Numer Algor (2017) 76:695–707
DOI 10.1007/s11075-017-0277-1

ORIGINAL PAPER

Stability and convergence of a fully discrete local discontinuous Galerkin method for multi-term time fractional diffusion equations

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Received: 13 October 2016 / Accepted: 26 January 2017 / Published online: 9 February 2017

Abstract In this paper, a fully discrete local discontinuous Galerkin method for a class of multi-term time fractional diffusion equations is proposed and analyzed. Using local discontinuous Galerkin method in spatial direction and classical $L1$ approximation in temporal direction, a fully discrete scheme is established. By choosing the numerical flux carefully, we prove that the method is unconditionally stable and convergent with order $O(h^{k+1} + (\Delta t)^{2-\alpha})$, where k, h, and Δt are the degree of piecewise polynomial, the space, and time step sizes, respectively. Numerical examples are carried out to illustrate the effectiveness of the numerical scheme.

Keywords Multi-term time fractional diffusion equations · Time fractional derivative · Local discontinuous Galerkin method · Stability · Convergence

Mathematics Subject Classification (2010) 65M12 · 65M06 · 35S10

This work is supported by the High-Level Personal Foundation of Henan University of Technology (2013BS041), Plan For Scientific Innovation Talent of Henan University of Technology (2013CXRC12), the National Natural Science Foundation of China (11461072,11601124), Foundation of Henan Educational Committee (15A110015,15A110018), and China Postdoctoral Science Foundation funded project (2015M572115).

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1 Introduction

Fractional calculus, which might be considered as an extension of classical calculus, attracts much attention in recent decades [14]. Fractional order partial differential equations (FPDEs) have been frequently used to solve many scientific problems in various fields, such as quantitative finance, engineering, biology, chemistry, and hydrology [19].

The multi-term time fractional diffusion equation which was developed to improve the modeling accuracy of the single-term model for describing anomalous diffusion has been studied extensively from different aspects due to its extraordinary capability of modeling anomalous diffusion phenomena in highly heterogeneous aquifers and complex viscoelastic materials [1]. Some underlying processes can be more accurately and flexibly modeled by multi-term FPDEs. For example, in order to distinguish explicitly the mobile and immobile status of the solute using fractional dynamics, the authors [17] proposed a two-term fractional-order diffusion model for the total concentration in solute transport. When describing subdiffusive motion in velocity fields, the kinetic equation with two fractional derivatives of different orders appears also quite naturally [13]. For further discussions on the model for wave-type phenomena, the readers could refer to [10].

In this paper, we consider the following multi-term time fractional diffusion equation in the Caputo sense:

$$\begin{aligned} &P_{\alpha,\alpha_1,\cdots,\alpha_l}(D_t)u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \qquad x \in (a,b), t \in (0,T], \\ &u(x,0) = 0, \qquad x \in [a,b], \end{aligned} \tag{1.1}$$

where

$$P_{\alpha,\alpha_1,\cdots,\alpha_l}(D_t)u(x,t) = (D_t^{\alpha} + \sum_{i=1}^{l} d_i D_t^{\alpha_i})u(x,t);$$

here, $0 < \alpha_1 \leq \cdots \leq \alpha_l < \alpha < 1$ are the orders of the fractional derivatives, $d_i > 0, i = 1, 2, \ldots, l$. The Caputo fractional derivative $D_t^{\eta}, 0 < \eta < 1$ is defined by the following [14]:

$$D_t^{\eta}u(x,t) = \frac{1}{\Gamma(1-\eta)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^{\eta}}, \; t > 0, \; 0 < \eta < 1, \tag{1.2}$$

where $\Gamma(\cdot)$ denotes the gamma function. We do not pay attention to boundary condition in this paper; hence, the solution is considered to be either periodic or compactly supported.

There are some previous works on the numerical solutions of problems with multiple fractional derivatives. Liu et al. [11] studied a numerical scheme based on a fractional predictor-corrector method for the multi-term time fractional wave-diffusion equation. Ding and Nieto [4] gave some analytical solutions for multi-term time-space fractional reaction-diffusion equations on an infinite domain. Jiang et al. [7] derived analytical solutions for the multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a finite domain. Diethelm and Luchko [6] presented an algorithm for solving the multi-term linear fractional differential equations. Edwards et al. solved linear multi-term fractional differential equations based on a reduction of the problem to a system of ordinary and fractional differential equations each of order at most unity in [5]. Katsikadelis [9] presented a numerical method which is based on the concept of the analog equation to solve linear multi-term fractional differential equations. Meanwhile, finite element methods [3, 8, 12, 22, 23], spectral methods [2, 24], and various finite difference methods [15, 16] are commonly used. Although some numerical schemes have been proposed for problems involving multi-term fractional derivatives, studies on the efficient and higher-order numerical methods are still in their early stages.

The remainder of this paper is organized as follows. In Section 2, some basic notations and theoretic results are introduced. In Section 3, we construct our fully discrete local discontinuous Galerkin method for the multi-term time fractional diffusion equation, and we prove the stability and error estimate in Section 4. Some numerical results are provided in Section 5, and the conclusion is included in the final section.

2 Notations and auxiliary results

Assume the following mesh covers the computational domain:

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b,$$

the cells are denoted by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \cdots N$, and the cell lengths by $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $1 \le j \le N$, $h = \max\limits_{1\le j\le N} \Delta x_j$.

Denote by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ the values of u at $x_{j+1/2}$, from the right cell I_{j+1} and from the left cell I_j, respectively.

The piecewise polynomial space V_h^k is defined as the space of polynomials of the degree up to k in each cell I_j, i.e.,

$$V_h^k = \{v : v \in P^k(I_j), x \in I_j, j = 1, 2, \cdots N\}.$$

For error estimates, we will be using two projections in one dimension $[a, b]$, denoted by $\mathcal{P} : H^{k+1}(D) \to V_h^k$, i.e., for each j,

$$\int_{I_j} (\mathcal{P}\omega(x) - \omega(x))v(x) = 0, \forall v \in P^k(I_j), \tag{2.1}$$

and special projection $\mathcal{P}^{\pm}: H^{k+1}(D) \rightarrow V_h^k$, i.e., for each j,

$$\int_{I_j} (\mathcal{P}^{+}\omega(x) - \omega(x))v(x) = 0, \forall v \in P^{k-1}(I_j),$$

$$\text{and} \quad \mathcal{P}^{+}\omega(x^{+}_{j-\frac{1}{2}}) = \omega(x_{j-\frac{1}{2}})$$

$$\int_{I_j} (\mathcal{P}^{-}\omega(x) - \omega(x))v(x) = 0, \forall v \in P^{k-1}(I_j),$$

$$\text{and} \quad \mathcal{P}^{-}\omega(x^{-}_{j+\frac{1}{2}}) = \omega(x_{j+\frac{1}{2}}). \tag{2.2}$$

For the above projections $\mathcal{P}$ and $\mathcal{P}^{\pm}$, when $\omega(t) \in H^{k+1}(D)$, we have [18, 20, 21],

$$\|\omega^e\| + h\|\omega^e\|_{\infty} + h^{\frac{1}{2}}\|\omega^e\|_{\tau_h} \leq Ch^{k+1}, \tag{2.3}$$

where $\omega^e = \mathcal{P}\omega - \omega$ or $\omega^e = \mathcal{P}^{\pm}\omega - \omega$. $\|\omega^e\|$ and $\|\omega^e\|_{\infty}$ denote the L^2 norm and L^{∞} norm, respectively. τ_h is the union of the interface point of every elements, and $\|\omega^e\|_{\tau_h}$ is the L^2-norm on τ_h, which is defined by the following:

$$\|\omega^e\|_{\tau_h} = \Big(\sum_{1\leq i\leq N} ((\omega^e)^{+}_{i+\frac{1}{2}})^2 + ((\omega^e)^{-}_{i+\frac{1}{2}})^2\Big)^{\frac{1}{2}}.$$

In the present paper, we use C to denote a positive constant which may have a different value in each occurrence. The scalar inner product on $L^2(D)$ be denoted by $(\cdot, \cdot)_D$, and the associated norm by $\|\cdot\|_D$. If $D = \Omega$, we drop D.

3 The schemes

In this section, we introduce the numerical scheme for the solution of (1.1).

We divide the interval $[0, T]$ uniformly with a time step size $\Delta t = T/M$, $M \in \mathbb{N}$, $t_n = n\Delta t, n = 0, 1, \cdots, M$ be the mesh points.

When $0 < \gamma < 1$, we know

$$\begin{aligned} D_t^{\gamma} u(x, t_n) = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} & \Big(u(x, t_n) + \sum_{k=1}^{n-1} (b^{\gamma}_{n-k} - b^{\gamma}_{n-k-1}) u(x, t_k) \\ & - b^{\gamma}_{n-1} u(x, t_0)\Big) + R^n_{\gamma}, \end{aligned}$$

where R^n_{γ} is the truncation error. $b_i^{\gamma} = (i+1)^{1-\gamma} - i^{1-\gamma}$, and we know

$$1 = b_0^{\gamma} > b_1^{\gamma} > b_2^{\gamma} > \cdots > b_n^{\gamma} > 0, b_n^{\gamma} \rightarrow 0 (n \rightarrow \infty), \sum_{i=1}^{n} (b^{\gamma}_{i-1} - b^{\gamma}_i) + b^{\gamma}_n = 1. \tag{3.1}$$

Then, we can obtain the following:

$$\begin{aligned}P_{\alpha,\alpha_1,\cdots,\alpha_l}(D_t)u(x,t) \;=\; & \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}((1+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(\Delta t)^{\alpha-\alpha_i})u(x,t_n)\\ & +\sum_{k=1}^{n-1}(b_{n-k}^{\alpha}-b_{n-k-1}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k}^{\alpha_i}-b_{n-k-1}^{\alpha_i})\\ & \times(\Delta t)^{\alpha-\alpha_i})u(x,t_k)-(b_{n-1}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\\ & \times u(x,t_0))+R_{\alpha,\alpha_1,\cdots,\alpha_l}^{n} \end{aligned} \tag{3.2}$$

Rewrite (1.1) as a first-order system:

$$p=u_x, \quad P_{\alpha,\alpha_1,\cdots,\alpha_l}(D_t)u(x,t)=p_x+f(x,t). \tag{3.3}$$

Let $u_h^n, p_h^n \in V_h^k$ be the approximation of $u(\cdot,t_n)$, $p(\cdot,t_n)$, respectively, $f^n(x)=f(x,t_n)$. We define a fully discrete local discontinuous Galerkin scheme as follows: find $u_h^n, p_h^n \in V_h^k$, such that for all test functions $v, \xi \in V_h^k$,

$$\begin{aligned} & \beta_0\int_\Omega u_h^n v dx+\beta_1(\int_\Omega p_h^n v_x dx-\sum_{j=1}^{N}((\widehat{p_h^n}v^-)_{j+\frac{1}{2}}-(\widehat{p_h^n}v^+)_{j-\frac{1}{2}}))\\ =&\sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha}-b_{n-k}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i}-b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\int_\Omega u_h^k v dx\\ & +(b_{n-1}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\int_\Omega u_h^0 v dx+\beta_1\int_\Omega f^n v dx,\\ & \int_\Omega p_h^n\xi dx+\int_\Omega u_h^n\xi_x dx-\sum_{j=1}^{N}((\widehat{u_h^n}\xi^-)_{j+\frac{1}{2}}-(\widehat{u_h^n}\xi^+)_{j-\frac{1}{2}})=0, \end{aligned} \tag{3.4}$$

where $\beta_0=1+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(\Delta t)^{\alpha-\alpha_i}$, $\beta_1=(\Delta t)^{\alpha}\Gamma(2-\alpha)$.

The initial conditions u_h^0 is taken as the L^2 projections of $u(,0)$

$$\int_\Omega u_h^0 v dx=\int_\Omega \mathcal{P}u(x,0)v dx=\int_\Omega u_0(x)v dx,$$
$$\forall v\in V_h^k.$$

The "hat" terms in (3.4) in the cell boundary terms from integration by parts are the so-called 'numerical fluxes, which are single-valued functions defined on the edges and should be designed based on different guiding principles for different PDEs to ensure stability. It turns out that we can take the simple choices such that

$$\widehat{u_h^n}=(u_h^n)^-, \; \widehat{p_h^n}=(p_h^n)^+. \tag{3.5}$$

We remark that the choice for the fluxes (3.5) is not unique. In fact the crucial part is taking $\widehat{u_h^n}$ and $\widehat{p_h^n}$ from opposite sides [21].

4 Stability and convergence

In order to simplify the notations without losing of generality, we consider the case $f=0$ in its numerical analysis.

Theorem 4.1 *For periodic or compactly supported boundary conditions, the fully discrete LDG scheme (3.4) is unconditionally stable, and there exists a positive constant C depending on u, T, such that*

$$\|u_h^n\| \leq \|u_h^0\|, \quad n=1,2\cdots,M.$$

Proof Taking the test functions $v=u_h^n, \xi=\beta_1 p_h^n$ in scheme (3.4), and with the fluxes choice (3.5), we obtain the following:

$$\begin{aligned}
&\beta_0\|u_h^n\|^2+\beta_1\|p_h^n\|^2+\beta_1\sum_{j=1}^{N}(\Psi(u_h^n,p_h^n)_{j+\frac{1}{2}}-\Psi(u_h^n,p_h^n)_{j-\frac{1}{2}}+\Theta(u_h^n,p_h^n)_{j-\frac{1}{2}})\\
&\leq\sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha}-b_{n-k}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i}-b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\|u_h^k\|\|u_h^n\|\\
&+(b_{n-1}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\|u_h^0\|\|u_h^n\|, \qquad (4.1)
\end{aligned}$$

here,

$$\begin{aligned}
\Psi(u_h^n,p_h^n)&=(p_h^n)^-(u_h^n)^--\widehat{p_h^n}(u_h^n)^--\widehat{u_h^n}(p_h^n)^-,\\
\Theta(u_h^n,p_h^n)&=(p_h^n)^-(u_h^n)^--(p_h^n)^+(u_h^n)^+-\widehat{p_h^n}(u_h^n)^-+\widehat{p_h^n}(u_h^n)^+-\widehat{u_h^n}(p_h^n)^-.\\
&\quad+\widehat{u_h^n}(p_h^n)^+.
\end{aligned}$$

After some calculation, we can easily obtain $\Theta(u_h^n,p_h^n)=0$.

Then, based on the (4.1), we can get the following:

$$\begin{aligned}
\beta_0\|u_h^n\|\leq&\sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha}-b_{n-k}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i}-b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\|u_h^k\|\\
&+(b_{n-1}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\|u_h^0\|. \qquad (4.2)
\end{aligned}$$

We will prove Theorem 4.1 by mathematical induction. Let $n=1$ in (4.2), we can obtain the following:

$$\beta_0\|u_h^n\|\leq(b_0^{\alpha}+\sum_{i=1}^{l}\tfrac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_0^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\|u_h^0\|.$$

Notice that $b_0^{\alpha}=b_0^{\alpha_i}=1$, we can obtain $\|u_h^1\|\leq\|u_h^0\|$.

Now, suppose the following inequality holds $\|u_h^m\|\leq\|u_h^0\|, m=1,2,3\cdots,n-1$.

We need to prove $\|u_h^n\| \le \|u_h^0\|$.
It follows from (4.2) that

$$\begin{aligned}\beta_0\|u_h^n\| &\le \sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha}-b_{n-k}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i}-b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\|u_h^0\| \\ &\quad +(b_{n-1}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\|u_h^0\| \\ &= \beta_0\|u_h^0\|. \end{aligned} \tag{4.3}$$

Consequently, we have

$$\|u_h^n\| \le \|u_h^0\|.$$

This finishes the proof of the stability result. □

Lemma 4.1 [23] *For each $t \in (0,T]$, If $u_{tt} \in L^2(D)$, then there holds*

$$\|R_{\alpha,\alpha_1,\cdots,\alpha_l}^n\| \le C \max_{0\le t\le T}\|u_{tt}\|(\Delta t)^{2-\alpha}.$$

Lemma 4.2 [23] *If $\psi^n \ge 0, n = 1, 2, \cdots, N$, $\psi^0 = 0$, $d_i > 0, i = 1, 2, \ldots, l$,*

$$\begin{aligned}\beta_0\psi^n &\le \sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha}-b_{n-k}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i}-b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\psi^k \\ &\quad +(b_{n-1}^{\alpha}+\sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\psi^0+\chi, \end{aligned} \tag{4.4}$$

then we have

$$\psi^n \le C(\Delta t)^{\alpha}\chi$$

where C is a positive constant independent of h and Δt.

Theorem 4.2 *Let $u(x,t_n)$ be the exact solution of problem (1.1), which is sufficiently smooth such that $u(t) \in H^{m+1}(D)$ with $0 \le m \le k+1$. Let u_h^n be the numerical solution of the fully discrete LDG scheme (3.4), then there holds the following error estimate:*

$$\|u(x,t_n)-u_h^n\| \le C(h^{k+1}+(\Delta t)^{2-\alpha}), n = 1,\cdots,M, \tag{4.5}$$

where C is a constant depending on u, T, α.

Proof Denote

$$\begin{aligned} e_u^n &= u(x,t_n)-u_h^n = \mathcal{P}^-e_u^n-(\mathcal{P}^-u(x,t_n)-u(x,t_n)), \\ e_p^n &= p(x,t_n)-p_h^n = \mathcal{P}^+e_p^n-(\mathcal{P}^+p(x,t_n)-p(x,t_n)). \end{aligned} \tag{4.6}$$

With the fluxes (3.5), we can obtain the error equation:

$$
\begin{aligned}
\beta_0 \int_\Omega e_u^n v dx \; + \; \beta_1 (\int_\Omega e_p^n v_x dx - \sum_{j=1}^{N} (((e_p^n)^+ v^-)_{j+\frac{1}{2}} - ((e_p^n)^+ v^+)_{j-\frac{1}{2}}))& \\
- \sum_{k=1}^{n-1} (b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l} \frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i (b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})& \\
\times \int_\Omega e_u^k v dx - (b_{n-1}^{\alpha} + \sum_{i=1}^{l} \frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i b_{n-1}^{\alpha_i} (\Delta t)^{\alpha-\alpha_i}) \int_\Omega e_u^0 v dx& \\
+ \int_\Omega e_p^n \xi dx + \int_\Omega e_u^n \xi_x dx - \sum_{j=1}^{N} (((e_u^n)^- \xi^-)_{j+\frac{1}{2}} - ((e_u^n)^- \xi^+)_{j-\frac{1}{2}})& \\
+ \beta_1 \int_\Omega R_n v dx = 0.&
\end{aligned}
\tag{4.7}
$$

Using (4.6), the error equation (4.7) can be written as follows:

$$
\begin{aligned}
&\beta_0 \int_\Omega \mathcal{P}^- e_u^n v dx + \beta_1 (\int_\Omega \mathcal{P}^+ e_p^n v_x dx - \sum_{j=1}^{N} (((\mathcal{P}^+ e_p^n)^+ v^-)_{j+\frac{1}{2}} - ((\mathcal{P}^+ e_p^n)^+ v^+)_{j-\frac{1}{2}})) \\
&+ \int_\Omega \mathcal{P}^+ e_p^n \xi dx + \int_\Omega \mathcal{P}^- e_u^n \xi_x dx - \sum_{j=1}^{N} (((\mathcal{P}^- e_u^n)^- \xi^-)_{j+\frac{1}{2}} - ((\mathcal{P}^- e_u^n)^- \xi^+)_{j-\frac{1}{2}}) \\
= &\sum_{k=1}^{n-1} (b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l} \frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i (b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i}) \int_\Omega \mathcal{P}^- e_u^k v dx \\
&+ (b_{n-1}^{\alpha} + \sum_{i=1}^{l} \frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i b_{n-1}^{\alpha_i} (\Delta t)^{\alpha-\alpha_i}) \int_\Omega \mathcal{P}^- e_u^0 v dx - \beta_1 \int_\Omega R_n v dx \\
&+ \beta_0 \int_\Omega (\mathcal{P}^- u(x,t_n) - u(x,t_n)) v dx + \beta_1 (\int_\Omega (\mathcal{P}^+ p(x,t_n) - p(x,t_n)) v_x dx \\
&- \sum_{j=1}^{N} (((\mathcal{P}^+ p(x,t_n) - p(x,t_n))^+ v^-)_{j+\frac{1}{2}} - ((\mathcal{P}^+ p(x,t_n) - p(x,t_n))^+ v^+)_{j-\frac{1}{2}})) \\
&+ \int_\Omega (\mathcal{P}^+ p(x,t_n) - p(x,t_n)) \xi dx + \int_\Omega (\mathcal{P}^- u(x,t_n) - u(x,t_n)) \xi_x dx \\
&- \sum_{j=1}^{N} (((\mathcal{P}^- u(x,t_n) - u(x,t_n))^- \xi^-)_{j+\frac{1}{2}} - ((\mathcal{P}^- u(x,t_n) - u(x,t_n))^- \xi^+)_{j-\frac{1}{2}}) \\
&- \sum_{k=1}^{n-1} (b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l} \frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i (b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i}) \\
&\times \int_\Omega (\mathcal{P}^- u(x,t_k) - u(x,t_k)) v dx \\
&- (b_{n-1}^{\alpha} + \sum_{i=1}^{l} \frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i b_{n-1}^{\alpha_i} (\Delta t)^{\alpha-\alpha_i}) \int_\Omega (\mathcal{P}^- u(x,t_0) - u(x,t_0)) v dx.
\end{aligned}
\tag{4.8}
$$

Taking the test functions $v = \mathcal{P}^- e_u^n, \xi = \beta_1 \mathcal{P}^+ e_p^n$ in (4.8), using the properties (2.1)–(2.2), then the following equality holds:

$$\begin{aligned}
&\beta_0\|\mathcal{P}^- e_u^n\|^2 dx + \beta_1\|\mathcal{P}^+ e_p^n\|^2 dx\\
&= \sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i (b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i}) \int_\Omega \mathcal{P}^- e_u^k \mathcal{P}^- e_u^n dx\\
&\quad + (b_{n-1}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i}) \int_\Omega \mathcal{P}^- e_u^0 \mathcal{P}^- e_u^n dx - \beta_1 \int_\Omega R_n \mathcal{P}^- e_u^n dx\\
&\quad + \beta_1 \int_\Omega (\mathcal{P}^+ p(x,t_n) - p(x,t_n))\mathcal{P}^+ e_p^n dx + FHS. \qquad (4.9)
\end{aligned}$$

Denote

$$\begin{aligned}
FHS = &\beta_0 \int_\Omega (\mathcal{P}^- u(x,t_n) - u(x,t_n))\mathcal{P}^- e_u^n dx\\
&- \sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i (b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\\
&\times \int_\Omega (\mathcal{P}^- u(x,t_k) - u(x,t_k))\mathcal{P}^- e_u^n dx\\
&- (b_{n-1}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i}) \int_\Omega (\mathcal{P}^- u(x,t_0) - u(x,t_0))\mathcal{P}^- e_u^n dx.
\end{aligned}$$

After some manual calculation, we have

$$\begin{aligned}
FHS = -\Delta t \sum_{k=0}^{n-1}(b_k^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i b_k^{\alpha_i}(\Delta t)^{\alpha-\alpha_i} \times \int_\Omega \partial_t(\mathcal{P}^- u(x,t_{n-k})\\
- u(x,t_{n-k}))\mathcal{P}^- e_u^n dx),
\end{aligned}$$

here,

$$\partial_t \varphi(x,t_k) = \frac{\varphi(x,t_k) - \varphi(x,t_{k-1})}{\Delta t}.$$

We know that

$$\|\partial_t(\mathcal{P}^- u(x,t_{n-k}) - u(x,t_{n-k}))\| \le C h^{k+1}.$$

Based on the fact that

$$\sum_{k=0}^{n-1}(b_k^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i b_k^{\alpha_i}(\Delta t)^{\alpha-\alpha_i}) = n^{1-\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)} d_i n^{1-\alpha_i},$$

then we can obtain the following:

$$|FHS| \leq Ch^{k+1}(n^{1-\alpha}\Delta t + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i n^{1-\alpha_i}\Delta t)\|\mathcal{P}^- e_u^n\|$$
$$\leq Ch^{k+1}(\Delta t)^{\alpha}(T^{1-\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i T^{1-\alpha_i})\|\mathcal{P}^- e_u^n\|. \tag{4.10}$$

Noting the fact that

$$ab \leq \frac{1}{2\beta_0}a^2 + \frac{\beta_0}{2}b^2,$$

and applying the Cauchy-Schwarz inequality and Lemma 1, we obtain the following:

$$\begin{aligned}\beta_0\|\mathcal{P}^- e_u^n\| \leq &\sum_{k=1}^{n-1}(b_{n-k-1}^{\alpha} - b_{n-k}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i(b_{n-k-1}^{\alpha_i} - b_{n-k}^{\alpha_i})(\Delta t)^{\alpha-\alpha_i})\\ &\times\|\mathcal{P}^- e_u^k\| + (b_{n-1}^{\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i b_{n-1}^{\alpha_i}(\Delta t)^{\alpha-\alpha_i})\|\mathcal{P}^- e_u^0\|\\ &+C(\Delta t)^2 + Ch^{k+1}(\Delta t)^{\alpha} + Ch^{k+1}(\Delta t)^{\alpha}\\ &\times(T^{1-\alpha} + \sum_{i=1}^{l}\frac{\Gamma(2-\alpha)}{\Gamma(2-\alpha_i)}d_i T^{1-\alpha_i}).\end{aligned} \tag{4.11}$$

Table 1 Spatial accuracy test using piecewise P^k polynomials when $\alpha = 0.8, \alpha_1 = 0.2, \Delta t = 0.000001, T = 1$

	N	L^2-error	order	L^∞-error	order
	5	0.265460542121009	–	0.624399289671560	–
P^0	10	0.129685346887392	1.03	0.314518957308930	0.99
	15	8.610572614783776E-002	1.01	0.209947911764219	1.00
	20	6.450257933806432E-002	1.00	0.157530522492345	0.99
	5	6.748428947093195E-002	–	0.253808413729305	–
P^1	10	1.715293753861337E-002	1.98	6.816273541844342E-002	1.94
	15	7.936473886583322E-003	1.90	3.201472916365267E-002	1.96
	20	4.569824563015863E-003	1.92	1.862475210320156E-002	1.88
	5	7.114330932342767E-003	–	3.385374126729135E-002	–
P^2	10	9.142927498685679E-004	2.96	4.442104626417648E-003	2.93
	15	2.787005892244741E-004	2.93	1.381802446713825E-003	2.88
	20	1.224089295751527E-004	2.86	6.246187894427804E-004	2.76

By Lemma 2, we can obtain the following result immediately:

$$\|\mathcal{P}^{-}e_u^n\| \leq C(h^{k+1} + (\Delta t)^{2-\alpha}).$$

Thus, Theorem 4.2 follows by the triangle inequality and the interpolation property (2.3). □

5 Numerical examples

In this section, some numerical experiments are presented to illustrate the efficiency and numerical accuracy of the proposed fully discrete local discontinuous Galerkin method.

Example. Consider the original multi-term time fractional diffusion (1.1) in $\Omega = (0, 1)$, and take

$$f(x,t) = (\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{2-\alpha_1}}{\Gamma(3-\alpha_1)} + 4\pi^2 t^2)\sin(2\pi x),$$

then the exact solution is

$$u(x,t) = t^2 \sin(2\pi x).$$

First in Tables 1 and 2, we show the numerical convergence orders of the scheme (3.4) in space with the fixed and sufficiently small step sizes $\Delta t = 0.000001$ and the varying $h = 1/5, 1/10, 1/15, 1/20$, respectively, the numerical errors and convergence orders in L^2-norm and L^∞-norm are recorded. From these tables, we can see that the errors attain $(k+1)$-th order of accuracy for piecewise P^k polynomials.

Table 2 Spatial accuracy test using piecewise P^k polynomials when $\alpha = 0.9, \alpha_1 = 0.3, \Delta t = 0.000001, T = 1$

	N	L^2-error	order	L^∞-error	order
	5	0.265436414563494	–	0.624339889320916	–
P^0	10	0.129733877632282	1.03	0.314632196570233	0.99
	15	8.615022293262845E-002	1.01	0.210039976913946	1.00
	20	6.454380669560460E-002	1.00	0.157603864836982	1.00
	5	6.750343889056618E-002	–	0.254358800577037	–
P^1	10	1.721002495151978E-002	1.97	6.866955985919898E-002	1.89
	15	8.053653114254383E-003	1.87	3.250579429668865E-002	1.84
	20	4.635466124246632E-003	1.92	1.901234343396593E-002	1.86
	5	7.243347269448695E-003	–	3.416090143330275E-002	–
P^2	10	9.244431741019006E-004	2.97	4.390161886827248E-003	2.96
	15	2.840891557845535E-004	2.91	1.382357734460025E-003	2.85
	20	1.237034202200009E-004	2.89	6.141065203388816E-004	2.82

6 Conclusion

The major contribution of this work lies in the construction of an unconditionally stable fully discrete local discontinuous Galerkin method for solving a class of multi-term time fractional diffusion equations. The convergence results are also derived without restrictions on the step sizes in space and time. Numerical results are computed to show the convergence orders and the excellent numerical performance of the proposed method.

Acknowledgments The author wish to thank the anonymous referees for their valuable comments and suggestions.

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