-t_\lambda;\mu_\lambda)\};\hat{f}](t)=\mathbf{H}[\{(t_1^2;\mu_1),\ldots,(t_\lambda^2;\mu_\lambda)\};f^*](t^2).$$

Proof Set $d = \mu_1 + \cdots + \mu_\lambda$. Let us define $P(t) = \mathbf{H}[\{(t_1^2;\mu_1),\ldots,(t_\lambda^2;\mu_\lambda)\};f^*](t)$ and $Q(t) = P(t^2)$. Then, Q is an even polynomial of degree at most $2d-2$. Hence, it suffices to check that

$$Q^{(i)}(t_j) = \hat{f}^{(i)}(t_j), \quad Q^{(i)}(-t_j) = \hat{f}^{(i)}(-t_j), \quad 1 \le j \le \lambda, 0 \le i \le \mu_j - 1.$$

Since both Q and $\hat{f}$ are even functions, we need only to prove the equalities for derivatives at t_j. Fix $j \in \{1, \ldots, \lambda\}$. For $i = 0$, by definition, we have $Q(t_i) = f^*(t_i^2) = f(t_i) = \hat{f}(t_i)$. Next, we consider the case $i > 0$. For simplicity, we set $\varphi(t) = t^2$. By the Faa di Bruno formula [11], we obtain

$$\begin{aligned} Q^{(i)}(t_j) &= (P \circ \varphi)^{(i)}(t_j) \\ &= \sum \frac{i!}{k_1! \cdots k_i!} P^{(k)}(\varphi(t_j)) \left(\frac{\varphi'(t_j)}{1!}\right)^{k_1} \cdots \left(\frac{\varphi^{(i)}(t_j)}{i!}\right)^{k_i} \end{aligned} \tag{1}$$

where, in the second line, $k = k_1 + \cdots + k_i$ and the sum is over all values of $k_1, \ldots, k_i \in \mathbb{N}$ such that $k_1 + 2k_2 + \cdots + ik_i = i$. From the interpolation condition, we have

$$P^{(k)}(\varphi(t_j)) = P^{(k)}(t_j^2) = (f^*)^{(k)}(t_j^2) = (f^*)^{(k)}(\varphi(t_j)).$$

Substituting this into (1), we obtain

$$\begin{aligned} Q^{(i)}(t_j) &= \sum \frac{i!}{k_1! \cdots k_i!} (f^*)^{(k)}(\varphi(t_j)) \left(\frac{\varphi'(t_j)}{1!}\right)^{k_1} \cdots \left(\frac{\varphi^{(i)}(t_j)}{i!}\right)^{k_i} \\ &= (f^* \circ \varphi)^{(i)}(t_j) \\ &= f^{(i)}(t_j). \end{aligned}$$

The proof is complete. □

3 Hermite interpolation on straight lines

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 > 0$. We associate each affine polynomial $r(x, y) = \alpha x + \beta y - \gamma$ with the following differential operator

$$\mathscr{D}_r = -\beta \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y}.$$

Note that the derivative is the directional derivative along the direction vector of the line r. The higher order of derivatives is defined by

$$\mathscr{D}_r^i = \left(-\beta\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial y}\right)^i = \sum_{l=0}^{i}\binom{i}{l}(-\beta)^{i-l}\alpha^l\frac{\partial^i}{\partial x^{i-l}\partial y^l}, \quad i \geq 1.$$

Of course, D_r^0 is the identity operator. From now on, we always consider two forms of r, that is $r(x,y) = x - \gamma$ or $r(x,y) = \alpha x - y - \gamma$. Remark that

$$\mathscr{D}_r = \begin{cases} \frac{\partial}{\partial y} & \text{if} \quad r(x,y) = x - \gamma \\ \frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial y} & \text{if} \quad r(x,y) = \alpha x - y - \gamma. \end{cases}$$

3.1 Construction of polynomial interpolants

Our construction is inspired from [7]. Let $\nu_1, \ldots, \nu_d$ be positive integers. Let $\mathbf{a}_1, \ldots, \mathbf{a}_d$ be d distinct points on r . Let f be a function of class C^{ν_j-1} in a neighborhood of $\mathbf{a}_j$ for $j = 1, \ldots, d$. We want to find a precise polynomial $P \in \mathscr{S}$ such that

$$\mathscr{D}_r^i(P)(\mathbf{a}_j) = \mathscr{D}_r^i(f)(\mathbf{a}_j), \quad 1 \leq j \leq d, \quad 0 \leq i \leq \nu_j - 1. \tag{2}$$

Like in the univariate Hermite interpolation, we also identify the set $\mathbf{A} = \{\mathbf{b}_0, \ldots, \mathbf{b}_n\}$ of not necessarily distinct points on the straight line r with $\{(\mathbf{a}_1; \nu_1), \ldots, (\mathbf{a}_d; \nu_d)\}$, where the $\mathbf{a}_j$s are pairwise distinct and $\nu_1, \ldots, \nu_d$ are positive integers with $\nu_1 + \cdots + \nu_d = n + 1$. Sometime, we confuse $\mathbf{A}$ with the tuple $(\mathbf{b}_0, \ldots, \mathbf{b}_n)$. Hence, we use $\mathbf{A}$ in three different meanings throughout this paper: a set of pairs of nodes and multiplicities, a multipoint set, and a finite sequence of nodes.

Case 1 We consider the case $r(x,y) = x - \gamma$ with $\gamma \in \mathbb{R}$. Let $g(y)$ be the even extension of the function $f(\gamma, y)$ for $y > 0$, that is

$$g(y) = \begin{cases} f(\gamma, y) & \text{if} \quad y > 0, \\ f(\gamma, -y) & \text{if} \quad y < 0. \end{cases}$$

In this case, we assume that

$$\{\mathbf{a}_j = (\gamma, b_j) : j = 1, \ldots, d\} \subset r \cap (\mathbb{R} \times (0, \infty)) \quad \text{and} \quad \nu_1 + \cdots + \nu_d = \lfloor\frac{n}{2}\rfloor + 1. \tag{3}$$

We consider the Hermite interpolation scheme

$$B = \{(b_1; \nu_1), \ldots, (b_d; \nu_d), (-b_1; \nu_1), \ldots, (-b_d; \nu_d)\}.$$

We have

$$\frac{d^i\mathbf{H}[B; g](b_j)}{dy^i} = \frac{d^i g(b_j)}{dy^i}, \quad 1 \leq j \leq d, \quad 0 \leq i \leq \nu_j - 1.$$

Let us define

$$\mathbb{H}_r[\mathbf{A}; f](x,y) = \mathbf{H}[B; g](y), \quad \mathbf{A} = \{(\mathbf{a}_1; \nu_1), \ldots, (\mathbf{a}_d; \nu_d)\}. \tag{4}$$

By Proposition 1, we can write

$$\mathbb{H}_r[\mathbf{A}; f](x,y) = \mathbf{H}[\{(b_1^2; \nu_1), \ldots, (b_d^2; \nu_d)\}; g^*](y^2), \quad g^*(y) = g(\sqrt{y}). \tag{5}$$

Lemma 1 *Under the assumptions in (3), we have* $\mathbb{H}_r[\mathbf{A}; f] \in \mathscr{S}_n$ *and*

$$\mathscr{D}_r^i(\mathbb{H}_r[\mathbf{A}; f])(\mathbf{a}_j) = \mathscr{D}_r^i(f)(\mathbf{a}_j), \quad 1 \le j \le d, \quad 0 \le i \le \nu_j - 1. \tag{6}$$

Proof The second assertion is trivial since $\mathscr{D}_r^i = \frac{\partial}{\partial y}$. It is easy see that

$$\mathbf{H}[\{(b_1^2; \nu_1), \dots, (b_d^2; \nu_d)\}; g^*](y^2)$$

is an even polynomial of degree at most $2\lfloor \frac{n}{2} \rfloor$. Hence, it belongs to $\mathscr{S}_n$. From (5), we get $\mathbb{H}_r[\mathbf{A}; f] \in \mathscr{S}_n$, and the proof is complete. □

Case 2 We consider the case $r(x, y) = \alpha x - y - \gamma$. Assume that

$$\{\mathbf{a}_j = (a_j, \alpha a_j - \gamma) : j = 1, \dots, d\} \subset r \quad \text{and} \quad \nu_1 + \cdots + \nu_d = n + 1. \tag{7}$$

It is well-known that the Hermite interpolation polynomial of the function $h(x) = f(x, \alpha x - \gamma)$ at $C = \{(a_1; \nu_1), \dots, (a_d; \nu_d)\}$ also exists uniquely.

Lemma 2 *Let the assumptions in (7) hold. Then the polynomial* $\mathbb{H}_r[\mathbf{A}; f]$ *defined by*

$$\mathbb{H}_r[\mathbf{A}; f](x, y) = \mathbf{H}[C; h](x), \tag{8}$$

belongs to $\mathscr{S}_n$ *and satisfies the following relations*

$$\mathscr{D}_r^i(\mathbb{H}_r[\mathbf{A}; f])(\mathbf{a}_j) = \mathscr{D}_r^i(f)(\mathbf{a}_j), \quad 1 \le j \le d, \quad 0 \le i \le \nu_j - 1. \tag{9}$$

Proof Since $\mathbf{H}[C; g]$ is a univariate polynomial of degree at most n in x, we have $\mathbb{H}_r[\mathbf{A}; f] \in \mathscr{S}_n$. Note that $\mathscr{D}_r = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y}$. Relation (9) follows directly from the interpolation conditions,

$$\mathscr{D}_r^i(\mathbb{H}_r[\mathbf{A}; f])(\mathbf{a}_j) = \frac{d^i \mathbf{H}[C; h]}{dx^i}(a_j) = \frac{d^i h}{dx^i}(a_j) = \mathscr{D}_r^i(f)(\mathbf{a}_j).$$

□

3.2 Some properties of polynomial interpolants

In this subsection, we always assume that r is defined in case 1 or case 2. The following result follows directly from the definition.

Lemma 3 *a) The map* $f \mapsto \mathbb{H}_r[\mathbf{A}; f]$*, defined on the space of sufficiently differentiable functions on an open set containing* $\mathbf{A}$*, is linear.*
b) If $P = \mathbb{H}_r[\mathbf{A}; f]$*, then* $P = \mathbb{H}_r[\mathbf{A}; P]$*.*

The next result shows a kind of Leibniz's formula .

Lemma 4 *For sufficiently differentiable functions* f *and* g*, we have*

$$\mathbb{H}_r[\mathbf{A}; fg] = \mathbb{H}_r\left[\mathbf{A}; f\mathbb{H}_r[\mathbf{A}; g]\right].$$

Proof From (4) and (8), we see that $\mathbb{H}_r[\mathbf{A}; f]$ is identical to a univariate Hermite interpolation polynomial. Hence, the assertions follows directly from the similar properties of univariate Hermite interpolation. For convenience to the reader, we prove the lemma for case 2. We keep the notations given in this case. Let us set $u(x) = g(x, \alpha x - \gamma)$. Then

$$\mathbb{H}_r[\mathbf{A}; fg](x, y) = \mathbf{H}[C; hu](x), \quad \mathbb{H}_r[\mathbf{A}; f\mathbb{H}_r[\mathbf{A}; g]](x, y) = \mathbf{H}[C; h\mathbf{H}[C; u]](x).$$

For each $1 \le j \le d$ and $0 \le i \le \nu_j - 1$, the Leibniz rule and interpolation conditions imply that

$$\begin{aligned}(h\mathbf{H}[C; u])^{(i)}(a_j) &= \sum_{l=0}^{i} \binom{i}{l} h^{(i-l)}(a_j)\,(\mathbf{H}[C; u])^{(l)}(a_j) = \sum_{l=0}^{i} \binom{i}{l} h^{(i-l)}(a_j) u^{(l)}(a_j) \\ &= (hu)^{(i)}(a_j).\end{aligned}$$

Hence, $\mathbf{H}[C; hu] = \mathbf{H}[C; h\mathbf{H}[C; u]]$, which proves the assertion. □

Using Theorem 2, we conclude from (5) and (8) the following result.

Lemma 5 *Let K be a closed segment on $r \cap (\mathbb{R} \times (0, \infty))$ when $r(x, y) = x - \gamma$ and on r when $r(x, y) = \alpha x - y - \gamma$. Let f be a bivariate function such that it is of class C^n in a neighborhood Ω of K. Then the polynomial $\mathbb{H}_r[\mathbf{A}; f]$ depends continuously on the interpolation points $\mathbf{A} \in K^{n+1}$, i.e.,*

$$\lim_{\|\mathbf{A}^k - \mathbf{A}\| \to 0} \mathbb{H}_r[\mathbf{A}^k; f] = \mathbb{H}_r[\mathbf{A}; f], \tag{10}$$

where $\mathbf{A}^k = \left(\mathbf{b}_0^k, \dots, \mathbf{b}_n^k\right)$, $\mathbf{A} = (\mathbf{b}_0, \dots, \mathbf{b}_n)$ and

$$\|\mathbf{A}^k - \mathbf{A}\| = \max\{\|\mathbf{b}_i^k - \mathbf{b}_i\| : i = 0, \dots, n\}.$$

Furthermore, if $\{f_k\} \subset C^n(\Omega)$ converges to f in $C^n(\Omega)$, then

$$\lim_{\|\mathbf{A}^k - \mathbf{A}\| \to 0} \mathbb{H}_r[\mathbf{A}^k; f_k] = \mathbb{H}_r[\mathbf{A}; f]. \tag{11}$$

Here $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^2$.

3.3 Interpolation spaces

Definition 1 Let $\mathbf{A} = \{(\mathbf{a}_1; \nu_1), \cdots, (\mathbf{a}_d; \nu_d)\} \subset r$ and f be a sufficiently differentiable function. Let P be a bivariate polynomial. We write $P \in \mathscr{I}_r[\mathbf{A}; f]$ if the following relation holds

$$\mathscr{D}_r^i(P)(\mathbf{a}_j) = \mathscr{D}_r^i(f)(\mathbf{a}_j), \quad 1 \le j \le d, \quad 0 \le i \le \nu_j - 1.$$

From Section 3.1, we see that the set $\mathscr{I}_r[\mathbf{A}; f]$ always contains $\mathbb{H}_r[\mathbf{A}; f]$. The following result can be regarded as a weak Leibniz property.

Lemma 6 *Let* $\mathbf{A} = \{(\mathbf{a}_1; \nu_1), \cdots, (\mathbf{a}_d; \nu_d)\} \subset r$. *Let* P, Q *be bivariate polynomials such that* $Q(\mathbf{a}_i) \neq 0$ *for* $i = 1, \ldots, d$ *and* $PQ \in \mathscr{I}_r[\mathbf{A}; 0]$, *i.e.,*

$$\mathscr{D}_r^i(PQ)(\mathbf{a}_j) = 0, \quad 1 \leq j \leq d, \quad 0 \leq i \leq \nu_j - 1.$$

Then $P \in \mathscr{I}_r[\mathbf{A}; 0]$, *i.e.,*

$$\mathscr{D}_r^i(P)(\mathbf{a}_j) = 0, \quad 1 \leq j \leq d, \quad 0 \leq i \leq \nu_j - 1. \tag{12}$$

Proof Let us fix $j \in \{1, \ldots, d\}$. From the definition of $\mathscr{D}_r$, we see at once that

$$\mathscr{D}_r(PQ)(\mathbf{a}_j) = Q(\mathbf{a}_j)\mathscr{D}_r P(\mathbf{a}_j) + P(\mathbf{a}_j)\mathscr{D}_r Q(\mathbf{a}_j).$$

Hence, we can prove by induction that

$$\mathscr{D}_r^k(PQ)(\mathbf{a}_j) = \sum_{i=0}^{k} \binom{k}{i} \mathscr{D}_r^{k-i} P(\mathbf{a}_j)\mathscr{D}_r^i Q(\mathbf{a}_j).$$

The formula is called the Leibniz rule for $\mathscr{D}_r$. Since $P(\mathbf{a}_j)Q(\mathbf{a}_j) = 0$ and $Q(\mathbf{a}_j) \neq 0$, we have $P(\mathbf{a}_j) = 0$. Assume (12) holds up to $i < \nu_j - 1$; we will prove it for $i + 1$. Observe that

$$\begin{aligned} 0 = \mathscr{D}_r^{i+1}(PQ)(\mathbf{a}_j) &= Q(\mathbf{a}_j)\mathscr{D}_r^{i+1}(P)(\mathbf{a}_j) + \sum_{l=1}^{i+1} \binom{i+1}{l} \mathscr{D}_r^l(Q)(\mathbf{a}_j)\mathscr{D}_r^{i+1-l}(P)(\mathbf{a}_j) \\ &= Q(\mathbf{a}_j)\mathscr{D}_r^{i+1}(P)(\mathbf{a}_j). \end{aligned}$$

Hence, we get $\mathscr{D}_r^{i+1}(P)(\mathbf{a}_j) = 0$, and the proof is complete. □

4 A divisibility criterion

In this section, we give a divisibility criterion for symmetric polynomials.

Lemma 7 *Let* $\nu_1, \ldots, \nu_d$ *be positive integers such that* $\nu_1 + \cdots + \nu_d = n + 1$. *Let* $\mathbf{a}_1, \ldots, \mathbf{a}_d$ *be* d *distinct points on the straight line* r *with* $r(x, y) = \alpha x - y - \gamma$. *If* $P \in \mathscr{S}_n$ *satisfies the relations*

$$\mathscr{D}_r^i(P)(\mathbf{a}_j) = 0, \quad 1 \leq j \leq d, \quad 0 \leq i \leq \nu_j - 1$$

then P *is divisible by* R, *where* $R(x, y) = r(x, y)r(x, -y)$.

Proof The line r can be parameterized globally by $\rho(x) = (x, \alpha x - \gamma)$, $x \in \mathbb{R}$. Hence, we can find d distinct real numbers $a_1, \ldots, a_d$ such that $\mathbf{a}_i = (a_i, \rho(a_i))$ for $i = 1, \ldots, d$. Using the hypothesis and the chain rule, we obtain

$$\frac{d^i}{dx^i} P \circ \rho(a_j) = 0, \quad 1 \leq j \leq d, \quad 0 \leq i \leq \nu_j - 1. \tag{13}$$

Since the polynomial $P \circ \rho$ is of degree less than or equal to n in one variable, the uniqueness of univariate Hermite interpolation follows that $P \circ \rho = 0$. In other words, P restricted on r is identically zero. By the Bezout' theorem, r divides P. This enables us to find a polynomial Q such that $P(x, y) = r(x, y)Q(x, y)$. Replacing y by $-y$ and using the hypothesis $P \in \mathscr{S}_n$, we get $P(x, y) = r(x, -y)Q(x, -y)$. It follows that both $r(x, y)$ and $r(x, -y)$ divide $P(x, y)$. This completes the proof. □

Lemma 8 *Let $\nu_1, \ldots, \nu_d$ be positive integers such that $\nu_1 + \cdots + \nu_d = \lfloor \frac{n}{2} \rfloor + 1$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_d$ be d distinct points on $r \cap (\mathbb{R} \times (0, \infty))$ with $r(x, y) = x - \gamma$. If $P \in \mathscr{S}_n$ satisfies the relations*

$$\mathscr{D}_r^i(P)(\mathbf{a}_j) = 0, \quad 1 \le j \le d, \quad 0 \le i \le \nu_j - 1,$$

then P is divisible by r.

Proof We write $\mathbf{a}_j = (\gamma, b_j)$, $b_j > 0$ for $j = 1, \ldots, d$. The hypothesis gives

$$\frac{d^i}{dy^i} P(\gamma, y)\Big|_{y=b_j} = 0, \quad 1 \le j \le d, \quad 0 \le i \le \nu_j - 1. \tag{14}$$

Since $P(\gamma, -y) = P(\gamma, y)$, we get

$$\frac{d^i}{dy^i} P(\gamma, y)\Big|_{y=-b_j} = 0, \quad 1 \le j \le d, \quad 0 \le i \le \nu_j - 1. \tag{15}$$

The number of interpolation conditions in (14) and (15) equals to

$$2(\nu_1 + \cdots + \nu_d) = 2\lfloor \frac{n}{2} \rfloor + 2 \ge n + 1.$$

The uniqueness of univariate Hermite interpolation follows that $P(\gamma, y) = 0$ for every $y \in \mathbb{R}$. Hence, r divides P, and the proof is complete. □

5 Bivariate Hermite interpolation schemes

Let n, m be natural numbers with $m \le n$. For each $l = 0, \ldots, m$, let

$$R_l(x, y) = \begin{cases} r_l(x, y) = x - \gamma_l & \text{if} \quad k_l = 1, \\ r_l(x, y) r_l(x, -y), \quad r_l(x, y) = \alpha_l x - y - \beta_l & \text{if} \quad k_l = 2 \end{cases}$$

be a curve of degree $k_l \in \{1, 2\}$ with $\sum_{l=0}^m k_l = n + 1$. We define the integers n_l by the relation

$$n_0 = n, \quad n_l = n - \sum_{i=0}^{l-1} k_i, \quad l = 1, \ldots, m. \tag{16}$$

For each $l = 0, \ldots, m$, let $\mathbf{A}_l$ be a set of *not necessarily distinct points* on $r_l \setminus (R_0 \cup \cdots \cup R_{l-1})$ such that $\mathbf{A}_l \subset \mathbb{R} \times (0, \infty)$ with

$$\#\mathbf{A}_l = \lfloor \frac{k_l n_l}{2} \rfloor + 1 = \dim \mathscr{S}_{n_l} - \dim \mathscr{S}_{n_l - k_l}, \quad l = 0, \ldots, m.$$

Here $R_0 \cup \cdots \cup R_{l-1}$ is taken to be empty set when $l = 0$. We say that

$$\mathbf{X} = \mathbf{A}_0 \cup \cdots \cup \mathbf{A}_m$$

is a multipoint Berzolari-Radon set (MBR set for short) for $\mathscr{S}_n$ with lines $r_0, \ldots, r_m$.

Theorem 3 *Let* $\mathbf{X}$ *be a MBR set for* $\mathscr{S}_n$ *with lines* $r_0, \dots, r_m$*. Then, for any sufficiently differentiable function* f *defined on an open set containing* $\mathbf{X}$*, the symmetric interpolation problem*

$$P \in \mathscr{I}_{r_l}[\mathbf{A}_l; f], \quad l = 0, \dots, m, \tag{17}$$

has a unique solution $P \in \mathscr{S}_n$*. Moreover,*

$$P = P_0 + \cdots + P_m, \tag{18}$$

where

$$P_0 = \mathbb{H}_{r_0}[\mathbf{A}_0; f], \quad P_l = R_0 \cdots R_{l-1} \mathbb{H}_{r_l}\left[\mathbf{A}_l; \frac{f - P_0 - \cdots - P_{l-1}}{R_0 \cdots R_{l-1}}\right], \quad l = 1, \dots, m. \tag{19}$$

Proof We first show that the interpolation problem has a unique solution in $\mathscr{S}_n$. We learn some ideas of Bos and Calvi [4]. Since the number of interpolation conditions equals $\sum_{l=0}^{m} \#\mathbf{A}_l = \dim \mathscr{S}_n$, it is sufficient to prove that if $P \in \mathscr{S}_n$ satisfies the following relation

$$P \in \mathscr{I}_{r_l}[\mathbf{A}_l; 0], \quad l = 0, \dots, m, \tag{20}$$

then $P = 0$. Using Lemmas 7 and 8, we conclude from (20) for $l = 0$ that R_0 divides P. Hence, we can find a polynomial $Q_1 \in \mathscr{S}_{n-k_0} = \mathscr{S}_{n_1}$ such that $P = R_0 Q_1$. It follows that $R_0 Q_1 \in \mathscr{I}_{r_l}[\mathbf{A}_l; 0]$, $l = 1, \dots, m$. Since R_0 does not vanish on $\cup_{l=1}^{m} \mathbf{A}_l$, Lemma 6 gives

$$Q_1 \in \mathscr{I}_{r_l}[\mathbf{A}_l; 0], \quad l = 1, \dots, m. \tag{21}$$

Similarly, from (21) we deduce that Q_1 is divisible by R_1. Hence, $Q_1 = R_1 Q_2$ with $Q_2 \in \mathscr{S}_{n_2}$. We continue in this fashion to obtain

$$P = R_0 \cdots R_m Q_{m+1}, \quad Q_{m+1} \in \mathscr{S}.$$

Since $\deg P \le n$ and $\sum_{l=0}^{m} \deg R_l = n + 1$, we conclude from the last relation that $P = 0$.

Evidently, the polynomial P given in (18) belongs to $\mathscr{S}_n$. It is sufficient to show that

$$\mathbb{H}_{r_i}[\mathbf{A}_i; P] = \mathbb{H}_{r_i}[\mathbf{A}_i; f], \quad i = 0, \dots, m.$$

Remark that P_l vanishes on R_i for $i < l$. Hence $\mathbb{H}_{r_i}[\mathbf{A}_i; P_l] = 0$ for $i < l$. This enables us to write

$$\mathbb{H}_{r_i}[\mathbf{A}_i; P] = \sum_{l=0}^{m} \mathbb{H}_{r_i}[\mathbf{A}_i; P_l] = \sum_{l=0}^{i} \mathbb{H}_{r_i}[\mathbf{A}_i; P_l]. \tag{22}$$

For $i = 0$, Using Lemma 3, we have $\mathbb{H}_{r_0}[\mathbf{A}_0; P] = \mathbb{H}_{r_0}[\mathbf{A}_0; P_0] = P_0 = \mathbb{H}_{r_0}[\mathbf{A}_0; f]$. Now, assume that $1 \le i \le m$. For simplicity, we set $\Pi_i = R_0 \cdots R_{i-1}$ and $\Sigma_i = P_0 + \cdots + P_{i-1}$. Lemma 4 gives

$$\begin{aligned}\mathbb{H}_{r_i}[\mathbf{A}_i; P_i] &= \mathbb{H}_{r_i}\left[\mathbf{A}_i; \Pi_i \mathbb{H}_{r_i}[\mathbf{A}_i; \frac{f - \Sigma_i}{\Pi_i}]\right] \\ &= \mathbb{H}_{r_i}\left[\mathbf{A}_i; \Pi_i \frac{f - \Sigma_i}{\Pi_i}\right] \\ &= \mathbb{H}_{r_i}\left[\mathbf{A}_i; f - \Sigma_i\right] \\ &= \mathbb{H}_{r_i}\left[\mathbf{A}_i; f\right] - \mathbb{H}_{r_i}\left[\mathbf{A}_i; \Sigma_i\right].\end{aligned}$$

Combining the last relation with (22), we finally get

$$\mathbb{H}_{r_i}[\mathbf{A}_i; P] = \mathbb{H}_{r_i}[\mathbf{A}_i; P_i] + \mathbb{H}_{r_i}\left[\mathbf{A}_i; \Sigma_i\right] = \mathbb{H}_{r_i}\left[\mathbf{A}_i; f\right],$$

and the proof is complete. □

Definition 2 The polynomial P in Theorem 3 is called the Hermite interpolation polynomial of f at $\mathbf{X}$. We write

$$P = \mathscr{H}\left[\{(\mathbf{A}_0, r_0), \ldots, (\mathbf{A}_m, r_m)\}; f\right].$$

Remark that the Newton formula for P given in Theorem 3 does depend on the ordering of the sets $\mathbf{A}_l$ and the lines r_l.

From the Newton formula in Theorem 3, we obtain an algorithm to compute the polynomial $\mathscr{H}\left[\{(\mathbf{A}_0, r_0), \ldots, (\mathbf{A}_m, r_m)\}; f\right]$:
Step 1. Compute $P_0 = \mathbb{H}_{r_0}[\mathbf{A}_0; f]$ by using (5) and (8);
Step 2. Compute $P_l = R_0 \cdots R_{l-1}\mathbb{H}_{r_l}\left[\mathbf{A}_l; \frac{f - P_0 - \cdots - P_{l-1}}{R_0 \cdots R_{l-1}}\right]$ for $l = 1, \ldots, m$ respectively by using (5) and (8);
Step 3. Compute the sum $\mathscr{H}\left[\{(\mathbf{A}_0, r_0), \ldots, (\mathbf{A}_m, r_m)\}; f\right] = P_0 + \cdots + P_m$.
The following result shows that the polynomial $\mathscr{H}\left[\{(\mathbf{A}_0, r_0), \ldots, (\mathbf{A}_m, r_m)\}; f\right]$ depends continuously on the interpolation points. Here, we let the points of the $\mathbf{A}_l$ move on the segment on r_l but the lines r_l, $l = 0, \ldots, m$, are fixed.

Theorem 4 *Let K_l be a closed segment on $\left(r_l \setminus \cup_{i=0}^{l-1} R_i\right) \cap (\mathbb{R} \times (0, \infty))$. Then, for any sufficiently differentiable function f defined on an open set containing $\cup_{l=0}^{m} K_l$, the following mapping is continuous*

$$\begin{aligned} K_0^{s_0} \times \cdots \times K_m^{s_m} &\to \mathscr{S}_n \\ (\mathbf{A}_0, \ldots, \mathbf{A}_m) &\mapsto \mathscr{H}\left[\{(\mathbf{A}_0, r_0), \ldots, (\mathbf{A}_m, r_m)\}; f\right],\end{aligned}$$

where $\cup_{l=0}^{m}\mathbf{A}_l$ is regarded as a MBR set with respect to r_l, $l = 0, \ldots, m$. Here $s_l = \#\mathbf{A}_l = \lfloor \frac{k_l n_l}{2} \rfloor + 1$.

Proof Note that we work with tuples of points $\mathbf{A}_l$ rather than sets of points. The convergence of tuples is understood as in Lemma 5. Assume that $\mathbf{A}_l^k, \mathbf{A}_l \subset K_l$ such that $\lim_{k\to\infty} \|\mathbf{A}_l^k - \mathbf{A}_l\| = 0$ for $l = 0, \ldots, m$. We will show that

$$\lim_{k\to\infty} \mathscr{H}\left[\{(\mathbf{A}_0^k, r_0), \ldots, (\mathbf{A}_m^k, r_m)\}; f\right] = \mathscr{H}\left[\{(\mathbf{A}_0, r_0), \ldots, (\mathbf{A}_m, r_m)\}; f\right].$$

By Theorem 3, we can write

$$\mathscr{H}\left[\{(\mathbf{A}_0^k, r_0), \ldots, (\mathbf{A}_m^k, r_m)\}; f\right] = P_0^k + \cdots + P_m^k,$$

where

$$P_0^k = \mathbb{H}_{r_0}[\mathbf{A}_0^k; f], \quad P_l^k = R_0 \cdots R_{l-1}\mathbb{H}_{r_l}\left[\mathbf{A}_l^k; \frac{f - P_0^k - \cdots - P_{l-1}^k}{R_0 \cdots R_{l-1}}\right], \quad l = 1, \ldots, m.$$

Using Lemma 5, we can easily prove by induction that

$$\lim_{k\to\infty} P_0^k = \mathbb{H}_{r_0}[\mathbf{A}_0; f] = P_0$$

and

$$\lim_{k\to\infty} P_l^k = R_0 \cdots R_{l-1}\mathbb{H}_{r_l}\left[\mathbf{A}_l; \frac{f - P_0 - \cdots - P_{l-1}}{R_0 \cdots R_{l-1}}\right] = P_l, \quad l = 1, \ldots, m.$$

It follows that

$$\begin{aligned}\lim_{k\to\infty} \mathscr{H}\left[\{(\mathbf{A}_0^k, r_0), \ldots, (\mathbf{A}_m^k, r_m)\}; f\right] &= P_0 + \cdots + P_m \\ &= \mathscr{H}\left[\{(\mathbf{A}_0, r_0), \ldots, (\mathbf{A}_m, r_m)\}; f\right].\end{aligned}$$

□

Remark 1 Examining the proof of Theorem 3 (resp. Theorem 4), we see that the conclusions still hold when we take $\mathbf{A}_l$ (resp. K_l) on $r_l \setminus \cup_{i=0}^{l-1} R_i$ in the case where $r_l(x, y) = \alpha_l x - y - \gamma_l$.

Next, we study interpolation with antisymmetric polynomials. For bivariate smooth functions f and g at $\mathbf{a} \in r$ with $r(x, y) = \alpha x + \beta y - \gamma$, we recall the Leibniz rule for $\mathscr{D}_r = -\beta\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial y}$,

$$\mathscr{D}_r^k(fg)(\mathbf{a}) = \sum_{i=0}^{k} \binom{k}{i} \mathscr{D}_r^{k-i}(f)(\mathbf{a})\mathscr{D}_r^i(g)(\mathbf{a}).$$

Assume that $\mathbf{a} = (b, c) \in r$ with $c > 0$ and f is sufficiently smooth (with respect to $\mathscr{D}_r$) at $\mathbf{a}$. Set $h(x, y) = \frac{f(x,y)}{y}$. Then, for $k \geq 1$, the Leibniz rule for $\mathscr{D}_r$ implies $\mathscr{D}_r^k(f)(\mathbf{a}) = c\mathscr{D}_r^k(h)(\mathbf{a}) + k\alpha\mathscr{D}_r^{k-1}(h)(\mathbf{a})$. Hence

$$\mathscr{D}_r^k(h)(\mathbf{a}) = \frac{1}{c}\mathscr{D}_r^k(f)(\mathbf{a}) - \frac{k\alpha}{c}\mathscr{D}_r^{k-1}(h)(\mathbf{a}). \tag{23}$$

Lemma 9 *Let f, p be bivariate functions that are ν-times differentiable (with respect to $\mathscr{D}_r$) at $\mathbf{a} \in r$ with $\mathbf{a} = (b, c)$, $c > 0$. Let $q(x, y) = yp(x, y)$ and $h(x, y) = \frac{f(x,y)}{y}$. If*

$$\mathscr{D}_r^k q(\mathbf{a}) = \mathscr{D}_r^k f(\mathbf{a}), \quad k = 0, \ldots, \nu,$$

then

$$\mathscr{D}_r^k p(\mathbf{a}) = \mathscr{D}_r^k h(\mathbf{a}), \quad k = 0, \ldots, \nu.$$

Proof The proof is by induction on k. The assertion is trivial for $k = 0$ since $cp(\mathbf{a}) = q(\mathbf{a}) = f(\mathbf{a}) = ch(\mathbf{a})$. Assuming the assertion to hold for $k - 1 < \nu$, we will prove it for k. Using Leibniz's rule for $q(x, y) = yp(x, y)$ and the induction hypothesis, we obtain

$$\mathscr{D}_r^k f(\mathbf{a}) = \mathscr{D}_r^k q(\mathbf{a}) = c\mathscr{D}_r^k(p)(\mathbf{a}) + k\alpha\mathscr{D}_r^{k-1}(p)(\mathbf{a}) = c\mathscr{D}_r^k(p)(\mathbf{a}) + k\alpha\mathscr{D}_r^{k-1}(h)(\mathbf{a}).$$

From the last relation and (23) it follows that

$$\mathscr{D}_r^k(p)(\mathbf{a}) = \frac{1}{c}\mathscr{D}_r^k(f)(\mathbf{a}) - \frac{k\alpha}{c}\mathscr{D}_r^{k-1}(h)(\mathbf{a}) = \mathscr{D}_r^k(h)(\mathbf{a}),$$

which establishes the desired relation. □

Lemma 9 shows that the antisymmetric Hermite interpolation problem for $\mathscr{A}_{n+1}$ does reduce to the symmetric Hermite interpolation problem for $\mathscr{S}_n$. Theorem 3 immediately implies the following result.

Corollary 1 *Let $\mathbf{X} \subset \mathbb{R} \times (0, \infty)$ be a MBR set for $\mathscr{S}_n$ with lines $r_0, \ldots, r_m$. Then for any sufficiently differentiable function f defined on an open set containing $\mathbf{X}$, the antisymmetric interpolation problem*

$$P \in \mathscr{I}_{r_l}[\mathbf{A}_l; f], \quad l = 0, \ldots, m,$$

has a unique solution $P \in \mathscr{A}_{n+1}$. Moreover,

$$P(x, y) = y\left(P_0(x, y) + \cdots + P_m(x, y)\right),$$

where

$$P_0 = \mathbb{H}_{r_0}[\mathbf{A}_0; h], \quad P_l = R_0 \cdots R_{l-1}\mathbb{H}_{r_l}\left[\mathbf{A}_l; \frac{h - P_0 - \cdots - P_{l-1}}{R_0 \cdots R_{l-1}}\right], \quad l = 1, \ldots, m$$

and $h(x, y) = \frac{f(x,y)}{y}$.

Note that the continuity property for Hermite interpolation with antisymmetric polynomials also holds as in Theorem 4.

Finally, we analyze the general symmetric interpolation problem that is similar to [7, Section 3]. We reuse notations and convention given in [7]. Let us denote by T the transformation $T(x, y) = (x, -y)$. For each bijective affine transformation U of $\mathbb{R}^2$, we set $T^U = U^{-1}TU$. A bivariate function f is said to be symmetric with respect to T^U if

$$f(T^U(\xi, \eta)) = f(\xi, \eta).$$

We write $f \in \mathscr{S}(T^U)$. Note that if $f \in \mathscr{S}(T^U)$ then $f \circ U^{-1}$ is a symmetric function.

Let r be a straight line and $\mathbf{a} \in r$. Let $q \in \mathscr{S}(T^U)$ and g be a suitably defined function. Assume that the following relation holds

$$\mathscr{D}_r^i(q)(\mathbf{a}) = \mathscr{D}_r^i(g)(\mathbf{a}), \quad i = 0, \dots, \nu. \tag{24}$$

Let $\rho : \mathbb{R} \to \mathbb{R}^2$ be a linear parameterization of r such that $\rho(0) = \mathbf{a}$. It means that two coordinate functions of ρ are one-degree polynomials. It is not difficult to see that there exists a non-zero constant $C(r, \rho)$ depending only on r and ρ such that, for any suitably defined function f,

$$\mathscr{D}_r^i(f)(\mathbf{a}) = C^i(r, \rho)(f \circ \rho)^{(i)}(0), \quad i \geq 0.$$

Hence, relation (24) reduces to

$$(q \circ \rho)^{(i)}(0) = (g \circ \rho)^{(i)}(0), \quad i = 0, \dots, \nu. \tag{25}$$

Observe that $\tilde{\rho} := U \circ \rho$ is a linear parameterization of the line $\tilde{r} := r \circ U^{-1}$ and $\mathbf{b} = U(\mathbf{a}) = \tilde{\rho}(0)$ is on $\tilde{r}$. For $0 \leq i \leq \nu$, we have

$$\mathscr{D}_{\tilde{r}}^i(q \circ U^{-1})(\mathbf{b}) = C^i(\tilde{r}, \tilde{\rho})(q \circ U^{-1} \circ \tilde{\rho})^{(i)}(0) = C^i(\tilde{r}, \tilde{\rho})(q \circ \rho)^{(i)}(0).$$

Similarly

$$\mathscr{D}_{\tilde{r}}^i(g \circ U^{-1})(\mathbf{b}) = C^i(\tilde{r}, \tilde{\rho})(g \circ \rho)^{(i)}(0).$$

Combining the last two equations with (25), we obtain

$$\mathscr{D}_{\tilde{r}}^i(q \circ U^{-1})(\mathbf{b}) = \mathscr{D}_{\tilde{r}}^i(g \circ U^{-1})(\mathbf{b}), \quad 0 \leq i \leq \nu. \tag{26}$$

Note that $q \circ U^{-1} \in \mathscr{S}$ and relation (26) is similar to the Hermite interpolation condition for $\mathscr{S}$. Consequently, the above arguments enable us to transform a general symmetric interpolation problem into a symmetric interpolation problem in $\mathscr{S}$.

6 Some examples

In this section, we construct a MBR set of degree 3 and compute a Hermite interpolation polynomial. We also give an example concerning the continuity property of interpolation polynomials.

Example 1 Let $r_0(x, y) = x - 1$, $r_1(x, y) = x - 2$ and $r_2(x, y) = x + y$. Then $k_0 = k_1 = 1$, $k_2 = 2$ and $n = 3$. We also have $n_0 = 3$, $n_1 = 2$ and $n_2 = 1$. By definition, $R_0(x, y) = x - 1$, $R_1(x, y) = x - 2$ and $R_2(x, y) = x^2 - y^2$. Let us take $\mathbf{A}_0 = \{(1, 1), (1, 2)\}$, $\mathbf{A}_1 = \{(2, 1), (2, 1)\}$ and $\mathbf{A}_2 = \{(-1, 1), (-1, 1)\}$. Note that the two points in $\mathbf{A}_1$ are identical and so are in $\mathbf{A}_2$. The MBR set $\cup_{l=0}^2 \mathbf{A}_l$ is illustrated in Fig. 1. We will use Theorem 3 to find

$$P = \mathscr{H}\left[\{(\mathbf{A}_0, r_0), (\mathbf{A}_1, r_1), (\mathbf{A}_2, r_2)\}; f\right], \quad f(x, y) = x + x^2y^2 + xy^4.$$

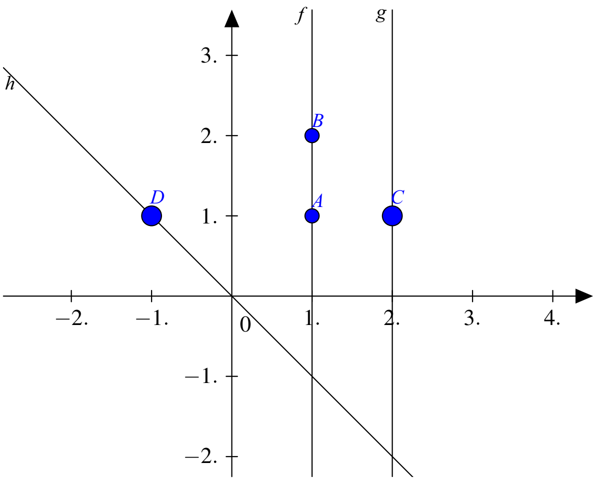

Fig. 1 The MBR set in Example 1

The interpolation conditions are

$$P(1,1)=f(1,1),\ P(1,2)=f(1,2),\ P(2,1)=f(2,1),\ \frac{\partial P(2,1)}{\partial y}=\frac{\partial f(2,1)}{\partial y},$$

$$P(-1,1)=f(-1,1),\ (\frac{\partial}{\partial x}-\frac{\partial}{\partial y})P(-1,1)=(\frac{\partial}{\partial x}-\frac{\partial}{\partial y})f(-1,1).$$

We have $P=P_0+P_1+P_2$, where

$$P_0=\mathbb{H}_{r_0}[\mathbf{A}_0;f],\quad P_1=R_0\mathbb{H}_{r_1}[\mathbf{A}_1;\frac{f-P_0}{R_0}],\quad P_2=R_0R_1\mathbb{H}_{r_2}[\mathbf{A}_2;\frac{f-P_0-P_1}{R_0R_1}].$$

From (5), we obtain

$$\mathbb{H}_{r_0}[\mathbf{A}_0;f](x,y)=\mathbf{H}[\{1,4\};f(1,\sqrt{y})](y^2)=6y^2-3.$$

We set $f_1(x,y)=\frac{f(x,y)-P_0(x,y)}{R_0(x,y)}=\frac{x+x^2y^2+xy^4-6y^2+3}{x-1}$. Then $f_1(2,y)=2y^4-2y^2+5$. Using relation (5) again, we get

$$\mathbb{H}_{r_1}[\mathbf{A}_1;f](x,y)=\mathbf{H}[\{1,1\};f_1(2,\sqrt{y})](y^2)=2y^2+3.$$

We define $f_2=\frac{f-P_0-P_1}{R_0R_1}$. Then $f_2(x,-x)=\frac{x^5+x^4-2x^3-4x^2-2x+6}{(x-1)(x-2)}$. From (8) it follows that

$$\mathbb{H}_{r_2}[\mathbf{A};f_2](x,y)=\mathbf{H}[\{-1,-1\};f_2(-x,x)](x)=x+2.$$

Combining above computations, we finally get

$$\begin{aligned}P(x,y)&=6y^2-3+(x-1)(2y^2+3)+(x-1)(x-2)(x+2)\\&=x^3-x^2+2xy^2-x+4y^2-2.\end{aligned}$$

Example 2 Let $r_0(x,y)=x$, $r_1(x,y)=x-y+1$. We have $k_0=1$, $k_1=2$ and $n=2$. By definition, we get $n_0=2$ and $n_1=1$. Obviously, $R_0(x,y)=x$,

$R_1(x, y) = (x+1)^2 - y^2$. Let us take $\mathbf{A}_0(\epsilon) = \{(0, \sqrt{1+\epsilon}), (0, \sqrt{1-\epsilon})\} \subset r_0$, $\mathbf{A}_1(\delta) = \{(\delta, 1+\delta), (\delta, 1+\delta)\} \subset r_1 \setminus R_0$ with $\epsilon, \delta > 0$ (see Fig. 2). We first compute

$$P = \mathscr{H}\left[\{(\mathbf{A}_0(\epsilon), r_0), (\mathbf{A}_1(\delta), r_1)\}\,;\, f\right] \quad \text{with} \quad f(x, y) = y^4.$$

Note that

$$P(0, \sqrt{1+\epsilon}) = f(0, \sqrt{1+\epsilon}),\, P(0, \sqrt{1-\epsilon}) = f(0, \sqrt{1-\epsilon}),\, ,$$

$$P(\delta, 1+\delta) = f(\delta, 1+\delta),\, (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})P(\delta, 1+\delta) = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})f(\delta, 1+\delta).$$

By Theorem 3, we can write $\mathscr{H}\left[\{(\mathbf{A}_0(\epsilon), r_0), (\mathbf{A}_1(\delta), r_1)\}\,;\, f\right] = P_0 + P_1$, where

$$P_0 = \mathbb{H}_{r_0}[\mathbf{A}_0(\epsilon);\, f], \qquad P_1 = R_0 \mathbb{H}_{r_1}[\mathbf{A}_1(\delta);\, \frac{f - P_0}{R_0}].$$

From (5), we see that

$$\mathbb{H}_{r_0}[\mathbf{A}_0(\epsilon);\, f](x, y) = \mathbf{H}[\{1-\epsilon, 1+\epsilon\};\, f(0, \sqrt{y})](y^2) = 2y^2 - 1 + \epsilon^2.$$

Let us set $f_1(x, y) = \frac{f(x,y) - P_0(x,y)}{R_0(x,y)}$. Then $g(x) := f_1(x, x+1) = x^3 + 4x^2 + 4x - \frac{\epsilon^2}{x}$. From (8), it follows that

$$\begin{aligned}
\mathbb{H}_{r_1}[\mathbf{A}_1(\delta);\, \frac{f - P_0}{R_0}](x, y) &= \mathbf{H}[\{\delta, \delta\};\, g(x)](x) \\
&= (\delta^3 + 4\delta^2 + 4\delta - \frac{\epsilon^2}{\delta}) + (3\delta^2 + 8\delta + 4 + \frac{\epsilon^2}{\delta^2})(x - \delta) \\
&= -(2\delta^3 + 4\delta^2 + \frac{2\epsilon^2}{\delta}) + (3\delta^2 + 8\delta + 4 + \frac{\epsilon^2}{\delta^2})x.
\end{aligned}$$

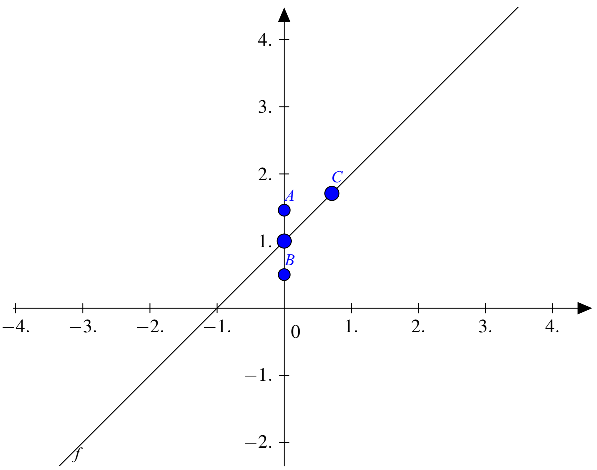

Fig. 2 The MBR set in Example 2

Combining above computations, we finally get

$$\begin{aligned}\mathscr{H}\left[\{(\mathbf{A}_0(\epsilon), r_0), (\mathbf{A}_1(\delta), r_1)\}; f\right](x, y) &= P_0(x, y) + P_1(x, y)\\ &= 2y^2 - 1 + \epsilon^2 - (2\delta^3 + 4\delta^2 + \frac{2\epsilon^2}{\delta})x\\ &+ (3\delta^2 + 8\delta + 4 + \frac{\epsilon^2}{\delta^2})x^2.\end{aligned}$$

We see that $\mathbf{A}_0(\epsilon) \to (1, 0)$ as $\epsilon \to 0$ and $\mathbf{A}_1(\delta) \to (1, 0)$ as $\delta \to 0$. If $\epsilon = \sqrt{\delta}$ then $\frac{\epsilon^2}{\delta^2} \to \infty$ as $\delta \to 0^+$. Hence, there does not exist

$$\lim_{\delta\to 0^+} \mathscr{H}\left[\left\{(\mathbf{A}_0(\sqrt{\delta}), r_0), (\mathbf{A}_1(\delta), r_1)\right\}; f\right].$$

On the other hand, it is easily seen that

$$\lim_{\delta\to 0^+} \mathscr{H}\left[\{(\mathbf{A}_0(t\delta), r_0), (\mathbf{A}_1(\delta), r_1)\}; f\right] = 2y^2 + (t^2 + 4)x^2 - 1, \quad t > 0.$$

It follows when all interpolation points tend to a unique point, the limit of the Hermite interpolation could not exist. In some case, when the limit exists, it depends on the *speed* of the coalescence. Remark that the coalescences of points as in this example were considered in [8].

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