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# Modified complex-symmetric and skew-Hermitian splitting iteration method for a class of complex-symmetric indefinite linear systems

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**Abstract** In this paper, based on the complex-symmetric and skew-Hermitian splitting (CSS) of the coefficient matrix, a modified complex-symmetric and skew-Hermitian-splitting (MCSS) iteration method is presented to solve a class of complex-symmetric indefinite linear systems from the classical state-space formulation of frequency analysis of the degree-of-freedom discrete system. The convergence properties of the MCSS method are obtained. The corresponding MCSS preconditioner is proposed and some useful properties of the preconditioned matrix are established. Numerical experiments are reported to verify the efficiency of both the MCSS method and the MCSS preconditioner.

**Keywords** Complex-symmetric linear system · Complex-symmetric matrix · Skew-Hermitian matrix · Matrix splitting · CSS method · Convergence

**Mathematics Subject Classification (2010)** 65F10 · 65F50

## 1 Introduction

Consider the equations of motion of the degree-of-freedom linear system

$$M\ddot{q} + C\dot{q} + Kq = p, \tag{1.1}$$

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where $q$ is the configuration vector, $p$ is the vector of generalized components of dynamic forces, matrices $M$, $K$, and $C$ are the inertia, stiffness, and viscous damping matrices, respectively. Complex harmonic excitation at the driving circular frequency $\omega > 0$, i.e., of the type $p(t) = fe^{i\omega t}$, admits the steady state solution $q(t) = \tilde{q}(\omega)e^{i\omega t}$, where $\tilde{q}$ solves the linear system $E(\omega)\tilde{q}(\omega) = f$ and $E(\omega)$ is the dynamic impedance matrix. Substituting $q(t) = \tilde{q}(\omega)e^{i\omega t}$ into (1.1) leads to the following matrix $E(\omega)$

$$E(\omega) = -\omega^2 M + i\omega C + K,$$

which leads to the complex-symmetric linear system

$$[-\omega^2 M + K + i\omega C]\tilde{q}(\omega) = f, \tag{1.2}$$

where $M$, $K$, and $C$ are real symmetric positive definite matrices, and $i = \sqrt{-1}$ denotes the imaginary unit. When the driving circular frequency $\omega$ is sufficiently large, $-\omega^2 M + K$ in (1.2) is symmetric indefinite. This implies that the linear system (1.2) is a complex-symmetric indefinite linear system. At present, system such as (1.2) attracts considerable attention because it comes from many actual problems in scientific computing and engineering applications, such as Helmholtz equations [1–4]. One can also see [5–12] for more examples and additional references.

In recent years, many efficient numerical methods for solving complex-symmetric linear system have been developed in the literatures. One can deal with one of its several $2n \times 2n$ equivalent real formulations blow

$$\begin{bmatrix} -\omega^2 M + K & -\omega C \\ \omega C & -\omega^2 M + K \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{1.3}$$

to obtain the solution of complex-symmetric linear system in [8, 9, 13, 14]. For other real equivalent formulations, one can see [8, 9]. In [8], the nonsymmetric Krylov subspace methods combining with standard incomplete LU (ILU) preconditioner to solve this formulation can perform reasonably well. Further, based on the different real equivalent formulations, the different types of block preconditioners are discussed in [9] and argued that if either the real or the symmetric part of the coefficient matrix is positive semidefinite, block preconditioners for real equivalent formulations may be a useful alternative to preconditioners for the original complex formulation. Based on the following parameter-dependent formulation

$$\begin{bmatrix} -\omega^2 M + K - \alpha\omega C & \omega\sqrt{1+\alpha^2} C \\ \omega\sqrt{1+\alpha^2} C & \omega^2 M - K - \alpha\omega C \end{bmatrix} \begin{bmatrix} x_1 \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ d \end{bmatrix}, \alpha > 0, \tag{1.4}$$

a class of “C-to-R” methods has been proposed in [13]. Essentially, the “C-to-R” method is a preconditioned iteration method applied to a Schur complement reduction of a bilaterally transformed variant of the block two-by-two linear system (1.4). In the implementations, the corresponding Schur complement linear system can be solved efficiently by using the preconditioner $-\omega^2 M + K + \alpha\omega C$ [13, 14].

For solving the linear system (1.2) efficiently, based on the Hermitian and skew-Hermitian splitting of the coefficient matrix, a class of Hermitian and skew-Hermitian splitting (HSS) iteration method is designed in [15]. To improve the efficiency of the HSS method, the alternately iterating preconditioned HSS (PHSS) method [5] is developed. At each step of the PHSS iteration, it not only needs to solve the linear sub-system with Hermitian positive definite matrix $\alpha V + H$ but also needs to solve the linear sub-system with the shifted skew-Hermitian matrix $\alpha V + S$, where matrices $H$ and $S$, respectively, are Hermitian and skew-Hermitian parts of the coefficient matrix in (1.3) and matrix $V$ is Hermitian positive definite, when the coefficient matrix in (1.3) is positive definite. The shifted skew-Hermitian system, however, can be much more problematic. To overcome this drawback, the modified HSS (MHSS) method [16] and the preconditioned MHSS (PMHSS) method [6, 17] are proposed. A considerable advantage of the PMHSS method consists in the fact that the solution of linear system with the shifted skew-Hermitian matrix is avoided and only two linear sub-systems with the real and symmetric positive definite coefficient matrix $-\omega^2 M + K + \alpha\omega C$ need to be solved when the preconditioning matrix $V$ is chosen to matrix $\omega C$ in [17].

When $-\omega^2 M + K$ is symmetric positive definite, the aforementioned methods are often popular, such as the PMHSS method [17] and the "C-to-R" method [13, 14, 17]. When $-\omega^2 M + K$ is symmetric indefinite, these two numerical methods may breakdown. "C-to-R" iteration method may be a risk because the preconditioner $-\omega^2 M + K + \alpha\omega C$ may be singular (such as, in theory, choosing the values of $\omega$ and $\alpha$ satisfies $\omega^2 M = K + \alpha\omega C$). The PMHSS method is invalid because the spectral radius of the iteration matrix of the corresponding iteration method may be very close to 1 or even larger than 1 when matrix $-\omega^2 M + K + \alpha\omega C$ may be very close to singular or singular. To avoid the symmetric indefinite matrix $-\omega^2 M + K$ in (1.2), the complex-symmetric and skew-Hermitian splitting (CSS) iteration method [18] is established by the complex-symmetric and skew-Hermitian splitting (CSS) of the coefficient matrix from the classical state-space formulation of frequency analysis of discrete dynamic linear systems [10]. The convergence properties of the CSS method are obtained. Whereas, in theory, it only shows in [18] that the spectral radius of the iteration matrix of the CSS method is less than or equal to one. In this paper, to overcome their shortcomings and improve the convergence rate of the CSS method, by introducing a proper matrix for the classical state-space formulation of frequency analysis of the degree-of-freedom discrete system and combining with the CSS method, we establish a modified CSS (MCSS) method for solving the complex-symmetric linear system (1.2). The convergence properties of the MCSS method are discussed.

The remainder of the paper is organized as follows. In Section 2, the MCSS method is established. In Section 3, we discuss the convergence of the MCSS method and the eigenproperties of the MCSS-preconditioned matrix. In Section 4, the results of numerical experiments from the degree-of-freedom linear system and Helmholtz equations are reported. Finally, in Section 5, we give some conclusions to end the paper.

## 2 The MCSS method

In this section, we design the MCSS method for solving the complex-symmetric linear system (1.2). To this end, the system (1.2) can be rewritten in the classical state-space formulation as follows:

$$Ax \equiv \begin{bmatrix} i\omega K & -K \\ K & C+i\omega M \end{bmatrix} \begin{bmatrix} \tilde{q}(\omega) \\ i\omega\tilde{q}(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}. \tag{2.1}$$

Let

$$P = \begin{bmatrix} (1-i)I & 0 \\ \frac{1}{\omega}I & Q \end{bmatrix},$$

where matrix $Q$ is real symmetric positive definite, $I$ denotes an identity matrix with proper sizes here and in the subsequent discussions. By left-multiplying matrix $P$ for the complex linear system (2.1), we obtain

$$\mathcal{A}x \equiv \begin{bmatrix} (1+i)\omega K & (i-1)K \\ iK+QK & -\frac{K}{\omega}+QC+i\omega QM \end{bmatrix} \begin{bmatrix} \tilde{q}(\omega) \\ i\omega\tilde{q}(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ Qf \end{bmatrix} \equiv b. \tag{2.2}$$

*Remark 2.1* When $M$, $K$, and $C$ are real and symmetric positive definite matrices and $\omega \neq 0$, matrix $\mathcal{A}$ in (2.2) is nonsingular because matrix $P$ is nonsingular and matrix $A$ is nonsingular as well in [18]. The potential advantage of the coefficient matrix in (2.2) is the fact that the indefinite symmetric matrix $-\omega^2 M + K$ can be avoided. Here, we can easily choose the proper matrix $Q$ to keep matrix $-\frac{K}{\omega}+QC$ positive definite. For example, if we take $Q = \frac{2K}{\omega}C^{-1}$, then $-\frac{K}{\omega}+QC = \frac{K}{\omega}$ is symmetric positive definite. Even if $Q = I$, the symmetric positive definite matrix $-\frac{K}{\omega}+C$ in (2.2) is often encountered in [5, 6, 18–20]. In particular, when the driving circular frequency $\omega$ is sufficiently large, we can also obtain a symmetric positive definite matrix $-\frac{K}{\omega}+C$. For simplicity, without loss of generality, we assume $Q = I$ throughout this paper.

Based on the following matrix splitting of the coefficient matrix $\mathcal{A}$ in (2.2)

$$\mathcal{A} = \begin{bmatrix} (1+i)\omega K & 0 \\ 0 & -\frac{K}{\omega}+C+i\omega M \end{bmatrix} + \begin{bmatrix} 0 & (i-1)K \\ (1+i)K & 0 \end{bmatrix} \equiv \mathcal{P}+\mathcal{S}, \tag{2.3}$$

where matrix $\mathcal{P}$ is complex-symmetric and matrix $\mathcal{S}$ is skew-Hermitian, i.e., the matrix splitting (2.3) of matrix $\mathcal{A}$ is a complex-symmetric and skew-Hermitian matrix splitting. Since matrix $\mathcal{S}$ is skew-Hermitian, all its eigenvalues of matrix $\mathcal{S}$ are pure imaginary numbers. Here, we consider that the driving circular frequency $\omega$ is sufficiently large such that matrix $-\frac{K}{\omega}+C$ is symmetric positive definite. In this case, matrix $\mathcal{P}$ is positive definite.

Based on (2.3), we have

$$(\alpha I + \mathcal{P})x = (\alpha I - \mathcal{S})x + b \tag{2.4}$$

and

$$(\alpha I + \mathcal{S})x = (\alpha I - \mathcal{P})x + b. \tag{2.5}$$

By alternately iterating between two systems of fixed-point equations (2.4) and (2.5), we present a modified CSS (MCSS) method to solve the system of linear equations (2.2) and describe as follows.

**The MCSS method** Let $\alpha > 0$ and $x^{(0)} \in \mathbb{C}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty}$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + \mathcal{P})x^{(k+\frac{1}{2})} = (\alpha I - \mathcal{S})x^{(k)} + b, \\ (\alpha I + \mathcal{S})x^{(k+1)} = (\alpha I - \mathcal{P})x^{(k+\frac{1}{2})} + b. \end{cases} \tag{2.6}$$

Since matrix $\mathcal{P}$ is a complex-symmetric matrix, the MCSS method does not belong to a class of the classical HSS methods [15] because of $\mathcal{P}^* \neq \mathcal{P}$, nor belong to a class of the classical NSS methods [22] because of $\mathcal{P}^*\mathcal{P} \neq \mathcal{P}\mathcal{P}^*$. When matrix $\mathcal{P}$ is positive definite, the MCSS method belongs to the class of the PSS methods [23].

Evidently, at each step of the MCSS iteration, we need to solve two linear subsystems, whose coefficient matrices, respectively, are the matrix $\alpha I + \mathcal{P}$ and the matrix $\alpha I + \mathcal{S}$. Since matrices $\alpha I + \mathcal{P}$ and $\alpha I + \mathcal{S}$ are positive definite, there can employ some Krylov subspace methods (such as GMRES) to solve the linear systems.

After straightforward derivations, the MCSS method can be reformulated into the standard form

$$x^{(k+1)} = M_\alpha x^{(k)} + N_\alpha b, k = 0, 1, 2, \ldots \tag{2.7}$$

where

$$M_\alpha = (\alpha I + \mathcal{S})^{-1}(\alpha I - \mathcal{P})(\alpha I + \mathcal{P})^{-1}(\alpha I - \mathcal{S})$$

is the iteration matrix of the MCSS method and

$$N_\alpha = 2\alpha(\alpha I + \mathcal{S})^{-1}(\alpha I + \mathcal{P})^{-1}.$$

In addition, if we introduce matrices

$$B_\alpha = \frac{1}{2\alpha}(\alpha I + \mathcal{P})(\alpha I + \mathcal{S}) \text{ and } C_\alpha = \frac{1}{2\alpha}(\alpha I - \mathcal{P})(\alpha I - \mathcal{S}),$$

then

$$A = B_\alpha - C_\alpha \text{ and } M_\alpha = B_\alpha^{-1} C_\alpha. \tag{2.8}$$

Therefore, one can readily verify that the MCSS method can be induced by the matrix splitting $A = B_\alpha - C_\alpha$. From (2.8), we know that the splitting matrix $B_\alpha$ can be used as a preconditioner, which can be called the MCSS preconditioner, to improve the convergence rate of GMRES. When $B_\alpha$ is used as a preconditioner, the multiplicative factor $\frac{1}{2\alpha}$ has no effect on the preconditioned system, and it can be deleted.

## 3 Convergence analysis

In this section, we prove that under suitable conditions, the MCSS method (2.6) converges to the unique solution of (2.1) for any initial guess. To establish the convergence theory of the MCSS method, the following lemma is required.

**Lemma 3.1** ([15]) *Let* $A \in \mathbb{C}^{n\times n}$, $A = M_i - N_i$ $(i = 1, 2)$ *be two splittings of* $A$, *and* $x^{(0)} \in \mathbb{C}^n$ *be a given initial vector. If* $\{x^{(k)}\}$ *is a two-step iteration sequence defined by*

$$\begin{cases} M_1 x^{(k+\frac{1}{2})} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{(k+\frac{1}{2})} + b, \end{cases}$$

$k = 0, 1, \ldots,$ *then*

$$x^{(k+1)} = M_2^{-1}N_2M_1^{-1}N_1x^{(k)} + M_2^{-1}(I + N_2M_1^{-1})b, \ k = 0, 1, \ldots.$$

*Moreover, if the spectral radius* $\rho(M_2^{-1}N_2M_1^{-1}N_1) < 1$, *then the iterative sequence* $\{x^{(k)}\}$ *converges to the unique solution* $x^* \in \mathbb{C}^n$ *of the system (1.2) for all initial vectors* $x^{(0)} \in \mathbb{C}^n$.

Concerning the convergence property of the stationary MCSS iteration method, we have the following theorem.

**Theorem 3.1** *Let* $\mathcal{A} = \mathcal{P} + \mathcal{S} \in \mathbb{C}^{n\times n}$ *with*

$$\mathcal{P} = \begin{bmatrix} (1+i)\omega K & 0 \\ 0 & -\frac{K}{\omega} + C + i\omega M \end{bmatrix} \text{ and } \mathcal{S} = \begin{bmatrix} 0 & (i-1)K \\ (i+1)K & 0 \end{bmatrix},$$

*where* $\omega$, $M$, $K$ *and* $C$ *are previously defined. If matrix* $-\frac{K}{\omega} + C$ *is positive definite, then for* $\alpha > 0$, *the iteration matrix* $M_\alpha$ *of the MCSS method is*

$$M_\alpha = (\alpha I + \mathcal{S})^{-1}(\alpha I - \mathcal{P})(\alpha I + \mathcal{P})^{-1}(\alpha I - \mathcal{S}), \tag{3.1}$$

*and its spectral radius* $\rho(M_\alpha)$ *of the MCSS iteration matrix satisfies*

$$\rho(M_\alpha) < 1 \textit{ for } \forall \alpha > 0.$$

*Proof* Setting in Lemma 3.1

$$M_1 = \alpha I + \mathcal{P}, N_1 = \alpha I - \mathcal{S}, M_2 = \alpha I + \mathcal{S} \text{ and } N_1 = \alpha I - \mathcal{P}$$

and considering that $\alpha I + \mathcal{P}$ and $\alpha I + \mathcal{S}$ are nonsingular for $\alpha > 0$ , we obtain (3.1).

Using the convergence theorem of the PSS method (Theorem 2.3 in [23]), it is not difficult to find that the results in Theorem 3.1 hold. □

If the extreme eigenvalues of the matrix $P$ are known, then the value of $\alpha$ which minimizes the upper bound can be obtained. This fact is precisely stated as the following corollary, one can see [22] for more details.

**Corollary 3.1** *Let the conditions of Theorem 3.1 be satisfied and* $\lambda(\mathcal{P})$ *denote the spectral set of matrix* $\mathcal{P}$. *Then,*

$$M_\alpha \leq \sigma_\alpha < 1,$$

*where*

$$\sigma_\alpha = \max_{\lambda_j \in \lambda(\mathcal{P})} \frac{|\alpha - \lambda_j|}{|\alpha + \lambda_j|} = \max_{\gamma_j + i\eta_j \in \lambda(\mathcal{P})} \frac{\sqrt{(\alpha - \gamma_j)^2 + \eta_j^2}}{\sqrt{(\alpha + \gamma_j)^2 + \eta_j^2}}.$$

*Moreover, if* $\gamma_{\min}$ *and* $\gamma_{\max}$, $\eta_{\min}$, *and* $\eta_{\max}$ *are the lower and the upper bounds of the real, the absolute values of the imaginary parts of the eigenvalues of the matrix* $\mathcal{P}$, *respectively, and* $\Omega = [\gamma_{\min}, \gamma_{\max}] \times [\eta_{\min}, \eta_{\max}]$, *then*

$$\alpha^* = \arg\min_{\alpha} \left\{ \max_{(\gamma,\eta)\in\Omega} \frac{\sqrt{(\alpha-\gamma_j)^2+\eta_j^2}}{\sqrt{(\alpha+\gamma_j)^2+\eta_j^2}} \right\}$$
$$= \begin{cases} \sqrt{\gamma_{\min}\gamma_{\max}-\eta_{\max}^2} & \text{for } \eta_{\max} < \sqrt{\dfrac{\gamma_{\min}(\gamma_{\max}-\gamma_{\min})}{2}}, \\ \sqrt{\gamma_{\min}^2+\eta_{\max}^2} & \text{for } \eta_{\max} \geq \sqrt{\dfrac{\gamma_{\min}(\gamma_{\max}-\gamma_{\min})}{2}} \end{cases}$$

*and*

$$\sigma_{\alpha^*} = \begin{cases} \sqrt{\dfrac{\gamma_{\min}+\gamma_{\max}-2\sqrt{\gamma_{\min}\gamma_{\max}-\eta_{\max}^2}}{\gamma_{\min}+\gamma_{\max}+2\sqrt{\gamma_{\min}\gamma_{\max}-\eta_{\max}^2}}} & \text{for } \eta_{\max} < \sqrt{\dfrac{\gamma_{\min}(\gamma_{\max}-\gamma_{\min})}{2}}, \\ \sqrt{\dfrac{\sqrt{\gamma_{\min}^2+\eta_{\max}^2}-\gamma_{\min}}{\sqrt{\gamma_{\min}^2+\eta_{\max}^2}+\gamma_{\min}}} & \text{for } \eta_{\max} \geq \sqrt{\dfrac{\gamma_{\min}(\gamma_{\max}-\gamma_{\min})}{2}}. \end{cases}$$

Some remarks on Theorem 3.1 and Corollary 3.1 are given below.

- The convergence rate of the MCSS method is bounded by $\sigma_\alpha$, which depends on the spectrum of the positive definite matrix $\mathcal{P}$, and does not depend on the spectrum of the skew-Hermitian matrix $\mathcal{S}$ and nor on the eigenvectors of the matrices $\mathcal{P}$, $\mathcal{S}$, and $\mathcal{A}$.
- From Theorem 3.1 and Corollary 3.1, it makes sense to seek $\alpha$ to make $\rho(M_\alpha)$ as small as possible, but the determination of the optimal parameter $\alpha$ is a nontrivial task. Even so, Corollary 3.1 is still helpful for us to choose an effective parameter $\alpha$ for the MCSS method.
- If we use the proper matrix $V$ (such as symmetric positive definite matrix $V$) instead of $I$ in (2.6), then this yields the preconditioned MCSS (PMCSS) method. If we take

  $$V = \begin{bmatrix} (1-\frac{i}{\alpha})\omega K & 0 \\ 0 & -\frac{K}{\omega}+C-\frac{i}{\alpha}\omega M \end{bmatrix},$$

  then $\alpha V + \mathcal{P}$ is symmetric positive definite. In this case, we can solve the corresponding linear sub-system either exactly by a sparse Cholesky factorization or inexactly by conjugated gradient scheme. Since matrix $\alpha V + \mathcal{S}$ is positive definite, the solution of linear sub-system with matrix $\alpha V + \mathcal{S}$ can be obtained by Krylov space methods (such as GMRES).
- From Theorem 3.1, the preconditioned matrix $B_\alpha^{-1}\mathcal{A}$ is positive definite. It is because all the eigenvalues of the MCSS-preconditioned matrix lie in the interior of the disk of radius 1 centered at the point $(1, 0)$.

With respect to the eigenvalue distribution of the MCSS-preconditioned matrix $B_\alpha^{-1}\mathcal{A}$, we have the following theorem. One can see [18] for details.

**Theorem 3.2** *Let the conditions of Theorem 3.1 be satisfied. Then all the eigenvalues of* $B_\alpha^{-1}\mathcal{A}$ *satisfy* $|1-|\lambda||\leq 1$.

## 4 Numerical experiments

In this section, numerical experiments are used to verify the performance of the MCSS method and the MCSS preconditioner. At the same time, we compare the MCSS method with the CSS method, the HSS method [15], the MHSS method [16], the SHNS method [19], and the GPMHSS method (Method 2.1) [24] and also compare their deuterogenic preconditioners.

In our numerical experiments, $M = I$ and $C = \omega C_V + C_H$, where $C_V = 5I$ and $C_H = \mu K$ with $\mu = 0.02$, and $K$ is the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh size $h = \frac{1}{m+1}$. The matrix $K \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$ with $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. In addition, the right-hand side vector $b$ is to be adjusted such that $b = (1 + i)\mathcal{A}e$ ($e = (1, 1, \ldots, 1)^T$).

All tests are started from the zero vector and these six methods terminate if the relative residual error satisfies $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} < 10^{-6}$. In the following table, "IT" denotes the number of iteration steps. "CPU" denotes the construction time (in seconds). "–" denotes that the iteration sequences do not converge within 1000 iterations.

First, we test that the MCSS method is regarded as a solver. We compare the MCSS method with the CSS, HSS, MHSS, SHNS, and GPMHSS methods to solve the complex symmetric linear system (2.1) under the bigger $\omega$. For the coefficient matrix $A$ in (2.1), the HSS method in [15] is established below

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \quad (\alpha > 0),$$

where

$$H = \frac{1}{2}(A + A^*) = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \text{and } S = \frac{1}{2}(A - A^*) = \begin{bmatrix} i\omega K & -K \\ K & i\omega M \end{bmatrix}.$$

Clearly, $A = H + S$. Further, we set $Z = \frac{1}{2i}(A - A^*)$. Then,

$$A = H + iZ \tag{4.1}$$

and $S = iZ$. Therefore, the corresponding MHSS method in [16] is described as follows:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - iZ)x^{(k)} + b, \\ (\alpha I + Z)x^{(k+1)} = (\alpha I + iH)x^{(k+\frac{1}{2})} - ib, \end{cases} \quad (\alpha > 0).$$

Based on (4.1), we can establish the SHNS method and the GPMHSS method as well. Specifically, one can see [19, 24] for more details.

Next, we need to choose the proper parameter for these six methods. Since the parameter $\alpha$ plays an important role in the HSS method and its other versions (such as MHSS, SHNS, and GPMHSS), along with CSS, an accurate approximation to the optimal value of parameter $\alpha$ may significantly speed up the convergence rate of these iteration method. Although this is very difficult, many researchers have devoted to estimate the value of the parameter $\alpha$ and have obtained many valuable results, see [25–29]. By adopting a reasonable and simple optimization principle, Chen in [29] derived a cubic polynomial equation to estimate the parameter $\alpha$ for the HSS method. Numerical results show that the strategy for computing the optimal parameter in [29] is better than that in [15, 28]. Based on this, noting that the minimal singular value of the matrix $S$ is 0 in our experiments, we estimate the value of the parameter $\alpha$ using the method proposed in [29], i.e.,

$$2\alpha^3 + (\lambda_{\max} + \lambda_{\min} - \frac{\sigma_{\max}^2}{\lambda_{\max} - \lambda_{\min}})\alpha^2 - 2\frac{\sigma_{\max}^2 \lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}\alpha - \frac{\sigma_{\max}^2 \lambda_{\min}^2}{\lambda_{\max} - \lambda_{\min}} = 0, \quad (4.2)$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimal and the maximal eigenvalues of the matrix $H$, and $\sigma_{\max}$ is the maximal singular value of the matrix $S$. Since the eigenvalues of the matrix $\mathcal{P}$ in the MCSS are complex, the minimal and the maximal value of the absolute value of all the eigenvalues of matrices $\mathcal{P}$ instead of the corresponding $\lambda_{\min}$ and $\lambda_{\max}$ in (4.2) is to estimate the value of the parameter $\alpha$ for the MCSS method, as well as the matrix $\mathcal{C}$ in the CSS method [18]. In this case, the corresponding number of the iteration steps and CPU times with varying $\omega$ are listed in Table 1.

Some remarks on Table 1 are given below.

- When the GPMHSS method in [24] is applied, the choice of the preconditioner $P$ is $P = I$. The goal of this choice is to tend a fair comparison with MCSS, CSS, HSS, MHSS, and SHNS.
- When using (4.2) to obtain the value of the parameter for the HSS and MHSS methods, we find that this strategy for the choice of the parameter for the HSS and MHSS methods is invalid because this way leads to the non-convergence of the HSS and MHSS methods. In this case, Table 1 does not list the numerical results of the HSS and MHSS methods.

**Table 1** $\alpha$, IT, and CPU for MCSS, CSS, and GPMHSS with $n = 512$

| | $\omega$ | 100 | 200 | 300 | 500 |
|---|---|---|---|---|---|
| | $\alpha$ | 5.1524 | 5.1261 | 5.1171 | 5.1099 |
| MCSS | IT | 9 | 6 | 6 | 5 |
| | CPU | 0.078 | 0.063 | 0.047 | 0.047 |
| | $\alpha$ | 5.1543 | 5.1265 | 5.1173 | 5.11 |
| CSS | IT | 8 | 6 | 6 | 5 |
| | CPU | 0.062 | 0.047 | 0.032 | 0.032 |
| | $\alpha$ | 9.6900e+3 | 1.9371e+4 | 2.9054e+4 | 5 4.8419e+4 |
| GPMHSS | IT | 124 | 122 | 121 | 121 |
| | CPU | 1.25 | 1.094 | 1.079 | 1.125 |

- When this strategy in (4.2) for the choice of the parameter for the SHNS method is also invalid because of $\lambda_{max} = \lambda_{min}$. This implies that we do not obtain the value of the parameter for the SHNS method. In this case, Table 1 does not present the numerical results of the SHNS method.
- Numerical results in Table 1 show that the performance of the MCSS and CSS methods is almost the same. Whereas, there is a certain risk for the CSS method. It is reason that the spectral radius of the iteration matrix of the CSS method is less than or equal to one in [18].

From Table 1, the convergence rate of MCSS, CSS, and GPMHSS depends on the parameter $\alpha$ and the driving circular frequency $\omega$. The numerical results in Table 1 show that the MCSS method solving this class of complex linear system (2.1) is feasible and competitive.

In order to further contrast the performance of the above six methods, the value of iteration parameter $\alpha$ is selected by the statement on the choice of the iteration parameter [21], that is to say, experience suggests that in most applications and for an appropriate scaling of the problem, a "small" value of $\alpha$ (usually between 0.01 and 0.5) may give good results. The numerical results are reported in Tables 2 and 3.

In Tables 2 and 3, we show the number of iterations and CPU times for MCSS, CSS, HSS, MHSS, and SHNS methods. In our numerical computations, we find that the GPMHSS method is not convergent, Tables 2 and 3 do not list its numerical results. But beyond that, the other five methods are convergent. From Tables 2 and 3, the performance of the MCSS and CSS methods is almost the same, the HSS and MHSS methods too. MCSS, CSS, HSS, and MHSS outperform SHNS under the convergent condition. Fixing the mesh size with $\omega$ and $\alpha$ increasing, a trend of the number of iterations of MCSS, CSS, and SHNS reduces and a trend of the number of iterations of the HSS and MHSS methods grows. These numerical results in Tables 2 and 3 further show that the MCSS method for solving this class of complex linear system (2.1) is very competitive under certain conditions.

**Table 2** $\alpha$, IT, and CPU for MCSS, CSS, and GPMHSS with $n = 512$

| | $\omega$ | 100 | 200 | 300 | 500 | 1000 |
|---|---|---|---|---|---|---|
| | $\alpha$ | 0.01 | 0.05 | 0.1 | 0.5 | 1 |
| MCSS | IT | 8 | 6 | 6 | 5 | 4 |
| | CPU | 0.157 | 0.093 | 0.109 | 0.094 | 0.078 |
| CSS | IT | 8 | 6 | 6 | 5 | 4 |
| | CPU | 0.079 | 0.062 | 0.047 | 0.047 | 0.032 |
| HSS | IT | 43 | 45 | 46 | 50 | 50 |
| | CPU | 0.296 | 0.297 | 0.297 | 0.375 | 0.343 |
| MHSS | IT | 43 | 45 | 46 | 50 | 50 |
| | CPU | 0.218 | 0.203 | 0.219 | 0.266 | 0.219 |
| SHNS | IT | 59 | 56 | 54 | 52 | 50 |
| | CPU | 0.5 | 0.484 | 0.453 | 0.407 | 0.45 |

**Table 3** $\alpha$, IT, and CPU for MCSS, CSS, and GPMHSS with $n = 2048$

| | $\omega$ | 100 | 200 | 300 | 500 | 1000 |
|---|---|---|---|---|---|---|
| | $\alpha$ | 0.01 | 0.05 | 0.1 | 0.5 | 1 |
| MCSS | IT | 14 | 8 | 7 | 6 | 5 |
| | CPU | 2.953 | 1.687 | 1.453 | 1.328 | 1.11 |
| CSS | IT | 11 | 8 | 6 | 6 | 5 |
| | CPU | 0.938 | 0.609 | 0.531 | 0.516 | 0.407 |
| HSS | IT | 40 | 42 | 43 | 45 | 45 |
| | CPU | 1.375 | 1.328 | 1.485 | 1.484 | 1.422 |
| MHSS | IT | 40 | 42 | 43 | 45 | 45 |
| | CPU | 0.969 | 1.062 | 1.016 | 1.016 | 1.094 |
| SHNS | IT | 53 | 51 | 49 | 48 | 45 |
| | CPU | 2.703 | 2.688 | 2.5 | 2.375 | 2.234 |

In the sequel, the MCSS iteration is used as a preconditioner with GMRES(20), comparing with the CSS, HSS, GPMHSS, MHS, and SHNS preconditioners. When comparing these six preconditioners, the choice of parameter $\alpha$ is similar to the front. Specifically, see Tables 4 and 5.

In our numerical computations, the GMRES(20) method terminates if the relative residual error satisfies $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} < 10^{-6}$. In Tables 4 and 5, "$P_{MCSS}$" denotes the MCSS-preconditioned GMRES(20) method, "$P_{CSS}$" denotes the CSS-preconditioned GMRES(20) method, "$P_{HSS}$" denotes the HSS-preconditioned GMRES(20) method, "$P_{GPMHSS}$" denotes the GPMHSS-preconditioned GMRES(20) method, "$P_{MHSS}$"

**Table 4** IT and CPU for MCSS-, CSS-, HSS-, GPMHSS-, MHSS- and SHNS-preconditioned GMRES(20) with $n = 512$

| | $\omega$ | 100 | 50 | 30 | 20 | 1 |
|---|---|---|---|---|---|---|
| | $\alpha$ | 0.01 | 0.05 | 0.1 | 0.5 | 1 |
| $P_{MCSS}$ | IT | 1 | 1 | 1 | 3 | 8 |
| | CPU | 0.032 | 0.047 | 0.109 | 0.11 | 0.187 |
| $P_{CSS}$ | IT | 2 | 2 | 4 | 4 | 11 |
| | CPU | 0.156 | 0.141 | 0.234 | 0.219 | 0.453 |
| $P_{HSS}$ | IT | 1 | 7 | – | 20 | 8 |
| | CPU | 0.031 | 0.094 | – | 0.25 | 0.093 |
| $P_{GPMHSS}$ | IT | 1 | 1 | 1 | 1 | 10 |
| | CPU | 0.031 | 0.031 | 0.047 | 0.047 | 0.125 |
| $P_{MHSS}$ | IT | 1 | 7 | – | 20 | 8 |
| | CPU | 0.047 | 0.078 | – | 0.39 | 0.203 |
| $P_{SHNS}$ | IT | 1 | 1 | 1 | 1 | 15 |
| | CPU | 0.45 | 0.484 | 0.453 | 0.407 | 0.5 |

**Table 5** IT and CPU for MCSS-, CSS-, HSS-, GPMHSS-, MHSS-, and SHNS-preconditioned GMRES(20) with $n = 2048$

| | $\omega$ | 100 | 50 | 30 | 20 | 1 |
|---|---|---|---|---|---|---|
| | $\alpha$ | 0.01 | 0.05 | 0.1 | 0.5 | 1 |
| $P_{MCSS}$ | IT | 1 | 1 | 1 | 1 | 5 |
| | CPU | 1.953 | 0.609 | 1.813 | 1.062 | 4.828 |
| $P_{CSS}$ | IT | 2 | 2 | 2 | 3 | 10 |
| | CPU | 2.641 | 2.829 | 2.407 | 3.172 | 7.719 |
| $P_{HSS}$ | IT | 1 | – | – | 18 | 8 |
| | CPU | 0.172 | – | – | 1.328 | 1.047 |
| $P_{GPMHSS}$ | IT | 1 | 1 | 1 | 10 | 7 |
| | CPU | 0.234 | 0.235 | 0.219 | 0.985 | 0.734 |
| $P_{MHSS}$ | IT | 1 | – | – | 18 | 8 |
| | CPU | 0.172 | – | – | 1.328 | 0.672 |
| $P_{SHNS}$ | IT | 1 | 1 | 1 | 1 | 111 |
| | CPU | 0.219 | 0.219 | 0.266 | 0.25 | 9.172 |

denotes the MHSS-preconditioned GMRES(20) method, and "$P_{SHNS}$" denotes the SHNS-preconditioned GMRES(20) method.

In Tables 4 and 5, we report some numerical results for GMRES(20) preconditioned with MCSS, CSS, HSS, GPMHSS, MHSS, and SHNS. From these results, we observe when used as a preconditioner, the MCSS preconditioner is quite competitive in terms of convergence rate, robustness, and efficiency when Krylov subspace methods combining with these six preconditioners are applied to solve the linear system (2.1) under certain conditions. It is noted that when the HSS preconditioner and the MHSS preconditioner are applied to solve the linear system (2.1), breakdown may happen from Tables 4 and 5. Further, by a lot of numerical experiments, we find that the number of iterations of the MCSS-preconditioned GMRES(20) is stable with the grid increasing when the driving circular frequency $\omega$ exceeds 30. This implies that the efficiency of the MCSS preconditioner may be accepted when it is applied to solve the large sparse linear system (2.1) for the sufficiently large $\omega$.

Finally, we consider the complex-symmetric linear systems from the following Helmholtz equations

$$-\Delta u - \sigma_1 u + i\sigma_2 u = f,$$

where $\sigma_1$ and $\sigma_2$ are real coefficient functions, and $u$ satisfies Dirichlet boundary conditions in $D = [0, 1] \times [0, 1]$. The above equation describes the propagation of damped time-harmonic waves. We take $H$ the five-point centered difference matrix approximating the negative Laplacian operator on an uniform mesh with mesh size $h = \frac{1}{m+1}$. The matrix $H \in \mathbb{R}^{n\times n}$ possesses the tensor-product form $H = B_m \otimes I + I \otimes B_m$ with $B_m = h^{-2}\cdot\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Hence, $H$ is an $n \times n$

**Table 6** IT and CPU of MCSS-preconditioned GMRES(20) for Eq. (4.1)

| | $\alpha$ | 0.01 | 0.05 | 0.1 | 0.5 | 1 |
|---|---|---|---|---|---|---|
| $n = 512$ | IT | 3 | 3 | 3 | 3 | 3 |
| | CPU | 0.235 | 0.219 | 0.219 | 0.187 | 0.125 |
| $n = 2048$ | IT | 2 | 2 | 2 | 2 | 2 |
| | CPU | 2.484 | 2.64 | 2.578 | 2.297 | 2.81 |
| $n = 3528$ | IT | 1 | 1 | 1 | 1 | 1 |
| | CPU | 5.859 | 6.031 | 5.969 | 5.828 | 5.68 |

block-tridiagonal matrix, with $n = m^2$. This leads to the complex-symmetric linear system (1.2) of the form

$$[(H - \sigma_1 I) + i\sigma_2 I]x = b,$$

which is equal to

$$\mathcal{A}x \equiv \begin{bmatrix} (1+i)H & (i-1)H \\ (1+i)H & -H + \sigma_2 + i\sigma_1 \end{bmatrix} \begin{bmatrix} x \\ ix \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}. \tag{4.3}$$

To keep matrix $-H + \sigma_2 I$ symmetric positive definite and matrix $H - \sigma_1 I$ symmetric indefinite, some values of $\sigma_1$ and $\sigma_2$ need to be selected. For convenience, we set $\sigma_1 = 4h^{-2}$ and $\sigma_2 = 10h^{-2}$. In our computations, the right-hand side vector $b$ is adjusted to be $b = (1 + i)Ae$ $(e = (1, 1, \ldots, 1)^T)$.

By investigating the aforementioned numerical results, the MCSS method and the MCSS preconditioner, respectively, has certain advantages, compared with the other five methods and their deuterogenic preconditioners. The MCSS preconditioner combining with GMRES(20) outperforms the MCSS method. In this case, we only test the efficiency of the MCSS preconditioner for solving the linear system (4.3).

From Table 6, the MCSS preconditioner is quite competitive in terms of convergence rate, robustness, and efficiency when some Krylov subspace methods combining with the MCSS preconditioner are applied to solve the linear system (4.3) under certain conditions. Table 6 implies that the MCSS preconditioner maybe be suitable for the large sparse linear system (4.3) from Helmholtz equations.

## 5 Conclusion

In this paper, we have introduced the modified CSS (MCSS) method for a class of complex-symmetric indefinite linear system. Theoretical analysis shows that the MCSS method is unconditionally convergent under certain conditions. Numerical experiments illustrate the efficiency of both the MCSS method and the MCSS preconditioner. In particular, the resulting MCSS preconditioner leads to fast convergence when it is used to preconditioned Krylov subspace methods such as GMRES.

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