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ORIGINAL PAPER

Viscosity iterative algorithms for fixed point problems of asymptotically nonexpansive mappings in the intermediate sense and variational inequality problems in Banach spaces

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Abstract In this paper, we introduce a generalized viscosity algorithm for finding a fixed point of an asymptotically nonexpansive mapping in the intermediate sense which is also a solution to a variational inequality problem of two inverse-strongly monotone operators in 2-uniformly smooth and uniformly convex Banach spaces. Strong convergence theorems are given under suitable assumptions imposed on the parameters. The results obtained in this paper improve and extend many recent ones in the literature. Three numerical examples are also given to show the efficiency and implementation of our results.

Keywords Fixed point · Variational inequality · Asymptotically nonexpansive mapping in the intermediate sense · Banach spaces

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1 Introduction

Recently, variational inequality theory has become an important tool for solving many problems arising in several branches of pure and applied sciences, such as optimal control, mathematical programming, equilibrium problems, and signal recovery problems. For more details, we refer our readers to [3–16] and the references contained therein.

In this paper, we introduce a generalized viscosity algorithm for finding a common element of the set of fixed points of an asymptotically nonexpansive mapping in the intermediate sense and the set of solutions to variational inequality problems for two inverse-strongly monotone operators in 2-uniformly smooth and uniformly convex Banach spaces. Under appropriate conditions imposed on the parameters, we obtain strong convergence result of the sequence generated by our algorithm. Finally, we give three numerical examples to show that our iterative scheme is implementable, efficient and faster than some previously known schemes for solving variational inequality and fixed point problem in Hilbert spaces. Precisely, in the first and second numerical examples, we compare our iterative scheme with algorithm (11) of G. Cai et al. in [17] and algorithm (6) of Ceng et al. in [9], respectively. We show that our proposed algorithm is efficient and easy to implement. Also, the third numerical example gives a convergence result for two- and three-dimensional cases.

2 Definitions and preliminaries

Let C be a nonempty, closed, and convex subset of a real Banach space E and $T : C \to C$ be a mapping. We denote by $F(T)$ the set of fixed points of T (i.e., $F(T) := \{x \in C : x = Tx\}$). The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 , \ \left\|x^*\right\| = \|x\|\right\}, \ \forall x \in E.$$

It is easy to see that if E is a real Hilbert space, then $J = I$, where I is the identity mapping on E. When E is smooth, we know from [30] that J is single-valued, which we shall denote by j. Let $\{x_n\}$ be a sequence in E. In the sequel, we shall use $x_n \to x$(respectively, $x_n \rightharpoonup x, \ x_n \overset{*}{\rightharpoonup} x$) to denote strong(respectively, weak, weak*) convergence of the sequence $\{x_n\}$ to x. Now, we recall the following basic concepts and facts.

A mapping $f : C \to C$ is called a strict contraction, if there exists a constant $\delta \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \delta \|x - y\|, \ \forall\, x, y \in C. \tag{1}$$

A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \ \forall\, x, y \in C \text{ and } n \ge 1. \tag{2}$$

A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \forall\, x, y \in C. \tag{3}$$

A mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{\theta_n\} \subset [0, +\infty)$ with $\lim_{n\to\infty} \theta_n = 0$ such that

$$\left\|T^n x - T^n y\right\| \leq (1 + \theta_n) \|x - y\|, \forall n \geq 1,\ x, y \in C. \tag{4}$$

A mapping $T : C \to C$ is said to be asymptotically nonexpansive in the intermediate sense ([31, 32]) if T is continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{5}$$

It is easy to see that asymptotically nonexpansive mapping in the intermediate sense properly contains the class of strict contractions, the class of nonexpansive mappings and the class of asymptotically nonexpansive mappings.

Throughout this paper, we assume that

$$c_n = \max\left\{0, \sup_{x,y\in C} (\left\|T^n x - T^n y\right\| - \|x - y\|)\right\},$$

then $c_n \geq 0$ for all $n \in \mathbb{N}$, $c_n \to 0$ as $n \to \infty$ and (5) reduces to the relation

$$\left\|T^n x - T^n y\right\| \leq \|x - y\| + c_n \tag{6}$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Now we give two examples of asymptotically nonexpansive mapping in the intermediate sense.

Example 2.1 ([1]) Let $X = \mathbb{R}$ and $C = [0, 1]$. For all $x \in C$, we define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{4}\sqrt{\frac{1}{2} - x} + \frac{\sqrt{2}}{2}, & \text{if } x \in [0, \frac{1}{2}], \\ \sqrt{x}, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then,

- (i) T is asymptotically nonexpansive in the intermediate sense.
- (ii) T is continuous but not uniformly L-Lipschitzian, and hence T is not asymptotically nonexpansive.

Example 2.2 ([2]) Let $H = \mathbb{R}$ and $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and let $|k| < 1$. For each $x \in C$, we define

$$Tx = \begin{cases} kx \sin\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then,

- (i) T is asymptotically nonexpansive in the intermediate sense.
- (ii) T is not Lipschitzian; therefore, T is not asymptotically nonexpansive.

A mapping $A : C \to E$ is called to be accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \forall\, x, y \in C. \tag{7}$$

A mapping $A : C \to E$ is called to be α-inverse-strongly accretive if there exists $j(x-y) \in J(x-y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha \left\| Ax - Ay \right\|^2, \forall\, x, y \in C. \tag{8}$$

In a smooth Banach space, an operator A is said to be strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J(x)\rangle \geq \overline{\gamma}\, \|x\|^2,\ \|aI - bA\| = \sup_{\|x\|\leq 1} |\langle (aI - bA)x, J(x)\rangle|\ a \in [0,1],\ b \in [-1,1], \tag{9}$$

where I is the identity mapping and J is the normalized duality mapping.

Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $A : C \to H$ be a nonlinear mapping. The classical variational inequality is to find an $x^* \in C$ such that

$$\left\langle Ax^*, x - x^*\right\rangle \geq 0,\ \ \forall\, x \in C. \tag{10}$$

We denoted by $VI(A, C)$, the set of solutions to (10).

In [9], Ceng et al. studied the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0,\ \ \forall\, x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0,\ \ \forall\, x \in C, \end{cases} \tag{11}$$

which is called a general system of variational inequalities, where $A, B : C \to H$ are two mappings, $\lambda > 0$ and $\mu > 0$ are two constants. It is easy to see that problem (11) contains the classical variational inequality (10) as a special case.

For finding a common element in the set of solutions to problem (11) and the set of fixed points of a nonexpansive mapping T, Ceng et al. [9] introduced the following algorithm:

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n). \end{cases} \tag{12}$$

Strong convergence theorems were obtained under some suitable conditions on the parameters.

On the other hand, let C be a nonempty, closed, and convex subset of a real Banach space E and $A, B : C \to E$ be two operators. Recently, Yao et al. [15] studied the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0,\ \ \forall\, x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0,\ \ \forall\, x \in C, \end{cases} \tag{13}$$

For solving the problem (13), Yao et al. [15] considered the following iterative algorithm:

$$\begin{cases} u, x_0 \in C, \\ y_n = Q_C(x_n - Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - Ay_n), n \geq 0, \end{cases} \tag{14}$$

and obtained strong convergence results under some suitable conditions on the parameters.

In this paper, we consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0,\ \ \forall\, x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0,\ \ \forall\, x \in C, \end{cases} \tag{15}$$

which is called the system of more general variational inequalities in a real Banach space. If $\lambda = \mu = 1$, the problem (15) becomes problem (13).

Now, we recall some useful facts which are necessary for proving our main results.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) := \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E),\ \|y\| \le t\right\}.$$

A Banach space E is called to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Furthermore, Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^q$. It is well known that if E is q-uniformly smooth, then $q \le 2$ and E is uniformly smooth.

A Banach space E is called to be strictly convex, if x and y are not colinear, then $\|x + y\| < \|x\| + \|y\|$. Let $\delta_E(\epsilon)$ be the modulus of convexity of E defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\|, \|y\| \le 1,\ \|x-y\| \ge \epsilon\right\},$$

for all $\epsilon \in [0, 2]$. A Banach space E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \le 2$. It is known that L^p is uniformly smooth and uniformly convex Banach space, where $p > 1$. Precisely, L^p is $\min\{p, 2\}$-uniformly smooth and $\max\{p, 2\}$-uniformly convex for every $p > 1$.

Let C and D be nonempty subsets of a Banach space E such that C is nonempty, closed and convex and $D \subset C$. A mapping $P : C \to D$ is called to be sunny (see [18, 20]) if $P(x+t(x-P(x))) = P(x)$, $\forall\, x \in C$ and $t \ge 0$, whenever $x+t(x-P(x)) \in C$. A mapping $P : C \to D$ is called a retraction if $Px = x, \forall\, x \in D$. Moreover, P is said to be a sunny nonexpansive retraction from C onto D if P is a retraction from C onto D, which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction P from C onto D (see [33] for more details).

A duality mapping J is said to be weakly sequentially continuous (see [27, 28]), if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then $J(x_n) \overset{*}{\rightharpoonup} J(x)$. In [27], Gossez and Lami Dozo showed that a space with a weakly continuous duality mapping satisfies Opial's condition. Conversely, we know from [34] that if a space satisfies Opial's condition and has a uniformly Gâteaux differentiable norm, then it has a weakly continuous zero duality mapping.

Proposition 2.3 ([18]) *Let C be a closed and convex subset of a smooth Banach space E. Let D be a nonempty subset of C. Let $P : C \to D$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(a) *P is sunny and nonexpansive;*
(b) $\|Px - Py\|^2 \le \langle x - y, J(Px - Py)\rangle,\ \forall\, x, y \in C;$
(c) $\langle x - Px, J(y - Px)\rangle \le 0,\ \forall\, x \in C, y \in D.$

Proposition 2.4 (Theorem 4.1, [19]) *Let D be a closed and convex subset of a reflexive Banach space E with a uniformly Gâteaux differentiable norm. If C is a nonexpansive retract of D, then it is a sunny nonexpansive retract of D.*

Lemma 2.5 ([21]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \delta_n, \ n \geq 0,$$

where

(i) $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;
(iii) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([22]) *Let E be a uniformly convex Banach space, C a bounded, closed, and convex subset of E, and T a self-mapping of C which is asymptotically nonexpansive in the intermediate sense. If $\{x_\beta\}_{\beta\in\Lambda}$ is a net in C converging weakly to x and if* $\lim_{k\to\infty}(\limsup_{\beta\in\Lambda} \|x_\beta - T^k x_\beta\|) = 0$, *then $Tx = x$.*

Lemma 2.7 ([23]) *Assume that A is a strongly positive linear bounded operator on a smooth Banach space E with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then, $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.8 ([24]) *Let E be a real smooth and uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \|x\|^2 - 2\langle x, jy\rangle + \|y\|^2$, for all $x, y \in B_r$,where $B_r = \{z \in E : \|z\| \leq r\}$.*

Lemma 2.9 ([25],Lemma 2.1) *In a Banach space E, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \ x, y \in X,$$

where $j(x + y) \in J(x + y)$.

Lemma 2.10 ([26]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A : C \to E$ be a α-inverse-strongly accretive. Then, the following inequality holds*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 - 2\lambda(\alpha - K^2\lambda)\|Ax - Ay\|^2.$$

In particular, if $0 < \lambda \leq \frac{\alpha}{K^2}$, then $I - \lambda A$ is nonexpansive, where K is the 2-uniformly smoothness constant of E (i.e., K is a positive constant (see [39]) satisfying

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\|Ky\|^2, \ x, y \in E.$$

Lemma 2.11 ([26]) *Let C be a nonempty, closed, and convex subset of a real 2-uniformly smooth Banach space E. Assume that C is a sunny nonexpansive retract of E. Let P_C be the sunny nonexpansive retraction from E onto C. Let the mapping A :*

$C \to E$ be α-inverse-strongly accretive and let $B : C \to E$ be β-inverse-strongly accretive. Let $G : C \to C$ be a mapping defined by

$$G(x) = P_C\left[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)\right], \ \forall x \in C.$$

If $0 < \lambda \leq \frac{\alpha}{K^2}$ and $0 < \mu \leq \frac{\beta}{K^2}$, then $G : C \to C$ is nonexpansive, where K is the 2-uniformly smoothness constant of E.

Lemma 2.12 ([26]) *Let C be a nonempty, closed, and convex subset of a real 2-uniformly smooth Banach space E. Assume that C is a sunny nonexpansive retract of E. Let P_C be the sunny nonexpansive retraction from E onto C. Let $A, B : C \to E$ be two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (2) if and only if $x^* = P_C(y^* - \lambda Ay^*)$, where $y^* = P_C(x^* - \mu Bx^*)$, that is $x^* = Gx^*$, where G is defined by Lemma 2.11.*

Lemma 2.13 ([29]) *Let C be a nonempty, bounded, and closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then x is a fixed point of T.*

3 Main results

In this section, we give strong convergence analysis of approximation of a fixed point of an asymptotically nonexpansive mapping in the intermediate sense which is also a solution to general variational inequality problem (15). Our result in this paper is more applicable than the previous results on general variational inequality problem (15) and fixed point problem since our algorithm solves both general variational inequality problem (15) and fixed point problem at the same time.

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a 2-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous duality mapping and C a sunny nonexpansive retract of E. Let P_C be the sunny nonexpansive retraction from E to C. Let the mappings $A, B : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense with $F(T) \cap F(G) \neq \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.11. Let $f : C \to C$ be a strict contraction with coefficient $\gamma \in [0, 1)$ and $F : C \to C$ be a strongly positive linear bounded operator with the coefficient $\bar{\gamma}$ such that $0 < \gamma < \bar{\gamma}\theta$ and $0 < \theta \leq \|F\|^{-1}$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$, where c_n is defined by (6). Pick any $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - \mu Bx_n), \\ y_n = P_C(z_n - \lambda Az_n), \\ x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n\theta F)T^n y_n], \end{cases} \tag{16}$$

where $0 < \lambda < \frac{\alpha}{K^2}$ *and* $0 < \mu < \frac{\beta}{K^2}$*, where* K *is the 2-uniformly smooth constant appeared in* [39]. *Suppose that* $\{\alpha_n\}$ *is a real sequence in* $[0, 1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

If $\sum_{n=1}^{\infty} \left\| T^{n+1} y_n - T^n y_n \right\| < \infty$*, then* $\{x_n\}$ *converges strongly to* $q \in F(T) \cap F(G)$*, which is also the solution of the variational inequality:*

$$\langle f(q) - \theta F q, j(p - q)\rangle \leq 0 \qquad \forall\, p \in F(T) \cap F(G).$$

Proof We first show that $\{x_n\}$ is bounded. By condition (i), we may assume, without loss of generality, that $\alpha_n\theta \leq \|F\|^{-1}$. It follows from Lemma 2.7 that

$$\|I - \alpha_n\theta F\| \leq 1 - \alpha_n\bar{\gamma}\theta.$$

Take $x^* \in F(T) \cap F(G)$. By Lemma 2.12, we obtain

$$x^* = P_C[P_C(x^* - \mu B x^*) - \lambda A P_C(x^* - \mu B x^*)].$$

Let $y^* = P_C(x^* - \mu B x^*)$, then $x^* = P_C(y^* - \lambda A y^*)$. It follows from Lemma 2.11 that

$$\begin{aligned} \left\| y_n - x^* \right\| &= \left\| G x_n - G x^* \right\| \\ &\leq \left\| x_n - x^* \right\|. \end{aligned} \tag{17}$$

Combining (16) and (17), we have

$$\begin{aligned} \left\| x_{n+1} - x^* \right\| &= \left\| P_C[\alpha_n f(x_n) + (I - \alpha_n\theta F)T^n y_n] - P_C x^* \right\| \\ &\leq \left\| \alpha_n(f(x_n) - \theta F x^*) + (I - \alpha_n\theta F)(T^n y_n - x^*) \right\| \\ &\leq \alpha_n \left\| f(x_n) - f(x^*) \right\| + \alpha_n \left\| f(x^*) - \theta F x^* \right\| + (1 - \alpha_n\bar{\gamma}\theta)(\left\| y_n - x^* \right\| + c_n) \\ &\leq \alpha_n\gamma \left\| x_n - x^* \right\| + \alpha_n \left\| f(x^*) - \theta F x^* \right\| + (1 - \alpha_n\bar{\gamma}\theta) \left\| x_n - x^* \right\| + c_n \\ &= [1 - \alpha_n(\bar{\gamma}\theta - \gamma)] \left\| x_n - x^* \right\| + \alpha_n(\bar{\gamma}\theta - \gamma)\frac{\|f(x^*) - \theta F x^*\|}{\bar{\gamma}\theta - \gamma} + c_n \\ &\leq \max\{\left\| x_n - x^* \right\|, \frac{\|f(x^*) - \theta F x^*\|}{\bar{\gamma}\theta - \gamma}\} + c_n, \end{aligned}$$

By induction, we get

$$\left\| x_n - x^* \right\| \leq \max\left\{ \left\| x_1 - x^* \right\|, \frac{\|f(x^*) - \theta F x^*\|}{\bar{\gamma}\theta - \gamma} \right\} + \sum_{n=1}^{\infty} c_n,$$

which implies that $\{x_n\}$ is bounded. By (17), we have that $\{y_n\}$ is also bounded.

Next, we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Indeed, we observe

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|P_C(z_{n+1}-\lambda Az_{n+1})-P_C(z_n-\lambda Az_n)\|\\ &\leq \|(I-\lambda A)z_{n+1}-(I-\lambda A)z_n\|\\ &\leq \|z_{n+1}-z_n\|\\ &= \|P_C(x_{n+1}-\mu Bx_{n+1})-P_C(x_n-\mu Bx_n)\|\\ &\leq \|(I-\mu B)x_{n+1}-(I-\mu B)x_n\|\\ &\leq \|x_{n+1}-x_n\|. \qquad (18)\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_{n+1}-x_n\|\\ &= \left\|P_C[\alpha_n f(x_n)+(I-\alpha_n\theta F)T^ny_n]-P_C[\alpha_{n-1}f(x_{n-1})+(I-\alpha_{n-1}\theta F)T^{n-1}y_{n-1}]\right\|\\ &\leq \left\|\alpha_n f(x_n)+(I-\alpha_n\theta F)T^ny_n-\alpha_{n-1}f(x_{n-1})-(I-\alpha_{n-1}\theta F)T^{n-1}y_{n-1}\right\|\\ &= \|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})(f(x_{n-1})-\theta FT^ny_{n-1})+(I-\alpha_n\theta F)(T^ny_n-T^ny_{n-1})\\ &\quad +(I-\alpha_{n-1}\theta F)(T^ny_{n-1}-T^{n-1}y_{n-1})\|\\ &\leq \alpha_n\gamma\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\left\|f(x_{n-1})-\theta FT^ny_{n-1}\right\|+(1-\alpha_n\bar{\gamma}\theta)(\|y_n-y_{n-1}\|+c_n)\\ &\quad +(1-\alpha_{n-1}\bar{\gamma}\theta)\left\|T^ny_{n-1}-T^{n-1}y_{n-1}\right\|\\ &\leq [1-\alpha_n(\bar{\gamma}\theta-\gamma)]\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_1+c_n+\left\|T^ny_{n-1}-T^{n-1}y_{n-1}\right\|,\end{aligned}$$

where

$$M_1=\sup_{n\geq 2}\left\|f(x_{n-1})-\theta FT^ny_{n-1}\right\|.$$

Putting

$$\delta_n=|\alpha_n-\alpha_{n-1}|M_1+c_n+\left\|T^ny_{n-1}-T^{n-1}y_{n-1}\right\|,\quad \sigma_n=0,$$

we see (by conditions (i), (ii), $\sum_{n=1}^{\infty}c_n<\infty$ and $\sum_{n=1}^{\infty}\left\|T^{n+1}y_n-T^ny_n\right\|<\infty$) from Lemma 2.5 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (19)$$

Next, we show that $\lim_{n\to\infty}\|y_n-Ty_n\|=0$. It follows from Lemma 2.10 that

$$\begin{aligned}\left\|z_n-y^*\right\|^2 &= \left\|Q_C(x_n-\mu Bx_n)-Q_C(x^*-\mu Bx^*)\right\|^2\\ &\leq \left\|x_n-x^*-\mu(Bx_n-Bx^*)\right\|^2\\ &\leq \left\|x_n-x^*\right\|^2-2\mu(\beta-K^2\mu)\left\|Bx_n-Bx^*\right\|^2. \qquad (20)\end{aligned}$$

and

$$\begin{aligned}\|y_n - x^*\|^2 &= \|Q_C(z_n - \lambda A z_n) - Q_C(y^* - \lambda A y^*)\|^2 \\ &\le \|z_n - y^* - \lambda(Az_n - Ay^*)\|^2 \\ &\le \|z_n - y^*\|^2 - 2\lambda(\alpha - K^2\lambda)\|Az_n - Ay^*\|^2 . \end{aligned} \tag{21}$$

Substituting (20) into (21), we obtain

$$\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 - 2\mu(\beta - K^2\mu)\|Bx_n - Bx^*\|^2 - 2\lambda(\alpha - K^2\lambda)\|Az_n - Ay^*\|^2. \tag{22}$$

Let $v_n = \alpha_n f(x_n) + (I - \alpha_n \theta F)T^n y_n$ for all $n \in \mathbb{N}$. By Lemma 2.9, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|P_C[\alpha_n f(x_n) + (I - \alpha_n\theta F)T^n y_n] - x^*\|^2 \\ &\le \|v_n - x^*\|^2 \\ &= \|\alpha_n(f(x_n) - \theta F T^n y_n) + (T^n y_n - x^*)\|^2 \\ &\le \|T^n y_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - \theta F T^n y_n, j(v_n - x^*)\rangle \\ &\le (\|y_n - x^*\| + c_n)^2 + 2\alpha_n \|f(x_n) - \theta F T^n y_n\| \|v_n - x^*\| \\ &\le \|y_n - x^*\|^2 + c_n(2\|y_n - x^*\| + c_n) + 2\alpha_n \|f(x_n) - \theta F T^n y_n\| \|v_n - x^*\| \\ &\le \|y_n - x^*\|^2 + c_n M_2 + \alpha_n M_3, \end{aligned} \tag{23}$$

where

$$M_2 = \sup_{n\ge 1}\{2\|y_n - x^*\| + c_n\},\ \ M_3 = \sup_{n\ge 2}\{2\|f(x_n) - \theta F T^n y_n\| \|v_n - x^*\|\}.$$

Combining (22) and (23), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le\ &\|x_n - x^*\|^2 - 2\mu(\beta - K^2\mu)\|Bx_n - Bx^*\|^2 - 2\lambda(\alpha - K^2\lambda)\|Az_n - Ay^*\|^2 \\ &+ c_n M_2 + \alpha_n M_3, \end{aligned}$$

which implies

$$\begin{aligned} &2\mu(\beta - K^2\mu)\|Bx_n - Bx^*\|^2 + 2\lambda(\alpha - K^2\lambda)\|Az_n - Ay^*\|^2 \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + c_n M_2 + \alpha_n M_3 \\ &\le \|x_n - x_{n+1}\| (\|x_n - x^*\| + \|x_{n+1} - x^*\|) + c_n M_2 + \alpha_n M_3. \end{aligned} \tag{24}$$

Since $0 < \lambda < \frac{\alpha}{K^2}$, $0 < \mu < \frac{\beta}{K^2}$, $\lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} \alpha_n = 0$ and (19), we obtain by (24)

$$\lim_{n\to\infty} \|Bx_n - Bx^*\| = 0,\ \lim_{n\to\infty} \|Az_n - Ay^*\| = 0. \tag{25}$$

Let $r_1 = \sup_{n\geq 0}\{\|z_n - y^*\|, \|x_n - x^*\|\}$. It follows from Proposition 2.3 and Lemma 2.8 that

$$\begin{aligned}
&\|z_n - y^*\|^2 \\
&= \|P_C(x_n - \mu Bx_n) - P_C(x^* - \mu Bx^*)\|^2 \\
&\leq \langle x_n - \mu Bx_n - (x^* - \mu Bx^*), j(z_n - y^*)\rangle \\
&= \langle x_n - x^*, j(z_n - y^*)\rangle + \mu\langle Bx^* - Bx_n, j(z_n - y^*)\rangle \\
&\leq \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) + \mu\langle Bx^* - Bx_n, j(z_n - y^*)\rangle,
\end{aligned}$$

where $g_1 : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function such that $g_1(0) = 0$. Consequently, we have

$$\begin{aligned}
\|z_n - y^*\|^2 &\leq \|x_n - x^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) + 2\mu\langle Bx^* - Bx_n, j(z_n - y^*)\rangle \\
&\leq \|x_n - x^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) + 2\mu\|Bx_n - Bx^*\|\|z_n - y^*\|. \quad (26)
\end{aligned}$$

Let $r_2 = \sup_{n\geq 0}\{\|z_n - y^*\|, \|y_n - x^*\|\}$. Again by Proposition 2.3 and Lemma 2.8, we have

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|P_C(z_n - \lambda Az_n) - P_C(y^* - \lambda Ay^*)\|^2 \\
&\leq \langle z_n - \lambda Az_n - (y^* - \lambda Ay^*), j(y_n - x^*)\rangle \\
&= \langle z_n - y^*, j(y_n - x^*)\rangle + \lambda\langle Ay^* - Az_n, j(y_n - x^*)\rangle \\
&\leq \frac{1}{2}(\|z_n - y^*\|^2 + \|y_n - x^*\|^2 - g_2(\|z_n - y_n + (x^* - y^*)\|) + \lambda\langle Ay^* - Az_n, j(y_n - x^*)\rangle,
\end{aligned}$$

where $g_2 : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing, and convex function such that $g_2(0) = 0$. Therefore, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \|z_n - y^*\|^2 - g_2(\|z_n - y_n + (x^* - y^*)\|) + 2\lambda\langle Ay^* - Az_n, j(y_n - x^*)\rangle \\
&\leq \|z_n - y^*\|^2 - g_2(\|z_n - y_n + (x^* - y^*)\|) + 2\lambda\|Az_n - Ay^*\|\|y_n - x^*\|. \quad (27)
\end{aligned}$$

Substituting (26) into (27), we obtain

$$\begin{aligned}
\|y_n - x^*\|^2 \leq\ & \|x_n - x^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) + 2\mu\|Bx_n - Bx^*\|\|z_n - y^*\| \\
& - g_2(\|z_n - y_n + (x^* - y^*)\|) + 2\lambda\|Az_n - Ay^*\|\|y_n - x^*\|. \quad (28)
\end{aligned}$$

Substituting (28) into (23), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq\ & \|x_n - x^*\|^2 - g_1(\|x_n - z_n - (x^* - y^*)\|) + 2\mu\|Bx_n - Bx^*\|\|z_n - y^*\| \\
& - g_2(\|z_n - y_n + (x^* - y^*)\|) + 2\lambda\|Az_n - Ay^*\|\|y_n - x^*\| + c_n M_2 + \alpha_n M_3.
\end{aligned}$$

This implies that

$$\begin{aligned}
&g_1(\|x_n - z_n - (x^* - y^*)\|) + g_2(\|z_n - y_n + (x^* - y^*)\|) \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\mu \|Bx_n - Bx^*\| \|z_n - y^*\| \\
&\quad + 2\lambda \|Az_n - Ay^*\| \|y_n - x^*\| + c_n M_2 + \alpha_n M_3 \\
&\le \|x_n - x_{n+1}\| (\|x_n - x^*\| + \|x_{n+1} - x^*\|) + 2\mu \|Bx_n - Bx^*\| \|z_n - y^*\| \\
&\quad + 2\lambda \|Az_n - Ay^*\| \|y_n - x^*\| + c_n M_2 + \alpha_n M_3. \qquad (29)
\end{aligned}$$

Noticing $\lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} \alpha_n = 0$, (19) and (25), we have

$$\lim_{n\to\infty} g_1(\|x_n - z_n - (x^* - y^*)\|) = 0, \lim_{n\to\infty} g_2(\|z_n - y_n + (x^* - y^*)\|) = 0.$$

According to the properties of g_1 and g_2, we obtain

$$\lim_{n\to\infty} \|x_n - z_n - (x^* - y^*)\| = 0, \lim_{n\to\infty} \|z_n - y_n + (x^* - y^*)\| = 0. \qquad (30)$$

This implies that

$$\begin{aligned}
\|x_n - y_n\| &\le \|x_n - z_n - (x^* - y^*)\| + \|z_n - y_n + (x^* - y^*)\| \\
&\to 0 \text{ as } n \to \infty. \qquad (31)
\end{aligned}$$

By (19) and $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\begin{aligned}
\|x_n - T^n y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n y_n\| \\
&= \|x_n - x_{n+1}\| + \|P_C[\alpha_n f(x_n) + (I - \alpha_n \theta F)T^n y_n] - P_C T^n y_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - \theta F T^n y_n\| \\
&\to 0 \text{ as } n \to \infty. \qquad (32)
\end{aligned}$$

It follows from (31) and (32) that

$$\begin{aligned}
\|y_n - T^n y_n\| &\le \|y_n - x_n\| + \|x_n - T^n y_n\| \\
&\to 0 \text{ as } n \to \infty. \qquad (33)
\end{aligned}$$

We observe

$$\|y_n - Ty_n\| \le \|y_n - T^n y_n\| + \|T^n y_n - T^{n+1} y_n\| + \|T^{n+1} y_n - Ty_n\|.$$

By condition $\sum_{n=1}^{\infty} \|T^{n+1} y_n - T^n y_n\| < \infty$, (33) and the fact that T is continuous, we obtain

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0. \qquad (34)$$

Next, we show that

$$\limsup_{n\to\infty} \langle f(q) - \theta F(q), j(x_n - q)\rangle \le 0, \qquad (35)$$

where $q = P_{F(T)\cap F(G)}(f + I - \theta F)(q)$. In fact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(q) - \theta F(q), j(x_n - q)\rangle = \lim_{i\to\infty} \langle f(q) - \theta F(q), j(x_{n_i} - q)\rangle.$$

Now, we prove that $P_{F(T)\cap F(G)}(f+I-\theta F)$ is a strict contraction. In fact, for any $x, y \in C$, it follows from Lemma 2.7 that

$$\begin{aligned}
&\left\| P_{F(T)\cap F(G)}(f+I-\theta F)(x) - P_{F(T)\cap F(G)}(f+I-\theta F)(y) \right\| \\
&\leq \|f(x)-f(y)\| + \|(I-\theta F)(x)-(I-\theta F)(y)\| \\
&\leq \gamma \|x-y\| + (1-\bar{\gamma}\theta)\|x-y\| \\
&= [1-(\bar{\gamma}\theta-\gamma)]\|x-y\|,
\end{aligned}$$

which implies that $P_{F(T)}(f+I-\theta F)$ is a contractive mapping. Banach's contraction mapping principle guarantees that $P_{F(T)}(f+I-\theta F)$ has a unique fixed point. Say $q \in C$, that is, $q = P_{F(T)\cap F(G)}(f+I-\theta F)(q)$. Since $\{x_n\}$ is a bounded in C, without loss of generality, we can assume that $x_{n_i} \rightharpoonup z \in C$. By (31), we know that $y_{n_i} \rightharpoonup z \in C$. From (34), we have that $\lim_{i\to\infty}\left\| y_{n_i} - T^m y_{n_i} \right\| = 0$ for all $m \in \mathbb{N}$. It follows from Lemma 2.6 that $z \in F(T)$. From Lemma 2.13 and (31), we obtain that $z \in F(G)$. Then, $z \in F(T) \cap F(G)$. Since E admits a weakly sequentially continuous duality mapping j and $\{x_n\}$ is bounded, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(q)-\theta F(q), j(x_n-q)\rangle &= \lim_{i\to\infty} \left\langle f(q)-\theta F(q), j(x_{n_i}-q)\right\rangle \\
&= \langle f(q)-\theta F(q), j(z-q)\rangle \leq 0,
\end{aligned}$$

which implies that (35) holds. By (19) and noticing that j is also norm-norm uniformly continuous on bounded subsets of C , we have that

$$\limsup_{n\to\infty} \langle f(q)-\theta F(q), j(x_{n+1}-q)\rangle \leq 0. \tag{36}$$

Finally, we prove that $x_n \to q$ as $n \to \infty$. It follows from $x_{n+1} = P_C v_n$ and Proposition 2.3 (c) that

$$\langle P_C v_n - v_n, j(P_C v_n - q)\rangle \leq 0,$$

which implies

$$\langle x_{n+1} - v_n, j(x_{n+1}-q)\rangle \leq 0.$$

Then, we have

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&= \langle x_{n+1}-v_n, j(x_{n+1}-q)\rangle + \langle v_n - q, j(x_{n+1}-q)\rangle \\
&\leq \langle v_n - q, j(x_{n+1}-q)\rangle \\
&= \left\langle \alpha_n(f(x_n)-\theta Fq) + [I-\alpha_n\theta F](T^n y_n - q), j(x_{n+1}-q)\right\rangle \\
&= \alpha_n \langle f(x_n)-f(q), j(x_{n+1}-q)\rangle + \alpha_n \langle f(q)-\theta Fq, j(x_{n+1}-q)\rangle \\
&\quad + \left\langle [I-\alpha_n\theta F](T^n y_n - q), j(x_{n+1}-q)\right\rangle \\
&\leq \alpha_n\gamma \|x_n-q\|\|x_{n+1}-q\| + (1-\alpha_n\bar{\gamma}\theta)(\|y_n-q\|+c_n)\|x_{n+1}-q\| \\
&\quad + \alpha_n \langle f(q)-\theta Fq, j(x_{n+1}-q)\rangle \\
&\leq [1-\alpha_n(\bar{\gamma}\theta-\gamma)]\|x_n-q\|\|x_{n+1}-q\| + c_n M_4 + \alpha_n \langle f(q)-\theta Fq, j(x_{n+1}-q)\rangle \\
&\leq \frac{1-\alpha_n(\bar{\gamma}\theta-\gamma)}{2}[\|x_n-q\|^2 + \|x_{n+1}-q\|^2] + c_n M_4 + \alpha_n \langle f(q)-\theta Fq, j(x_{n+1}-q)\rangle \\
&\leq \frac{1}{2}\|x_{n+1}-q\|^2 + \frac{1-\alpha_n(\bar{\gamma}\theta-\gamma)}{2}\|x_n-q\|^2 + c_n M_4 + \alpha_n \langle f(q)-\theta Fq, j(x_{n+1}-q)\rangle,
\end{aligned}$$

which implies

$$\|x_{n+1}-q\|^2 \le [1-\alpha_n(\bar{\gamma}\theta-\gamma)]\|x_n-q\|^2+\alpha_n(\bar{\gamma}\theta-\gamma)\frac{2\langle f(q)-\theta Fq, j(x_{n+1}-q)\rangle}{\bar{\gamma}\theta-\gamma}+2c_nM_4, \tag{37}$$

where

$$M_4=\sup_{n\ge 1}\|x_{n+1}-q\|.$$

Apply Lemma 2.5 to (37), we have $x_n \to q$ as $n \to \infty$. This completes the proof. □

The following results can be easily deduced from Theorem 3.1. We omit the details.

Corollary 3.2 *Let C be a nonempty, closed, and convex subset of a 2-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous duality mapping and C a sunny nonexpansive retract of E. Let P_C be the sunny nonexpansive retraction from E to C. Let the mappings $A, B : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense with $F(T)\cap F(G) \ne \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.11. Let $f : C \to C$ be a strict contraction with coefficient $\gamma \in [0,1)$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$, where c_n is defined by (6). Pick any $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n-\mu Bx_n),\\ y_n = P_C(z_n-\lambda Az_n),\\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)T^n y_n, \end{cases} \tag{38}$$

where $0<\lambda<\frac{\alpha}{K^2}$ and $0<\mu<\frac{\beta}{K^2}$. Suppose that $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(ii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$.

If $\sum_{n=1}^{\infty}\left\|T^{n+1}y_n-T^n y_n\right\|<\infty$. Then, $\{x_n\}$ converges strongly to $q \in F(T)\cap F(G)$, which is also the solution of the variational inequality:

$$\langle f(q)-q, j(p-q)\rangle \le 0 \qquad \forall p \in F(T)\cap F(G).$$

Corollary 3.3 *Let C be a nonempty, closed, and convex subset of a 2-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous duality mapping and C a sunny nonexpansive retract of E. Let P_C be the sunny nonexpansive retraction from E to C. Let the mappings $A, B : C \to E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $T : C \to C$ be an asymptotically nonexpansive mapping with $F(T)\cap F(G)\ne\emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.11. Let $f : C \to C$ be a strict contraction with coefficient $\gamma \in [0,1)$ and $F : C \to C$ be a strongly positive linear* *bounded operator with the coefficient* $\bar{\gamma}$ *such that* $0 < \gamma < \bar{\gamma}\theta$ *and* $0 < \theta \leq \|F\|^{-1}$. *Pick any* $x_1 \in C$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = P_C(z_n - \lambda A z_n), \\ x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \theta F) T^n y_n], \end{cases} \tag{39}$$

where $0 < \lambda < \frac{\alpha}{K^2}$ *and* $0 < \mu < \frac{\beta}{K^2}$. *Suppose that* $\{\alpha_n\}$ *is a real sequence in* $[0, 1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

If $\sum_{n=1}^{\infty} \left\|T^{n+1} y_n - T^n y_n\right\| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $q \in F(T) \cap F(G)$, *which is also the solution of the variational inequality:*

$$\langle f(q) - \theta F q, j(p - q)\rangle \leq 0 \qquad \forall p \in F(T) \cap F(G).$$

Corollary 3.4 *Let C be a nonempty, closed, and convex subset of a real Hilbert space* H. *Let the mappings* $A, B : C \to H$ *be* α*-inverse-strongly accretive and* β*-inverse-strongly accretive, respectively. Let* $T : C \to C$ *be an asymptotically nonexpansive mapping with* $F(T) \cap F(G) \neq \emptyset$, *where* $G : C \to C$ *is a mapping defined by Lemma 2.11. Let* $f : C \to C$ *be a strict contraction with coefficient* $\gamma \in [0, 1)$ *and* $F : C \to C$ *be a strongly positive linear bounded operator with the coefficient* $\bar{\gamma}$ *such that* $0 < \gamma < \bar{\gamma}\theta$ *and* $0 < \theta \leq \|F\|^{-1}$. *Pick any* $x_1 \in C$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = P_C(z_n - \lambda A z_n), \\ x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \theta F) T^n y_n], \end{cases} \tag{40}$$

where $0 < \lambda < 2\alpha$ *and* $0 < \mu < 2\beta$. *Suppose that* $\{\alpha_n\}$ *is a real sequence in* $[0, 1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

If $\sum_{n=1}^{\infty} \left\|T^{n+1} y_n - T^n y_n\right\| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $q \in F(T) \cap F(G)$, *which is also the solution of the variational inequality:*

$$\langle f(q) - \theta F q, p - q\rangle \leq 0 \qquad \forall p \in F(T) \cap F(G).$$

Remark 3.5 Theorem 3.1 improves and extends Theorem 3.3 of Cai et al. [17] in the following aspects:

(i) From asymptotically nonexpansive mapping to asymptotically nonexpansive mapping in the intermediate sense.
(ii) We add a strongly positive linear bounded operator in our iterative algorithm.
(iii) The assumption of $\{\alpha_n\}$ of Theorem 3.1 is different from Theorem 3.3 of Cai et al. [17].

According to the proof of Theorem 3.1, we know that $\{y_n\}$ is bounded. We now give some examples of mappings that satisfy the condition $\sum_{n=1}^{\infty} \left\| T^{n+1} y_n - T^n y_n \right\| < \infty$ of Theorem 3.1.

Example 3.6 Let $T : C \to C$ be a strict contraction with a constant $\beta \in (0, 1)$ and let $\{x_n\}$ be a bounded sequence in C, then

$$\left\| T^{n+1} x_n - T^n x_n \right\| \leq \beta^n \left\| T x_n - x_n \right\| \leq \beta^n K_1,$$

where K_1 is a constant such that $K_1 = \sup_{n \geq 1} \|T x_n - x_n\|$. Then, we have

$$\sum_{n=1}^{\infty} \left\| T^{n+1} x_n - T^n x_n \right\| \leq \sum_{n=1}^{\infty} \beta^n K_1 < \infty.$$

Example 3.7 Let C be a nonempty, closed, and convex subset of a Banach space. Define mapping $T : C \to C$ as $T^n x = (1 + \frac{2}{n})x$ for any $x \in C$. It is easy to see that T is asymptotically nonexpansive mapping in the intermediate sense. Let $\{x_n\}$ be a bounded sequence in C, we observe

$$\left\| T^{n+1} x_n - T^n x_n \right\| = \frac{2}{n(n+1)} \|x_n\| \leq \frac{2}{n^2} \|x_n\| \leq \frac{2}{n^2} K_2,$$

where K_2 is a constant such that $K_2 = \sup_{n \geq 1} \|x_n\|$. Hence, we obtain

$$\sum_{n=1}^{\infty} \left\| T^{n+1} x_n - T^n x_n \right\| \leq \sum_{n=1}^{\infty} \frac{2}{n^2} K_2 < \infty.$$

Example 3.8 Define a mapping $T : \mathbb{R} \to \mathbb{R}$ as $T^n x = x + \frac{1}{n}$ for all $x \in \mathbb{R}$. Then, for any $x, y \in \mathbb{R}$, we have

$$|T^n x - T^n y| = |x + \frac{1}{n} - y - \frac{1}{n}| = |x - y|.$$

So T is asymptotically nonexpansive mapping in the intermediate sense. Moreover, for all $x \in \mathbb{R}$, we obtain

$$|T^{n+1} x - T^n x| = |x + \frac{1}{n+1} - x - \frac{1}{n}| = \frac{1}{n(n+1)} \leq \frac{1}{n^2}.$$

It follows that

$$\sum_{n=1}^{\infty} |T^{n+1} x - T^n x| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

4 Applications

Now, we give an application to variational inequality problem for strict pseudocontractive mappings.

A mapping $T : C \to C$ is called to be λ-strict pseudocontractive if there exists a fixed constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda \|(I-T)x - (I-T)y\|^2, \qquad (41)$$

for some $j(x-y) \in J(x-y)$ and for every $x, y \in C$. A simple computation shows that (41)s is equivalent to the following inequality:

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \lambda \|(I-T)x - (I-T)y\|^2 \qquad (42)$$

for some $j(x-y) \in J(x-y)$ and for every $x, y \in C$. Therefore, $I - T$ is λ-inverse-strongly accretive.

By Theorem 3.1, we can obtain the following results easily.

Theorem 4.1 *Let C be a nonempty, closed, and convex subset of a 2-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous duality mapping. Let P_C be the sunny nonexpansive retraction from E to C. Let the mappings $A, B : C \to C$ be α-strict pseudocontractive and β-strict pseudocontractive, respectively. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense with $F(T) \cap F(G) \neq \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.11. Let $f : C \to C$ be a strict contraction with coefficient $\gamma \in [0, 1)$ and $F : C \to C$ be a strongly positive linear bounded operator with the coefficient $\bar{\gamma}$ such that $0 < \gamma < \bar{\gamma}\theta$ and $0 < \theta \leq \|F\|^{-1}$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$, where c_n is defined by (6). Pick any $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1-\mu)x_n + \mu B x_n, \\ y_n = (1-\lambda)z_n + \lambda A z_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \theta F)T^n y_n], \end{cases} \qquad (43)$$

where $0 < \lambda < \frac{\alpha}{K^2}$ and $0 < \mu < \frac{\beta}{K^2}$. Suppose that $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

If $\sum_{n=1}^{\infty} \left\|T^{n+1} y_n - T^n y_n\right\| < \infty$. Then, $\{x_n\}$ converges strongly to $q \in F(T) \cap F(G)$, which is also the solution of the variational inequality:

$$\langle f(q) - \theta F q, j(p-q)\rangle \leq 0 \qquad \forall p \in F(T) \cap F(G).$$

Theorem 4.2 *Let C be a nonempty, closed, and convex subset of a 2-uniformly smooth and uniformly convex Banach space E which admits a weakly sequentially continuous duality mapping. Let P_C be the sunny nonexpansive retraction from E to C. Let the mappings $A, B : C \to C$ be α-strict pseudocontractive and β-strict pseudocontractive, respectively. Let $T : C \to C$ be an asymptotically nonexpansive mapping with $F(T) \cap F(G) \neq \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 2.11. Let $f : C \to C$ be a strict contraction with coefficient $\gamma \in [0, 1)$ and $F : C \to C$ be a strongly positive linear bounded operator with the coefficient $\bar{\gamma}$*

such that $0 < \gamma < \bar{\gamma}\theta$ *and* $0 < \theta \leq \|F\|^{-1}$. *Pick any* $x_1 \in C$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = (1-\mu)x_n + \mu B x_n, \\ y_n = (1-\lambda)z_n + \lambda A z_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \theta F)T^n y_n], \end{cases} \tag{44}$$

where $0 < \lambda < \frac{\alpha}{K^2}$ *and* $0 < \mu < \frac{\beta}{K^2}$. *Suppose that* $\{\alpha_n\}$ *is a real sequence in* $[0, 1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

If $\sum_{n=1}^{\infty} \left\|T^{n+1}y_n - T^n y_n\right\| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $q \in F(T) \cap F(G)$, *which is also the solution of the variational inequality:*

$$\langle f(q) - \theta F q, j(p-q)\rangle \leq 0 \qquad \forall p \in F(T) \cap F(G).$$

5 Numerical example

Numerical examples of the problem considered in Section 3 (Theorem 3.1) are given in this section. The stability, effectiveness, and easy implementation of the algorithm (3.1) considered in Theorem 3.1 is demonstrated and comparison is made with algorithm (11) in Cai et al. [17]. All codes were written in Matlab 2014b and run on Dell $i-5$ Dual-Core laptop.

Example 5.1 Let $E = L_2([0, 1])$ and C be defined as $C := \{g \in L_2([0, 1]) : \int_0^1 t g(t)dt \geq 0\}$. Suppose that $A, B : C \to E$ are defined by $(Ax)(t) := ax(t) = (Bx)(t)$, where $0 < a < 1$. Furthermore, we define $T : C \to C$ by $(Tx)(t) := bx(t)$, for some $0 < b < 1$. Then, for all $x, y \in C$, we have

$$\begin{aligned} \langle Ax - Ay, x - y\rangle &= \langle ax - ay, x - y\rangle \\ &= a\langle x - y, x - y\rangle = a||x - y||^2 \\ &\geq a^2||x - y||^2 = ||ax - ay||^2 \\ &= ||Ax - Ay||^2, \end{aligned}$$

which implies that A (and hence B) is inverse-strongly accretive. Also, for every $x, y \in C$, we have

$$\begin{aligned} ||T^n x - T^n y|| &= ||b^n(x - y)|| \\ &= b^n||x - y|| \\ &\leq (1 + b^n)||x - y||, \end{aligned}$$

which implies that T is asymptotically nonexpansive with $k_n = 1 + b^n, \quad \forall n \geq 1$. Furthermore, it is easy to see that $\lim_{n\to\infty} ||T^{n+1}x_n - T^n x_n|| = 0$ for any bounded sequence $\{x_n\} \subset C$ since for some $M > 0$, we have that

$$\begin{aligned} 0 \leq ||T^{n+1}x_n - T^n x_n|| &= ||b^n x_n(b - 1)|| \\ &= b^n(1 - b)||x_n|| \leq b^n(1 - b)M. \end{aligned}$$

Table 1 Example 5.1, Case I: Algorithm 45 and Algorithm (11) in Cai et al. [17]

	(λ, μ)	(0.01, 0.01)	(0.19, 0.19)	(0.19, 0.01)	(0.01, 0.19)
Alg. 45	No. of iterations	7	7	7	7
	cpu (time)	0.0040	0.0044	0.0041	0.0041
G. Cai et al. Alg. (11)	No. of iterations	17	17	17	17
	cpu (time)	0.0055	0.0062	0.0068	0.0064

Next, we observe that problem (15) in the context of our choice of operators A and B here has solution $p = 0$, a.e on $[0, 1]$. Similarly, $F(T) = \{x \in C : x = 0, a.e.\}$. Hence, $0 \in F(T) \cap F(G)$. Now, using the algorithm 16 of Theorem 3.1, we choose $\alpha_n = \frac{1}{n+1}$. Then, all the assumptions are satisfied. Clearly, $(fx)(t) = \frac{1}{2}x(t)$ is a contraction with $\gamma = \frac{1}{2}$. We then have the metric projection P_C as

$$P_C(x) = x - \frac{\langle c, x\rangle}{||c||_2^2}.$$

By our choices above with $\theta = 1$ and $F = I$, our iterative algorithm 16 reduces to the following algorithm:

$$\begin{cases} z_n = P_C[(1-\mu a)x_n], \quad n \geq 1, \\ y_n = P_C[(1-\lambda a)z_n], \\ x_{n+1} = P_C\left[\frac{1}{2(n+1)}x_n + \left(1 - \frac{1}{n+1}\right)b^n y_n\right]. \end{cases} \tag{45}$$

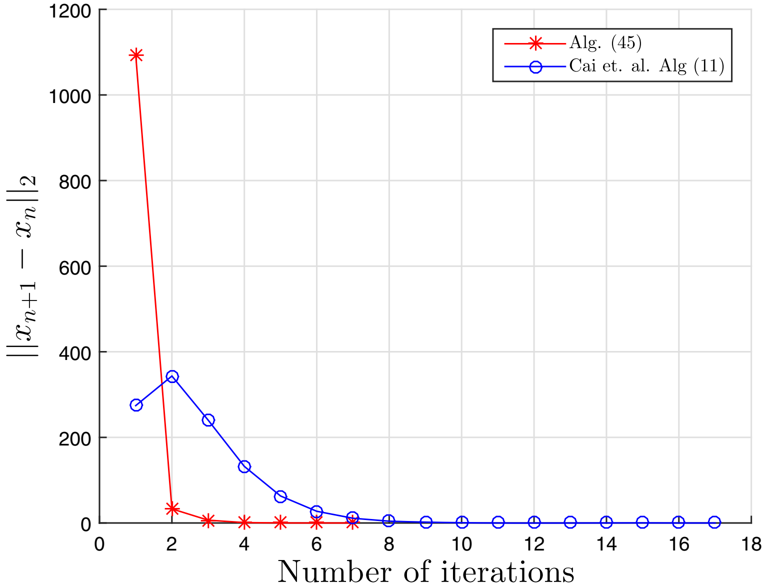

Fig. 1 Example 5.1, Case I with $(\lambda, \mu) = (0.01, 0.01)$

We compare our algorithm (45) and algorithm (11) in Cai et al. [17]. Different choices of x_1, λ, and μ are used with $\frac{||x_{n+1}-x_n||}{||x_2-x_1||} < 10^{-6}$ as stopping criterion for fixed $a = b = 0.1$.

Case I: Consider $x_1 = 3e^{2t}$ and different $(\lambda, \mu) = (0.01, 0.01)$, $(\lambda, \mu) = (0.19, 0.19)$, $(\lambda, \mu) = (0.19, 0.01)$ and $(\lambda, \mu) = (0.01, 0.19)$. The numerical results are displayed in Table 1 and the graphs are given in Figs. 1, 2, 3 and 4.

Case II: Consider $x_1 = 3t^2 + 2t + 5$ and different $(\lambda, \mu) = (0.01, 0.01)$, $(\lambda, \mu) = (0.19, 0.19)$, $(\lambda, \mu) = (0.19, 0.01)$, and $(\lambda, \mu) = (0.01, 0.19)$. The numerical results are displayed in Table 2 and the graphs are given in Figs. 5, 6, 7 and 8.

Remark 5.2 1. The numerical results from Example 5.1 above show that both the algorithm (45) and Algorithm (11) in Cai et al. [17] are very efficient, consistent across different choices of λ and μ. Irrespective of the choice of initial guess, there is no significant difference in the number of iterations and the cpu time taken for both algorithms.

2. Clearly from Tables 1–2 and Figs. 1–8 obtained for the Example 5,1, the proposed algorithm is faster and has fewer number of iterations compare to Algorithm (11) in Cai et al. [17].
3. This Example 5.1 also displays the simple nature of the implementation of the proposed algorithm and so, it can easily be applied.

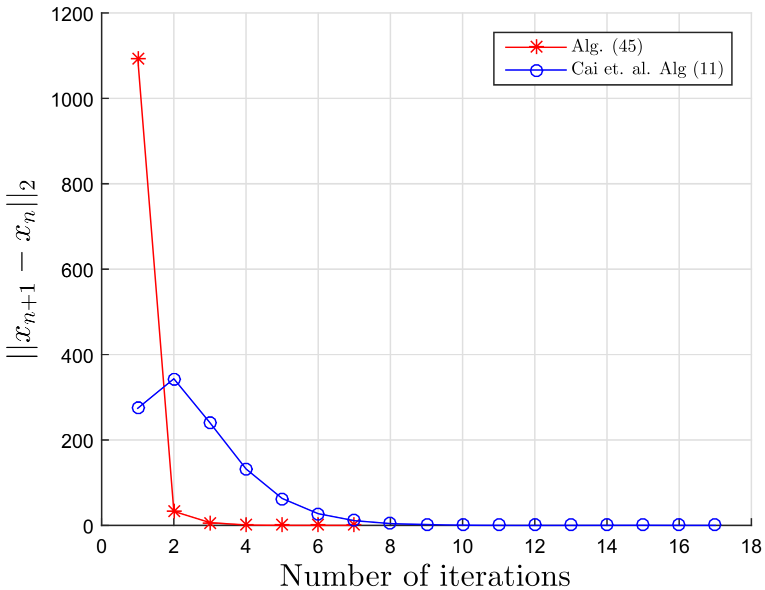

Fig. 2 Example 5.1, Case I with $(\lambda, \mu) = (0.19, 0.19)$

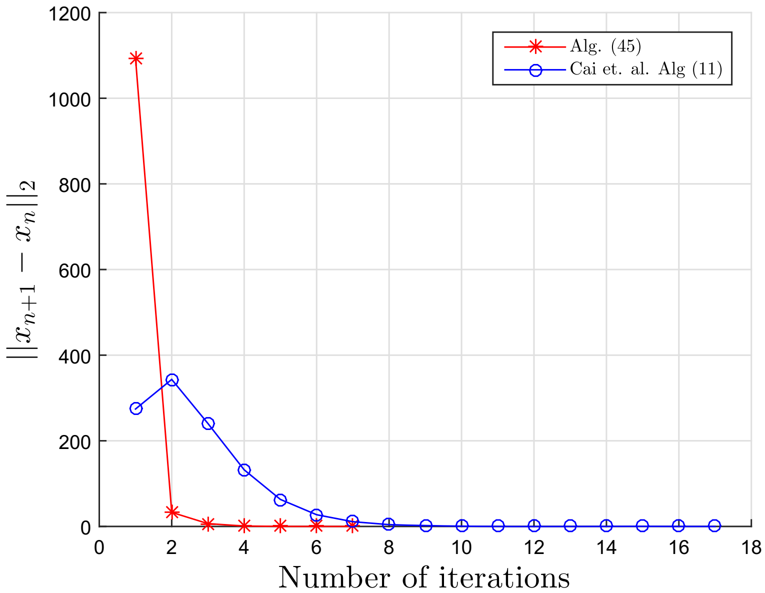

Fig. 3 Example 5.1, Case I with $(\lambda, \mu) = (0.19, 0.01)$

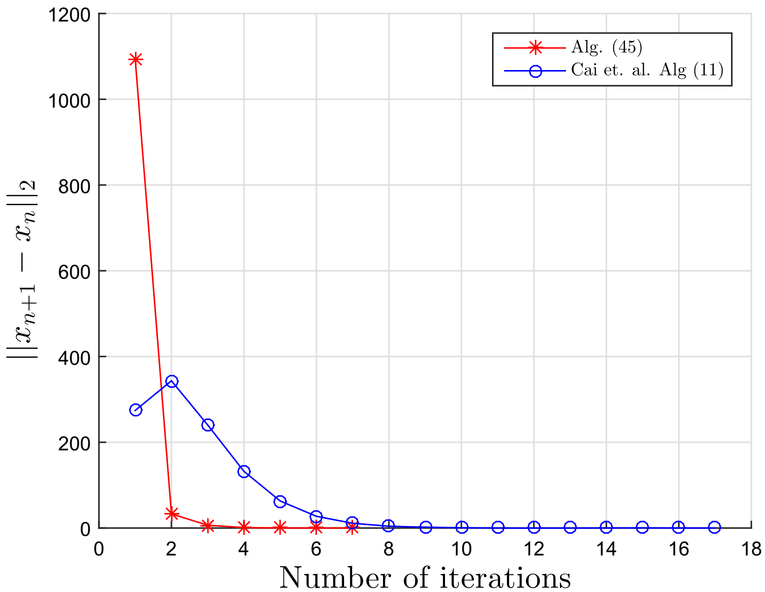

Fig. 4 Example 5.1, Case I with $(\lambda, \mu) = (0.01, 0.19)$

Table 2 Example 5.1, Case II: Algorithm 45 and Algorithm (11) in Cai et al. [17]

	(λ, μ)	(0.01, 0.01)	(0.19, 0.19)	(0.19, 0.01)	(0.01, 0.19)
Alg. 45	No. of iterations	8	8	8	8
	cpu (time)	0.0046	0.0047	0.0048	0.0047
G. Cai et al. Alg. (11)	No. of iterations	17	17	17	17
	cpu (time)	0.0066	0.0067	0.0055	0.0064

Example 5.3 We next compare in real Hilbert space, our proposed algorithm (3.1) with algorithm (12) studied by Ceng et al. [9].

Let $E = L_2([0, 1])$ and C be defined as $C := \{g \in L_2([0, 1]) : \int_0^1 tg(t)dt \geq 0\}$. Suppose that $A, B : C \to E$ are defined by $(Ax)(t) := ax(t) = (Bx)(t)$, where $0 < a < 1$. Furthermore, we define $S = T : C \to C$ by $(Tx)(t) := bx(t)$, for some $0 < b < 1$. Then, it is easy to see that A and B are inverse-strongly monotone and S and T are nonexpansive mapping with $F(S) = F(T)$. Also, by Example 5.1, T satisfies the condition $\lim_{n\to\infty} ||T^{n+1}x_n - T^n x_n|| = 0$ for any bounded sequence $\{x_n\} \subset C$. Furthermore, $F(T) = \{x \in C : x = 0, a.e.\}$ and $0 \in F(T) \cap F(G)$. Now, using the algorithm (16) of Theorem 3.1, we choose $\alpha_n = \frac{1}{n+1}$. Then, all the assumptions are satisfied. Let $f := u$, where u is a fixed constant. Recall that the metric projection P_C is

$$P_C(x) = x - \frac{\langle c, x\rangle}{||c||_2^2}.$$

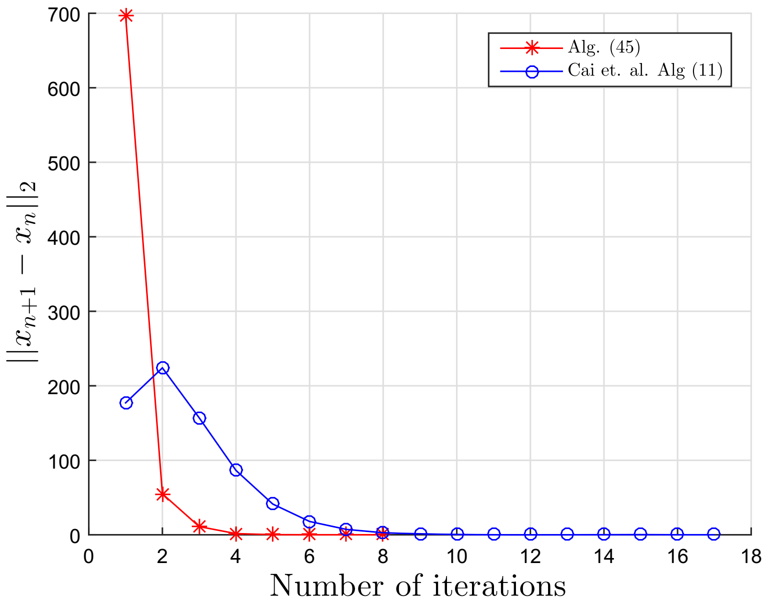

Fig. 5 Example 5.1, Case II with $(\lambda, \mu) = (0.01, 0.01)$

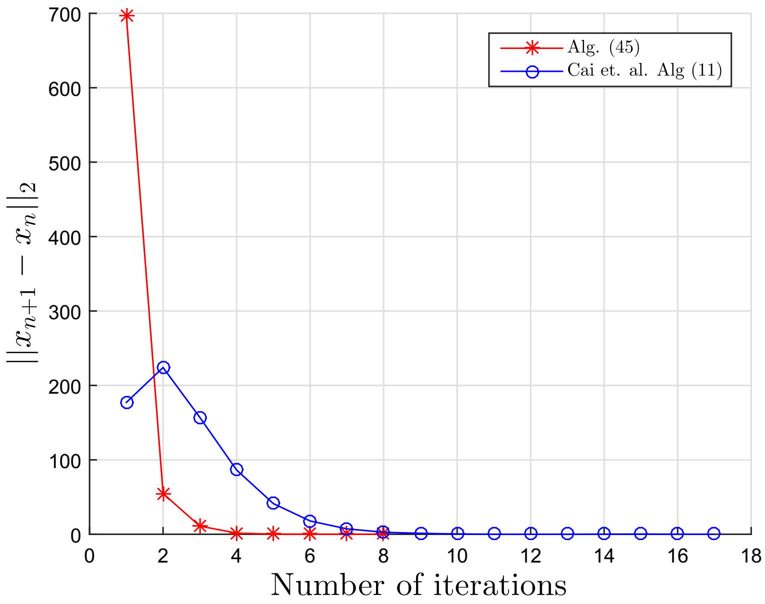

Fig. 6 Example 5.1, Case II with $(\lambda, \mu) = (0.19, 0.19)$

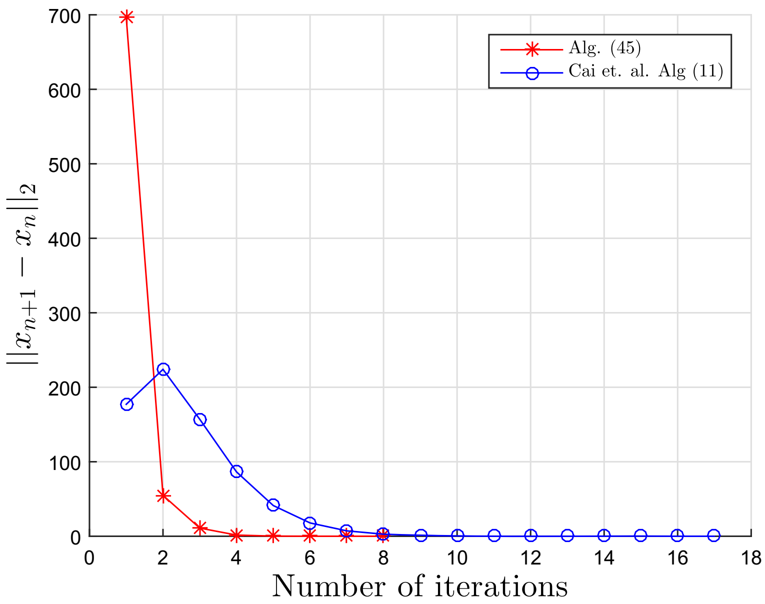

Fig. 7 Example 5.1, Case II with $(\lambda, \mu) = (0.19, 0.01)$

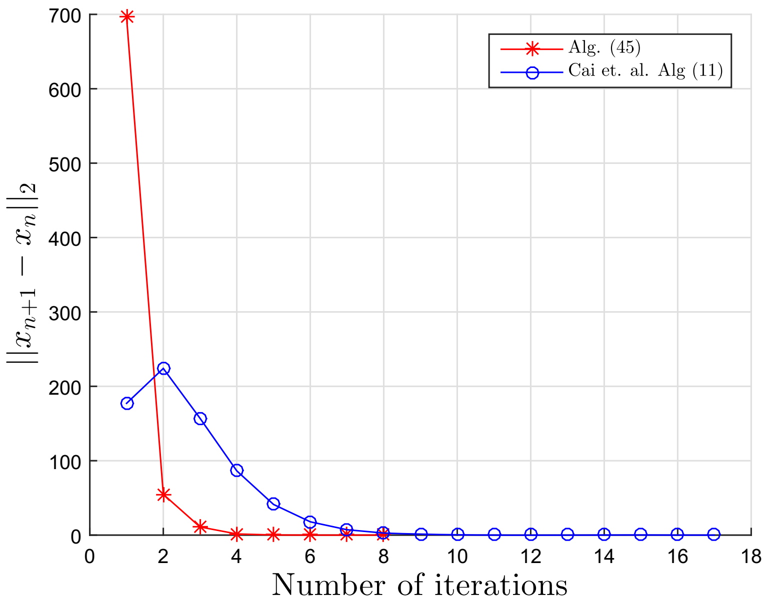

Fig. 8 Example 5.1, Case II with $(\lambda, \mu) = (0.01, 0.19)$

Choose $\theta = 1$ and $F = I$, our iterative algorithm (16) reduces to the following algorithm:

$$\begin{cases} z_n = P_C[(1-\mu a)x_n], \quad n \geq 1, \\ y_n = P_C[(1-\lambda a)z_n], \\ x_{n+1} = P_C\left[\frac{1}{n+1}u + \left(1 - \frac{1}{n+1}\right) b y_n\right]. \end{cases} \tag{46}$$

Furthermore with these choices, the algorithm (12) studied by Ceng et al. [9] becomes (with $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{4(n+1)}$ and $\gamma_n = \frac{3n}{4(n+1)}$)

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C[(1-\mu a)x_n], \\ x_{n+1} = \frac{1}{n+1}u + \frac{n}{4(n+1)}x_n + \frac{3n}{4(n+1)} b P_C[(1-\lambda a)y_n]. \end{cases} \tag{47}$$

Table 3 Example 5.3, Case I: Comparison between our proposed algorithm 46 and Ceng et al. algorithm 47

	(λ, μ)	$(0.01, 0.01)$	$(0.19, 0.19)$	$(0.19, 0.01)$	$(0.01, 0.19)$
Proposed Alg. 46	No. of iterations	6	6	6	6
	cpu (time)	0.017825	0.017018	0.015655	0.015826
Ceng et al. Alg. 47	No. of iterations	292	291	292	291
	cpu (time)	0.74563	0.76279	0.71296	0.70151

We compare our algorithm (46) and algorithm (47) with different choices of x_1, λ and μ are used with $\frac{||x_{n+1}-x_n||}{||x_2-x_1||} < 10^{-6}$ as stopping criterion for fixed $a = b = 0.1$ and $u = 2t$.

Case I: Consider $x_1 = t^3 + 5t^2 - t + 20$ and different $(\lambda, \mu) = (0.01, 0.01)$, $(\lambda, \mu) = (0.19, 0.19)$, $(\lambda, \mu) = (0.19, 0.01)$ and $(\lambda, \mu) = (0.01, 0.19)$. The numerical results are displayed in Table 3 and the graphs are given in Figs. 9, 10, 11, and 12.

Case II: Consider $x_1 = 3t^2 + 2t + 5$ and different $(\lambda, \mu) = (0.01, 0.01)$, $(\lambda, \mu) = (0.19, 0.19)$, $(\lambda, \mu) = (0.19, 0.01)$ and $(\lambda, \mu) = (0.01, 0.19)$. The numerical results are displayed in Table 4 and the graphs are given in Figs. 13, 14, 15, and 16.

Remark 5.4 1. This Example 5.3 also support the arguments as in Example 5.1 regarding our proposed in the sense that it is very efficient, reliable, consistent across different choices of λ and μ, and very fast with very small number of iterations required for convergence.

2. We can see from the comparison Tables 3 and 4 and the Fig. 9, 10, 11, 12, 13, 14, 15, and 16 obtained for this Example 5.3 that our proposed algorithm is faster (about 400 % faster) and has fewer number of iterations compare to Ceng et al. [9] Algorithm 47.

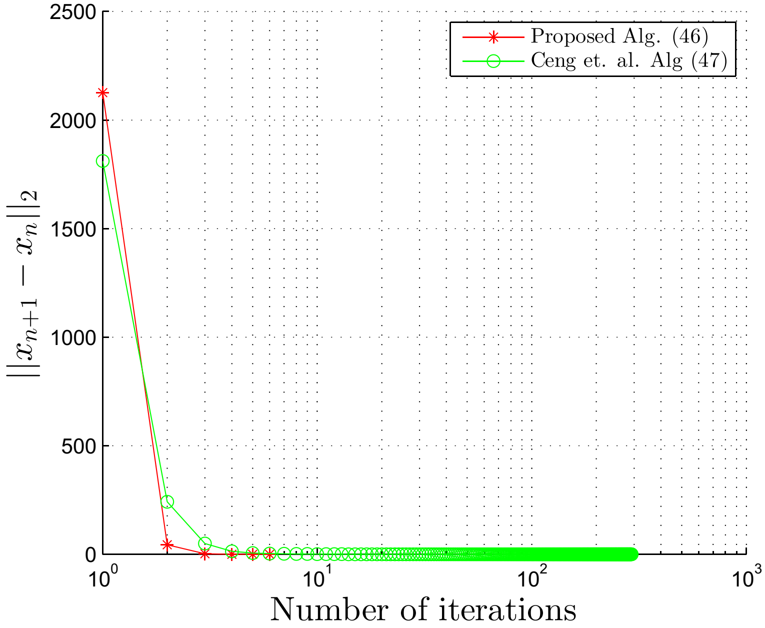

Fig. 9 Example 5.3, Case I with $(\lambda, \mu) = (0.01, 0.01)$

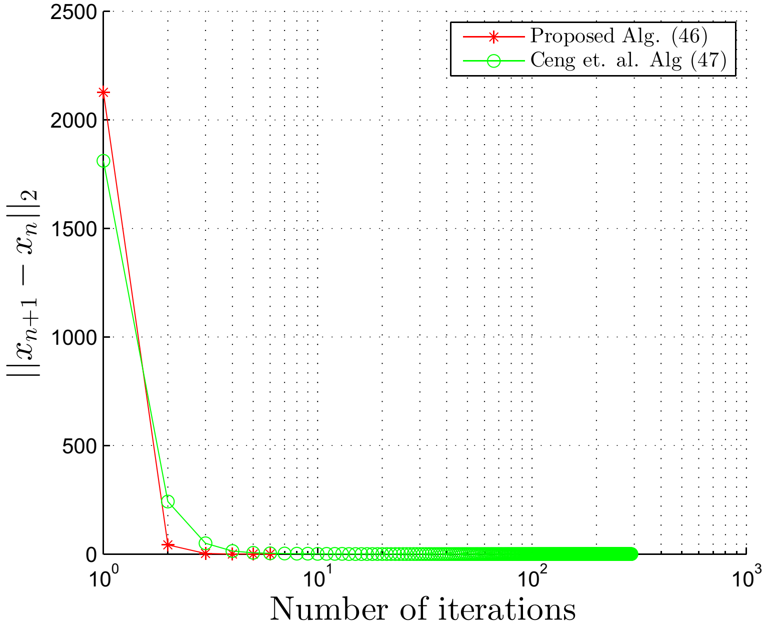

Fig. 10 Example 5.3, Case I with $(\lambda, \mu) = (0.19, 0.19)$

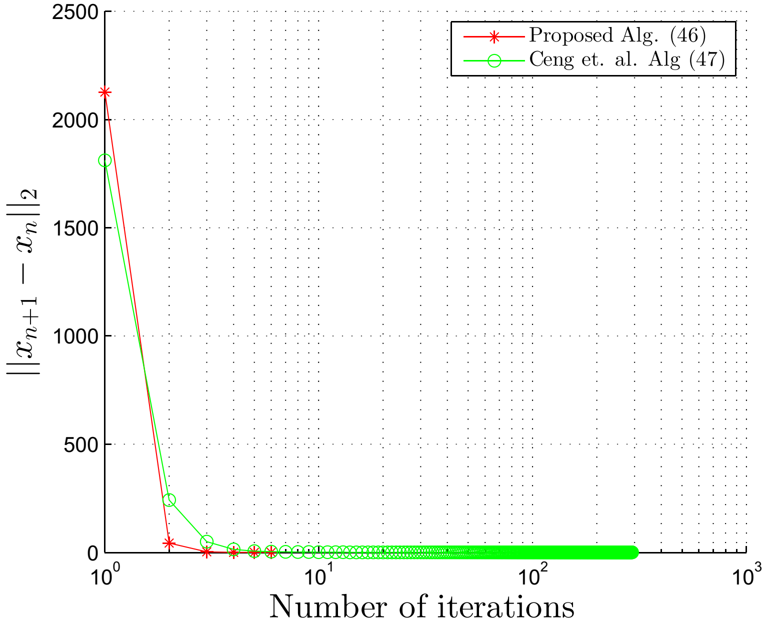

Fig. 11 Example 5.3, Case I with $(\lambda, \mu) = (0.19, 0.01)$

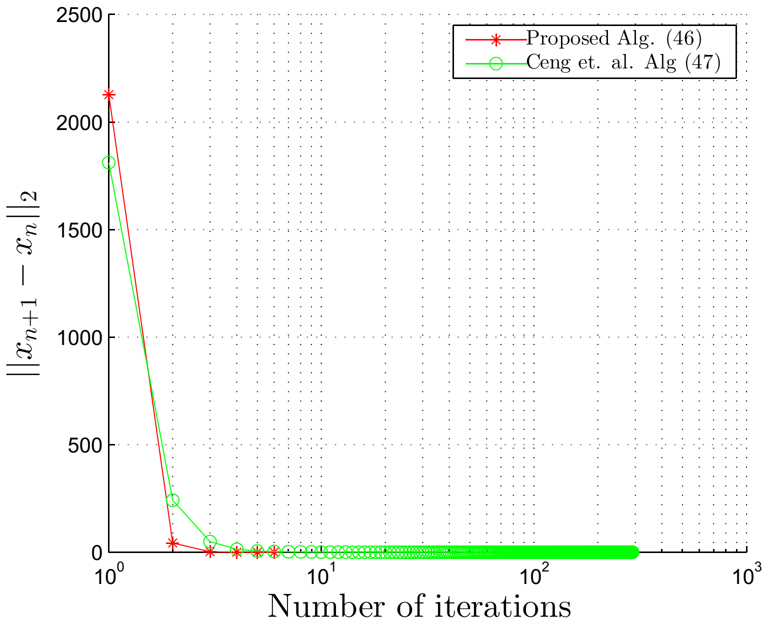

Fig. 12 Example 5.3, Case I with $(\lambda, \mu) = (0.01, 0.19)$

Example 5.5 Let inner product $< \cdot, \cdot >: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$$

and the usual norm $\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$ is defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$ for all $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. For all $x \in \mathbb{R}$, let $T, A, B, F, f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T^n\mathbf{x} = \frac{n+1}{n}\mathbf{x}$, $A\mathbf{x} = \frac{1}{4}\mathbf{x}$, $B\mathbf{x} = \frac{1}{5}\mathbf{x}$, $F\mathbf{x} = \mathbf{x}$ and $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}$, respectively. Let $\alpha_n = \frac{1}{2n}$ for all $n \in \mathbb{N}$ and $\lambda = \frac{1}{5}, \mu = \frac{1}{4}, \theta = \frac{1}{2}$. Let $\{x_n\}$ be a sequence generated by (16), then $\{\mathbf{x}_n\}$ converges strongly to 0.

It is easy to see that $F(T) \cap F(G) = \{0\}$, where G is defined by Lemma 2.11. We rewrite (16) as follows:

$$\mathbf{x}_{n+1} = \frac{1444n^2 + 1483n - 361}{1600n^2}\mathbf{x}_n. \tag{48}$$

Choosing $\mathbf{x}_1 = (1, 2, 3)$ in (48), we have the following numerical results in Figs. 17 and 18.

Table 4 Example 5.3, Case II: Comparison between our proposed algorithm 46 and Ceng et al. algorithm 47

	(λ, μ)	(0.01, 0.01)	(0.19, 0.19)	(0.19, 0.01)	(0.01, 0.19)
Proposed Alg. 46	No. of iterations	6	6	6	6
	cpu (time)	0.017115	0.016997	0.016146	0.018949
Ceng et al. Alg. 47	No. of iterations	392	392	392	392
	cpu (time)	0.98679	0.94996	0.99117	0.97754

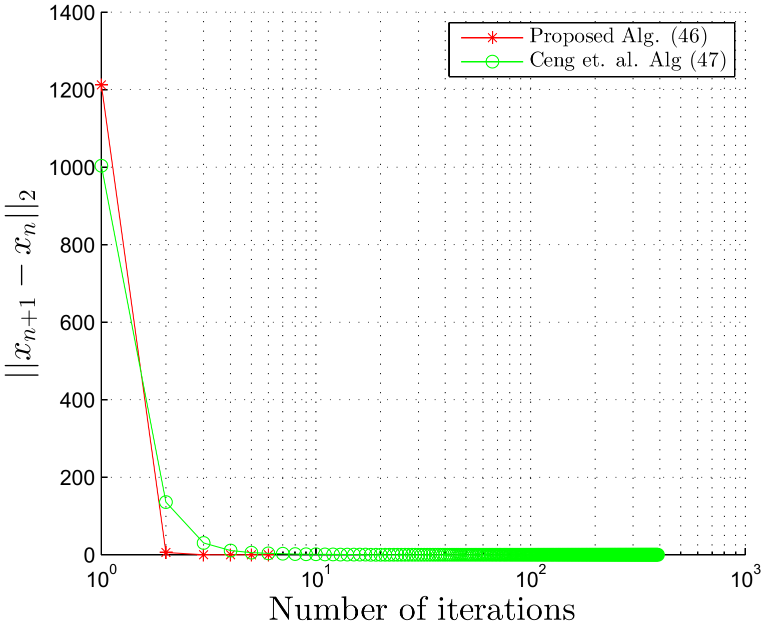

Fig. 13 Example 5.3, Case II with $(\lambda, \mu) = (0.01, 0.01)$

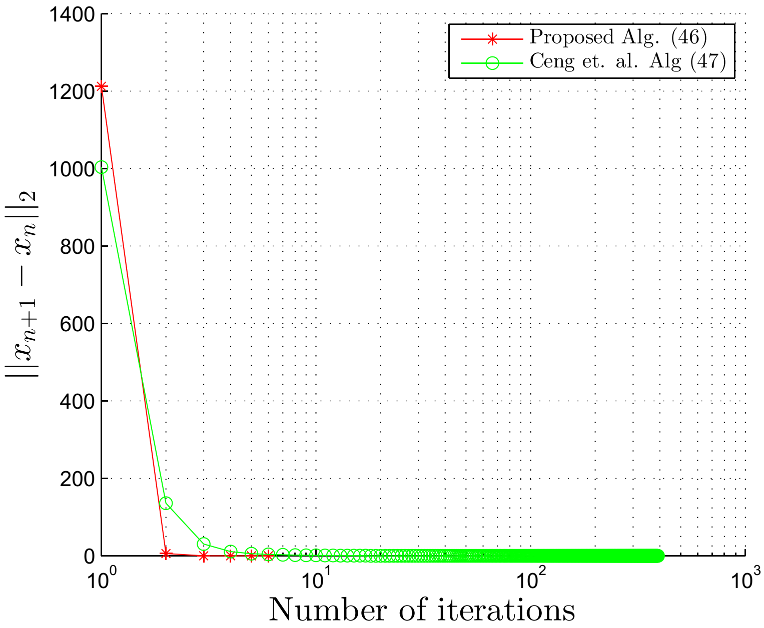

Fig. 14 Example 5.3, Case II with $(\lambda, \mu) = (0.19, 0.19)$

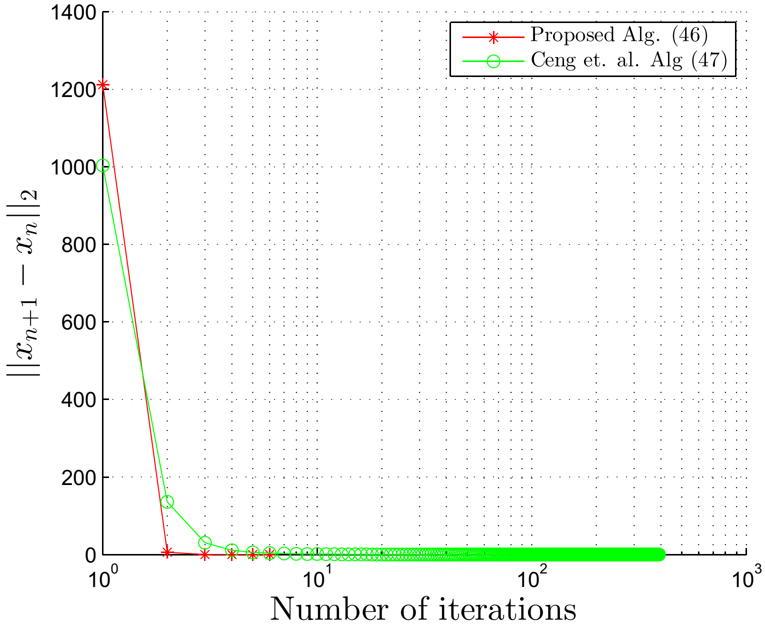

Fig. 15 Example 5.3, Case II with $(\lambda, \mu) = (0.19, 0.01)$

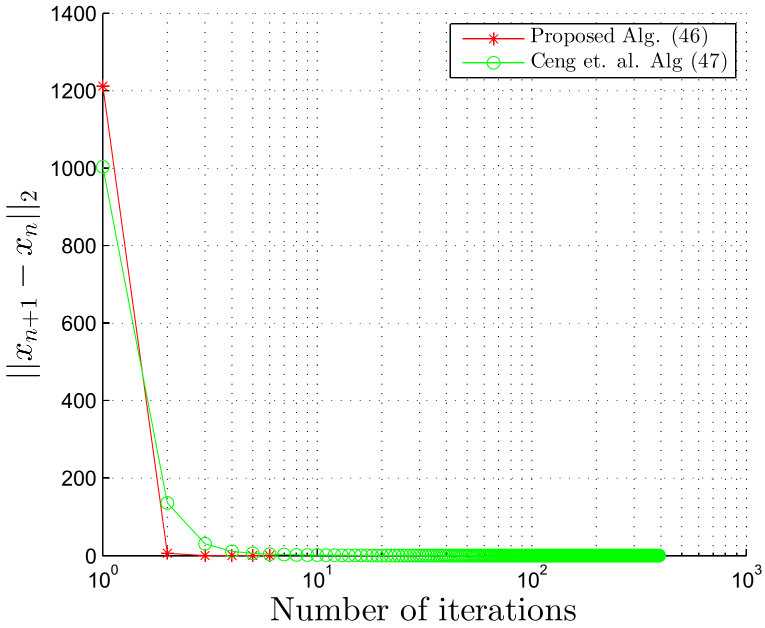

Fig. 16 Example 5.3, Case II with $(\lambda, \mu) = (0.01, 0.19)$

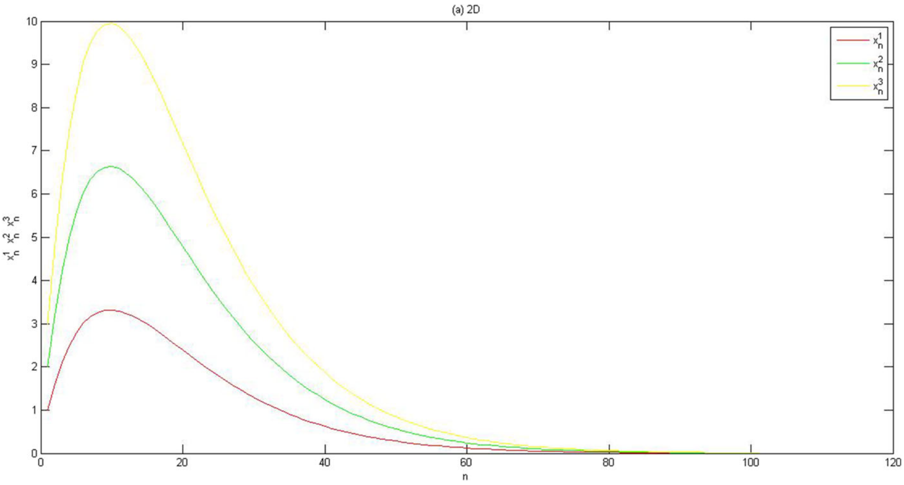

Fig. 17 Example 5.5: Two Dimension

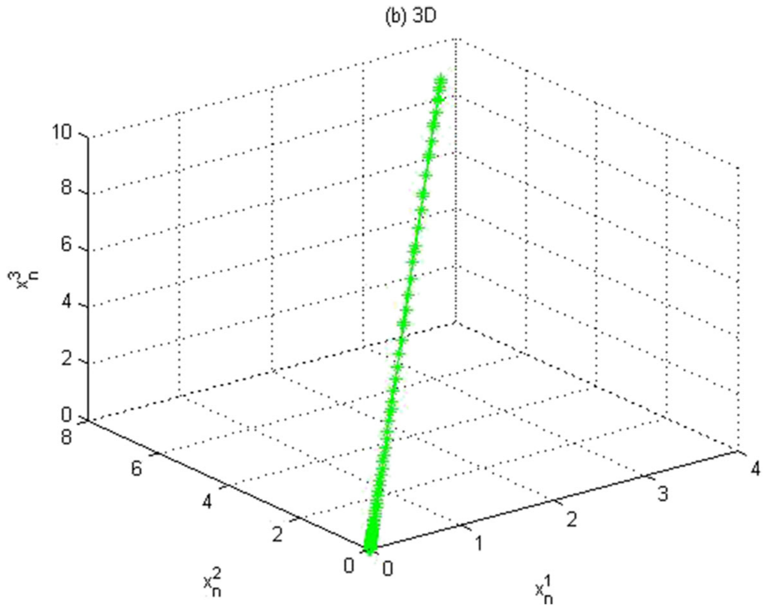

Fig. 18 Example 5.5: Three Dimension

6 Conclusions

Recently, the algorithms for the existence of common fixed points for a finite family of nonexpansive mappings has been studied by many authors (see [35–38] and the references therein). For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings [37]. The problem of finding an optimal point that minimizes a given cost function over the set of common fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance [36]. A simple algorithmic solution to the problem of minimizing a quadratic function over the set of common fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation [38]. On the other hand, variational inequality theory has many applications in pure and applied sciences. There are some numerical methods for solving variational inequality problems and related optimization problems. An important method to solve variational inequality problem is translating into fixed point problem. In this paper, by using a modified extragradient method, we study a generalized viscosity algorithm for finding a common element for the set of fixed points of one asymptotically nonexpansive mapping in the intermediate sense and the set of solutions of variational inequality problems for two inverse-strongly monotone operators in 2-uniformly smooth and uniformly convex Banach spaces. We also give three numerical examples to support our main results.

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