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The parameterized upper and lower triangular splitting methods for saddle point problems

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Abstract In this paper, we propose a class of parameterized upper and lower triangular splitting (denoted by PULTS) methods for solving nonsingular saddle point problems. The eigenvalues and eigenvectors of iteration matrix of the proposed iteration methods are analyzed. It is shown that the proposed methods converge to the unique solution of linear equations under certain conditions. Besides, the optimal iteration parameters and corresponding convergence factors are obtained with some special cases of the PULTS methods. Numerical experiments are presented to confirm the theoretical results, which implies that PULTS methods are effective and feasible for saddle point problems.

Keywords Saddle point problems · Parameterized upper and lower triangular splitting · Iteration method · Convergence

Mathematics Subject Classification (2010) 65F10 · 65F08 · 65F50

1 Introduction

In this paper, we consider the following large and sparse saddle point problems of the form:

$$\begin{pmatrix} A & B \\ B^{\top} & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \tag{1.1}$$

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where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ is a matrix of full column rank, $p \in R^m$ and $q \in R^n$ $(m \geq n)$ are two given vectors, $B^\top$ denote the transpose of B, these assumptions guarantee the existence and uniqueness of the solution of saddle point problems (1.1). Many scientific computing and engineering applications can derive linear systems structured as (1.1), such as computational fluid dynamics, image reconstruction, mixed finite element approximation of elliptic PDEs, constrained least-squares problem, network computer graphics, and so forth. See [1–7] and references therein.

Although the direct methods are very attractive in the form of preconditioners embedded in an iterative framework for the saddle point problems (1.1), iteration methods become more efficient than direct methods when the matrices A and B are large and sparse. While matrix B of (1.1) is rank deficient, then we call linear (1.1) a singular saddle point problem. Many authors proposed a variety of iteration methods for singular saddle point problems. Yang et al. [14] proposed the Uzawa-HSS method to solve singular saddle point problems; the Uzawa-HSS method converges to a solution of the singular saddle point problem under certain conditions. A number of other efficient iteration methods have been proposed for singular saddle point problems, including the HSS-type methods [16, 17], parameterized Uzawa (PU) methods [15], Uzawa-type methods [18, 19], the matrix splitting iteration methods [20], and Krylov subspace methods [8].

While rank $(B) = n$, then linear system (1.1) is called the nonsingular saddle point problems. A number of efficient iteration methods have been studied in the literature, such as null space methods [9], Uzawa-type methods [21], HSS method and its variants [22, 23], matrix splitting iteration method [24] and Krylov subspace methods [25], and so on. Bai and Wang [10] studied the parameterized inexact Uzawa (PIU) methods for solving the nonsingular saddle point problems. Chen and Jiang [11] generalized the PIU method and presented the generalized PIU method. Moreover, Liang and Zhang [12] presented some variants of the accelerated parameterized inexact Uzawa (VAPIU) methods for nonsingular saddle point problems based on the SOR and SSOR splitting of coefficient matrix of linear system (1.1). Bai et al. [13] proposed the GSOR method for nonsingular saddle point problems, which includes the classical Uzawa method [26] and SOR-like method [27] as special cases.

Recently, Zheng and Ma [29] proposed the upper and lower triangular (ULT) splitting iteration method for solving the nonsingular saddle point problems. In order to have a faster computing speed, we generalized the ULT iteration method by introducing a new parameter. In this paper, we presented the parameterized ULT splitting methods for the nonsingular saddle point problems; the proposed methods are based on parameterized upper and lower triangular splitting of coefficient matrix of nonsingular saddle point problems (1.1). We call the new method as PULTS methods for simplicity. The convergence of the new methods are analyzed. The optimal iteration parameters and corresponding convergence factors are obtained with some special cases of the PULTS iteration methods. Numerical experiments are provided to confirm the theoretical results and illustrate the effectiveness of the new methods.

The paper is organized as follows: In Section 2, we propose the parameterized upper and lower splitting methods for nonsingular saddle point problems. In Section 3, we devote to investigate the convergence property for the proposed methods. Numerical results are presented in Section 4 to show the effectiveness of the new methods. Finally, some conclusions are given in Section 5.

2 The PULTS methods

In this section, we first propose the PULTS iteration methods to solve the saddle point problem (1.1). Evidently, the linear system (1.1) can be rewritten as

$$\begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{2.1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ is a matrix of full column rank. Let

$$\widehat{A} = \begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix}, z = \begin{pmatrix} x \\ y \end{pmatrix}, b = \begin{pmatrix} p \\ -q \end{pmatrix},$$

then (2.1) can be expressed as

$$\widehat{A}z = b. \tag{2.2}$$

For the coefficient matrix $\widehat{A}$ of the linear (2.1), we make the following matrix splitting:

$$\begin{aligned} \widehat{A} = \begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix} &= \begin{pmatrix} A & O \\ -B^\top & \alpha Q \end{pmatrix} - \begin{pmatrix} O & -B \\ O & \alpha Q \end{pmatrix} := M_1 - N_1 \\ &= \begin{pmatrix} A & B \\ O & \beta Q \end{pmatrix} - \begin{pmatrix} O & O \\ B^\top & \beta Q \end{pmatrix} := M_2 - N_2, \end{aligned} \tag{2.3}$$

where Q is a symmetric and positive definite matrix and parameters α and β are positive real numbers. Obviously, matrix M_1, M_2 are both invertible matrices. We can find that M_1, N_2 are lower triangular matrices and M_2, N_1 are upper triangular matrices.

Analogously to the classical alternating direction implicit iteration method, we proposed the PULTS iteration methods to solve the linear (2.1) by making use of the matrix splitting (2.3). And it follows that:

The PULTS methods Given an initial vectors $z^{(0)} \in R^{m+n}$, and two positive real parameters α, β, for $k = 0, 1, 2, \cdots$, until the iteration sequence $\{z^k\}$ converges to the exact solution of the linear (2.1), compute

$$\begin{cases} M_1 z^{(k+\frac{1}{2})} = N_1 z^{(k)} + b \\ M_2 z^{(k+1)} = N_2 z^{(k+\frac{1}{2})} + b \end{cases}$$

The convergence analysis of PULTS methods will be given in Section 3 of this paper. On the basis of (2.3) and the above PULTS methods, we have proposed we can obtain the following specific algorithmic procedures of the PULTS methods.

Specific algorithmic procedures of the PULTS method Let $x^{(0)} \in R^m$, $y^{(0)} \in R^n$, Q is a symmetric and positive definite matrix, and we have an initial guess $z^{(0)} =$

$(x^{(0)\top}, y^{(0)\top})^\top$. For $k = 0, 1, 2, \cdots$, until the iteration sequence $\{(x^{(k)\top}, y^{(k)\top})^\top\}$ converges to the exact solution of the linear (2.1), we can compute the iterate $z^{(k+1)} = (x^{(k+1)\top}, y^{(k+1)\top})^\top$ according to the following procedure:

$$\begin{cases} x^{\left(k+\frac{1}{2}\right)} = x^{(k)} + A^{-1}(p - Ax^{(k)} - By^{(k)}) \\ y^{\left(k+\frac{1}{2}\right)} = y^{(k)} + \frac{1}{\alpha}Q^{-1}(B^\top x^{\left(k+\frac{1}{2}\right)} - q) \\ y^{(k+1)} = y^{\left(k+\frac{1}{2}\right)} + \frac{1}{\beta}Q^{-1}(B^\top x^{\left(k+\frac{1}{2}\right)} - q) \\ x^{(k+1)} = A^{-1}(p - By^{(k+1)}) \end{cases} \tag{2.4}$$

Obviously, the PULTS iteration methods will reduce to ULT splitting iteration method when $\alpha = \beta = 1$.

3 Convergence analysis of the PULTS methods

In this section, we turn to study the convergence rate of the PULTS iteration methods. Moreover, necessary and sufficient conditions for the convergence of the PULTS methods are also provided.

Theorem 3.1 *Let $A \in R^{m\times m}$ and $Q \in R^{n\times n}$ are symmetric and positive definite matrix, $B \in R^{m\times n}$ is a column full rank matrix. Then, the iteration matrix $T(\alpha, \beta)$ of the PULTS iteration methods is given by*

$$T(\alpha,\beta) = M_2^{-1}N_2M_1^{-1}N_1 = \begin{pmatrix} O & -A^{-1}B[I - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B] \\ O & I - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B \end{pmatrix}.$$

Suppose λ is an eigenvalue of the iteration matrix $T(\alpha, \beta)$ of the PULTS method, $\lambda = 0$ with multiple m. The remaining n eigenvalues of $T(\alpha, \beta)$ satisfy the following equation:

$$\lambda - 1 + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\mu_i = 0, \tag{3.1}$$

here, μ_i $(i = 1, 2, \cdots, n)$ are the eigenvalues of the matrix $Q^{-1}B^\top A^{-1}B$. So,we know that

$$\rho(T(\alpha,\beta)) = \max\left\{\left|1 - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\mu_{\min}\right|, \left|1 - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\mu_{\max}\right|\right\},$$

where $\mu_{\max}$ and $\mu_{\min}$ are the largest and smallest eigenvalues of the matrix $Q^{-1}B^\top A^{-1}B$, respectively.

Proof According to the equations

$$\begin{cases} M_1 z^{\left(k+\frac{1}{2}\right)} = N_1 z^{(k)} + b, \\ M_2 z^{(k+1)} = N_2 z^{\left(k+\frac{1}{2}\right)} + b, \end{cases}$$

the PULTS methods can be descried as follows:

$$\begin{aligned} z^{(k+1)} &= M_2^{-1}N_2M_1^{-1}N_1z^{(k)} + M_2^{-1}(I + N_2M_1^{-1})b \\ &= T(\alpha,\beta)z^{(k)} + M(\alpha,\beta)^{-1}b, \end{aligned} \tag{3.2}$$

where the iteration matrix of the PULTS iteration methods is

$$\begin{aligned} T(\alpha,\beta) &= M_2^{-1}N_2M_1^{-1}N_1 \\ &= \begin{pmatrix} A & B \\ O & \beta Q \end{pmatrix}^{-1} \begin{pmatrix} O & O \\ B^\top & \beta Q \end{pmatrix} \begin{pmatrix} A & O \\ -B^\top & \alpha Q \end{pmatrix}^{-1} \begin{pmatrix} O & -B \\ O & \alpha Q \end{pmatrix} \\ &= \begin{pmatrix} A^{-1} & -\frac{1}{\beta}A^{-1}BQ^{-1} \\ O & \frac{1}{\beta}Q^{-1} \end{pmatrix} \begin{pmatrix} O & O \\ B^\top & \beta Q \end{pmatrix} \begin{pmatrix} A^{-1} & O \\ \frac{1}{\alpha}Q^{-1}B^\top A^{-1} & \frac{1}{\alpha}Q^{-1} \end{pmatrix} \begin{pmatrix} O & -B \\ O & \alpha Q \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{\beta}A^{-1}BQ^{-1}B^\top & -A^{-1}B \\ \frac{1}{\beta}Q^{-1}B & I \end{pmatrix} \begin{pmatrix} O & -A^{-1}B \\ O & I - \frac{1}{\alpha}Q^{-1}B^\top A^{-1}B \end{pmatrix} \\ &= \begin{pmatrix} O & -A^{-1}B\left[I - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B\right] \\ O & I - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B \end{pmatrix}, \end{aligned}$$

If λ is an eigenvalue of the $T(\alpha,\beta)$, then we have

$$\begin{aligned} \det(\lambda I - T(\alpha,\beta)) &= \det \begin{pmatrix} \lambda I_m & A^{-1}B[I_n - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B] \\ O & (\lambda - 1)I_n + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B \end{pmatrix} \\ &= \lambda^m \det\left((\lambda - 1)I_n + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B\right) \\ &= 0. \end{aligned}$$

Here, we denote the m-by-m and n-by-n identity matrix by I_m and I_n. Then, $\lambda = 0$ is eigenvalue of the iteration matrix with multiply m, and we can easily find that the remaining n eigenvalues of the iteration matrix $T(\alpha,\beta)$ satisfy (3.1). It is easy to see that matrix $Q^{-1}B^\top A^{-1}B$ is similar to matrix $Q^{-\frac{1}{2}}B^\top A^{-1}BQ^{-\frac{1}{2}}$, and we can introduce matrix $Q^{-\frac{1}{2}}B^\top A^{-1}BQ^{-\frac{1}{2}}$ that is symmetric and positive definite, so μ_i are positive real ($i = 1, 2, \cdots, n$). Finally, according to the definition of spectral radius, we have

$$\rho(T(\alpha,\beta)) = \max\left\{\left|1 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\mu_{\min}\right|, \left|1 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\mu_{\max}\right|\right\}.$$

This completes the proof. □

Corollary 3.1 *The PULTS iteration schemes* (3.2) *can be induced by the matrix splitting* $\widehat{A} = M(\alpha,\beta) - N(\alpha,\beta)$, *where*

$$N(\alpha,\beta) = \begin{pmatrix} O & O \\ O & \frac{\alpha\beta}{\alpha+\beta}Q - B^\top A^{-1}B \end{pmatrix}.$$

Proof According to (3.2), we have

$$
\begin{aligned}
M(\alpha,\beta) &= M_1(M_1+N_2)^{-1}M_2\\
&= \begin{pmatrix} A & O\\ -B^\top & \alpha Q\end{pmatrix}\begin{pmatrix} A^{-1} & O\\ O & \frac{1}{\alpha+\beta}Q^{-1}\end{pmatrix}\begin{pmatrix} A & B\\ O & \beta Q\end{pmatrix}\\
&= \begin{pmatrix} A & B\\ -B^\top & \frac{\alpha\beta}{\alpha+\beta}Q - B^\top A^{-1}B\end{pmatrix},
\end{aligned}
$$

in fact, if we let

$$
\begin{aligned}
N(\alpha,\beta) &= M(\alpha,\beta) - \widehat{A}\\
&= \begin{pmatrix} O & O\\ O & \frac{\alpha\beta}{\alpha+\beta}Q - B^\top A^{-1}B\end{pmatrix},
\end{aligned}
$$

then

$$
\widehat{A} = M(\alpha,\beta) - N(\alpha,\beta),
$$

is a splitting of coefficient matrix $\widehat{A}$, and PULTS methods also can be induced by this matrix splitting. □

Theorem 3.2 *Let $A \in R^{m\times m}$ and $Q \in R^{n\times n}$ are symmetric and positive definite matrix; $B \in R^{m\times n}$ is a column full rank matrix. Suppose $\lambda \neq 0$ is an eigenvalue of iteration matrix $T(\alpha,\beta)$ of the PULTS iteration methods and $(u^\top, v^\top)^\top$ be the corresponding eigenvector, then $u = -A^{-1}Bv$ $(v \neq 0)$, where $u \in C^m$ and $v \in C^n$ are two complex vectors.*

Proof From the proof of Theorem 3.1, we have

$$
\begin{pmatrix} O & -A^{-1}B[I - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B]\\ O & I - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B\end{pmatrix}\begin{pmatrix} u\\ v\end{pmatrix} = \lambda\begin{pmatrix} u\\ v\end{pmatrix},
$$

this means

$$
\begin{cases} -A^{-1}B[I - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}B]v = \lambda u\\ v - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)Q^{-1}B^\top A^{-1}Bv = \lambda v.\end{cases} \tag{3.3}
$$

Substituting the second equality in (3.3) into the first equality in (3.3), we get $-\lambda A^{-1}Bv = \lambda u$. Due to $\lambda \neq 0$, it holds $u = -A^{-1}Bv$. Notice that when $v = 0$, we have $u = 0$. So, according to the definition of eigenvector, we have $v \neq 0$. The proof is completed. □

The following result will give the sufficient and necessary conditions of the convergence of the PULTS iteration methods.

Theorem 3.3 *If matrix $A \in R^{m\times m}$ and matrix $Q \in R^{n\times n}$ are symmetric and positive definite, $B \in R^{m\times n}$ is a column full rank matrix. Then, the PULTS iteration methods are convergent if and only if*

$$\mu_{\max} < \frac{2\alpha\beta}{\alpha+\beta}.$$

Proof Suppose $\lambda \neq 0$ is an eigenvalue of iteration matrix $T(\alpha,\beta)$ of the PULTS iteration methods. PULTS iteration methods are convergent if and only if $\rho(T(\alpha,\beta)) < 1$. Combining with linear equation (3.1), we can see that the PULTS iteration methods are convergent if and only if

$$|\lambda| = \left|1 - \Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)\mu_i\right| < 1.$$

it is equivalent to

$$0 < \mu_i < \frac{2\alpha\beta}{\alpha+\beta}.$$

From the proof of Theorem 3.1, we have μ_i is positive real, so the PULTS iteration methods are convergent if and only if $\mu_i < \frac{2\alpha\beta}{\alpha+\beta}$. It means $\mu_{\max} < \frac{2\alpha\beta}{\alpha+\beta}$. The proof is completed. □

From the proof of above theorem, we know PULTS iteration methods are convergent if and only if $\mu_{\max} < \frac{2\alpha\beta}{\alpha+\beta}$. Although the size of the $\mu_{\max}$ is closely related to the matrix Q, we can make the $\mu_{\max} < \frac{2\alpha\beta}{\alpha+\beta}$ by selecting the appropriate parameters α and β, this means we can ensure that PULTS iteration methods are convergent.

Corollary 3.2 *Suppose matrix $A \in R^{m\times m}$ is symmetric and positive definite, $B \in R^{m\times n}$ is a column full rank matrix. Let $Q = \theta I(\theta > 0)$, then the PULTS iteration methods are convergent if and only if*

$$\lambda_{\max}(B^\top A^{-1}B) < \frac{2\alpha\beta\theta}{\alpha+\beta},$$

Proof By using the conclusion of the above theorem, we know the PULTS iteration methods are convergent if and only if $\mu_{\max} < \frac{2\alpha\beta}{\alpha+\beta}$. If $Q = \theta I(\theta > 0)$, we can easily draw the conclusion. The proof is completed. □

The following theorem gives the optimal iteration parameter and corresponding convergence factors of the PULTS iteration methods for $Q = \theta I$ $(\theta > 0)$.

Theorem 3.4 *If matrix $A \in R^{m\times m}$ is symmetric and positive definite, $B \in R^{m\times n}$ is a column full rank matrix and $Q = \theta I$ $(\theta > 0)$. The $\gamma_{\max}$ and $\gamma_{\min}$ denote the largest and smallest eigenvalues of the matrix $B^\top A^{-1}B$, respectively. Then, optimal iteration parameters of the PULTS iteration methods are*

$$\theta_{\text{opt}} = \frac{(\alpha+\beta)(\gamma_{\max}+\gamma_{\min})}{2\alpha\beta}, \tag{3.4}$$

and corresponding convergence factors is

$$\rho(T_{\mathrm{opt}}(\alpha,\beta)) = \frac{\gamma_{\max} - \gamma_{\min}}{\gamma_{\max} + \gamma_{\min}}. \tag{3.5}$$

Proof Let $\lambda \neq 0$ be the eigenvalue of $T(\alpha,\beta)$, if $Q = \theta I(\theta > 0)$. Then, according to (3.1), we can have $\lambda = 1 - (\frac{1}{\alpha} + \frac{1}{\beta})\mu_i = 1 - (\frac{1}{\alpha} + \frac{1}{\beta})\frac{\gamma}{\theta}$; here, we denote the eigenvalue of the matrix $B^{\top}A^{-1}B$ by γ and we can easily see that γ is positive real. The selection of the optimal parameters θ_{opt} depends on the solution of the following problem

$$\min_{\theta} \max_{\gamma_{\min} \leq \gamma \leq \gamma_{\max}} |\lambda| = \min_{\theta} \max_{\gamma_{\min} \leq \gamma \leq \gamma_{\max}} \left| 1 - \frac{\alpha+\beta}{\alpha\beta\theta}\gamma \right|.$$

Evidently, we can see that the optimal parameter θ_{opt} is attained when

$$1 - \frac{\alpha+\beta}{\alpha\beta\theta}\gamma_{\min} = -\left(1 - \frac{\alpha+\beta}{\alpha\beta\theta}\gamma_{\max}\right),$$

through calculation have

$$\theta_{\mathrm{opt}} = \frac{(\alpha+\beta)(\gamma_{\max}+\gamma_{\min})}{2\alpha\beta},$$

which implies

$$\rho(T_{\mathrm{opt}}(\alpha,\beta)) = \frac{\gamma_{\max} - \gamma_{\min}}{\gamma_{\max} + \gamma_{\min}}.$$

This completes the proof. □

4 Numerical results

In this section, we will perform two numerical examples to examine the effectiveness of the PULTS iteration methods for solving the nonsingular saddle point problems (1.1), from the point of view of the number of iteration steps (denoted as "IT") and the elapsed CPU time in seconds (denoted as "CPU").

In actual computations, the initial vector was set to the zero vector. We choose the right-hand-side vector $b \in R^{m+n}$ such that the exact solution of the nonsingular saddle point problems (1.1) is $z = (1, 1, \ldots, 1)^{\top} \in R^{m+n}$. Moreover, all runs are terminated if $ERR \leq 10^{-6}$ or the number of the prescribed iteration steps $k_{\max} = 1000$ is exceeded, where

$$\mathrm{ERR} = \frac{\sqrt{||p - Ax^{(k)} - By^{(k)}||_2^2 + ||q - B^{\top}x^{(k)}||_2^2}}{\sqrt{||p - Ax^{(0)} - By^{(0)}||_2^2 + ||q - B^{\top}x^{(0)}||_2^2}}.$$

All numerical tests are carried out on the personal computer using MATLAB 2014a under the AMD A8-4500M 1.9GHz CPU and 4G RAM Win7 operating system.

Example 1 [28] Consider the following Stokes problems: find u and p such that

$$\begin{cases} -\nu\Delta u + \nabla p = \widetilde{f}, & \text{in} \quad \Omega, \\ \nabla \cdot u = \widetilde{g}, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \\ \int_\Omega p(x)dx = 0. & \end{cases} \tag{4.1}$$

Here, $\Omega = [0, 1] \times [0, 1] \subset R^2$, $\partial\Omega$ is the boundary of Ω, ν denotes the viscous coefficient of fluid, Δ denotes the componentwise Laplace operator, u is a vector-valued function, and p is a scalar function; they represent the velocity and pressure of fluid.

By discrete equation (4.1) with difference scheme, we obtain the following linear system

$$\begin{pmatrix} A & B \\ -B^\top & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{4.2}$$

where

$$A = \begin{pmatrix} I \otimes T + T \otimes I & O \\ O & I \otimes T + T \otimes I \end{pmatrix} \in R^{2l^2 \times 2l^2}, \quad B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in R^{2l^2 \times l^2},$$

which

$$T = \frac{1}{h^2} Tridiag(-1, 2, -1) \in R^{l \times l}, \quad F = \frac{1}{h} Tridiag(-1, 1, 0) \in R^{l \times l},$$

Table 1 Numerical results about different methods for Example 4.1

l		8	16	24	32
	α	0.79	0.78	0.81	0.88
	β	1.24	1.23	1.18	1.11
PULTS	IT	25	34	40	45
	CPU	0.019	0.202	1.027	5.171
	ERR	7.3786e-7	8.3976e-7	9.2608e-7	9.1304e-7
	IT	108	208	311	445
MSSOR	CPU	0.531	0.932	4.763	23.411
	ERR	7.0432e-7	8.9574e-7	9.3365e-7	9.2317e-7
	IT	25	34	40	45
ULT	CPU	0.020	0.227	1.172	5.965
	ERR	7.3786e-7	8.3976e-7	9.2608e-7	9.1304e-7
	IT	32	49	64	78
Uzawa	CPU	0.020	0.251	1.396	6.973
	ERR	7.3774e-7	9.6455e-7	9.7402e-7	9.1639e-7
	IT	25	36	45	52
GSOR	CPU	0.057	0.665	3.472	14.719
	ERR	6.2964e-7	8.5226e-7	8.0255e-7	9.5615e-7

with $\otimes$ being the Kronecker product and $h = \frac{1}{l+1}$ the denotes mesh size. Here, we set $m = 2l^2$ and $n = l^2$, then the total number of variables is $3l^2$.

We compared the PULTS iteration methods with the MSSOR [30] methods, ULT method [29] and the Uzawa method [26] as well as the GSOR methods [13]. The preconditioner matrix of the GSOR methods, Uzawa method and the MSSOR methods are taken as $B^\top \dot{A} B$, $B^\top A^{-1} B$ and $B^\top \ddot{A} B$, where $\dot{A} = tridiag(A)$ and $\ddot{A} = diag(A)$. The preconditioner matrix Q of the PULTS methods and ULT method are taken as θI $(\theta > 0)$. Moreover, the optimal parameters α and β are selected by computer.

In Table 1, we list the IT, CPU, and ERR of the PULTS, MSSOR, ULT,Uzawa, and GSOR methods in relation to different sizes of the coefficient matrix. From Table 1, we can see that PULTS methods perform better than Uzawa method and ULT method, perform very well as compared with MSOR methods and GSOR methods, since it requires much less CPU time and IT to achieve the stopping criterion than other methods.

In Fig. 1, we compare the spectral radius of the Uzawa methods with PULTS methods; we observe that the spectral radius of the iteration matrix of the PULTS methods is less than the spectral radius of the iteration matrix of the Uzawa methods

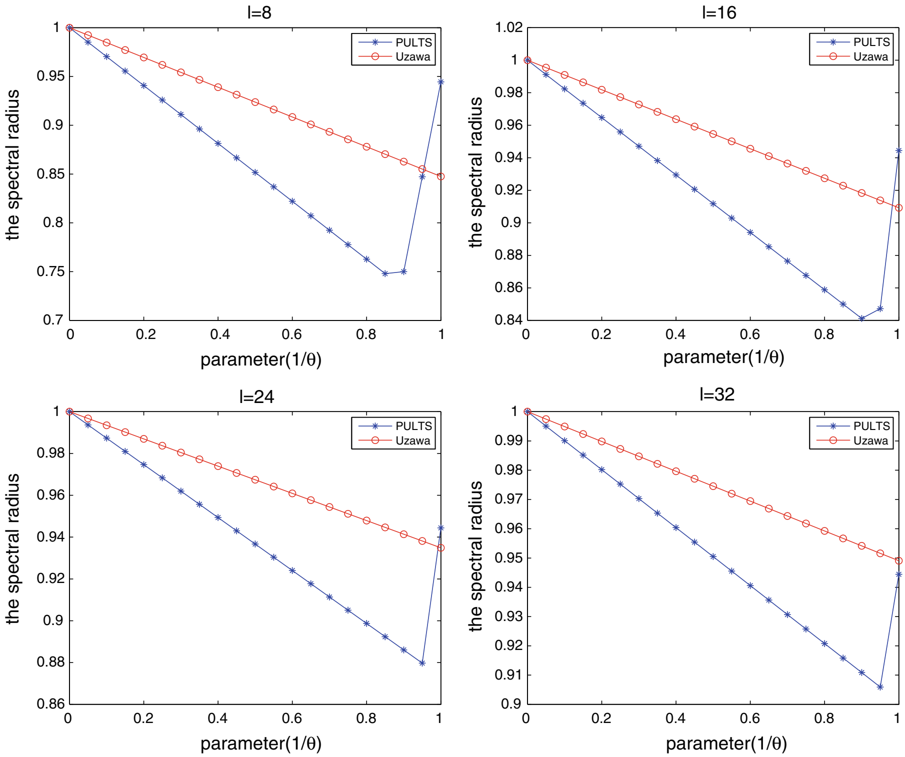

Fig. 1 The spectral radius of PULTS method and the spectral radius of Uzawa method of the iteration matrices for different l

in most cases. In order to demonstrate the trait of the PULTS methods, in Fig. 2, we also plot the eigenvalues distribution of the iteration matrices of the PULTS methods for different l when $\theta = \theta_{\text{opt}}$. As observed from Fig. 2, most eigenvalues of the iteration matrices are quite clustered; it means that the PULTS methods have good convergence properties.

Example 2 Consider the linear system like the form (1.1) with the following matrix blocks

$$A = (a_{ij})_{m\times m} = \begin{cases} a_{ij} = i + j, \; i = j, \\ a_{ij} = -\frac{1}{m}, \quad i \neq j, \end{cases}$$
$$B = (b_{ij})_{m\times n} = \begin{cases} b_{ij} = 1, \; i = j, \\ b_{ij} = 0, \; i \neq j, \end{cases}$$

For this numerical test, we compared the PULTS iteration methods with the Uzawa method [26]; the corresponding numerical results are listed in Table 2.

The preconditioner matrix Q of the PULTS and the Uzawa methods are taken as θI and $B^\top A^{-1}B$. From Table 2, we can see that PULTS methods perform better

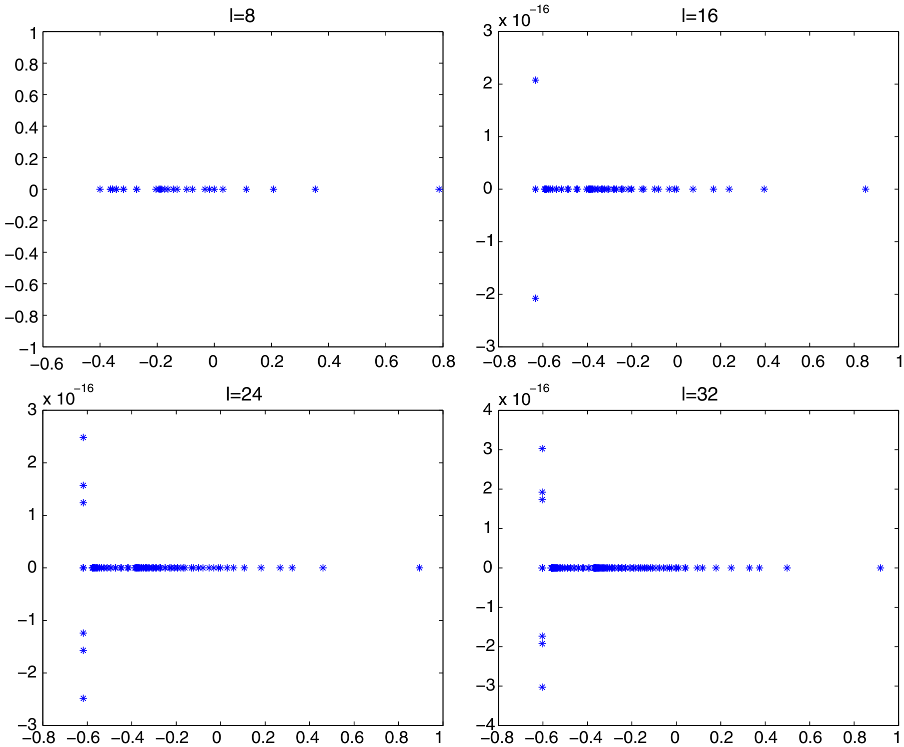

Fig. 2 The eigenvalue distributions of the iteration matrices of the PULTS methods for diffierent l when $\theta = \theta_{\text{opt}}$

Table 2 Numerical results about different methods for Example 2

m		128	200	512	800
n		64	100	256	400
	τ_{opt}	3.938	3.9603	3.9844	3.9900
	IT	228	323	646	874
Uzawa	CPU	0.2768	0.4915	2.5016	6.3837
	ERR	9.9525e-7	9.9083e-7	9.9295e-7	9.9989e-7
	α	1.11	1.15	1.18	1.25
	β	0.88	0.85	0.79	0.72
	θ_{opt}	0.5173	0.5166	0.5304	0.5486
PULTS	IT	182	252	464	591
	CPU	0.1024	0.2206	1.6362	4.3463
	ERR	9.9965e-07	9.8380e-07	9.9415e-07	9.9571e-7

than Uzawa method, since the PULTS iteration methods need much less CPU and IT to achieve the stopping criterion than Uzawa method, which further confirms the feasibility of PULTS methods.

5 Conclusions

In this paper, we studied a class of new iterative methods for large sparse nonsingular saddle point problems (1.1) based on the parametered upper and lower triangular splitting (PULTS) of the coefficient matrix. The property of eigenvectors and eigenvalues of the iteration matrix of PULTS iteration methods are analyzed. We verified that these new methods are convergent under some conditions; sufficient and necessary conditions for the convergence of PULTS methods are provided in the paper. Moreover, the optimal iteration parameters and corresponding convergence factors are obtained with some special cases of the PULTS methods. Numerical experiments are given to confirm the theoretical results, which implies that PULTS methods are effective and feasible for nonsingular saddle point problems.

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