Numer Algor (2017) 76:675–694
DOI 10.1007/s11075-017-0276-2

ORIGINAL PAPER

Recursive polynomial interpolation algorithm (RPIA)

Abderrahim Messaoudi[1] · **Hassane Sadok**[2]

Received: 23 November 2016 / Accepted: 26 January 2017 / Published online: 17 February 2017

Abstract Let $x_0, x_1, \cdots, x_n$ be a set of n+1 distinct real numbers (i.e., $x_i \neq x_j$ for $i \neq j$) and $y_0, y_1, \cdots, y_n$ be given real numbers; we know that there exists a unique polynomial $p_n(x)$ of degree n such that $p_n(x_i) = y_i$ for $i = 0, 1, \cdots, n$; p_n is the interpolation polynomial for the set $\{(x_i, y_i),\ i = 0, 1, \cdots, n\}$. The polynomial $p_n(x)$ can be computed by using the Lagrange method or the Newton method. This paper presents a new method for computing interpolation polynomials. We will reformulate the interpolation polynomial problem and give a new algorithm for giving the solution of this problem, the recursive polynomial interpolation algorithm (RPIA). Some properties of this algorithm will be studied and some examples will also be given.

Keywords Polynomial interpolation · Lagrange method · Newton method · Vandermonde matrix · Schur complement · Sylvester identity · Recursive interpolation algorithm

1 Introduction

In [2], Brezinski proposed two algorithms, called the recursive interpolation algorithm (RIA) and the recursive projection algorithm (RPA); some of their properties

✉ Abderrahim Messaoudi
abderrahim.messaoudi@gmail.com

Hassane Sadok
sadok@lmpa.univ-littoral.fr

1 Ecole Normale Supérieure, Mohammed V University in Rabat, Av. Mohammed Belhassan El Ouazzan, B.P. 5118, Takaddoum, Rabat, Morocco

2 L.M.P.A, Université du Littoral Côte d'Opale, 50 rue F. Buisson BP 699, 62228 Calais Cedex, France

are studied in [8]. These algorithms have been applied for implementing some vector sequence transformations, which can be expressed as a ratio of two determinants [2, 3]. They are connected to other methods used in numerical analysis [4]. In [9, 10], Messaoudi presents a unified approach to the majority of the existing algorithms for solving systems of linear equations. They are embedded in a general class of algorithms, the RIA where they correspond to particular choices of two parameters. The RIA contains essentially all possible algorithms with the following property: they can solve, in exact arithmetic, a linear system starting from an arbitrary point and in a number of iterations no greater than the number of equations. The majority of the direct and iterative methods proposed in the literature have this property and fall therefore into the RIA.

This algorithm (RIA) can be applied for the polynomial interpolation problem. Let $x_0, x_1, \cdots, x_n$ be $n+1$ distinct real numbers (i.e., $x_i \neq x_j$, for $i = 0, 1, \cdots, n$) and $y_0, y_1, \cdots, y_n$ be given real numbers, we know that there exists a unique polynomial $p_n(x)$ of degree n, such that $p_n(x_i) = y_i$, for $i = 0, 1, \cdots, n$. The polynomial $p_n(x)$ can be computed by using the Lagrange formula [1, 7, 13], it can also be computed by using the Newton formula [1, 7, 13].

In this paper, we will reformulate the polynomial interpolation problem and give a new algorithm, the recursive polynomial interpolation algorithm (RPIA), for giving $p_n(x)$. Some of its properties will be studied and some numerical examples will also be given. The paper is organized as follows. In Section 2, we recall the Schur complements [12], some of their properties [4, 5] and the Sylvester identity [11]. In Section 3, we recall the Lagrange interpolation problem, we also give a new formulation of the polynomial interpolation problem. We show how to construct the RPIA and prove some of its properties. Section 4 is concerned with some examples.

2 Schur complement and Sylvester's identity

First, let us recall the definition of the Schur complement [12] and give some of its properties [4, 5].

Definition 2.1 Let M be a matrix partitioned in four blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{2.1}$$

where the submatrix D is assumed to be nonsingular. The Schur complement of D in M, denoted by (M/D), is defined by

$$(M/D) = A - BD^{-1}C. \tag{2.2}$$

Let us now give some properties of the Schur complement. It is easy to show the following properties.

Proposition 2.1 *Let us assume that the matrix D is nonsingular and M is a square matrix, then we have*

$$|(M/D)| = \frac{|M|}{|D|}, \tag{2.3}$$

where $|Z|$ denote the determinant of the square matrix Z.

Proposition 2.2 *Let us assume that the matrix D is nonsingular, then we have*

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}/D\right) = \left(\begin{bmatrix} D & C \\ B & A \end{bmatrix}/D\right) = \left(\begin{bmatrix} B & A \\ D & C \end{bmatrix}/D\right) = \left(\begin{bmatrix} C & D \\ A & B \end{bmatrix}/D\right). \tag{2.4}$$

Proposition 2.3 *Assuming that the matrix D is nonsingular and $E*$ is a given operator such that $E * A$ is well defined, then*

$$\left(\begin{bmatrix} E * A & E * B \\ C & D \end{bmatrix}/D\right) = E * \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}/D\right). \tag{2.5}$$

Now we will give the Sylvester identity [9], *which is a particular case of the quotient property* [11].

Proposition 2.4 *(The Sylvester identity) Let M be the matrix defined by* (2.1) *and K be the matrix partitioned as follows:*

$$K = \begin{bmatrix} E & F & G \\ H & A & B \\ L & C & D \end{bmatrix}.$$

If the matrices A and M are nonsingular, then we have

$$\begin{aligned}(K/M) &= ((K/A)/(M/A)) \\ &= \left(\begin{bmatrix} E & F \\ H & A \end{bmatrix}/A\right) - \left(\begin{bmatrix} F & G \\ A & B \end{bmatrix}/A\right)(M/A)^{-1}\left(\begin{bmatrix} H & A \\ L & C \end{bmatrix}/A\right).\end{aligned} \tag{2.6}$$

We will use the Schur complement, its properties and the Sylvester identity for obtaining the RPIA.

3 Recursive polynomial interpolation algorithm

First we will recall the polynomial interpolation problem and give the Lagrange formula and Newton formula for building the interpolation polynomials. After we will give another formulation of the polynomial interpolation problem, we will show that the interpolation polynomials can be expressed as Schur complements and we will use the properties of the Schur complements and the Sylvester identity for giving the RPIA. Some properties of the RPIA will also be studied.

3.1 Polynomial interpolation

Given that n is a nonnegative integer, let $\mathcal{P}_n$ denote the set of all real-valued polynomials of degree $\leq n$, defined over the set $\mathbb{R}$ of real numbers. For $n \geq 1$, let $1, x, x^2, \cdots, x^n \in \mathcal{P}_n$, and suppose that x_i, $i = 0, 1, \cdots, n$, are distinct real numbers (i.e., $x_i \neq x_j$, for $i \neq j$) and y_i, $i = 0, 1, \cdots, n$, are given real numbers. The Lagrange interpolation problem, [1, 7, 13], is defined as follows:

Find $p_n(x) \in \mathcal{P}_n$ such that

$$p_n(x_i) = y_i, \quad i = 0, 1, \cdots, n. \tag{3.1}$$

The Lagrange interpolation polynomial is given by

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x), \tag{3.2}$$

with $L_i(x)$, for $i = 0, 1, \cdots, n$, the polynomial defined by

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}. \tag{3.3}$$

The set $\{(x_i, y_i) : i = 0, 1, \cdots, n\}$ is called the set of nodes. The polynomial $p_n(x)$ is called the Lagrange interpolation polynomial for the set $\{(x_i, y_i) : i = 0, 1, \cdots, n\}$. The Lagrange interpolation polynomials are usually better, but they are not convenient if a node is added or dropped from the set $\{(x_i, y_i) : i = 0, 1, \cdots, n\}$. For example, if (x_{n+1}, y_{n+1}) were added to the previous set, and we wished to compute the Lagrange polynomial of degree $n + 1$ that interpolated the set $\{(x_i, y_i) : i = 0, 1, \cdots, n+1\}$, then the polynomials, defined by (3.3), would all have to be recomputed. There is another representation of the interpolating polynomial that is very useful in this context, this is the Newton formula [1, 7, 13], which we now describe.

The Newton formula, for computing the interpolation polynomial $p_n(x)$ is given by

$$p_n(x) = y_0 + \sum_{i=1}^{n} [x_0, x_1, \cdots, x_i] N_i(x),$$

where $N_i(x)$ are the Newton polynomials defined by

$$\begin{cases} N_0(x) = 1, \quad \text{and for} \quad i = 1, \cdots, n, \\ N_i(x) = \displaystyle\prod_{j=0}^{i-1} (x - x_j), \end{cases} \tag{3.4}$$

and $[x_0, x_1, \cdots, x_i]$ are the divided differences of $x_0, x_1, \cdots, x_i$, which they are defined by the following formula

$$\begin{cases} [x_i] = y_i, \quad i = 0, 1, \cdots, n \\ [x_0, x_1, \cdots, x_i] = \dfrac{[x_1, x_2, \cdots, x_i] - [x_0, x_1, \cdots, x_{i-1}]}{x_i - x_0}, \quad i = 1, 2, \cdots, n. \end{cases}$$

The polynomial interpolation problem can also be solved by using the Vandermonde system. Let

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i, \tag{3.5}$$

then the relation (3.1) can be written in the matrix-vector form as follows

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}. \tag{3.6}$$

The coefficient matrix of (3.6), which we denote by V_n, is called Vandermonde matrix and is nonsingular if and only if the points x_i, for $i = 0, 1, \cdots, n$, are distinct.

Remark 3.1 The real numbers y_i, $i = 0, 1, \cdots, n$, can be given as the values of a real-valued function f, defined on a closed real interval $[a, b]$, at the distinct interpolation points $x_i \in [a, b]$, $i = 0, 1, \cdots, n$, then $p_n(x)$, defined by (3.1) is the interpolation polynomial of degree n for the function f.

3.2 Formulation of the RPIA

Now we give another formulation of the polynomial interpolation problem. As for the polynomial interpolation problem, we assume that $x_0, x_1, \cdots, x_n$ are given distinct real numbers and $y_0, y_1, \cdots, y_n$ are given real numbers. Then the polynomial interpolation problem can be defined as follows:

Let $q_i(x)$, for $i = 0, 1, \cdots, n$, be polynomials of $\mathcal{P}_n$ of degree i, we assume that $q_0(x) = 1$. Let c_i, for $i = 0, 1, \cdots, n$, be the functional defined by $c_i(g(x)) = g(x_i)$ for any given function g, which is usually called the Dirac functional and represented by δ_{x_i}. Find the polynomial $p_n(x) \in \mathcal{P}_n$, of degree n, such that

$$p_n(x) = \sum_{i=0}^{n} \alpha_i q_i(x), \tag{3.7}$$

and for $j = 0, 1, \cdots, n$

$$c_j(p_n(x)) = p_n(x_j) = y_j. \tag{3.8}$$

We will show how to solve this problem. The relation (3.8) can be written explicitly as follows

$$\begin{bmatrix} q_0(x_0) & q_1(x_0) & \cdots & q_n(x_0) \\ q_0(x_1) & q_1(x_1) & \cdots & q_n(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ q_0(x_n) & q_1(x_n) & \cdots & q_n(x_n) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}. \tag{3.9}$$

Denoting by $D_n = [q_j(x_i)]_{0\le i,j\le n}$ the matrix of linear system (3.9) and assuming that this matrix is nonsingular, then $p_n(x)$ exists and is unique. We get from (3.7)

$$p_n(x) = \left[\, q_0(x)\;\; q_1(x)\;\; \cdots\;\; q_n(x) \,\right] D_n^{-1} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}. \tag{3.10}$$

We see that $p_n(x)$ can be expressed as a Schur complement (2.2)

$$p_n(x) = -\left(\left[\begin{array}{c|cccc} 0 & q_0(x) & q_1(x) & \cdots & q_n(x) \\ y_0 & q_0(x_0) & q_1(x_0) & \cdots & q_n(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_n & q_0(x_n) & q_1(x_n) & \cdots & q_n(x_n) \end{array} \right] / D_n \right). \tag{3.11}$$

For computing $p_n(x)$ recursively we need the following.

Definition 3.1 D_n is said to be strongly nonsingular matrix if $|D_m| \neq 0$, for $m = 0, \cdots, n$.

Now we will show that D_n is a strongly nonsingular matrix.

Lemma 3.1 *For $m = 1, \cdots, n$, we have*

$$|D_m| = \beta_m \prod_{0\le i<j\le m} (x_j - x_i), \tag{3.12}$$

where $\beta_m = \prod_{i=1}^{m} a_i$ and a_i is the coefficient of x^i in $q_i(x)$. D_n is a strongly nonsingular matrix.

Proof We will show this result by induction. For $m = 1$, we have $q_0(x) = 1$ and $q_1(x)$ is a polynomial of degree 1, then

$$|D_1| = \begin{vmatrix} 1 & q_1(x_0) \\ 1 & q_1(x_1) \end{vmatrix} = q_1(x_1) - q_1(x_0) = a_1(x_1 - x_0) = \beta_1(x_1 - x_0),$$

then the property is true for $m = 1$. Assume that it is true for $1 < m < n$, i.e.

$$|D_m| = \beta_m \prod_{0\le i<j\le m} (x_j - x_i), \quad \beta_m = \prod_{i=1}^{m} a_i.$$

For $m+1$ we consider the following polynomial $p(x_0, x_1, \cdots, x_m, x)$ defined by

$$p(x_0, x_1, \cdots, x_m, x) = \begin{vmatrix} 1 & q_1(x_0) & \cdots & q_m(x_0) & q_{m+1}(x_0) \\ 1 & q_1(x_1) & \cdots & q_m(x_1) & q_{m+1}(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & q_1(x_m) & \cdots & q_m(x_m) & q_{m+1}(x_m) \\ 1 & q_1(x) & \cdots & q_m(x) & q_{m+1}(x) \end{vmatrix},$$

we see that $p(x_0, x_1, \cdots, x_m, x)$ is a polynomial of degree $m+1$ and $x_0, x_1, \cdots, x_m$ are the zeros of this polynomial. Then $p(x_0, x_1, \cdots, x_m, x)$ can be written as follows

$$p(x_0, x_1, \cdots, x_m, x) = \mu \prod_{i=0}^{m} (x - x_i),$$

where μ is the coefficient of x^{m+1} in $p(x_0, x_1, \cdots, x_m, x)$. μ is given by

$$\mu = a_{m+1} \begin{vmatrix} 1 & q_1(x_0) & \cdots & q_m(x_0) \\ 1 & q_1(x_1) & \cdots & q_m(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & q_1(x_m) & \cdots & q_m(x_m) \end{vmatrix} = a_{m+1}|D_m| = a_{m+1}\beta_m \prod_{0 \le i < j \le m} (x_j - x_i).$$

So if we choose $x = x_{m+1}$ in $p(x_0, x_1, \cdots, x_m, x)$, we get

$$p(x_0, x_1, \cdots, x_m, x_{m+1}) = |D_{m+1}| = \beta_{m+1} \prod_{0 \le i < j \le m+1} (x_j - x_i).$$

We see that D_m is a nonsingular matrix if and only if $x_0, x_1, \cdots, x_m$ are distinct. $\square$

We set for $m = 0, \cdots, n$

$$p_m(x) = -\left(\left[\begin{array}{c|cccc} 0 & q_0(x) & q_1(x) & \cdots & q_m(x) \\ y_0 & q_0(x_0) & q_1(x_0) & \cdots & q_m(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_m & q_0(x_m) & q_1(x_m) & \cdots & q_m(x_m) \end{array} \right] / D_m \right), \tag{3.13}$$

and for $m = 0, \cdots, n-1$ and for $i > m$, we define the following auxiliary polynomials by

$$g_{m,i}(x) = \left(\left[\begin{array}{c|cccc} q_i(x) & q_0(x) & q_1(x) & \cdots & q_m(x) \\ q_i(x_0) & q_0(x_0) & q_1(x_0) & \cdots & q_m(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_i(x_m) & q_0(x_m) & q_1(x_m) & \cdots & q_m(x_m) \end{array} \right] / D_m \right), \tag{3.14}$$

with $g_{-1,i}(x) = q_i(x)$.

For computing recursively $p_m(x)$ we need the following results.

Lemma 3.2 *For $m = 1, \cdots, n$, we have*

$$(D_m/D_{m-1}) = c_m(g_{m-1,m}(x)) = g_{m-1,m}(x_m) = \frac{|D_m|}{|D_{m-1}|}, \tag{3.15}$$

with $D_0 = c_0(q_0(x)) = 1$.

Proof As the Schur complement (D_m/D_{m-1}) is a scalar and using the relation (2.3) we get

$$(D_m/D_{m-1}) = |(D_m/D_{m-1})| = \frac{|D_m|}{|D_{m-1}|}.$$

Using relations (2.4), (2.5), and (3.14), we have

$$\begin{aligned}
(D_m/D_{m-1}) &= \left(\left[\begin{array}{ccc|c} q_0(x_0) & \cdots & q_{m-1}(x_0) & q_m(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) & q_m(x_{m-1}) \\ q_0(x_m) & \cdots & q_{m-1}(x_m) & q_m(x_m) \end{array} \right] / D_{m-1} \right) \\
&= \left(\left[\begin{array}{c|ccc} q_m(x_m) & q_0(x_m) & \cdots & q_{m-1}(x_m) \\ q_m(x_0) & q_0(x_0) & \cdots & q_{m-1}(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ q_m(x_{m-1}) & q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) \end{array} \right] / D_{m-1} \right) \\
&= c_m \left(\left(\left[\begin{array}{c|ccc} q_m(x) & q_0(x) & \cdots & q_{m-1}(x) \\ q_m(x_0) & q_0(x_0) & \cdots & q_{m-1}(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ q_m(x_{m-1}) & q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) \end{array} \right] / D_{m-1} \right) \right) \\
&= c_m(g_{m-1,m}(x)) = g_{m-1,m}(x_m).
\end{aligned}$$

□

Proposition 3.1 *For $m = 0, \cdots, n$ we have*

$$p_m(x) = p_{m-1}(x) + \frac{y_m - p_{m-1}(x_m)}{g_{m-1,m}(x_m)} g_{m-1,m}(x), \tag{3.16}$$

with $p_{-1}(x) = 0$ and $g_{m-1,m}(x)$ is the polynomial defined by (3.14).

Proof Applying the Sylvester identity to $p_m(x)$, we obtain

$$p_m(x) = -\left(\begin{bmatrix} 0 & q_0(x) & \cdots & q_{m-1}(x) & q_m(x) \\ y_0 & q_0(x_0) & \cdots & q_{m-1}(x_0) & q_m(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{m-1} & q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) & q_m(x_{m-1}) \\ y_m & q_0(x_m) & \cdots & q_{m-1}(x_m) & q_m(x_m) \end{bmatrix} / D_m\right)$$

$$= -\left(\begin{bmatrix} 0 & q_0(x) & \cdots & q_{m-1}(x) \\ y_0 & q_0(x_0) & \cdots & q_{m-1}(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_{m-1} & q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) \end{bmatrix} / D_{m-1}\right)$$

$$+\left(\begin{bmatrix} q_0(x) & \cdots & q_{m-1}(x) & q_m(x) \\ q_0(x_0) & \cdots & q_{m-1}(x_0) & q_m(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) & q_m(x_{m-1}) \end{bmatrix} / D_{m-1}\right) \times (D_m/D_{m-1})^{-1}$$

$$\times \left(\begin{bmatrix} y_0 & q_0(x_0) & \cdots & q_{m-1}(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_{m-1} & q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) \\ y_m & q_0(x_m) & \cdots & q_{m-1}(x_m) \end{bmatrix} / D_{m-1}\right).$$

Then using the relations (2.4), (2.5), (3.13), (3.14) and the relation (3.15) given by the lemma 3.2 we get

$$p_m(x) = p_{m-1}(x) + \frac{y_m - p_{m-1}(x_m)}{g_{m-1,m}(x_m)} g_{m-1,m}(x).$$

□

Proposition 3.2 *For $m = 0, \cdots, n-1$, and $i > m$, we have*

$$g_{m,i}(x) = g_{m-1,i}(x) - \frac{g_{m-1,i}(x_m)}{g_{m-1,m}(x_m)} g_{m-1,m}(x), \tag{3.17}$$

with $g_{-1,i}(x) = q_i(x)$.

Proof Applying the Sylvester identity to $g_{m,i}(x)$, we obtain

$$g_{m,i}(x) = \left(\left[\begin{array}{c|ccc|c} q_i(x) & q_0(x) & \cdots & q_{m-1}(x) & q_m(x) \\ q_i(x_0) & q_0(x_0) & \cdots & q_{m-1}(x_0) & q_m(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_i(x_{m-1}) & q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) & q_m(x_{m-1}) \\ q_i(x_m) & q_0(x_m) & \cdots & q_{m-1}(x_m) & q_m(x_m) \end{array} \right] / D_m \right)$$

$$= \left(\left[\begin{array}{c|ccc} q_i(x) & q_0(x) & \cdots & q_{m-1}(x) \\ q_i(x_0) & q_0(x_0) & \cdots & q_{m-1}(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ q_i(x_{m-1}) & q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) \end{array} \right] / D_{m-1} \right)$$

$$- \left(\left[\begin{array}{ccc|c} q_0(x) & \cdots & q_{m-1}(x) & q_m(x) \\ q_0(x_0) & \cdots & q_{m-1}(x_0) & q_m(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) & q_m(x_{m-1}) \end{array} \right] / D_{m-1} \right) \times (D_m / D_{m-1})^{-1}$$

$$\times \left(\left[\begin{array}{c|ccc} q_i(x_0) & q_0(x_0) & \cdots & q_{m-1}(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ q_i(x_{m-1}) & q_0(x_{m-1}) & \cdots & q_{m-1}(x_{m-1}) \\ q_i(x_m) & q_0(x_m) & \cdots & q_{m-1}(x_m) \end{array} \right] / D_{m-1} \right),$$

Then, using the relations (2.4), (2.5) and the relation (3.15) given by the lemma 3.2 we obtain

$$g_{m,i}(x) = g_{m-1,i}(x) - \frac{g_{m-1,i}(x_m)}{g_{m-1,m}(x_m)} g_{m-1,m}(x).$$

□

So using the relation (3.17) we can compute $g_{m-1,m}(x)$ recursively as follows.

Algorithm 1

$g_{-1,m}(x) = q_m(x)$;
for $i = 0, ..., m-1$

$$g_{i,m}(x) = g_{i-1,m}(x) - \frac{g_{i-1,m}(x_i)}{g_{i-1,i}(x_i)} g_{i-1,i}(x);$$

end i.

Then applying the relation (3.16) and the algorithm 1, we get the RPIA.

Algorithm 2 the RPIA

$p_{-1}(x) = 0;$
for $m = 0, \ldots, n$
  $g_{-1,m}(x) = q_m(x);$
  if $m > 1$
    for $i = 0, \ldots, m-1$
      $$g_{i,m}(x) = g_{i-1,m}(x) - \frac{g_{i-1,m}(x_i)}{g_{i-1,i}(x_i)} g_{i-1,i}(x);$$
    end i,
  end if,
  $$p_m(x) = p_{m-1}(x) + \frac{y_m - p_{m-1}(x_m)}{g_{m-1,m}(x_m)} g_{m-1,m}(x);$$
end m.

Remark 3.2

1. If we choose $q_i(x) = x^i$, for $i = 0, 1, \cdots, n$, D_n will be the Vandermonde matrix V_n, which is a strongly nonsingular matrix, so using the relations (3.11) and (2.3), $p_n(x)$ can be expressed as a ratio of two determinants

$$p_n(x) = -\frac{\begin{vmatrix} 0 & 1 \; x \; \cdots \; x^n \\ y_0 & \\ \vdots & V_n \\ y_n & \end{vmatrix}}{|V_n|},$$

this ratio is a element of $\mathcal{P}_n$, which is obtained by expanding the numerator with respect to its first row by using the classical rule for expanding a determinant.

2. If $q_i(x) = N_i(x)$, for $i = 0, 1, \cdots, n$, where $N_i(x)$ are the Newton polynomials defined by (3.4), then $D_n = [N_j(x_i)]_{0 \le i, j \le n}$ will be a lower triangular matrix with $N_i(x_i) \neq 0$, for $i = 0, 1, \cdots, n$, as diagonal elements, and D_n is a strongly nonsingular matrix, and

$$p_n(x) = \begin{bmatrix} 1 & N_1(x) & \cdots & N_n(x) \end{bmatrix} D_n^{-1} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{bmatrix},$$

where $\begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \end{bmatrix}^T$ are given by

$$\begin{cases} \alpha_0 = y_0, \quad \text{and for} \quad i = 1, 2, \cdots, n, \\ \alpha_i = \left(y_i - \sum_{j=0}^{i-1} N_j(x_i)\alpha_j \right) / N_i(x_i). \end{cases} \tag{3.18}$$

3. If we assume that the polynomials $q_i(x)$, for $i = 0, 1, \cdots, n$, are of degree n, and if we choose $q_i(x) = L_i(x)$, then the matrix $D_n = [L_j(x_i)]_{0\leq i,j\leq n}$ will be the identity matrix, and using the relation (3.10), we obtain $p_n(x) = \sum_{i=0}^{n} y_i L_i(x)$.

3.3 Some properties of the RPIA

We will give some properties of the RPIA. We will show that apart from a multiplicative constant, the polynomials $\{g_{i-1,i}(x); i = 0, 1, \cdots, n\}$, generated by the Algorithm 1, are the Newton polynomials, $N_i(x)$ for $i = 0, 1, \cdots, n$, given by the relation (3.4).

Proposition 3.3 *We have*

(1) The RPIA is well defined (i.e. no break-down).
(2) $c_m(g_{j,i}(x)) = g_{j,i}(x_m) = 0$, *for* $i > j \geq m$.
(3) $c_m(p_i(x)) = c_m(p_n(x)) = y_m$, *for* $i = 0, \cdots, n$ *and* $m = 0, \cdots, i$.
(4) $g_{j,i}(x)$ *is a linear combination of the polynomials* $q_0(x), \cdots, q_j(x), q_i(x)$.
(5) The polynomials $g_{-1,0}(x), g_{0,1}(x), \cdots, g_{n-1,n}(x)$ *generated by the RPIA are linearly independent.*

Proof

(1) The RPIA is well defined (i.e. no break-down) if $c_i(g_{i-1,i}(x)) = g_{i-1,i}(x_i) \neq 0$, for $i = 0, 1, \cdots, n$. As D_n is a strongly nonsingular matrix and $g_{i-1,i}(x_i) = |D_i|/|D_{i-1}| \neq 0$, then the RPIA is well defined because we have no division by zero.

(2) Using the relation (3.14) and (2.3), we remark that $c_m(g_{j,i}(x))$ can be expressed as a ratio of two determinants

$$c_m(g_{j,i}(x)) = g_{j,i}(x_m) = \left|\begin{array}{c|cccc} q_i(x_m) & q_0(x_m) & \cdots & q_j(x_m) \\ q_i(x_0) & q_0(x_0) & \cdots & q_j(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ q_i(x_j) & q_0(x_j) & \cdots & q_j(x_j) \end{array}\right| /|D_j|,$$

and for $m = 0, 1, \cdots, j$, the numerator of this ration will be zero.

(3) If we use the relations (3.13) and (2.3), $c_m(p_i(x)) = p_i(x_m)$ can be expressed as a ratio of two determinants

$$p_i(x_m) = -\left(\left[\begin{array}{c|cccc} 0 & q_0(x_m) & \cdots & q_i(x_m) \\ y_0 & q_0(x_0) & \cdots & q_i(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_i & q_0(x_i) & \cdots & q_i(x_i) \end{array}\right] / D_i\right)$$

$$= -\left|\begin{array}{c|cccc} 0 & q_0(x_m) & \cdots & q_i(x_m) \\ y_0 & q_0(x_0) & \cdots & q_i(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_i & q_0(x_i) & \cdots & q_i(x_i) \end{array}\right| /|D_i|,$$

then for $m = 0, 1, \cdots, i$, expanding the numerator of this ratio with respect to its first column and permuting the first row with the m^{th} row, we obtain

$$p_i(x_m) = -(-1)^{m-1}(-1)^m y_m = (-1)^{2m} y_m = y_m = p_n(x_m) = c_m(p_n(x)).$$

(4) For $i=1, 2, \cdots, n$, and $j=0, 1, \cdots, i-1$, and using the relation (3.17), we get

$$\begin{aligned} g_{j,i}(x) &= g_{j-1,i}(x) - \frac{g_{j-1,i}(x_j)}{g_{j-1,j}(x_j)} g_{j-1,j}(x) \\ &= g_{-1,i}(x) - \sum_{m=0}^{j} \frac{g_{m-1,i}(x_m)}{g_{m-1,m}(x_m)} g_{m-1,m}(x) \\ &= q_i(x) - \sum_{m=0}^{j} \frac{g_{m-1,i}(x_m)}{g_{m-1,m}(x_m)} g_{m-1,m}(x), \end{aligned}$$

then using again this relation it is easy to see that for $m = 0, 1, \cdots, j$, $g_{m-1,m}(x)$, which is a polynomial of degree m, is a combination of $q_0(x), \cdots, q_m(x)$, and the result follows.

(5) As $g_{m-1,m}(x)$ is a polynomial of degree m, then $g_{-1,0}(x), g_{0,1}(x), \cdots,$ $g_{n-1,n}(x)$ are linearly independent.

□

Remark 3.3 From (4) and (5) of Proposition 3.3 we see that the polynomials $\{q_0(x), \cdots, q_m(x)\}$ and $\{g_{-1,0}(x), \cdots, g_{m-1,m}(x)\}$ generate the same subspace, and the process defined by Algorithm 1, used for computing the polynomials $g_{m-1,m}(x)$, for $m = 0, \cdots, n$, can be interpreted as a process for constructing a new basis of $\mathcal{P}_n$ from the old basis $q_m(x)$.

Now we will give some other properties of the RPIA. For $m = 0, \cdots, n$, and for any real function $g(x)$ we denote by $c_m * g(x) = c_m(g(x)) = g(x_m)$, we also set

$$C_m = \left[\, c_0* \; c_1* \; \cdots \; c_m* \,\right],$$

$$Q_m(x) = \left[\, q_0(x) \; q_1(x) \; \cdots \; q_m(x) \,\right],$$

$$S_m = Q_m(x)[C_m^T Q_m(x)]^{-1} C_m^T.$$

Remark 3.4 It is easy to see that, for $m = 0, \cdots, n$, we have

$$D_m = C_m^T Q_m(x),$$

$$p_m(x) = S_m p_n(x), \tag{3.19}$$

$$g_{m,i}(x) = (1 - S_m) q_i(x), \quad for \;\; i > m. \tag{3.20}$$

Proposition 3.4 *We have*

(1) $S_m^2 = S_m$,
(2) $S_m S_i = S_i S_m = S_i$, *if* $m \geq i$.

Proof

(1). We have

$$S_m^2 = Q_m(x)[C_m^T Q_m(x)]^{-1} C_m^T Q_m(x)[C_m^T Q_m(x)]^{-1} C_m^T$$
$$= Q_m(x)[C_m^T Q_m(x)]^{-1} C_m^T.$$

(2). Let us remark that for $i = 0, \cdots, m$, we have

$$D_m^{-1} C_m^T Q_i(x) = E_{m,i} = \left[\, e_1 \;\, e_2 \;\, \cdots \;\, e_i \,\right],$$

where $e_j \in \mathbb{R}^m$ is the j^{th} element of the canonical basis of $\mathbb{R}^m$. Then we obtain

$$S_m S_i = Q_m(x) D_m^{-1} C_m^T Q_i(x) D_i^{-1} C_i^T = Q_m(x) E_{m,i} D_i^{-1} C_i^T = Q_i(x) D_i^{-1} C_i^T = S_i.$$

For $S_i S_m$, let us remark that

$$C_i^T Q_m(x) D_m^{-1} = E_{i,m} = \left[\begin{array}{c|c} I_i & 0 \end{array}\right],$$

and $E_{i,m} C_m^T = C_i^T$, Then we have

$$S_i S_m = Q_i(x) D_i^{-1} C_i^T Q_m(x) D_m^{-1} C_m^T = Q_i(x) D_i^{-1} C_i^T = S_i.$$

□

Now we set, for $m = 0, \cdots, n$

$$G_m(x) = \left[\, g_{-1,0}(x) \;\, g_{0,1}(x) \;\, \cdots \;\, g_{m-1,m}(x) \,\right], \tag{3.21}$$

$$D'_m = [c_i(g_{j-1,j}(x))]_{0 \le i,j \le m} = C_m^T G_m(x), \tag{3.22}$$

$$S'_m = G_m(x)[C_m^T G_m(x)]^{-1} C_m^T = G_m(x) D_m'^{-1} C_m^T. \tag{3.23}$$

Proposition 3.5 *$D'_n = C_n^T G_n(x)$ is a lower triangular strongly nonsingular matrix and we have for $m = 0, 1, \cdots, n$*

$$S'_m = S_m, \tag{3.24}$$

$$p_m(x) = S'_m p_n(x), \tag{3.25}$$

$$g_{m,i}(x) = (1 - S'_m) q_i(x), \quad for \;\; i > m. \tag{3.26}$$

Proof Using (2) of Proposition 3.3 we see that $D'_n = C_n^T G_n(x)$ is a lower triangular matrix and using the relation (3.15), D'_n is a strongly nonsingular matrix. (3.24) of this proposition will be proved by induction. For $m = 0$ we have $S'_0 = S_0$ because $g_{-1,0}(x) = q_0(x)$. Assume now that (3.24) is true for $m - 1$, with $m \geq 1$, we will prove it for m. First les us consider for $m = 1, \cdots, n$, the matrix partitioned as follows

$$D'_m = C_m^T G_m(x) = \begin{bmatrix} C_{m-1}^T \\ c_m* \end{bmatrix} \begin{bmatrix} G_{m-1}(x) & g_{m-1,m}(x) \end{bmatrix}$$

$$= \begin{bmatrix} D'_{m-1} & 0 \\ c_m * G_{m-1}(x) & c_m(g_{m-1,m}(x)) \end{bmatrix}.$$

Then we get

$$D'^{-1}_m = \begin{bmatrix} D'^{-1}_{m-1} & 0 \\ -\dfrac{c_m * G_{m-1}(x) D'^{-1}_{m-1}}{c_m(g_{m-1,m}(x))} & \dfrac{1}{c_m(g_{m-1,m}(x))} \end{bmatrix}. \tag{3.27}$$

We have from (3.23) and (3.27)

$$\begin{aligned} S'_m =& G_m(x) D'^{-1}_m C_m^T \\ =& \begin{bmatrix} G_{m-1}(x) & g_{m-1,m}(x) \end{bmatrix} \begin{bmatrix} D'^{-1}_{m-1} & 0 \\ -\dfrac{c_m * G_{m-1}(x) D'^{-1}_{m-1}}{c_m(g_{m-1,m}(x))} & \dfrac{1}{c_m(g_{m-1,m}(x))} \end{bmatrix} \begin{bmatrix} C_{m-1}^T \\ c_m* \end{bmatrix} \\ =& \begin{bmatrix} G_{m-1}(x) & g_{m-1,m}(x) \end{bmatrix} \begin{bmatrix} D'^{-1}_{m-1} C_{m-1}^T \\ -\dfrac{c_m * G_{m-1}(x) D'^{-1}_{m-1} C_{m-1}^T - c_m*}{c_m(g_{m-1,m}(x))} \end{bmatrix} \\ =& S'_{m-1} + \frac{g_{m-1,m}(x) c_m*}{c_m(g_{m-1,m}(x))}(1 - S'_{m-1}). \end{aligned}$$

For S_m, if we set

$$u_m = C_{m-1}^T q_m(x) = [\, c_0(q_m(x)) \;\; c_1(q_m(x)) \;\; \cdots \;\; c_{m-1}(q_m(x)) \,]^T,$$

and

$$v_m^T = c_m * Q_{m-1}(x) = [\, c_m(q_0(x)) \;\; c_m(q_1(x)) \;\; \cdots \;\; c_m(q_{m-1}(x)) \,],$$

then we get

$$D_m^{-1} = [C_m^T Q_m(x)]^{-1} = \begin{bmatrix} D_{m-1} & u_m \\ v_m^T & c_m(q_m(x)) \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} D_{m-1}^{-1} + D_{m-1}^{-1} u_m (D_m/D_{m-1})^{-1} v_m^T D_{m-1}^{-1} & -D_{m-1}^{-1} u_m (D_m/D_{m-1})^{-1} \\ -(D_m/D_{m-1})^{-1} v_m^T D_{m-1}^{-1} & (D_m/D_{m-1})^{-1} \end{bmatrix},$$

then using (3.15), (3.20), (3.27) and the fact that $S'_{m-1} = S_{m-1}$, we get

$$
\begin{aligned}
S_m &= Q_m(x) D_m^{-1} C_m^T = \left[\, Q_{m-1}(x)\;\; q_m(x) \,\right] D_m^{-1} \begin{bmatrix} C_{m-1}^T \\ \\ c_m * \end{bmatrix} \\
&= S_{m-1} + (D_m/D_{m-1})^{-1}(1 - S_{m-1}) q_m(x) c_m * (1 - S_{m-1}) \\
&= S'_{m-1} + \frac{g_{m-1,m}(x) c_m *}{c_m(g_{m-1,m}(x))}(1 - S'_{m-1}) \\
&= S'_m .
\end{aligned}
$$

Equations (3.25) and (3.26) of the proposition follow from (3.19), (3.20) and (3.24). □

Remark 3.5 From (4) of Proposition 3.3, $g_{i-1,i}(x)$ is a polynomial of degree i, and from (2) of the same proposition $g_{i-1,i}(x_m) = 0$, for $m = 0, 1, \cdots, i-1$, so

$$
g_{i-1,i}(x) = \mu_i \prod_{j=0}^{i-1}(x - x_j) = \mu_i N_i(x).
$$

Then if we set $Y_n = \left[\, y_0\;\; y_1\;\; \cdots\;\; y_n \,\right]^T$, and using the relation (3.25), we get

$$
\begin{aligned}
p_n(x) &= S'_n p_n(x) \\
&= G_n(x) D_n'^{-1} Y_n \\
&= \sum_{i=0}^{n} \alpha_i g_{i-1,i}(x),
\end{aligned}
$$

where D'_n is a lower triangular matrix and $\alpha = \left[\, \alpha_0\;\; \alpha_1\;\; \cdots\;\; \alpha_n \,\right]^T$ is the solution of the system $D'_n \alpha = Y_n$.

4 Some examples

Example 4.1 For this example we choose $n = 4$ and $q_m(x) = x^m$, for $m = 0, \cdots, 4$. The set $\{(x_m, y_m) : m = 0, 1, \cdots, 4\}$ is chosen as follows

m :	0	1	2	3	4
x_m :	-2	-1	0	$\frac{1}{2}$	1
y_m :	1	$\frac{1}{2}$	-2	0	$\frac{3}{4}$

Polynomials $g_{m-1,m}(x)$ and $p_m(x)$obtained by applying Algorithm 2 are given as follows, we drop the intermediate calculations. For $g_{m-1,m}(x)$

$$
\begin{aligned}
&g_{-1,0}(x) = 1;\\
&g_{0,1}(x) = x + 2;\\
&g_{1,2}(x) = (x + 2)(x + 1);\\
&g_{2,3}(x) = (x + 2)(x + 1)x;\\
&g_{3,4}(x) = (x + 2)(x + 1)x(x - \tfrac{1}{2});
\end{aligned}
$$

and for $p_m(x)$

$$
\begin{aligned}
&p_0(x) = y_0 = 1;\\
&p_1(x) = -\tfrac{1}{2}x;\\
&p_2(x) = -x^2 - \tfrac{7}{2}x - 2;\\
&p_3(x) = \tfrac{32}{15}x^3 + \tfrac{27}{5}x^2 + \tfrac{23}{30}x - 2;\\
&p_4(x) = -\tfrac{37}{20}x^4 - \tfrac{299}{120}x^3 + \tfrac{179}{40}x^2 + \tfrac{157}{60}x - 2;
\end{aligned}
$$

The matrix $D'_4 = [c_i(g_{j-1,j}(x))]_{0\leq i,j\leq 4}$, which is a lower triangular matrix, is as follows

$$
D'_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 1 & \frac{5}{2} & \frac{15}{4} & \frac{15}{8} & 0 \\ 1 & 3 & 6 & 6 & 3 \end{bmatrix},
$$

if we set

$$
p_4(x) = \sum_{m=0}^{4} \alpha_m g_{m-1,m}(x), \quad \text{and} \quad Y_4 = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 & y_4 \end{bmatrix}^T,
$$

then we get

$$
D_4'^{-1} Y_4 = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -1 \\ \frac{32}{15} \\ -\frac{37}{20} \end{bmatrix}.
$$

We see that the polynomials $g_{m-1,m}(x)$, for $m = 0, 1, \cdots, n$, obtained by applying Algorithm 2, are exactly the Newton polynomials $N_m(x)$.

Example 4.2 For this example, we choose $n = 5$ and the polynomials $q_m(x)$, for $m = 0, 1, \cdots, 5$, will be the Hermite polynomials [6].

The set $\{(x_m, y_m) : m = 0, 1, \cdots, 5\}$ and the polynomials $q_m(x)$ are given in the following

m	x_m	y_m	$q_m(x)$
0	-1	-2	1
1	$-\frac{1}{2}$	1	x
2	$\frac{1}{2}$	0	$x^2 - 1$
3	1	$-\frac{2}{3}$	$x^3 - 3x$
4	$\frac{5}{2}$	$\frac{3}{4}$	$x^4 - 6x^2 + 3$
5	3	2	$x^5 - 10x^3 + 15x$

Then applying Algorithm 2, we get for $g_{m-1,m}(x)$

$$\begin{aligned}
&g_{-1,0}(x) = 1;\\
&g_{0,1}(x) = x + 1;\\
&g_{1,2}(x) = (x + 1)(x + \tfrac{1}{2});\\
&g_{2,3}(x) = (x + 1)(x + \tfrac{1}{2})(x - \tfrac{1}{2});\\
&g_{3,4}(x) = (x + 1)(x + \tfrac{1}{2})(x - \tfrac{1}{2})(x - 1);\\
&g_{3,4}(x) = (x + 1)(x + \tfrac{1}{2})(x - \tfrac{1}{2})(x - 1)(x - \tfrac{5}{2});
\end{aligned}$$

and for $p_m(x)$

$$p_0(x) = y_0 = -2;$$

$$p_1(x) = 6x + 4;$$

$$p_2(x) = -\frac{14}{3}x^2 - x + \frac{5}{3};$$

$$p_3(x) = \frac{20}{9}x^3 - \frac{22}{9}x^2 - \frac{14}{9}x + \frac{10}{9};$$

$$p_4(x) = -\frac{191}{378}x^4 + \frac{20}{9}x^3 - \frac{2741}{1512}x^2 - \frac{14}{9}x + \frac{1489}{1512};$$

$$p_5(x) = \frac{79}{945}x^5 - \frac{5}{7}x^4 + \frac{1601}{756}x^3 - \frac{391}{252}x^2 - \frac{5801}{3780}x + \frac{235}{252}.$$

For this example the matrix $D'_5 = [c_i(g_{j-1,j}(x))]_{0\le i,j\le 5}$, is as follows

$$D'_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & \frac{3}{2} & \frac{3}{2} & 0 & 0 & 0 \\ 1 & 2 & 3 & \frac{3}{2} & 0 & 0 \\ 1 & \frac{7}{2} & \frac{21}{2} & 21 & \frac{63}{2} & 0 \\ 1 & 4 & 14 & 35 & 70 & 35 \end{bmatrix},$$

if we set

$$p_5(x) = \sum_{m=0}^{5} \alpha_m g_{m-1,m}(x), \quad \text{and} \quad Y_5 = \left[\, y_0 \ y_1 \ y_2 \ y_3 \ y_4 \ y_5 \,\right]^T,$$

then we get

$$D_5'^{-1}Y_5 = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ -\frac{14}{3} \\ \frac{20}{9} \\ -\frac{191}{378} \\ \frac{79}{945} \end{bmatrix}.$$

We see that the polynomials $g_{m-1,m}(x)$, for $m = 0, 1, \cdots, 5$, obtained by applying Algorithm 2, are exactly the Newton polynomials $N_m(x)$.

Example 4.3 For this example, we choose $n = 4$ and the polynomials $q_m(x)$, for $m = 0, 1, \cdots, 4$, will be the Chebychev polynomials of the second kind [6].

The set $\{(x_m, y_m) : m = 0, 1, \cdots, 5\}$ and the polynomials $q_m(x)$ are given as follows

m	x_m	y_m	$q_m(x)$
0	-3	-1	1
1	$-\frac{3}{2}$	1	$2x$
2	$\frac{1}{2}$	$-\frac{2}{3}$	$4x^2 - 1$
3	1	$\frac{1}{3}$	$8x^3 - 4x$
4	2	$\frac{3}{4}$	$16x^4 - 12x^2 + 1$

Then applying Algorithm 2, we get for $g_{m-1,m}(x)$

$$\begin{aligned} &g_{-1,0}(x) = 1; \\ &g_{0,1}(x) = 2(x+3); \\ &g_{1,2}(x) = 4(x+3)(x+\tfrac{3}{2}); \\ &g_{2,3}(x) = 8(x+3)(x+\tfrac{3}{2})(x-\tfrac{1}{2}); \\ &g_{3,4}(x) = 16(x+3)(x+\tfrac{3}{2})(x-\tfrac{1}{2})(x-1); \end{aligned}$$

and for $p_m(x)$

$$p_0(x) = y_0 = -1;$$

$$p_1(x) = \tfrac{4}{3}x + 3;$$

$$p_2(x) = -\tfrac{13}{21}x^2 - \tfrac{61}{42}x + \tfrac{3}{14};$$

$$p_3(x) = \tfrac{46}{105}x^3 + \tfrac{17}{15}x^2 - \tfrac{7}{15}x - \tfrac{27}{35};$$

$$p_4(x) = -\tfrac{67}{315}x^4 - \tfrac{1}{5}x^3 + \tfrac{271}{180}x^2 - \tfrac{103}{210}x - \tfrac{5}{4};$$

For this example the matrix $D_4' = [c_i(g_{j-1,j}(x))]_{0\le i,j\le 4}$, is as follows

$$D_4' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 7 & 28 & 0 & 0 \\ 1 & 8 & 40 & 40 & 0 \\ 1 & 10 & 70 & 210 & 420 \end{bmatrix},$$

if we set

$$p_4(x) = \sum_{m=0}^{5} \alpha_m g_{m-1,m}(x), \quad \text{and} \quad Y_5 = \left[\, y_0 \ y_1 \ y_2 \ y_3 \ y_4 \,\right]^T,$$

then we get

$$D_4'^{-1} Y_4 = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{2}{3} \\ -\frac{13}{84} \\ \frac{23}{420} \\ -\frac{67}{5040} \end{bmatrix}.$$

We see that apart from a multiplicative constant, the polynomials $g_{m-1,m}(x)$, for $m = 0, 1, \cdots, 5$, obtained by applying Algorithm 2, are the Newton polynomials $N_m(x)$.

Application of the RPIA to the Hermite polynomial interpolation is under investigation.

Acknowledgments We are grateful to Professor C. Brezinski for his help and encouragement. We would like to thank the referee for his helpful comments and valuable suggestions.

References

1. Atteia, M., Pradel, M.: Eléments d'analyse numérique, CEPADUES-Editions (1990)
2. Brezinski, C.: Recursive interpolation, extrapolation and projection. J. Comput. Appl. Math. **9**, 369–376 (1983)
3. Brezinski, C.: Some determinantal identities in a vector space, with applications. In: Werner, H., Bunger, H.J. (eds.) Padé Approximation and its Applications, Bad-Honnef, 1983, Lecture Notes in Mathematics, vol. 1071, pp. 1-11. Springer, Berlin (1984)
4. Brezinski, C.: Other manifestations of the Schur complement. Linear Algebra Appl. **111**, 231–247 (1988)
5. Cottle, R.W.: Manifestations of the Schur complement. Linear Algebra Appl. **8**, 189–211 (1974)
6. Gautschi, W.: Orthogonal polynomials computation and approximation. Oxford University Press (2004)
7. Golub, G.H., Ortega, J.M.: Scientific computing and differential equations, an introduction to numerical methods, Academic Press (1992)
8. Messaoudi, A.: Some properties of the recursive projection and interpolation algorithms. IMA J. Numer. Anal. **15**, 307–318 (1995)
9. Messaoudi, A.: Recursive interpolation Algorithm : a forMalism for linear equations-I: Direct methods. J. Comp. Appl. Math. **76**, 13–30 (1996)
10. Messaoudi, A.: Recursive interpolation Algorithm : a forMalism for linear equations-II: Iterative methods. J. Comp. Appl. Math. **76**, 31–53 (1996)
11. Ouellette, D.V.: Schur complements and statistics. Linear Algebra Appl. **36**, 187–295 (1981)
12. Schur, I.: Potenzreihn im innern des einheitskreises. J. Reine. Angew. Math. **147**, 205–232 (1917)
13. Süli, E., Mayers, D.: An introduction to numerical analysis. Cambridge University Press (2003)