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A generalized variant of the deteriorated PSS preconditioner for nonsymmetric saddle point problems

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Abstract Based on the variant of the deteriorated positive-definite and skew-Hermitian splitting (VDPSS) preconditioner developed by Zhang and Gu (BIT Numer. Math. 56:587–604, 2016), a generalized VDPSS (GVDPSS) preconditioner is established in this paper by replacing the parameter α in (2,2)-block of the VDPSS preconditioner by another parameter β. This preconditioner can also be viewed as a generalized form of the VDPSS preconditioner and the new relaxed HSS (NRHSS) preconditioner which has been exhibited by Salkuyeh and Masoudi (Numer. Algorithms, 2016). The convergence properties of the GVDPSS iteration method are derived. Meanwhile, the distribution of eigenvalues and the forms of the eigenvectors of the preconditioned matrix are analyzed in detail. We also study the upper bounds on the degree of the minimum polynomial of the preconditioned matrix. Numerical experiments are implemented to illustrate the effectiveness of the GVDPSS

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preconditioner and verify that the GVDPSS preconditioned generalized minimal residual method is superior to the DPSS, relaxed DPSS, SIMPLE-like, NRHSS, and VDPSS preconditioned ones for solving saddle point problems in terms of the iterations and computational times.

Keywords Saddle point problem · Generalized VDPSS preconditioner · GMRES · Preconditioning · Spectral properties

Mathematics Subject Classification (2010) 65F10 · 65F15 · 65F50

1 Introduction

Consider the following large and sparse saddle point problem

$$\mathcal{A}u = \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$ is a nonsymmetric positive definite (i.e., its symmetric part is positive definite), $B \in \mathbb{R}^{n\times m}$ has full row rank, and $p \in \mathbb{C}^m$ and $q \in \mathbb{C}^n$ are given vectors with $n \leq m$. The above assumptions show that the coefficient matrix $\mathcal{A}$ is nonsingular and (1) has unique solution. This kind of linear system is important and frequently arose in a variety of scientific and engineering applications, such as mixed or hybrid finite element approximations of second-order elliptic problems, computational fluid dynamics, constrained optimization, electronic networks and computer graphics, optimal control, weighted least squares problems, and so forth; see [2, 18, 20, 28] and references therein.

Due to the fact that the matrices A and B in (1) are large and sparse, iterative methods are preferable for solving the saddle point problem (1) in terms of storage requirements and computing time. A number of effective iteration methods have been developed for solving the saddle point problem (1), and their numerical properties have been studied in the literature, such as SOR-like methods [13, 15, 31, 32], Uzawa-type methods [13, 15, 19, 29, 43], Hermitian and skew-Hermitian splitting (HSS) methods [8] and its variants [3, 6, 7, 9, 10], RPCG iteration methods [11, 14], and so forth.

Meanwhile, Krylov subspace iteration methods [37] are a class of effective methods for solving such systems of linear equations. However, Krylov subspace methods without good preconditioner often suffer from slow convergence or even stagnation when they are applied to the nonsymmetric saddle point problem (1) as $\mathcal{A}$ is usually ill-conditioned. In order to accelerate the convergence of the associated Krylov subspace method, the preconditioning technique is often used [2, 38]. A high-quality preconditioner plays a crucial role in guaranteeing the fast convergence rate of Krylov subspace methods. The preconditioner usually reduces the number of iteration steps required for convergence. In general, favorable convergence rates of the Krylov subspace methods are often associated with a clustering of most of the eigenvalues of the preconditioned matrix around 1 and away from 0 [17]. Moreover, convergence properties of the Krylov subspace methods are also dependent on the properties of the corresponding eigenvectors of the preconditioned matrix except for the case that the preconditioned matrix is symmetric [1, 5]. In recent years, considerable efforts have been invested in investigating the preconditioners for Krylov subspace methods, such as the generalized minimal residual (GMRES) method [37]. In [3, 41, 42], the authors applied the block-diagonal and block-triangular preconditioners. Bai et al. [12] and Dollar et al. [27] employed the constraint preconditioners. Bai, Golub et al. [6, 7, 10, 36], Zhang et al. [44, 45], Fan et al. [30], and Cao et al. [21, 24] derived the HSS-based preconditioners. On the basis of the shift-splitting of a matrix [16], Cao et al. [22, 23] and Chen et al. [25, 26] investigated the shift-splitting and generalized shift-splitting preconditioners for saddle point problems and so on. By combining the Semi implicit method for pressure linked equations (SIMPLE) preconditioner [33] with the relaxed deteriorated positive-definite and skew-Hermitian splitting (RDPSS) preconditioner presented by Cao et al. [21], Zhang and Zhang newly constructed the SIMPLE-like (SL) preconditioners [35] for saddle point problems. Very recently, Zhou et al. [46] proposed the modified shift-splitting (MSS) preconditioner for non-symmetric saddle point problems. For more details, we refer the readers to [18] for a comprehensive survey of existing approaches for solving saddle point problems.

For nonsymmetric saddle point problem (1), on the basis of the splitting of the coefficient matrix $\mathcal{A}$

$$\mathcal{A} = \mathcal{H} + \mathcal{S},$$

where

$$\mathcal{H} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix}.$$

Pan et al. [36] considered the following splitting of $\mathcal{A}$

$$\mathcal{A} = (\alpha I + \mathcal{H}) - (\alpha I - \mathcal{S}) = (\alpha I + \mathcal{S}) - (\alpha I - \mathcal{H}), \tag{2}$$

where $\alpha > 0$ and I is the identity matrix with appropriate size. With a quite similar strategy of the alternating iteration method, the following splitting iteration method was derived.

$$\begin{cases} (\alpha I + \mathcal{H})u^{(k+\frac{1}{2})} = (\alpha I - \mathcal{S})u^{(k)} + b, \\ (\alpha I + \mathcal{S})u^{(k+1)} = (\alpha I - \mathcal{H})u^{(k+\frac{1}{2})} + b, \end{cases} \quad k = 0, 1, 2, \cdots. \tag{3}$$

From the above iteration scheme, authors proposed the deteriorated positive-definite and skew-Hermitian splitting (DPSS) preconditioner as follows

$$\mathcal{P}_{DPSS} = \frac{1}{2\alpha}(\alpha I + \mathcal{H})(\alpha I + \mathcal{S}) = \frac{1}{2\alpha}\begin{pmatrix} \alpha I + A & 0 \\ 0 & \alpha I \end{pmatrix}\begin{pmatrix} \alpha I & B^T \\ -B & \alpha I \end{pmatrix} \tag{4}$$

for non-Hermitian saddle point problems. Since the factor $\frac{1}{2}$ has no effect on the preconditioned systems, we replace $\frac{1}{2\alpha}$ by $\frac{1}{\alpha}$. From (1) and (4), we see that the disparity between the preconditioner $\mathcal{P}_{DPSS}$ and the matrix $\mathcal{A}$ is

$$\mathcal{R}_{DPSS} = \mathcal{P}_{DPSS} - \mathcal{A} = \begin{pmatrix} \alpha I & \frac{1}{\alpha}AB^T \\ 0 & \alpha I \end{pmatrix}. \tag{5}$$

A general criterion for an efficient preconditioner is that it should be as close as possible to the coefficient matrix $\mathcal{A}$ [44]. To get a closer approximation to the coefficient matrix $\mathcal{A}$ than the DPSS preconditioner, recently, Zhang and Gu [44] employed the variant of the deteriorated PSS (VDPSS) preconditioner for the nonsymmetric saddle point problem (1) as follows:

$$\mathcal{P}_{VDPSS} = \frac{1}{\alpha}\begin{pmatrix} A & 0 \\ 0 & \alpha I \end{pmatrix}\begin{pmatrix} \alpha I & B^T \\ -B & \alpha I \end{pmatrix} = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \alpha I \end{pmatrix}. \tag{6}$$

The difference between $\mathcal{P}_{VDPSS}$ and $\mathcal{A}$ is given by

$$\mathcal{R}_{\text{VDPSS}} = \mathcal{P}_{\text{VDPSS}} - \mathcal{A} = \begin{pmatrix} 0 & \frac{1}{\alpha}AB^T - B^T \\ 0 & \alpha I \end{pmatrix}. \tag{7}$$

It has been mentioned in [44] that although the (1,2)-block in the matrix (7) is different from that of (5), the (1,1)-block in (7) vanishes, and therefore, it means that $\mathcal{P}_{\text{VDPSS}}$ gives a better approximation to $\mathcal{A}$ for the same α. However, it should be noted that the (1,2)-block in (7) tends to $+\infty$ whereas the (2,2)-block approaches to 0 as $\alpha \to 0_+$. In addition, it may be a large α such that (1,2)-block in (7) tends to 0, while (2,2)-block in (7) can become large in this case. Thus, the α is still needed to be found in order to balance the proportion of the two sub-blocks.

Motivated by this situation and inspired by the idea of [30], an additional new parameter will be introduced to overcome the above difficulty. That is, we replace the parameter α in (2,2)-block in $\mathcal{P}_{\text{VDPSS}}$ by another parameter β and a new preconditioner which is referred to as the generalized VDPSS (GVDPSS) preconditioner is derived in this paper. Besides, the spectral properties of the GVDPSS preconditioner and the upper bounds on the degree of the minimal polynomial of the preconditioned matrix are investigated. Numerical experiments are implemented to confirm the effectiveness of the GVDPSS preconditioned GMRES method for nonsymmetric saddle point problems.

The framework of this paper is organized as follows. Section 2 introduces the new proposed preconditioner, i.e., the GVDPSS preconditioner and derives the convergence properties of the GVDPSS iteration method. The spectral properties of the GVDPSS preconditioner and the upper bounds of the degree of the minimal polynomial of the preconditioned matrix are discussed in Section 3. We analyze some implementation aspects about the preconditioner $\mathcal{P}_{\text{GVDPSS}}$ in Section 4. Section 5 is devoted to performing numerical examples to examine the feasibility and effectiveness of the GVDPSS preconditioned GMRES method for nonsymmetric saddle point problems and illustrate that the GVDPSS preconditioned GMRES method has superiority compared with the DPSS, RDPSS, SL, NRHSS, and VDPSS preconditioned GMRES methods for solving saddle point problems. Finally, the paper is ended with some conclusions in Section 6.

2 The generalized variant of the deteriorated PSS preconditioner

Since the spectral distribution of the preconditioned matrix relates closely to the convergence rate of Krylov subspace methods, it is expected that the preconditioned saddle point matrix has desired eigenvalue distribution like tightly clustered spectra or positive real spectra; see [4, 38].

In order to obtain the better preconditioner for the nonsymmetric saddle point problems, based on the coefficient matrix splitting in [6] and (2), we have the following matrix splitting

$$\mathcal{A} = (\Omega + \mathcal{H}) - (\Omega - \mathcal{S}) = (\Omega + \mathcal{S}) - (\Omega - \mathcal{H}),$$

where $\mathcal{H}$ and $\mathcal{S}$ are defined as in (2), and

$$\Omega = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}$$

with $\alpha, \beta > 0$ and I is the identity matrix with appropriate size.

With a quite similar strategy utilized in (3), we may establish the following splitting iteration method

$$\begin{cases} (\Omega + \mathcal{H})u^{(k+\frac{1}{2})} = (\Omega - \mathcal{S})u^{(k)} + b, \\ (\Omega + \mathcal{S})u^{(k+1)} = (\Omega - \mathcal{H})u^{(k+\frac{1}{2})} + b, \end{cases} \quad k = 0, 1, 2, \cdots. \tag{8}$$

Note that the splitting iteration (8) can be derived from the matrix splitting

$$\mathcal{A} = \mathcal{M} - \mathcal{N},$$

where $\mathcal{M} = \frac{1}{2}\Omega^{-1}(\Omega + \mathcal{H})(\Omega + \mathcal{S})$ and $\mathcal{N} = \frac{1}{2}\Omega^{-1}(\Omega - \mathcal{H})(\Omega - \mathcal{S})$. After proper manipulations, we have

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} \alpha^{-1} I & 0 \\ 0 & \beta^{-1} I \end{pmatrix}\begin{pmatrix} \alpha I + A & 0 \\ 0 & \beta I \end{pmatrix}\begin{pmatrix} \alpha I & B^T \\ -B & \beta I \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha I + A & \frac{1}{\alpha}(\alpha I + A)B^T \\ -B & \beta I \end{pmatrix}. \tag{9}$$

To get a better approximation of the coefficient matrix $\mathcal{A}$, we replace the shift term αI in (1,1)-block of the second matrix in (9) with zero matrix and change the factor $\frac{1}{2}$ to 1, then a new preconditioner which is referred to as the GVDPSS preconditioner is constructed as follows:

$$\mathcal{P}_{\text{GVDPSS}} = \begin{pmatrix} \alpha^{-1} I & 0 \\ 0 & \beta^{-1} I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & \beta I \end{pmatrix}\begin{pmatrix} \alpha I & B^T \\ -B & \beta I \end{pmatrix} = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix}, \tag{10}$$

where $\alpha > 0$ and $\beta > 0$ are two given constants.

The difference between $\mathcal{P}_{\text{GVDPSS}}$ and $\mathcal{A}$ is

$$\mathcal{R}_{\text{GVDPSS}} = \mathcal{P}_{\text{GVDPSS}} - \mathcal{A} = \begin{pmatrix} 0 & (\frac{1}{\alpha}A - I)B^T \\ 0 & \beta I \end{pmatrix}. \tag{11}$$

From (10), we observe that the GVDPSS preconditioner can be regarded as generalized versions of the VDPSS preconditioner derived by Zhang and Gu [44] and the new relaxed HSS (NRHSS) preconditioner proposed by Salkuyeh and Masoudi [40]. That is to say, when $\beta = \alpha$ and $\alpha = 1$, the GVDPSS preconditioner reduces to the VDPSS preconditioner and NRHSS preconditioner, respectively. In addition, we find that the nonzero (2,2)-block tends to the null matrix as the parameter $\beta \to 0_+$ and α can be chosen properly such that (1,2)-block in (11) tends to zero matrix as much as possible. Specially, as we mentioned when α tends to $+\infty$, (2,2)-block in $\mathcal{R}_{\text{VDPSS}}$ becomes unbounded, while we can choose $\beta \to 0_+$ for the GVDPSS preconditioner such that (2,2)-block in $\mathcal{R}_{\text{GVDPSS}}$ tends to zero matrix. This indicates that, for large values of α, the GVDPSS preconditioner is much closer to the coefficient matrix $\mathcal{A}$ than the VDPSS preconditioner in some degree, and the GVDPSS preconditioner with proper parameters α and β should be a better approximation to the coefficient matrix $\mathcal{A}$ than the VDPSS and NRHSS preconditioners due to the independence of the parameters α and β. Therefore, the GVDPSS preconditioner is expected to be a better preconditioner than the VDPSS and NRHSS preconditioners, and the merit of the GVDPSS preconditioner will be stressed by the numerical experiments.

Actually, the GVDPSS preconditioner $\mathcal{P}_{\text{GVDPSS}}$ can be induced by a fixed-point iteration, which is based on the following splitting of the matrix $\mathcal{A}$:

$$\mathcal{A} = \mathcal{P}_{\text{GVDPSS}} - \mathcal{R}_{\text{GVDPSS}} = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix} - \begin{pmatrix} 0 & (\frac{1}{\alpha}A - I)B^T \\ 0 & \beta I \end{pmatrix}. \tag{12}$$

Based on the above splitting, we can construct a new iterative method, called the GVDPSS iteration method, which is defined as follows:

The GVDPSS iteration method Let $\alpha > 0$ and $\beta > 0$ be two given constants. Given an initial guess $(x^{(0)^T}, y^{(0)^T})^T$. For $k = 0, 1, 2, \cdots$, until $(x^{(k)^T}, y^{(k)^T})^T$ converges, compute

$$\begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix} \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0 & (\frac{1}{\alpha}A - I)B^T \\ 0 & \beta I \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} p \\ -q \end{pmatrix}. \tag{13}$$

Hence, the GVDPSS iteration method can be written in the following fixed-point form:

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = M_{\alpha,\beta} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + c, \tag{14}$$

where

$$M_{\alpha,\beta} = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix}^{-1} \begin{pmatrix} 0 & (\frac{1}{\alpha}A - I)B^T \\ 0 & \beta I \end{pmatrix}$$

is the iteration matrix and

$$c = \begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix}^{-1} \begin{pmatrix} p \\ -q \end{pmatrix}.$$

The fixed-point iteration (14) converges to the solution $u = \mathcal{A}^{-1}b$ for arbitrary initial guesses $u^{(0)} = (x^{(0)^T}, y^{(0)^T})^T$ and right-hand sides b if and only if $\rho(M_{\alpha,\beta}) < 1$, where $\rho(T)$ denotes the spectral radius of T. Now, we discuss the convergence of the GVDPSS iteration method. We start with some lemmas which will be useful in our proofs.

Lemma 2.1 *Let*

$$\mathcal{P}_1 = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} I & \frac{1}{\alpha}B^T \\ -B & \beta I \end{pmatrix}.$$

Here, $\mathcal{P}_2$ has the block-triangular factorization

$$\mathcal{P}_2 = \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \tilde{A} \end{pmatrix} \begin{pmatrix} I & \frac{1}{\alpha}B^T \\ 0 & I \end{pmatrix}$$

with $\tilde{A} = \beta I + \frac{1}{\alpha}BB^T$. Then the form of $\mathcal{P}_{GVDPSS}^{-1}$ is given by

$$\begin{aligned} \mathcal{P}_{GVDPSS}^{-1} = \mathcal{P}_2^{-1}\mathcal{P}_1^{-1} &= \begin{pmatrix} I & -\frac{1}{\alpha}B^T \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \tilde{A}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A^{-1} - \frac{1}{\alpha}B^T\tilde{A}^{-1}BA^{-1} & -\frac{1}{\alpha}B^T\tilde{A}^{-1} \\ \tilde{A}^{-1}BA^{-1} & \tilde{A}^{-1} \end{pmatrix}. \end{aligned} \tag{15}$$

Lemma 2.2 *Let $A \in \mathbb{R}^{m\times m}$ be positive definite and $B \in \mathbb{R}^{n\times m}$ be of full row rank. If $x \in \mathbb{C}^n \neq 0$ and $x^*BA^{-1}B^Tx = a + bi$, then $a > 0$.*

Proof Since $x \in \mathbb{C}^n \neq 0$ and $x^*BA^{-1}B^Tx = a + bi$, it holds that $x^*BA^{-T}B^Tx = a - bi$, and therefore, $x^*B(A^{-1} + A^{-T})B^Tx = 2a$. Owing to the fact that A^{-1} is positive definite, $A^{-1} + A^{-T}$ is symmetric positive definite, which together with $x \neq 0$ and $rank(B) = n$ gives $a > 0$. This completes the proof. □

Lemma 2.3 [46] *If S is a skew-Hermitian matrix, then iS (i is the imaginary unit) is a Hermitian matrix and u^*Su is a purely imaginary number or zero for all $u \in \mathbb{C}^m$.*

Theorem 2.1 *Let $A \in \mathbb{R}^{m\times m}$ be nonsymmetric positive definite and $B \in \mathbb{R}^{n\times m}$ be full of row rank. Then the GVDPSS iteration method is convergent if and only if the parameters α and β satisfy*

$$\alpha > 0,\ \beta > \max\left\{\frac{a_1}{2} + \frac{b_1^2}{2a_1} - \frac{c_1}{\alpha}, 0\right\}, \tag{16}$$

where

$$\frac{x^*BA^{-1}B^Tx}{x^*x} = a_1 + ib_1,\ \frac{x^*BB^Tx}{x^*x} = c_1 \tag{17}$$

and x is an eigenvector corresponding to an eigenvalue of the matrix $\tilde{A}^{-1}BA^{-1}B^T$.

Proof By making use of Lemma 2.1, we derive

$$\begin{aligned}\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A} &= I - \mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{R}_{\text{GVDPSS}}\\ &= I - \begin{pmatrix} A^{-1} - \frac{1}{\alpha}B^T\tilde{A}^{-1}BA^{-1} & -\frac{1}{\alpha}B^T\tilde{A}^{-1} \\ \tilde{A}^{-1}BA^{-1} & \tilde{A}^{-1}\end{pmatrix}\begin{pmatrix} 0 & (\frac{1}{\alpha}A - I)B^T \\ 0 & \beta I\end{pmatrix}\\ &= \begin{pmatrix} I_m & K_1 \\ 0 & K_2\end{pmatrix},\end{aligned} \tag{18}$$

where $K_1 = -(\frac{1}{\alpha}B^T - A^{-1}B^T - \frac{1}{\alpha^2}B^T\tilde{A}^{-1}BB^T + \frac{1}{\alpha}B^T\tilde{A}^{-1}BA^{-1}B^T - \frac{\beta}{\alpha}B^T\tilde{A}^{-1})$ and $K_2 = \tilde{A}^{-1}BA^{-1}B^T$. As a result, the iteration matrix $M_{\alpha,\beta}$ can be rewritten by

$$M_{\alpha,\beta} = \mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{R}_{\text{GVDPSS}} = I - \mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A} = \begin{pmatrix} 0 & -K_1 \\ 0 & I_n - K_2\end{pmatrix}. \tag{19}$$

Hence, if μ is an eigenvalue of the matrix $M_{\alpha,\beta}$, then $\mu = 0$ or $\mu = 1 - \lambda$, where λ is an eigenvalue of the matrix K_2. Therefore, there exists a vector $x \neq 0$ such that

$$K_2 x = \tilde{A}^{-1}BA^{-1}B^T x = \lambda x,$$

which can be written as

$$BA^{-1}B^T x = \lambda\left(\beta I + \frac{1}{\alpha}BB^T\right)x. \tag{20}$$

As we know that the vector $x \neq 0$, then the definition $\frac{x^*}{x^*x}$ does make sense. Premultiplying (20) with $\frac{x^*}{x^*x}$ and utilizing the symbols defined as in (17) yield

$$\lambda = \frac{x^*BA^{-1}B^T x}{\beta x^*x + \frac{1}{\alpha}x^*BB^T x} = \frac{a_1 + ib_1}{\beta + \frac{c_1}{\alpha}} = \frac{\alpha(a_1 + ib_1)}{\alpha\beta + c_1}. \tag{21}$$

To ensure the convergence of the GVDPSS iteration method, it must hold that

$$|\mu| = \left|1 - \frac{\alpha(a_1 + ib_1)}{\alpha\beta + c_1}\right| = \left|\frac{(\alpha\beta + c_1 - \alpha a_1) - i\alpha b_1}{\alpha\beta + c_1}\right| < 1.$$

After some manipulations, we obtain

$$\alpha > 0,\ \beta > \frac{a_1}{2} + \frac{b_1^2}{2a_1} - \frac{c_1}{\alpha}. \tag{22}$$

Combining $\beta > 0$ and (22), we obtain the conditions for the convergence of the GVDPSS iteration method. □

According to Theorem 2.1, the following sufficient conditions for the convergence of the GVDPSS iteration method can be obtained immediately.

Corollary 2.1 *Assume the conditions in Theorem 2.1 are satisfied. Then, the GVDPSS iteration method for solving saddle point problem* (1) *is convergent if the parameters* α *and* β *satisfy:*

$$\alpha > 0,\ \beta > \left\{ \frac{\rho(\tilde{H})}{2} + \frac{\rho(\tilde{S})^2}{2\lambda_{\min}(\tilde{H})} - \frac{\sigma_{\min}^2}{\alpha}, 0 \right\}, \tag{23}$$

where

$$\tilde{H} = \frac{B(A^{-1} + A^{-T})B^T}{2},\ \tilde{S} = \frac{B(A^{-1} - A^{-T})B^T}{2},$$

$\lambda_{\min}(.)$ *and* $\rho(.)$ *denote the minimum eigenvalue and spectral radius of a matrix, respectively, and* $\sigma_{\min}$ *is the minimum singular value of the matrix* B*. In particular, if* A *is symmetric definite, the conditions* (23) *reduce to*

$$\alpha > 0,\ \beta > \left\{ \frac{\rho(BA^{-1}B^T)}{2} - \frac{\sigma_{\min}^2}{\alpha}, 0 \right\}. \tag{24}$$

Proof Denote the function

$$f(a_1, b_1, c_1, \alpha) = \frac{a_1}{2} + \frac{b_1^2}{2a_1} - \frac{c_1}{\alpha}.$$

By utilizing Lemma 2.2 and the symbols defined as in (17), it is easily seen that $a_1, c_1 > 0$ and $b_1^2 \geq 0$ and it is not difficult to verify that the function $f(a_1, b_1, c_1, \alpha)$ is monotonically increasing about b_1^2 and monotonically decreasing about c_1. It follows from (17) that

$$a_1 = \frac{x^*B(A^{-1} + A^{-T})B^T x}{2x^*x} = \frac{x^*\tilde{H}x}{x^*x},\ b_1 = \frac{x^*B(A^{-1} - A^{-T})B^T x}{2ix^*x} = \frac{x^*\tilde{S}x}{ix^*x} = -\frac{x^*i\tilde{S}x}{x^*x}.$$

Since $\tilde{H}$ and $\tilde{S}$ are Hermitian and skew-Hermitian matrices, respectively, it holds that $\lambda_{\min}(\tilde{H}) \leq a_1 \leq \rho(\tilde{H})$, $0 \leq b_1^2 \leq \rho(\tilde{S})^2$ and $c_1 \geq \sigma_{\min}^2$ by Lemma 2.3. Then an upper bound for $f(a_1, b_1, c_1, \alpha)$ is derived as follows:

$$f(a_1, b_1, c_1, \alpha) \leq \frac{\rho(\tilde{H})}{2} + \frac{\rho(\tilde{S})^2}{2\lambda_{\min}(\tilde{H})} - \frac{\sigma_{\min}^2}{\alpha}. \tag{25}$$

By making use of Theorem 2.1, inequalities (23) and (25), it can be seen that the GVDPSS iteration method is convergent if α and β satisfy (23). Specifically, if A is symmetric positive definite, the conditions (24) can be directly derived from (23) as $\tilde{H} = BA^{-1}B^T$ and $\tilde{S} = 0$ in this case. □

It is worthy noting that the GVDPSS iteration method contains two parameters α and β, and how to select the optimal parameters α and β for the GVDPSS iteration method is very practical and meaningful. However, it is a difficult task for us at present, and this problem needs to be investigated in the future.

3 The spectral properties of the preconditioned matrix $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$

The eigenvalue and eigenvector distributions of the preconditioned matrix relate closely to the convergence rate of Krylov subspace methods. Therefore, it is meaningful to investigate the spectral properties of the preconditioned matrix $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$. In the following, we will deduce the eigenvalue distribution of the preconditioned matrix $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$. Besides, the eigenvectors and the upper bounds of the minimal polynomial of the preconditioned matrix $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ are also discussed.

Theorem 3.1 *Let the GVDPSS preconditioner be defined as in (10), then the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ has eigenvalue 1 of algebraic multiplicity at least m. The real and imaginary parts of the remaining eigenvalues of the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ satisfy*

$$\frac{\alpha\sigma_m^2\lambda_{\min}(\hat{H})}{\alpha\beta+\sigma_m^2} \le Re(\lambda) \le \frac{\alpha\sigma_1^2\lambda_{\max}(\hat{H})}{\alpha\beta+\sigma_1^2},\ |Im(\lambda)| \le \frac{\alpha\sigma_1^2\rho(\hat{S})}{\alpha\beta+\sigma_1^2},$$

where σ_m and σ_1 are the minimum and maximum singular values of the matrix B, respectively, and $\hat{H} = \frac{A^{-1}+(A^{-1})^T}{2}$ and $\hat{S} = \frac{A^{-1}-(A^{-1})^T}{2}$ are the symmetric and skew-symmetric parts of the matrix A^{-1}, respectively.

Proof It follows from (18) that

$$\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A} = \begin{pmatrix} I_m & K_1 \\ 0 & K_2 \end{pmatrix}, \tag{26}$$

where $K_1 = -(\frac{1}{\alpha}B^T - A^{-1}B^T - \frac{1}{\alpha^2}B^T\tilde{A}^{-1}BB^T + \frac{1}{\alpha}B^T\tilde{A}^{-1}BA^{-1}B^T - \frac{\beta}{\alpha}B^T\tilde{A}^{-1})$ and $K_2 = \tilde{A}^{-1}BA^{-1}B^T$.

Equation (26) implies that the eigenvalues of the preconditioned matrix $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ are given by 1 with algebraic multiplicity as least m. The remaining non-unit eigenvalues of $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ are the solution of the eigenvalue problem

$$\tilde{A}^{-1}BA^{-1}B^Tu = \lambda u, \tag{27}$$

where $\tilde{A} = \beta I + \frac{1}{\alpha}BB^T$, which can be precisely rewritten as the generalized eigenvalue problem

$$BA^{-1}B^Tu = \lambda\left(\beta I + \frac{1}{\alpha}BB^T\right)u. \tag{28}$$

It is evident that $u \neq 0$. Without loss of generality, we assume $\|u\|_2 = 1$. Premultiplying (28) with u^* results in

$$u^*BA^{-1}B^Tu = \lambda\left(\beta + \frac{1}{\alpha}u^*BB^Tu\right). \tag{29}$$

Let $v = B^Tu$, then (29) reduces to

$$v^*A^{-1}v = \lambda\left(\beta + \frac{1}{\alpha}v^*v\right), \tag{30}$$

which implies that

$$v^*(A^{-1})^T v = \bar{\lambda}\left(\beta + \frac{1}{\alpha} v^* v\right). \tag{31}$$

Owing to the positive definiteness of the matrix A^{-1}, we obtain $\hat{H} = \frac{A^{-1}+(A^{-1})^T}{2}$ is symmetric positive definite. Combining (30) with (31) yields

$$\frac{v^*\hat{H}v}{v^*v} = Re(\lambda)\left(\frac{\beta}{v^*v} + \frac{1}{\alpha}\right), \quad \frac{v^*\hat{S}v}{v^*v} = Im(\lambda)i\left(\frac{\beta}{v^*v} + \frac{1}{\alpha}\right). \tag{32}$$

By making use of Lemma 2.3, it is not difficult to verify that

$$\lambda_{\min}(\hat{H}) \le \frac{v^*\hat{H}v}{v^*v} \le \lambda_{\max}(\hat{H}), \ \frac{v^*i\hat{S}v}{v^*v} \le \rho(\hat{S}), \ \sigma_m^2 \le v^*v = u^*BB^Tu \le \sigma_1^2. \tag{33}$$

Here, σ_m and σ_1 are the minimum and maximum singular values of the matrix B, respectively. By applying (32) and (33), we obtain

$$\frac{\alpha\sigma_m^2\lambda_{\min}(\hat{H})}{\alpha\beta + \sigma_m^2} \le Re(\lambda) \le \frac{\alpha\sigma_1^2\lambda_{\max}(\hat{H})}{\alpha\beta + \sigma_1^2}, \ |Im(\lambda)| \le \frac{\alpha\sigma_1^2\rho(\hat{S})}{\alpha\beta + \sigma_1^2}.$$

This proof is completed. □

Remark 3.1 From (30), the non-unit eigenvalues of the preconditioned matrix $\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A}$ satisfy

$$\lambda = \frac{v^*A^{-1}v}{\beta + \frac{1}{\alpha}v^*v} = \frac{\alpha v^*A^{-1}v}{\alpha\beta + v^*v}. \tag{34}$$

Let $0 < \alpha_0 < +\infty$ and $0 < \beta_0 < +\infty$. By making use of (34), we derive the following conclusions:

(a) Let $(\alpha, \beta) \to (0, \beta_0)$, then $\lambda \to 0$.
(b) Let $(\alpha, \beta) \to (+\infty, \beta_0)$, then $\lambda \to \frac{1}{\beta_0}v^*A^{-1}v$.
(c) Let $(\alpha, \beta) \to (\alpha_0, \beta_0)$, then $\lambda \to \frac{\alpha_0 v^*A^{-1}v}{\alpha_0\beta_0 + v^*v}$.
(d) Let $(\alpha, \beta) \to (0, +\infty)$, then $\lambda \to 0$.
(e) Let $(\alpha, \beta) \to (\alpha_0, +\infty)$, then $\lambda \to 0$.
(f) Let $(\alpha, \beta) \to (+\infty, +\infty)$, then $\lambda \to 0$.
(g) Let $(\alpha, \beta) \to (0, 0)$, then $\lambda \to 0$.
(h) Let $(\alpha, \beta) \to (\alpha_0, 0)$, then $\lambda \to \frac{\alpha_0 v^*A^{-1}v}{v^*v}$.

Let (λ, x) be an eigenpair of the matrix $(\beta I + \frac{1}{\alpha}BB^T)^{-1}BA^{-1}B^T$. In the same manner applied in the proof of Theorem 2.1, it has

$$\lambda = \frac{x^*BA^{-1}B^Tx}{\beta x^*x + \frac{1}{\alpha}x^*BB^Tx} = \frac{a_1 + ib_1}{\beta + \frac{c_1}{\alpha}} = \frac{\alpha(a_1 + ib_1)}{\alpha\beta + c_1}. \tag{35}$$

It is not difficult to verify that

$$\lambda \rightarrow \begin{cases} 0, \quad \alpha \rightarrow 0_+ \text{ or } \beta \rightarrow +\infty, \\ \frac{a_1}{\beta_0} + i\frac{b_1}{\beta_0}, \quad \alpha \rightarrow +\infty \text{ and } \beta \rightarrow \beta_0 \ (0 < \beta_0 < +\infty), \\ \frac{\alpha_0 a_1 + i\alpha_0 b_1}{c_1}, \quad \beta \rightarrow 0_+ \text{ and } \alpha \rightarrow \alpha_0 \ (0 < \alpha_0 < +\infty), \\ \frac{\alpha_0 a_1 + i\alpha_0 b_1}{\alpha_0\beta_0 + c_1}, \quad \alpha \rightarrow \alpha_0 \text{ and } \beta \rightarrow \beta_0 \ (0 < \alpha_0 < +\infty, 0 < \beta_0 < +\infty). \end{cases}$$

We summarize the above results in the following theorem.

Theorem 3.2 *The eigenvalues of the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ tend to scatter near the points $(1, 0)$ and $(0, 0)$ as $\alpha \rightarrow 0_+$ or $\beta \rightarrow +\infty$, tend to scatter near the points $(1, 0)$ and $(\frac{a_1}{\beta_0}, \frac{b_1}{\beta_0})$ as $\alpha \rightarrow +\infty$ and $\beta \rightarrow \beta_0$ $(0 < \beta_0 < +\infty)$, and tend to scatter near the points $(1, 0)$ and $(\frac{\alpha_0 a_1}{c_1}, \frac{\alpha_0 b_1}{c_1})$ as $\beta \rightarrow 0_+$ and $\alpha \rightarrow \alpha_0$ $(0 < \alpha_0 < +\infty)$. In addition, the eigenvalues of $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ tend to scatter near the points $(1, 0)$ and $(\frac{\alpha_0 a_1}{\alpha_0\beta_0 + c_1}, \frac{\alpha_0 b_1}{\alpha_0\beta_0 + c_1})$ as $\alpha \rightarrow \alpha_0$ and $\beta \rightarrow \beta_0$ $(0 < \alpha_0 < +\infty, 0 < \beta_0 < +\infty)$, where a_1, b_1, and c_1 are defined as in (17).*

It is easy to see that $a_1 > 0$ and $c_1 > 0$ by virtue of Lemma 2.2 and the definitions of a_1, c_1. By making use of (35), we have

$$Re(\lambda) = \frac{\alpha a_1}{\alpha\beta + c_1} > 0, \tag{36}$$

where $Re(\lambda)$ denotes the real part of λ. It follows immediately from (36) that all eigenvalues of $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ have positive real parts. Thus, the eigenvalues of $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ lie in a positive box, which may result in faster convergence of Krylov subspace acceleration. In particular, when A is a symmetric positive definite, $b_1 = 0$, and therefore, the eigenvalues of $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ locate in a positive real interval.

Recalling that the convergence of Krylov subspace methods is not only dependent on the eigenvalue distribution of the preconditioned matrix but also on the corresponding eigenvectors of the preconditioned matrix [1, 5]. We next discuss the eigenvector distribution of $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ in the following theorem.

Theorem 3.3 *Let the GVDPSS preconditioner $\mathcal{P}_{GVDPSS}$ be defined as in (10), if the matrix $I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B$ is nonsingular, then the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ has $m + k$ $(0 \le k \le n)$ linearly independent eigenvectors. If the matrix $I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B$ is singular, then the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ has $m + i + j$ $(0 \le i + j \le n)$ linearly independent eigenvectors. There are*

1) *m eigenvectors of the form $\begin{pmatrix} u_l \\ 0 \end{pmatrix}$ $(l = 1, 2, \cdots, m)$ that correspond to the eigenvalue 1, where u_l $(l = 1, 2, \cdots, m)$ are arbitrary linearly independent vectors.*

2) *If the matrix* $I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B$ *is nonsingular,* k $(0 \leq k \leq n)$ *eigenvectors of the form* $\begin{pmatrix} \frac{(I-\frac{1}{\alpha}B^T\tilde{A}^{-1}B)A^{-1}B^T}{\lambda-1}v_l^1 \\ v_l^1 \end{pmatrix}$ *that correspond to the eigenvalues* $\lambda \neq 1$*, where* $v_l^1 \neq 0$ *and* v_l^1 $(1 \leq l \leq k)$ *satisfy* $BA^{-1}B^T v_l^1 = \lambda(\beta I + \frac{1}{\alpha}BB^T)v_l^1$.
3) *If the matrix* $I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B$ *is singular,* i $(0 \leq i \leq n)$ *eigenvectors* $\begin{pmatrix} u_l^2 \\ v_l^2 \end{pmatrix}$ $(1 \leq l \leq i)$ *that correspond to the eigenvalue 1, where* $v_l^2 \neq 0$, v_l^2 $(1 \leq l \leq i)$ *satisfy* $(I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B)A^{-1}B^T v_l^2 = 0$ *and* $BA^{-1}B^T v_l^2 = (\beta I + \frac{1}{\alpha}BB^T)v_l^2$*, and* u_l^2 *are arbitrary vectors,* j $(0 \leq j \leq n)$ *eigenvectors of the form* $\begin{pmatrix} \frac{(I-\frac{1}{\alpha}B^T\tilde{A}^{-1}B)A^{-1}B^T}{\lambda-1}v_l^3 \\ v_l^3 \end{pmatrix}$ $(1 \leq l \leq j)$ *that correspond to the eigenvalues* $\lambda \neq 1$*, where* $v_l^3 \neq 0$ *and* v_l^3 $(1 \leq l \leq j)$ *satisfy* $BA^{-1}B^T v_l^3 = \lambda(\beta I + \frac{1}{\alpha}BB^T)v_l^3$.

Proof Let λ be an eigenvalue of the preconditioned matrix $\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A}$ and $\begin{pmatrix} u \\ v \end{pmatrix}$ be the corresponding eigenvector. To derive the eigenvector distribution, we consider the following generalized eigenvalue problem:

$$\begin{pmatrix} I_m & K_1 \\ 0 & K_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix}, \tag{37}$$

where $K_1 = -(\frac{1}{\alpha}B^T - A^{-1}B^T - \frac{1}{\alpha^2}B^T\tilde{A}^{-1}BB^T + \frac{1}{\alpha}B^T\tilde{A}^{-1}BA^{-1}B^T - \frac{\beta}{\alpha}B^T\tilde{A}^{-1}) = (I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B)A^{-1}B^T$ and $K_2 = \tilde{A}^{-1}BA^{-1}B^T$ with $\tilde{A} = \beta I + \frac{1}{\alpha}BB^T$. Equation (37) can be equivalently rewritten as

$$\begin{cases} (\lambda - 1)u = (I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B)A^{-1}B^T v, \\ BA^{-1}B^T v = \lambda(\beta I + \frac{1}{\alpha}BB^T)v. \end{cases} \tag{38}$$

If $\lambda = 1$ holds true, then from the first equation of (38), we can easily get $(I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B)A^{-1}B^T v = 0$. When $v = 0$, equation (38) is always true for the case of $\lambda = 1$. Hence, there are m linearly independent eigenvectors $\begin{pmatrix} u_l \\ 0 \end{pmatrix}$ $(l = 1, 2, \cdots, m)$ corresponding to the eigenvalue 1, where u_l $(l = 1, 2, \cdots, m)$ are arbitrary linearly independent vectors. When $v \neq 0$, and if $I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B$ is nonsingular, then it must be $\lambda \neq 1$. This contradicts to the assumption $\lambda = 1$; if $I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B$ is singular and there exists $v \neq 0$ which satisfies the second equation of (38) and $(I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B)A^{-1}B^T v = 0$, then there will be i $(1 \leq i \leq n)$ linearly independent eigenvectors $\begin{pmatrix} u_l^2 \\ v_l^2 \end{pmatrix}$ $(1 \leq l \leq i)$ corresponding to the eigenvalue 1, where u_l^2 $(1 \leq l \leq i)$ are arbitrary vectors and $v_l^2 \neq 0$ $(1 \leq l \leq i)$ satisfy $(I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B)A^{-1}B^T v_l^2 = 0$ and $BA^{-1}B^T v_l^2 = (\beta I + \frac{1}{\alpha}BB^T)v_l^2$.

Next, we consider the case $\lambda \neq 1$. If $v = 0$, then it follows from the first equation of (38) that $u = 0$, a contradiction. Hence, $v \neq 0$. If $I - \frac{1}{\alpha}B^T\tilde{A}^{-1}B$ is nonsingular and there exists $v \neq 0$ which satisfies the second equation of (38), then there are k

$(1 \leq k \leq n)$ linearly independent eigenvectors $\begin{pmatrix} u_l^1 \\ v_l^1 \end{pmatrix}$ $(1 \leq l \leq k)$ corresponding to the eigenvalues $\lambda \neq 1$, and the forms of u_l^1 $(1 \leq l \leq k)$ are:

$$u_l^1 = \frac{(I - \frac{1}{\alpha} B^T \tilde{A}^{-1} B) A^{-1} B^T}{\lambda - 1} v_l^1. \tag{39}$$

If $I - \frac{1}{\alpha} B^T \tilde{A}^{-1} B$ is singular and there exists $v \neq 0$ which satisfies the second equation of (38), there are j $(1 \leq j \leq n)$ linearly independent eigenvectors $\begin{pmatrix} u_l^3 \\ v_l^3 \end{pmatrix}$ $(1 \leq l \leq j)$ corresponding to the eigenvalues $\lambda \neq 1$. Here, $v_l^3 \neq 0$ $(1 \leq l \leq j)$ satisfy $BA^{-1}B^T v_l^3 = \lambda(\beta I + \frac{1}{\alpha} BB^T) v_l^3$ and the forms of u_l^3 $(1 \leq l \leq j)$ satisfy (39).

In the sequel, we show that the $m + k$ eigenvectors are linearly independent when the matrix $I - \frac{1}{\alpha} B^T \tilde{A}^{-1} B$ is nonsingular. Let $c^{(1)} = [c_1^{(1)}, c_2^{(1)}, \cdots, c_m^{(1)}]$ and $c^{(2)} = [c_1^{(2)}, c_2^{(2)}, \cdots, c_k^{(2)}]$ be two vectors with $0 \leq k \leq n$. Then, we need to show that

$$\begin{pmatrix} u_1 & \cdots & u_m \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_m^{(1)} \end{pmatrix} + \begin{pmatrix} u_1^1 & \cdots & u_k^1 \\ v_1^1 & \cdots & v_k^1 \end{pmatrix} \begin{pmatrix} c_1^{(2)} \\ \vdots \\ c_k^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{40}$$

holds if and only if the vectors $c^{(1)}$ and $c^{(2)}$ both are zero vectors. Recalling that in (40), the first matrix arises from the case $\lambda_l = 1$ $(l = 1, 2, \cdots, m)$ in 1), and the second matrix from the case $\lambda_l \neq 1$ $(l = 1, 2, \cdots, k)$ in 2). Multiplying both sides of (40) from left with $\mathcal{P}_{\mathrm{GVDPSS}}^{-1} \mathcal{A}$ leads to

$$\begin{pmatrix} u_1 & \cdots & u_m \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_m^{(1)} \end{pmatrix} + \begin{pmatrix} u_1^1 & \cdots & u_k^1 \\ v_1^1 & \cdots & v_k^1 \end{pmatrix} \begin{pmatrix} \lambda_1 c_1^{(2)} \\ \vdots \\ \lambda_k c_k^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{41}$$

Then, by subtracting (40) from (41), it holds that

$$\begin{pmatrix} u_1^1 & \cdots & u_k^1 \\ v_1^1 & \cdots & v_k^1 \end{pmatrix} \begin{pmatrix} (\lambda_1 - 1) c_1^{(2)} \\ \vdots \\ (\lambda_k - 1) c_k^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the eigenvalues $\lambda_l \neq 1$ and $\begin{pmatrix} u_l^1 \\ v_l^1 \end{pmatrix}$ $(1 \leq l \leq k)$ are linearly independent, we infer that $c_l^{(2)} = 0$ $(l = 1, 2, \cdots, k)$. Because of the linear independence of u_l $(l = 1, 2, \cdots, m)$, it follows that $c_l^{(1)} = 0$ $(l = 1, 2, \cdots, m)$. Therefore, the $m + k$ eigenvectors are linearly independent.

Finally, we verify the $m + i + j$ eigenvectors are linearly independent when the matrix $I - \frac{1}{\alpha} B^T \tilde{A}^{-1} B$ is singular. Let $c^{(1)} = [c_1^{(1)}, c_2^{(1)}, \cdots, c_m^{(1)}]$, $c^{(2)} =$

$[c_1^{(2)}, c_2^{(2)}, \cdots, c_i^{(2)}]$, and $c^{(3)} = [c_1^{(3)}, c_2^{(3)}, \cdots, c_j^{(3)}]$ be three vectors with $0 \le i, j \le n$, and

$$\begin{pmatrix} u_1 & \cdots & u_m \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_m^{(1)} \end{pmatrix} + \begin{pmatrix} u_1^2 & \cdots & u_i^2 \\ v_1^2 & \cdots & v_i^2 \end{pmatrix} \begin{pmatrix} c_1^{(2)} \\ \vdots \\ c_i^{(2)} \end{pmatrix} + \begin{pmatrix} u_1^3 & \cdots & u_j^3 \\ v_1^3 & \cdots & v_j^3 \end{pmatrix} \begin{pmatrix} c_1^{(3)} \\ \vdots \\ c_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (42)$$

It is necessary for us to prove that (42) holds if and only the vectors $c^{(1)}$, $c^{(2)}$, and $c^{(3)}$ are all zero vectors, where the first matrix consists of the eigenvectors corresponding to the eigenvalue 1 for the case 1), and the second and the third matrices consist of those for the case 3). Premultiplying (42) with $\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A}$ and going through the same algebraic operations as before, we also obtain

$$\begin{pmatrix} u_1^3 & \cdots & u_j^3 \\ v_1^3 & \cdots & v_j^3 \end{pmatrix} \begin{pmatrix} (\lambda_1 - 1)c_1^{(3)} \\ \vdots \\ (\lambda_j - 1)c_j^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Inasmuch as $\lambda_l \neq 1$ and $\begin{pmatrix} u_l^3 \\ v_l^3 \end{pmatrix}$ $(1 \le l \le j)$ are linearly independent, it must necessarily that $c_l^{(3)} = 0$ $(l = 1, 2, \cdots, j)$. As the vectors v_l^2 $(l = 1, 2, \cdots, i)$ are also linearly independent, we have $c_l^{(2)} = 0$ $(l = 1, 2, \cdots, i)$. Thus, (42) can be simplified to

$$\begin{pmatrix} u_1 & \cdots & u_m \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_m^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since u_l $(l = 1, 2, \cdots, m)$ are linearly independent, we have $c_l^{(1)} = 0$ $(l = 1, 2, \cdots, m)$. As a result, it holds that the $m + i + j$ eigenvectors are linearly independent. □

The GMRES method will terminate when the degree of the minimum polynomial is attained [39]. In particular, the degree of the minimum polynomial is equal to the dimension of the corresponding Krylov subspace [38]. Next theorem provides some analysis to the dimension of the Krylov subspace $K(\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A}, b)$.

Theorem 3.4 *Let the GVDPSS preconditioner be defined as in (10), then the degree of minimal polynomial of preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ is at most $n + 1$. Thus, the dimension of the Krylov subspace $K(\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}, b)$ is at most $n + 1$. In particular, if the matrix K_2 has k $(1 \le k \le n)$ distinct eigenvalues μ_i $(1 \le i \le k)$, of respective multiplicity δ_i with $\sum_{i=1}^{k} \delta_i = n$, then the dimension of the Krylov subspace $K(\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}, b)$ is at most $k + 1$.*

Proof According to the form of the preconditioned matrix $\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A}$ in (26), it is evident that the characteristic polynomial of the preconditioned matrix $\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A}$ is

$$\mathcal{X}(x) = (x-1)^m \prod_{i=1}^{n}(x-\mu_i),$$

where μ_i $(i = 1, 2, \cdots, n)$ are the eigenvalues of the matrix K_2. Let $p(x) = (x-1)\prod_{i=1}^{n}(x-\mu_i)$. Hence,

$$\begin{aligned} p(\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}) &= (\mathcal{P}_{GVDPSS}^{-1}\mathcal{A} - I)\prod_{i=1}^{n}(\mathcal{P}_{GVDPSS}^{-1}\mathcal{A} - \mu_i I) \\ &= \begin{pmatrix} 0 & K_1 \prod_{i=1}^{n}(K_2 - \mu_i I_n) \\ 0 & (K_2 - I_n)\prod_{i=1}^{n}(K_2 - \mu_i I_n) \end{pmatrix}. \end{aligned}$$

Inasmuch as μ_i $(i = 1, 2, \cdots, n)$ are the eigenvalues of the matrix K_2, we have $\prod_{i=1}^{n}(K_2 - \mu_i I_n) = 0$ and therefore $p(\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}) = 0$, which implies that the degree of the minimal polynomial of the preconditioned matrix $\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A}$ is at most $n+1$.

In addition, if the matrix K_2 has k $(1 \le k \le n)$ distinct eigenvalues μ_i $(1 \le i \le k)$, of respective multiplicity δ_i with $\sum_{i=1}^{k}\delta_i = n$, we rewrite the characteristic polynomial $\mathcal{X}(\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A})$ as

$$\begin{aligned} &(\mathcal{P}_{GVDPSS}^{-1}\mathcal{A} - I)^{m-1}\prod_{i=1}^{k}(\mathcal{P}_{GVDPSS}^{-1}\mathcal{A} - \mu_i I)^{\delta_i - 1} \\ &\times (\mathcal{P}_{GVDPSS}^{-1}\mathcal{A} - I)\prod_{i=1}^{k}(\mathcal{P}_{GVDPSS}^{-1}\mathcal{A} - \mu_i I). \end{aligned}$$

Let $\Upsilon = (\mathcal{P}_{GVDPSS}^{-1}\mathcal{A} - I)\prod_{i=1}^{k}(\mathcal{P}_{GVDPSS}^{-1}\mathcal{A} - \mu_i I)$, then

$$\Upsilon = \begin{pmatrix} 0 & K_1 \prod_{i=1}^{k}(K_2 - \mu_i I_n) \\ 0 & (K_2 - I_n)\prod_{i=1}^{k}(K_2 - \mu_i I_n) \end{pmatrix}.$$

Since $\prod_{i=1}^{k}(K_2 - \mu_i I_n) = 0$, it is not difficult to verify that Υ is a zero matrix. This means that the dimension of the Krylov subspace $K(\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A}, b)$ is at most $k+1$. □

Remark 3.2 From Theorem 3.4 and the results in [38], we are easy to see that the Krylov subspace method such as the GMRES method is applied to a preconditioned matrix $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$, it will converge to the exact solution of the nonsymmetric saddle point problem (1) in $n+1$ or fewer iterations. Even in some special case (K_2 has few distinct eigenvalues), it will terminate in few iterations. This reveals the excellent acceleration effect of the GVDPSS preconditioner, which will be confirmed in the section of numerical experiments.

If the matrix A in (1) is symmetric positive definite, we analyze the behavior of $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ in the following theorems.

Theorem 3.5 *Let λ be an eigenvalue of the matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ and $(x^*, y^*)^*$ be the corresponding eigenvector. Then $x \neq 0$ and λ either is $\lambda = 1$ or $\lambda = \frac{\beta b_2 + \frac{1}{\alpha}c_2}{a_2}$ with*

$$a_2 = x^*(\beta I + \frac{1}{\alpha}B^T B)A(\beta I + \frac{1}{\alpha}B^T B)x,\ b_2 = x^* B^T Bx,\ c_2 = x^*(B^T B)^2 x.$$

Furthermore, if $\beta \to 0_+$ and $\alpha \to \alpha_0 (0 < \alpha_0 < +\infty)$, then either is $\lambda = 1$ or

$$\frac{\alpha_0}{\lambda_{\max}(A)} \le \lambda \le \frac{\alpha_0}{\lambda_{\min}(A)},$$

where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the minimum and maximum eigenvalues of A, respectively. In particular, if $\beta \to 0_+$ and $\alpha \to \frac{(B^T Bx)^* A(B^T Bx)}{(B^T Bx)^*(B^T Bx)}$, then $\lambda \to 1$; and the eigenvalues of $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ are clustering around $(1, 0)$ and $(0, 0)$ as $\beta \to +\infty$.

Proof Let $(x^*, y^*)^*$ be the eigenvector of the matrix $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ corresponding to the eigenvalue λ. Thus, we have

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \mathcal{P}_{GVDPSS}\begin{pmatrix} x \\ y \end{pmatrix},$$

which can be written as

$$\begin{cases} Ax + B^T y = \lambda Ax + \frac{\lambda}{\alpha}AB^T y, \\ -Bx = -\lambda Bx + \lambda\beta y, \end{cases}$$

i.e.,

$$\begin{cases} (\lambda - 1)Ax + (\frac{\lambda}{\alpha}A - I)B^T y = 0, \\ (\lambda - 1)Bx = \lambda\beta y. \end{cases} \tag{43}$$

Premultiplying the second equation of (43) with B^T yields

$$(\lambda - 1)B^T Bx = \lambda\beta B^T y. \tag{44}$$

Solving $B^T y$ from (44) and substituting it into the first equation of (43) give

$$\lambda\beta(\lambda - 1)Ax + \left(\frac{\lambda}{\alpha}A - I\right)(\lambda - 1)B^T Bx = 0. \tag{45}$$

We observe that $x \neq 0$. Otherwise, the second equation of (43) together with the fact that $\beta > 0$ implies $y = 0$ or $\lambda = 0$, a contradiction. Without loss of generality, we assume that x has been normalized so that $\|x\|_2 = 1$. We find after some easy algebraic manipulations

$$\lambda^2 A\left(\beta I + \frac{1}{\alpha}B^T B\right)x - \lambda\left(A\left(\beta I + \frac{1}{\alpha}B^T B\right) + B^T B\right)x + B^T Bx = 0. \quad (46)$$

Premultiplying the (46) with $x^*(\beta I + \frac{1}{\alpha}B^T B)$ yields

$$a_2\lambda^2 - \left(a_2 + \beta b_2 + \frac{1}{\alpha}c_2\right)\lambda + \beta b_2 + \frac{1}{\alpha}c_2 = 0. \quad (47)$$

By straightforwardly solving (47), we immediately obtain that the roots of (47) are $\lambda = 1$ and

$$\lambda = \frac{\beta b_2 + \frac{1}{\alpha}c_2}{a_2} = \frac{\beta b_2 + \frac{1}{\alpha}c_2}{\beta^2 x^* Ax + \frac{\beta}{\alpha}(x^* B^T BAx + x^* AB^T Bx) + \frac{1}{\alpha^2}x^* B^T BAB^T Bx}. \quad (48)$$

If $Bx = 0$, then it follows from the second equation of (43) that $y = 0$. Substituting $y = 0$ into the first equation of (43) results in $(\lambda - 1)Ax = 0$, and it is clear that $\lambda = 1$ by $Ax \neq 0$. Therefore, if $\beta \to 0_+$ and $\alpha \to \alpha_0$ $(0 < \alpha_0 < +\infty)$, then $\lambda = 1$ or

$$\lambda \to \frac{\frac{1}{\alpha_0}c_2}{\frac{1}{\alpha_0^2}x^* B^T BAB^T Bx} = \alpha_0\frac{(B^T Bx)^*(B^T Bx)}{(B^T Bx)^* A(B^T Bx)} = \alpha_0\frac{z^* z}{z^* Az}, \quad (49)$$

where $z = B^T Bx$. Since A is symmetric positive definite, $\lambda_{\min}(A) \leq \frac{z^* Az}{z^* z} \leq \lambda_{\max}(A)$, which implies that

$$\frac{\alpha_0}{\lambda_{\max}(A)} \leq \lambda \leq \frac{\alpha_0}{\lambda_{\min}(A)}.$$

Moreover, if $\beta \to 0_+$ and $\alpha \to \frac{(B^T Bx)^* A(B^T Bx)}{(B^T Bx)^*(B^T Bx)}$, from (49), it is not difficult to see that $\lambda \to 1$. If $\beta \to +\infty$, then it follows (48) and $x^* Ax > 0$ that $\lambda \to 0$. This proof is completed. □

Theorem 3.6 *Let $A \in \mathbb{R}^{m\times m}$ be symmetric positive definite, $B \in \mathbb{R}^{n\times m}$ be of full row rank, and α, β be two positive constants. We suppose that $sp(BB^T) \subseteq [\delta_1, \delta_n]$ and $sp(BA^{-1}B^T) \subseteq [\tau_1, \tau_n]$, where $sp(T)$ denotes the spectrum of the matrix T. Then the preconditioned matrix $\mathcal{P}_{GVDPSS}^{-1}\mathcal{A}$ has eigenvalue 1 with algebraic multiplicity at least m, and the remaining eigenvalues are real and located in the positive interval*

$$\left[\frac{\alpha\tau_1}{\alpha\beta + \delta_n}, \frac{\alpha\tau_n}{\alpha\beta + \delta_1}\right]. \quad (50)$$

Proof By Theorem 3.1, we see that the preconditioned matrix $\mathcal{P}_{\mathrm{GVDPSS}}^{-1}\mathcal{A}$ has eigenvalue 1 with algebraic multiplicity at least m, and the remaining eigenvalues are the same as those of the matrix $(\beta I + \frac{1}{\alpha}BB^T)^{-1}BA^{-1}B^T$. Since the matrix A is symmetric positive definite and B has full row rank, it is easy to observe that BB^T

and $BA^{-1}B^T$ are symmetric positive definite, then it holds that $\delta_n \geq \delta_1 > 0$ and $\tau_n \geq \tau_1 > 0$, which yields that

$$sp\left(\beta I + \frac{1}{\alpha}BB^T\right) \subseteq \left[\frac{\alpha\beta + \delta_1}{\alpha}, \frac{\alpha\beta + \delta_n}{\alpha}\right].$$

Inasmuch as $\beta I + \frac{1}{\alpha}BB^T$ is symmetric positive definite for $\alpha > 0$ and $\beta > 0$, we obtain

$$sp\left(\left(\beta I + \frac{1}{\alpha}BB^T\right)^{-1}\right) \subseteq \left[\frac{\alpha}{\alpha\beta + \delta_n}, \frac{\alpha}{\alpha\beta + \delta_1}\right]$$

and

$$sp\left(\left(\beta I + \frac{1}{\alpha}BB^T\right)^{-1} BA^{-1}B^T\right) \subseteq \left[\frac{\alpha\tau_1}{\alpha\beta + \delta_n}, \frac{\alpha\tau_n}{\alpha\beta + \delta_1}\right],$$

which proves the desired bound (50). □

4 Implementation aspects about $\mathcal{P}_{\text{GVDPSS}}$

In what follows, we will elaborate on specific implementation issues. At each step of the GVDPSS iteration or applying the GVDPSS preconditioner $\mathcal{P}_{\text{GVDPSS}}$ with a Krylov subspace method, we need to solve a linear system with $\mathcal{P}_{\text{GVDPSS}}$ as the coefficient matrix. Besides, the matrix $\mathcal{P}_{\text{GVDPSS}}$ involves the parameters α and β, which may affect the convergence of both GVDPSS iteration and GVDPSS preconditioner. Therefore, to implement the GVDPSS iteration or the GVDPSS preconditioner efficiently, two aspects need to be considered.

The first one is to choose the parameters α and β such that all the eigenvalues of the preconditioned matrix $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ are clustering around $(1, 0)$, which is a desirable property for Krylov subspace acceleration and can result in favorable convergence rates. Therefore, we need to research the optimal parameters, while this is a difficult task for us now. The second one is how to solve the linear system of the form

$$\begin{pmatrix} A & \frac{1}{\alpha}AB^T \\ -B & \beta I \end{pmatrix} z = r, \tag{51}$$

where $z = (z_1^T, z_2^T)^T$, $r = (r_1^T, r_2^T)^T$ with $z_1, r_1 \in \mathbb{R}^m$ and $z_2, r_2 \in \mathbb{R}^n$. It follows from the decomposition of $\mathcal{P}_{\text{GVDPSS}}^{-1}$ in (15) that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} I & -\frac{1}{\alpha}B^T \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \tilde{A}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{52}$$

Then, the following algorithmic version of the GVDPSS iteration method can be derived.

Algorithm 4.1 *For a given vector $r = (r_1^T, r_2^T)^T$, we can compute the vector $z = (z_1^T, z_2^T)^T$ in (51) from the following steps:*

(i) *Solve $At_1 = r_1$.*

(ii) *Solve* $(\beta I + \frac{1}{\alpha}BB^T)z_2 = Bt_1 + r_2$.
(iii) *Compute* $z_1 = t_1 - \frac{1}{\alpha}B^T z_2$.

Remark 4.1 From Algorithm 4.1, the main costs at each iteration for computing (51) are solving two sub-linear systems with coefficient matrices A and $\beta I + \frac{1}{\alpha}BB^T$, respectively. According to Section 1, we see that A is positive definite and $\beta I + \frac{1}{\alpha}BB^T$ is symmetric positive definite for $\alpha, \beta > 0$. Therefore, we can solve the system with the coefficient matrix A by the GMRES method inexactly or by the sparse LU factorization [21, 44] exactly, and the system with the coefficient matrix $\beta I + \frac{1}{\alpha}BB^T$ can be efficiently solved by the CG method inexactly or the Cholesky factorization exactly. In actual implementations, the inexact solvers can be used to reduce the cost of each iteration, but they will also lead to somewhat slower convergence [44]. Thus, we solve these two sub-linear systems exactly by the sparse LU factorization and the Cholesky factorization, respectively, in this paper.

5 Numerical experiments

In this section, we carry out some numerical examples to illustrate the effectiveness and show the advantages of the GVDPSS preconditioned GMRES method over the DPSS, RDPSS, SL, NRHSS, and VDPSS preconditioned GMRES methods from the point of view of both the number of iterations (denoted by IT) and the total computing times (in seconds, denoted by CPU). All numerical procedures are performed in Matlab 6.5 on a personal computer with Intel® Pentium® CPU G3240T 2.70 GHz, 2.0-G memory, and Windows 7 operating system.

Example 5.1 Consider the saddle point problem structured as (1) with the following coefficient sub-matrices [34]:

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2}, \quad B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times p^2},$$

$$T = \frac{v}{h^2}.\text{tridiag}(-1, 2, -1) + \frac{1}{2h}.\text{tridiag}(-1, 0, 1) \in \mathbb{R}^{p \times p}, \quad F = \frac{1}{h}.\text{tridiag}(-1, 1, 0) \in \mathbb{R}^{p \times p},$$

with $\otimes$ being the Kronecker product and $h = \frac{1}{p+1}$ the discretization mesh size.

In actual computations, we choose the right-hand side vector b so that the exact solution of the nonsymmetric saddle point problem (1) is $(1, 1, \cdots, 1)^T \in \mathbb{R}^{m+n}$. Besides, all computations for the DPSS, RDPSS, SL, VDPSS, and GVDPSS preconditioned GMRES methods are started from initial vector $\mathbf{x}^{(0)} = (x^{(0)T}, y^{(0)T})^T = (0, 0, \cdots, 0)^T$ and terminated if the current iterations satisfy

$$\text{RES} = \frac{\|b - \mathcal{A}\mathbf{x}^{(k)}\|_2}{\|b - \mathcal{A}\mathbf{x}^{(0)}\|_2} < 10^{-6}, \tag{53}$$

or the maximum prescribed number of iterations $k_{max} = 500$ is exceeded.

For this example, the parameters of the GVDPSS preconditioner are chosen as $\alpha \in [0.001, 1000]$ and $\beta \in [0.001, 1000000]$, and the matrix Q in the SL preconditioner is chosen as $Q = \text{diag}(A)$. We recall that $\mathcal{P}_{\text{VDPSS}}$ is a special case of

Table 1 Numerical results for the GVDPSS preconditioned GMRES method with $\upsilon = 1$ and $p = 24$

	β	100,000	10,000	1000	100	10	1	0.1	0.01	0.001
$\alpha = 1000$	IT	1	6	8	9	11	13	22	26	26
	CPU	1.7139	4.4648	6.0329	5.3799	6.8712	7.7596	13.3736	14.9744	15.4604
$\alpha = 100$	IT	1	6	8	9	12	20	24	24	24
	CPU	1.6613	4.3202	5.3107	5.7943	7.5440	12.1440	15.2225	15.1507	16.7059
$\alpha = 10$	IT	1	6	8	10	17	21	22	22	22
	CPU	1.8516	4.2822	5.6013	6.5757	10.0541	13.0361	12.8947	12.4641	12.1600
$\alpha = 1$	IT	1	6	8	13	17	17	17	17	17
	CPU	2.0067	4.4885	5.6859	8.8107	10.8194	9.9767	10.1627	9.9926	9.9731
$\alpha = 0.1$	IT	1	6	10	12	13	13	13	13	13
	CPU	2.2942	4.5533	6.4616	7.4698	7.9501	8.6732	8.1504	8.0137	9.2277
$\alpha = 0.01$	IT	1	5	7	7	7	7	7	7	7
	CPU	2.0165	4.4085	5.1948	4.9949	5.2139	5.0206	4.8460	5.5540	4.7615
$\alpha = 0.001$	IT	1	2	4	4	4	4	4	4	4
	CPU	1.9044	2.7293	3.8888	3.5226	3.3572	3.6689	3.5159	3.7762	4.1576

$\mathcal{P}_{\text{GVDPSS}}$ when $\alpha = \beta$. We use "DPSS", "RDPSS", "SL", "VDPSS," and "GVDPSS" to denote the GMRES method with the DPSS, RDPSS, SL, VDPSS, and GVDPSS preconditioners, respectively.

In Tables 1, 2 and 3, we list the results of the GVDPSS preconditioned GMRES method in terms of IT and CPU for saddle point problems with $\upsilon = 1$, $\upsilon = 0.1$, and $\upsilon = 0.01$, respectively, for $p = 24$. To further confirm the effectiveness of the

Table 2 Numerical results for the GVDPSS preconditioned GMRES method with $\upsilon = 0.1$ and $p = 24$

	β	100,000	10,000	1000	100	10	1	0.1	0.01	0.001
$\alpha = 1000$	IT	11	14	17	18	18	23	31	32	30
	CPU	7.8485	9.5388	13.9622	14.6132	14.6561	17.4538	21.9757	23.6766	22.2521
$\alpha = 100$	IT	11	14	17	18	19	26	28	26	26
	CPU	9.5310	11.7241	14.1449	13.4458	13.1775	21.0852	19.3895	20.9695	20.7821
$\alpha = 10$	IT	11	14	16	17	23	26	27	26	26
	CPU	9.5004	11.9236	13.1705	15.1589	18.5401	20.5643	21.2656	20.2722	20.0981
$\alpha = 1$	IT	11	14	15	20	23	23	22	22	22
	CPU	9.5402	11.6114	12.5356	15.8989	18.3681	16.9818	16.4073	16.8074	16.4695
$\alpha = 0.1$	IT	11	13	16	20	20	20	20	20	20
	CPU	9.5217	11.0436	9.4993	11.7432	11.4953	11.6866	11.4748	11.4523	11.7257
$\alpha = 0.01$	IT	10	11	14	13	14	14	14	14	14
	CPU	6.3085	7.0634	8.3966	7.6749	8.5599	8.4482	8.1955	8.2760	8.2875
$\alpha = 0.001$	IT	6	9	9	9	9	9	9	9	9
	CPU	4.2292	5.8238	5.7442	5.7969	6.0181	5.7448	5.8025	5.8968	5.7026

Table 3 Numerical results for the GVDPSS preconditioned GMRES method with $\upsilon = 0.01$ and $p = 24$

	β	1,000,000	100,000	1000	100	10	1	0.1	0.01	0.001
$\alpha = 1000$	IT	14	84	113	118	116	75	51	48	45
	CPU	8.5736	46.3976	60.3448	63.7857	62.4106	40.1047	31.8685	29.1744	25.5613
$\alpha = 100$	IT	14	84	111	102	62	43	40	37	36
	CPU	8.5005	44.5203	59.0186	55.0031	33.5885	23.6799	23.4273	21.6468	20.9897
$\alpha = 10$	IT	14	84	95	56	36	37	35	33	33
	CPU	8.6677	44.6429	50.7588	31.7057	19.8900	20.7095	19.7924	18.4853	18.5156
$\alpha = 1$	IT	14	82	54	31	34	34	33	33	33
	CPU	8.5974	44.0950	29.9970	17.8284	19.1303	19.0409	18.4896	18.2374	18.7299
$\alpha = 0.1$	IT	14	47	24	27	28	28	27	26	26
	CPU	8.4448	29.3747	13.8424	15.5976	15.7023	15.6775	15.3890	14.6626	14.5597
$\alpha = 0.01$	IT	11	18	20	22	21	20	20	20	20
	CPU	7.0144	10.9576	11.7384	12.6556	11.9811	11.6172	11.4191	11.4749	11.5607
$\alpha = 0.001$	IT	8	14	14	14	12	14	14	14	14
	CPU	6.9666	8.4969	8.2909	8.5329	7.4792	9.0658	8.4597	8.5450	8.4449

GVDPSS preconditioned GMRES method, numerical results of the RDPSS, SL, and GVDPSS preconditioned GMRES methods with respect to IT, CPU, and RES for saddle point problems with a fixed $\beta = 100000$ and different values of α for $\upsilon = 0.1$ are reported in Table 4.

In order to compare effects of the DPSS, VDPSS, and GVDPSS preconditioned GMRES methods in terms of the parameter α, we plot the IT of the three preconditioned GMRES methods with α from 0.01 to 1 with step size 0.01 in Fig. 1. We consider $\upsilon = 1, \beta = 10,000$, $\upsilon = 0.1, \beta = 100,000$ and $\upsilon = 0.01, \beta = 1,000,000$, respectively, in this figure. Figures 2, 3 and 4 depict the eigenvalue distributions of

Table 4 Numerical results for the three preconditioned GMRES methods with $\upsilon = 0.1$

Case		16×16			24×24			32×32		
	α	1	0.1	0.01	1	0.1	0.01	1	0.1	0.01
RDPSS	IT	19	16	13	22	20	14	26	21	16
	CPU	1.9083	1.1624	0.9795	12.6164	11.3498	8.2667	73.1306	60.5860	49.3416
	RES	8.82e-07	8.94e-07	9.35e-07	9.24e-07	9.46e-07	9.65e-07	8.26e-07	8.53e-07	8.11e-07
SL	IT	20	21	19	26	26	25	31	31	29
	CPU	1.5590	1.5521	1.3713	15.8169	15.0121	13.8840	86.7787	89.9106	86.6060
	RES	5.89e-07	7.46e-07	3.01e-07	6.31e-07	3.77e-07	6.70e-07	3.23e-07	2.70e-07	4.07e-07
GVDPSS	IT	11	10	9	11	11	10	11	11	10
	CPU	0.8840	0.7686	0.7203	7.6650	6.8658	6.8242	34.3038	34.3429	31.7524
	RES	5.50e-07	9.46e-07	9.95e-07	6.70e-07	5.86e-07	6.22e-07	7.32e-07	6.44e-07	6.83e-07

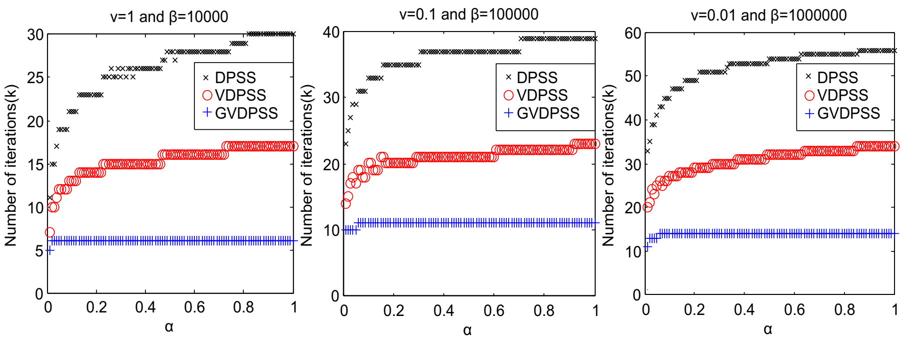

Fig. 1 Convergence curve of algorithms with varying α for $p = 24$

the original matrix $\mathcal{A}$ and the three preconditioned matrices for $v = 0.1$ with different values of α ($\alpha = 100$, $\alpha = 1$, and $\alpha = 0.01$), and we adopt $\beta = 100,000$ in the GVDPSS preconditioner. For more investigations, the eigenvalue distributions of $\mathcal{P}_{\text{VDPSS}}^{-1}\mathcal{A}$ and $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ with different values of α and β for $v = 0.1$ and $p = 24$ are displayed in Fig. 5. Additionally, we compare the eigenvalue distribution of $\mathcal{P}_{\text{VDPSS}}^{-1}\mathcal{A}$ for $\alpha = 1$ with that of $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ for $\alpha = 1$ and different values of β in Fig. 6.

From these tables and figures, we have the following observations:

- As observed in Tables 1, 2 and 3, the GVDPSS preconditioned GMRES method with the proper parameter β outperforms the VDPSS preconditioned GMRES method as it requires less IT and CPU times. This indicates that $\alpha = \beta$ is not the best choice. In addition, we can see that when α becomes small or β becomes large, the convergence rate of the GVDPSS preconditioned GMRES method

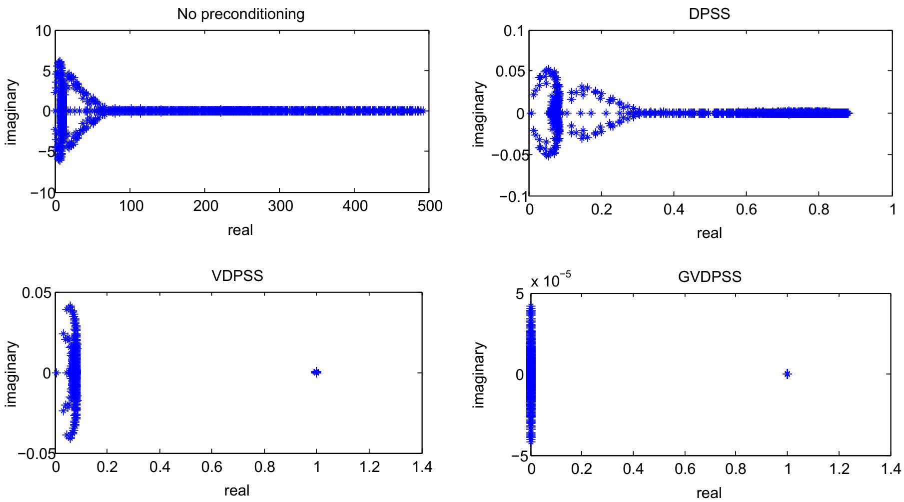

Fig. 2 The eigenvalue distribution of the four preconditioners for $\mathcal{A}$ when $p = 24$ and $\alpha = 100$ with $v = 0.1$

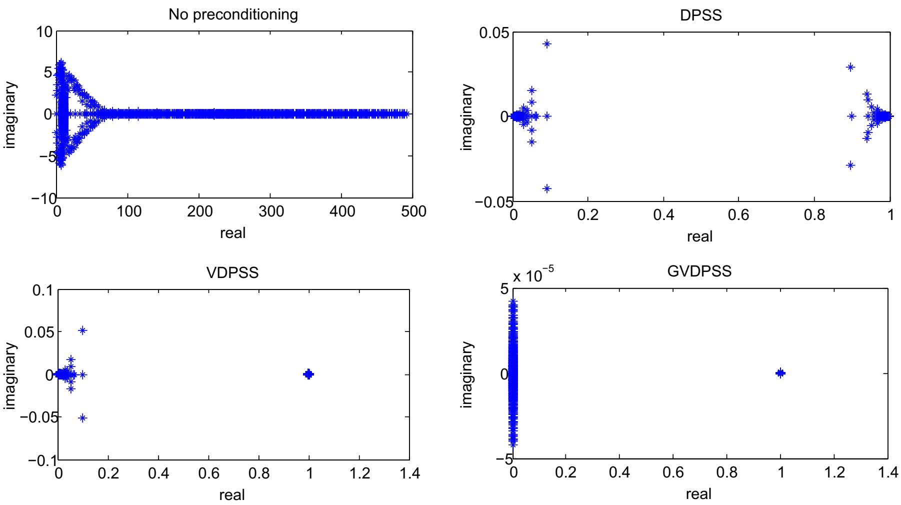

Fig. 3 The eigenvalue distribution of the four preconditioners for $\mathcal{A}$ when $p = 24$ and $\alpha = 1$ with $v = 0.1$

gradually becomes faster. By making use of Theorem 3.2, it holds that when the α becomes small or the β becomes large, the eigenvalues of the GVDPSS preconditioned matrix tend to clustered around two points $(0, 0)$ and $(1, 0)$, which means that the number of distinct eigenvalues of the GVDPSS preconditioned matrix is small, and it follows from Theorem 3.4 that the GVDPSS preconditioned GMRES method will terminate within a small number of steps and the rate of convergence will be rapid.

- By comparing the results in Table 4, it can be seen that the three preconditioned GMRES methods succeed in producing high-quality approximate solutions in all cases, while the GVDPSS preconditioned GMRES method outperforms the RDPSS and SL preconditioned GMRES methods in terms of IT and CPU times. Besides, numerical results in Table 4 show that the GVDPSS preconditioner with proper β is not sensitive to α and p.

- From Fig. 1, as we expected for Example 5.1, we see that the GVDPSS preconditioned GMRES method returns better numerical results than the other two methods. Another observation which can be posed here is that the GVDPSS preconditioner is not sensitive to the value of the parameter α, in the sense that the iteration count does not change dramatically. Hence, there may exist a fairly wide range of values of α that produce similar fast convergence results.

- Figures 2, 3 and 4 show that the eigenvalue distributions of the GVDPSS preconditioned matrix are well-clustered and clustered more closely than those of other three preconditioned matrices (without preconditioning situation can be regarded as a unit preconditioner).

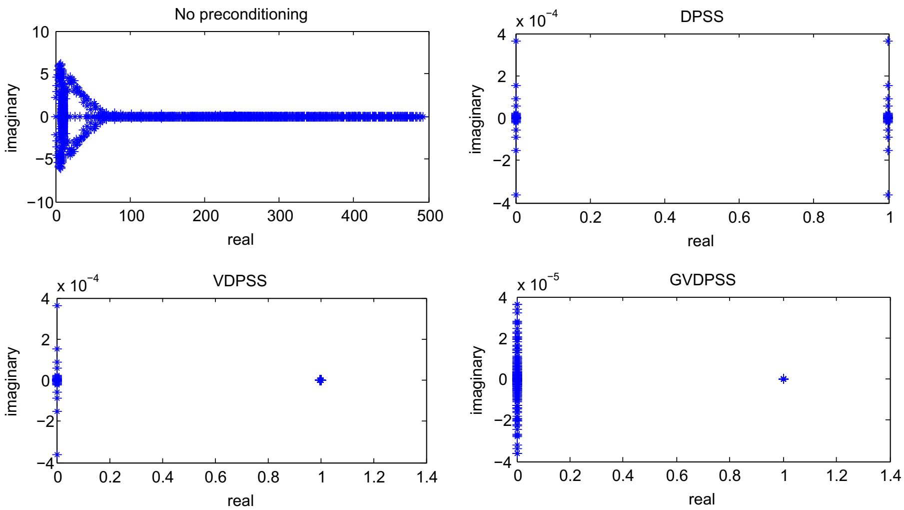

Fig. 4 The eigenvalue distribution of the four preconditioners for $\mathcal{A}$ when $p = 24$ and $\alpha = 0.01$ with $v = 0.1$

- As seen from Fig. 5, the eigenvalues of $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ are more clustered than those of $\mathcal{P}_{\text{VDPSS}}^{-1}\mathcal{A}$. This indicates that the GVDPSS preconditioner outperforms the VDPSS preconditioner, which is congruous with the results of Table 2.
- From Fig. 6, we observe that $\mathcal{P}_{\text{GVDPSS}}^{-1}\mathcal{A}$ with large β has much denser spectrum distribution. These observations imply that the GVDPSS preconditioner with proper β leads to much better performance than the VDPSS preconditioner for the GMRES method and it can act as an efficient preconditioner for saddle point problems.

Example 5.2 Consider the saddle point problem structured as (1) with the following coefficient sub-matrices [10]:

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2}, \quad B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times p^2},$$

$$T = \frac{v}{h^2}.\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{p \times p}, \quad F = \frac{1}{h}.\text{tridiag}(-1, 1, 0) \in \mathbb{R}^{p \times p},$$

with $\otimes$ being the Kronecker product and $h = \frac{1}{p+1}$ the discretization mesh size.

We compare the numerical results of the GVDPSS preconditioned GMRES method (denoted by "GVDPSS") with the NRHSS (denoted by "NRHSS") and VDPSS (denoted by "VDPSS") preconditioned GMRES methods by the initial vector $\mathbf{x}^{(0)} = (x^{(0)T}, y^{(0)T})^T = (0, 0, \cdots, 0)^T$.

In actual computations, we set the right-hand side vector $b = \mathcal{A}e^{m+n}$, where $e^{m+n} = (1, 1, \cdots, 1)^T \in \mathbb{R}^{m+n}$ and we adopt the parameters $\alpha \in [0.001, 1000]$ and $\beta \in [0.001, 1000000]$. All iteration processes are terminated if the current iterations

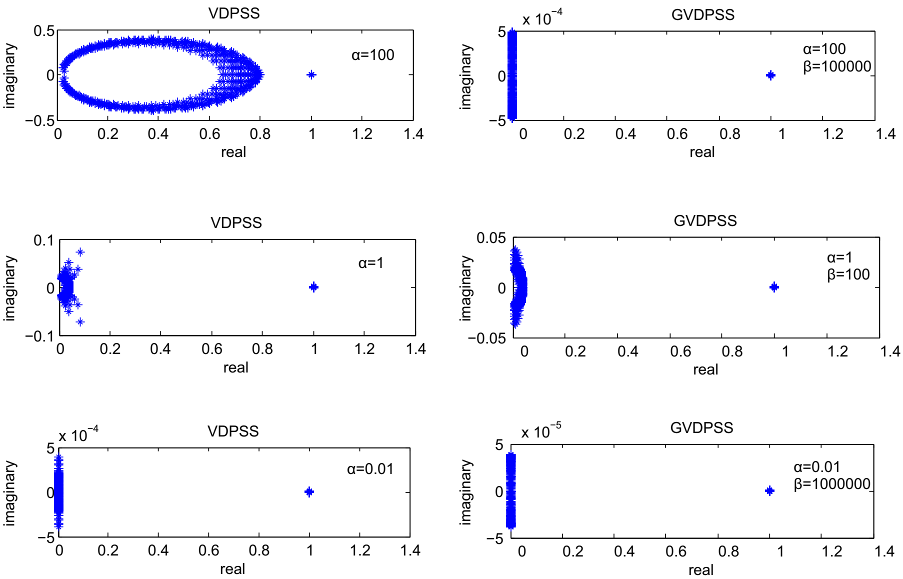

Fig. 5 The eigenvalue distributions of the VDPSS and GVDPSS preconditioned matrices for $\upsilon = 0.1$ with various α and β

$\mathbf{x}^{(k)}$ satisfy (53) or the maximum prescribed number of iterations $k_{max} = 500$ is exceeded. It is worthy noting that $\mathcal{P}_{\text{NRHSS}}$ is a special case of $\mathcal{P}_{\text{GVDPSS}}$ when $\alpha = 1$.

In Tables 5, 6 and 7, we list the numerical results with respect to IT and CPU for the GVDPSS preconditioned GMRES method for Example 5.2 with $\upsilon = 1$, $\upsilon = 0.1$, and $\upsilon = 0.01$, respectively, when $p = 24$. From these tables, we can conclude some observations as follows. Firstly, the GVDPSS preconditioned GMRES method with

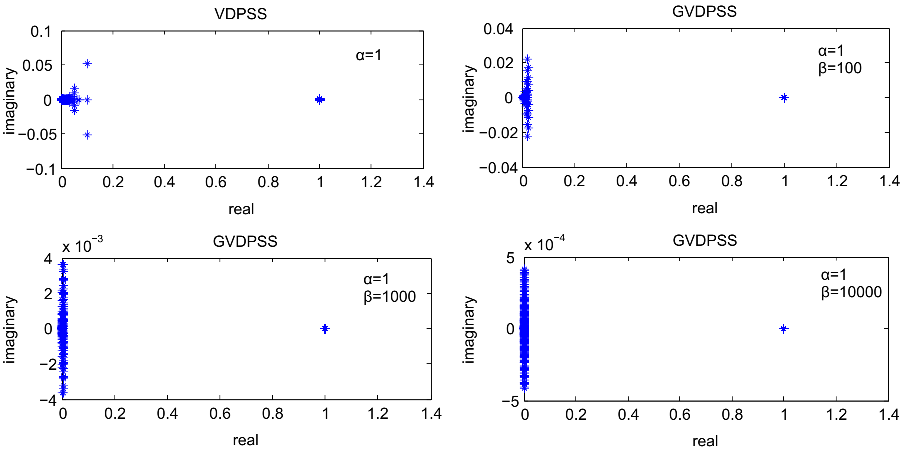

Fig. 6 The eigenvalue distributions of the VDPSS and GVDPSS preconditioned matrices for $\upsilon = 0.01$ and $p = 24$ with $\alpha = 1$ and various β

Table 5 Numerical results for the GVDPSS preconditioned GMRES method with $v = 1$ and $p = 24$

	β	100,000	10,000	1000	100	10	1	0.1	0.01	0.001
$\alpha = 1000$	IT	12	13	14	15	14	16	24	28	27
	CPU	7.8113	8.4007	8.1109	8.8716	8.5020	9.5757	13.7557	15.8431	15.2452
$\alpha = 100$	IT	12	13	14	15	18	26	29	27	27
	CPU	7.8435	8.2147	8.5684	9.1540	10.9894	14.9365	16.4701	15.1272	15.2587
$\alpha = 10$	IT	12	14	14	18	28	32	31	29	29
	CPU	8.2468	8.5649	8.8351	10.7273	15.9699	17.8846	17.4011	16.7404	16.2354
$\alpha = 1$	IT	12	14	17	25	29	29	29	29	28
	CPU	7.5519	8.5873	10.3029	14.2757	16.5848	16.3031	16.1276	16.2411	15.7956
$\alpha = 0.1$	IT	12	14	21	26	24	24	24	24	24
	CPU	8.0219	8.4633	12.2779	15.0582	13.7599	13.5899	13.6441	13.7281	13.5843
$\alpha = 0.01$	IT	12	17	19	19	19	19	19	19	19
	CPU	7.5776	10.1996	11.0534	11.1025	10.9963	11.0147	10.9751	11.0856	11.0754
$\alpha = 0.001$	IT	12	13	13	13	13	13	13	13	13
	CPU	7.5614	8.0011	7.9526	7.8986	7.9967	8.1247	7.9495	7.6570	7.9414

proper parameters α and β leads to much better numerical results than the VDPSS and NRHSS preconditioned GMRES methods as it requires less IT and CPU times. Secondly, the GVDPSS preconditioned GMRES method performs better when α becomes small and β becomes large. Thirdly, $\alpha = \beta$ and $\alpha = 1$ are not the best choices for the GVDPSS preconditioner.

Table 6 Numerical results for the GVDPSS preconditioned GMRES method with $v = 0.1$ and $p = 24$

	β	100,000	10,000	1000	100	10	1	0.1	0.01	0.001
$\alpha = 1000$	IT	10	11	11	13	13	18	27	31	30
	CPU	6.2198	6.7252	7.2278	7.2796	7.8076	10.2314	15.3845	17.2536	16.4706
$\alpha = 100$	IT	10	11	12	13	16	25	28	27	27
	CPU	6.3001	6.6917	7.2623	8.4718	10.7242	14.4231	15.3297	15.2308	15.1403
$\alpha = 10$	IT	10	11	12	16	24	28	27	27	27
	CPU	6.7175	6.8016	7.3179	9.6184	13.2259	16.5776	14.9875	15.1276	14.6431
$\alpha = 1$	IT	10	11	14	22	26	26	26	26	26
	CPU	6.4230	6.7906	8.4277	12.5615	14.8577	14.7954	14.5233	14.5359	14.9377
$\alpha = 0.1$	IT	10	12	18	22	23	23	23	23	23
	CPU	6.2812	7.3956	10.3204	12.8282	13.2038	12.9612	13.2454	13.0651	13.0898
$\alpha = 0.01$	IT	10	13	17	17	17	17	17	17	17
	CPU	6.3552	7.7748	9.7804	10.0527	9.8993	9.8578	9.7422	9.8262	10.0192
$\alpha = 0.001$	IT	10	11	11	11	11	11	11	11	11
	CPU	6.2891	6.9392	7.2923	6.8321	6.8480	6.7845	6.8787	6.7585	6.8782

Table 7 Numerical results for the GVDPSS preconditioned GMRES method with $v = 0.01$ and $p = 24$

	β	1,000,000	100,000	1000	100	10	1	0.1	0.01	0.001
$\alpha = 1000$	IT	7	8	11	11	13	20	32	35	35
	CPU	4.9020	5.4415	6.9786	7.2044	7.4657	11.1181	18.6025	21.5749	20.6574
$\alpha = 100$	IT	7	8	11	12	16	26	29	29	28
	CPU	4.9292	5.6570	6.8543	6.6029	9.5870	15.7063	17.1597	16.9284	15.9577
$\alpha = 10$	IT	7	8	11	13	20	25	25	25	25
	CPU	5.0905	5.2178	6.9554	7.9754	11.9678	15.0015	14.2441	14.3654	14.3516
$\alpha = 1$	IT	7	8	12	18	22	22	23	23	23
	CPU	4.7306	5.4082	7.8123	10.7065	12.8895	12.9706	13.6678	13.8716	13.9612
$\alpha = 0.1$	IT	7	8	14	18	21	21	21	21	21
	CPU	4.8779	5.5336	9.2694	10.6375	12.8437	13.8898	12.5921	12.9033	13.0654
$\alpha = 0.01$	IT	6	8	15	16	16	16	16	16	16
	CPU	4.4867	5.3750	9.0118	9.9970	10.8318	10.4923	10.7385	10.6914	10.3992
$\alpha = 0.001$	IT	6	8	12	12	12	12	12	12	12
	CPU	4.3447	5.2097	7.9352	8.1859	8.1304	8.0022	8.1526	8.0052	8.2025

To further confirm the superiority of the GVDPSS preconditioned GMRES method to the NRHSS and VDPSS preconditioned GMRES methods, we plot the IT of the three preconditioned GMRES methods with α from 0.01 to 1 with step size 0.01 in Fig. 7. We set $v = 1, \beta = 10,000$, $v = 0.1, \beta = 100,000$ and $v = 0.01, \beta = 1,000,000$, respectively, in this figure. From this figure, we note that the three preconditioned GMRES methods converge while the GVDPSS preconditioned GMRES method converges faster. Furthermore, it can be observed that the VDPSS preconditioner is more sensitive to the parameter α than the GVDPSS and NRHSS preconditioners.

In order to better investigate the performance of the NRHSS, VDPSS, and GVDPSS preconditioned GMRES methods, in Fig. 8, we plot the eigenvalue distributions of the NRHSS, VDPSS, and GVDPSS preconditioned matrices with varied parameters $\alpha = 1, 0.1$, and 0.01 when $v = 0.01$ and $p = 24$. Here, "NRHSS"

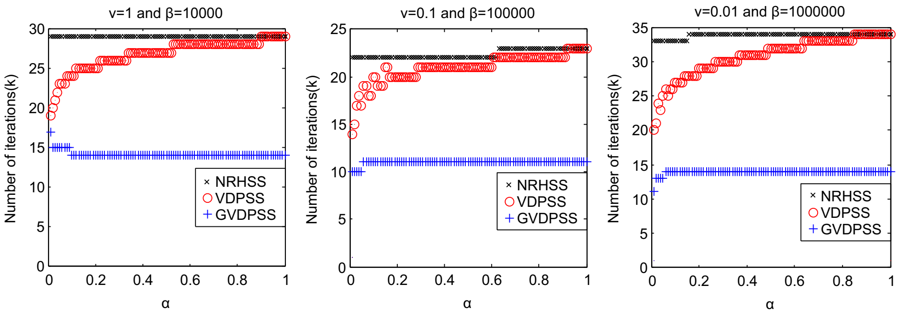

Fig. 7 Convergence curve of algorithms with varying α for $p = 24$

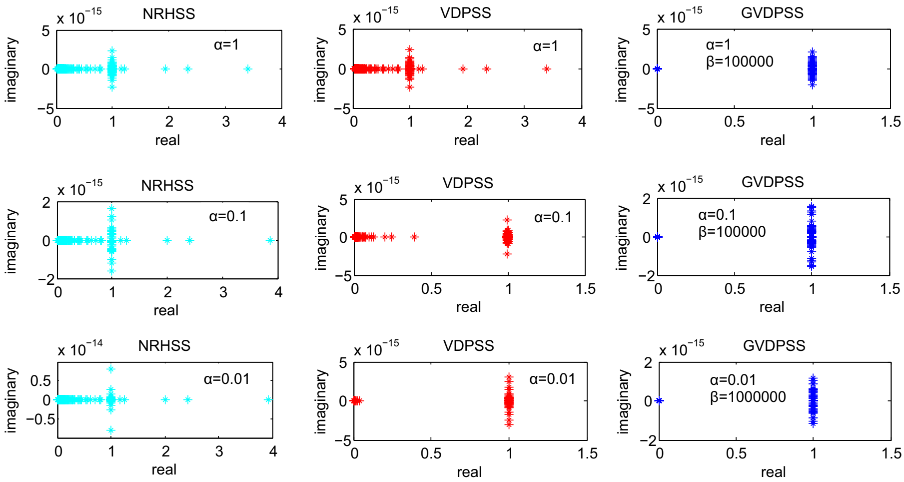

Fig. 8 The eigenvalue distributions of the three preconditioned matrices for $v = 0.01$ with various α and β

denotes the NRHSS preconditioned matrix, and the VDPSS and GVDPSS ones are denoted by "VDPSS" and "GVDPSS," respectively. By observation, we clearly find that the eigenvalue distributions of the GVDPSS preconditioned matrix with proper parameter β are more clustered than those of the other two preconditioned matrices. These observations imply that the GVDPSS preconditioned GMRES method performs much better than the NRHSS and VDPSS preconditioned GMRES methods. These facts are further confirmed by the numerical results listed in Table 7.

6 Conclusions

In this paper, we establish a generalized variant of the deteriorated PSS (GVDPSS) preconditioner for nonsymmetric saddle point problems. This new preconditioner is based on the VDPSS preconditioner [44], and it may result in more rapid convergence rate with suitable choices of the parameters α and β. In addition, the proposed preconditioner includes the VDPSS and NRHSS preconditioners exhibited in [44] and [40], respectively, as special cases. The convergence analyses of the GVDPSS iteration method for solving nonsymmetric saddle point problems are presented. Meanwhile, the distribution of eigenvalues, the forms of the eigenvectors, and the upper bounds on the degree of the minimum polynomial of the preconditioned matrix are analyzed in detail. Numerical experiments worked out in Section 5 (Tables 1, 2, 3, 4, 5, 6 and 7 and Figs. 1, 2, 3, 4, 5, 6, 7 and 8) reveal that the GVDPSS preconditioned GMRES method with suitable parameters has great superiority compared with DPSS, RDPSS, SL, NRHSS, and VDPSS preconditioned GMRES methods in terms of the iterations and CPU times, and illustrate that the GVDPSS preconditioned GMRES method is a very efficient method for solving the nonsymmetric saddle point problems.

However, we should mention that this new preconditioner involved two parameters α and β. How to choose the optimal parameters for the GVDPSS preconditioner is a very practical and interesting problem that needs to be further in-depth studied.

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References

1. Bai, Z.-Z.: Sharp error bounds of some Krylov subspace methods for non-Hermitian linear systems. Appl. Math. Comput. **109**, 273–285 (2000)
2. Bai, Z.-Z.: Structured preconditioners for nonsingular matrices of block two-by-two structures. Math. Comp. **75**, 791–815 (2006)
3. Bai, Z.-Z.: Optimal parameters in the HSS-like methods for saddle-point problems. Numer. Linear Algebra Appl. **16**, 447–479 (2009)
4. Bai, Z.-Z.: Eigenvalue estimates for saddle point matrices of Hermitian and indefinite leading blocks. J. Comput. Appl. Math. **237**, 295–306 (2013)
5. Bai, Z.-Z.: Motivations and realizations of Krylov subspace methods for large sparse linear systems. J. Comput. Appl Math. **283**, 71–78 (2015)
6. Bai, Z.-Z., Golub, G.H.: Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems. IMA J. Numer. Anal. **27**, 1–23 (2007)
7. Bai, Z.-Z., Golub, G.H., Lu, L.-Z., Yin, J.-F.: Block triangular and skew-Hermitian splitting methods for positive-definite linear systems. SIAM J. Sci. Comput. **26**, 844–863 (2005)
8. Bai, Z.-Z., Golub, G.H., Ng, M.K.: Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. SIAM J. Matrix Anal. Appl. **24**, 603–626 (2003)
9. Bai, Z.-Z., Golub, G.H., Ng, M.K.: On inexact Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. Linear Algebra Appl. **428**, 413–440 (2008)
10. Bai, Z.-Z., Golub, G.H., Pan, J.-Y.: Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems. Numer. Math. **98**, 1–32 (2004)
11. Bai, Z.-Z., Li, G.-Q.: Restrictively preconditioned conjugate gradient methods for systems of linear equations. IMA J. Numer. Anal. **23**, 561–580 (2003)
12. Bai, Z.-Z., Ng, M.K., Wang, Z.-Q.: Constraint preconditioners for symmetric indefinite matrices. SIAM J. Matrix Anal. Appl. **31**, 410–433 (2009)
13. Bai, Z.-Z., Parlett, B.N., Wang, Z.-Q.: On generalized successive overrelaxation methods for augmented linear systems. Numer. Math. **102**, 1–38 (2005)
14. Bai, Z.-Z., Wang, Z.-Q.: Restrictive preconditioners for conjugate gradient methods for symmetric positive definite linear systems. J. Comput. Appl Math. **187**, 202–226 (2006)
15. Bai, Z.-Z., Wang, Z.-Q.: On parameterized inexact Uzawa methods for generalized saddle point problems. Linear Algebra Appl. **428**, 2900–2932 (2008)
16. Bai, Z.-Z., Yin, J.-F., Sum, Y.-F.: A shift-splitting preconditioner for non-Hermitian positive definite matrices. J. Comput. Math. **24**, 539–552 (2006)
17. Benzi, M.: Preconditioning techniques for large linear systems: a survey. J. Comput. Phys. **182**, 418–477 (2002)
18. Benzi, M., Golub, G.H., Liesen, J.: Numerical solution of saddle point problems. Acta Numer. **14**, 1–137 (2005)
19. Bramble, J.H., Pasciak, J.E., Vassilev, A.T.: Analysis of the inexact Uzawa algorithm for saddle point problems. SIAM J. Numer Anal. **34**, 1072–1092 (1997)
20. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer-Verlag, New York and London (1991)
21. Cao, Y., Dong, J.-L., Wang, Y.-M.: A relaxed deteriorated PSS preconditioner for nonsymmetric saddle point problems from the steady Navier-Stokes equation. J. Comput. Appl. Math. **273**, 41–60 (2015)
22. Cao, Y., Du, J., Niu, Q.: Shift-splitting preconditioners for saddle point problems. J. Comput. Appl Math. **272**, 239–250 (2014)

23. Cao, Y., Li, S., Yao, L.-Q.: A class of generalized shift-splitting preconditioners for nonsymmetric saddle point problems. Appl. Math Lett. **49**, 20–27 (2015)
24. Cao, Y., Ren, Z.-R., Shi, Q.: A simplified, HSS preconditioner for generalized saddle point problems. BIT Numer. Math. **56**, 423–439 (2016)
25. Chen, C.-R., Ma, C.-F.: A generalized shift-splitting preconditioner for saddle point problems. Appl. Math. Lett. **43**, 49–55 (2015)
26. Chen, C.-R., Ma, C.-F.: A generalized shift-splitting preconditioner for singular saddle point problems. Appl. Math. Comput. **269**, 947–955 (2015)
27. Dollar, H.S., Wathen, A.J.: Approximate factorization constraint preconditioners for saddle-point matrices. SIAM J. Sci Comput. **27**, 1555–1572 (2006)
28. Elman, H.C.: Preconditioners for saddle point problems arising in computational fluid dynamics. Appl. Numer. Math. **43**, 75–89 (2002)
29. Elman, H.C., Golub, G.H.: Inexact and preconditioned Uzawa algorithms for saddle point problems. SIAM J. Numer. Anal. **31**, 1645–1661 (1994)
30. Fan, H.-T., Zhu, X.-Y.: A generalized relaxed positive-definite and skew-Hermitian splitting preconditioner for non-Hermitian saddle point problems. Appl. Math. Comput. **258**, 36–48 (2015)
31. Golub, G.H., Wu, X., Yuan, J.-Y.: SOR-like methods for augmented systems. BIT Numer. Math. **41**, 71–85 (2001)
32. Guo, P., Li, C.-X., Wu, S.-L.: A modified SOR-like method for the augmented systems. J. Comput. Appl. Math. **274**, 58–69 (2015)
33. Li, C., Vuik, C.: Eigenvalue analysis of the SIMPLE preconditioning for incompressible flow. Numer Linear Algebra Appl. **11**, 511–523 (2004)
34. Liang, Z.-Z., Zhang, G.-F.: PU-STS method for non-Hermitian saddle-point problems. Appl. Math. Lett. **46**, 1–6 (2015)
35. Liang, Z.-Z., Zhang, G.-F.: SIMPLE-like preconditioners for saddle point problems from the steady Navier-Stokes equations. J. Comput. Appl. Math. **302**, 211–223 (2016)
36. Pan, J.-Y., Ng, M.K., Bai, Z.-Z.: New preconditioners for saddle point problems. Appl. Math. Comput. **172**, 762–771 (2006)
37. Rozložník, M., Simoncini, V.: Krylov subspace methods for saddle point problems with indefinite preconditioning. SIAM J. Matrix Anal. Appl. **24**, 368–391 (2002)
38. Saad, Y. Iterative Methods for Sparse Linear Systems, 2nd edn. SIAM, Philadelphia (2003)
39. Saad, Y., Schultz, M.H.: GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems. SIAM J. Sci. Stat. Comput. **7**, 856–869 (1986)
40. Salkuyeh, D.K., Masoudi, M.: A new relaxed HSS preconditioner for saddle point problems. Numer. Algorithms doi:10.1007/s11075-016-0171-2 (2016)
41. Simoncini, V.: Block triangular preconditioners for symmetric saddle-point problems. Appl. Numer. Math. **49**, 63–80 (2004)
42. Sturler, E.D., Liesen, J.: Block-diagonal and constraint preconditioners for nonsymmetric indefinite linear systems. SIAM J. Sci. Comput. **26**, 1598–1619 (2005)
43. Yun, J.-H.: Variants of the Uzawa method for saddle point problem. Comput. Math. Appl. **65**, 1037–1046 (2013)
44. Zhang, J.-L., Gu, C.-Q.: A variant of the deteriorated PSS preconditioner for nonsymmetric saddle point problems. BIT Numer. Math. **56**, 587–604 (2016)
45. Zhang, J.-L., Gu, C.-Q., Zhang, K.: A relaxed positive-definite and skew-Hermitian splitting preconditioner for saddle point problems. Appl. Math. Comput. **249**, 468–479 (2014)
46. Zhou, S.-W., Yang, A.-L., Dou, Y., Wu, Y.-J.: The modified shift-splitting preconditioners for nonsymmetric saddle-point problems. Appl. Math. Lett. **59**, 109–114 (2016)