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The split common null point problem for generalized resolvents in two banach spaces

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Abstract In this paper, we consider the split common null point problem in two Banach spaces. Then, using the generalized resolvents of maximal monotone operators and the generalized projections, we prove a strong convergence theorem for finding a solution of the split common null point problem in two Banach spaces. It seems that such a theorem for generalized resolvents is the first of its kind outside Hilbert spaces.

Keywords Split common null point problem · Maximal monotone operator · Fixed point · Generalized projection · Generalized resolvent · Hybrid method · Duality mapping

Mathematics Subject Classifications 2010 47H05 · 47H09

1 Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear

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operator. Then the *split feasibility problem* (SFP) [8] is to find $z \in H_1$ such that $z \in D \cap T^{-1}Q$. There exists several generalizations of the SFP: the multiple set SFP (MASFP) where the sets D, Q consist of intersections of a finite number of convex sets [16] (the original reference to MSSFP is [9]), the split common fixed point problem (SCFPP) [11, 17], and the split common null point problem (SCNPP) [7]. (SCNPP) is as follows: given set-valued mappings $A : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$, and a bounded linear operators $T : H_1 \to H_2$, find a point $z \in H_1$ such that

$$z \in A^{-1}0 \cap T^{-1}(B^{-1}0),$$

where $A^{-1}0$ and $B^{-1}0$ are sets of null points of A and B, respectively. Defining $U = T^*(I - P_Q)T$ in the split feasibility problem, where T^* is the adjoint operator of T and P_Q is the metric projection of H_2 onto Q, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [3]. Furthermore, if $D \cap T^{-1}Q$ is nonempty, then $z \in D \cap T^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda T^*(I - P_Q)T)z, \tag{1.1}$$

where $\lambda > 0$ and P_D are the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and the split common null point problem; see, for instance, [3, 7, 10, 11, 16, 17, 32]. However, we have not found many results outside Hilbert spaces. The first extension of SFP to Banach spaces appears in [22]. This algorithm was later extended to MSSFP in [31]. A very recent contribution for the SFP is [23]. Takahashi [29] also solves the split common null point problem in Banach spaces. Let E be a strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call such a mapping P_C the metric projection of E onto C. Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r,$$

where J is the duality mapping on E. This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of A. Takahashi [27, 28] extended the result of (1.1) to Banach spaces. Furthermore, by using the methods of [18, 19, 24] and metric projections, Takahashi [29] proved a strong convergence theorem for metric resolvents of maximal monotone operators in two Banach spaces.

In this paper, motivated by Takahashi's theorem [29], we consider the split common null point problem with generalized resolvents of maximal monotone operators in two Banach spaces. Then using the generalized resolvents of maximal monotone operators and the generalized projections, we prove a strong convergence theorem for finding a solution of the split null point problem in two Banach spaces. The question of how to solve the split common null point problem for generalized resolvents in two Banach spaces was posed by [14].

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [12, 20].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. The norm of E is said to be *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. The classical L_p spaces for $1 < p < \infty$ are uniformly convex and uniformly smooth. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [12, 13, 20, 25, 26]. We know the following result:

Lemma 1 ([25]) *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle \ge 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi_E(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.2}$$

In the case of no misunderstanding, ϕ_E is denoted by ϕ. Observe that, in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2; \tag{2.3}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle; \tag{2.4}$$

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \tag{2.5}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{2.6}$$

The following lemma was proved by Kamimura and Takahashi [15].

Lemma 2 ([15]) *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty, closed and convex subset of a smooth, strictly convex, and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. For example, see [1, 2, 15].

Lemma 3 ([1, 2, 15]) *Let E be a smooth, strictly convex, and reflexive Banach space. Let C be a nonempty, closed, and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = \Pi_C x_1$;
(2) $\langle z - y, Jx_1 - Jz\rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to [6, 21]; see also [26, Theorem 3.5.4].

Theorem 4 ([6, 21]) *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$Jx \in Jx_r + rAx_r.$$

This equation has a unique solution x_r. In fact, it is obvious from Theorem 4 that there exists a solution x_r of $Jx \in Jx_r + rAx_r$. Assume that $Jx \in Ju + rAu$ and $Jx \in Jv + rAv$. Then there exist $w_1 \in Au$ and $w_2 \in Av$ such that $Jx = Ju + rw_1$ and $Jx = Jv + rw_2$. So, we have that

$$\begin{aligned} 0 &= \langle u - v, Jx - Jx \rangle \\ &= \langle u - v, Ju + rw_1 - (Jv + rw_2) \rangle \\ &= \langle u - v, Ju - Jv + rw_1 - rw_2 \rangle \\ &= \langle u - v, Ju - Jv \rangle + \langle u - v, rw_1 - rw_2 \rangle \\ &= \phi(u, v) + \phi(v, u) + r\langle u - v, w_1 - w_2 \rangle \\ &\geq \phi(u, v) + \phi(v, u) \end{aligned}$$

and hence $0 = \phi(u, v) = \phi(v, u)$. Since E is strictly convex, we have $u = v$. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the generalized resolvents of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [26].

3 Main result

In this section, using the generalized resolvents of maximal monotone operators and the generalized projections, we prove a strong convergence theorem for finding a solution of the split common null point problem in two Banach spaces. The following lemma was proved by Hojo and Takahashi [14].

Lemma 5 ([14]) *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ and Q_μ be the generalized resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $\lambda, \mu, r > 0$ and $z \in E$. Then the following are equivalent:*

(i) $z = J_\lambda J_E^{-1}\big(J_E z - rT^*(J_F Tz - J_F Q_\mu Tz)\big)$;
(ii) $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

Using the idea of (i) in Lemma 5, we can prove the following theorem which solves the split common null point problem for generalized resolvents of maximal monotone operators in two Banach spaces. Such a problem was posed by [14].

Theorem 6 *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ and Q_μ be the generalized resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F Q_{\mu_n} T x_n)\big), \\ y_n = J_{\lambda_n} z_n, \\ C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(T x_n, Q_{\mu_n} T x_n)\}, \\ D_n = \{z \in E : \langle y_n - z, J_E z_n - J_E y_n\rangle \geq 0\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2} \text{ and } 0 < b \leq \lambda_n, \mu_n \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = \Pi_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

Proof It is obvious that $C_n \cap D_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. To show that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n$ for all $n \in \mathbb{N}$, let us show that

$$2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(T x_n, Q_{\mu_n} T x_n)$$

for all $z \in T^{-1}(B^{-1}0)$ and $n \in \mathbb{N}$. In fact, we have that for all $z \in T^{-1}(B^{-1}0)$ and $n \in \mathbb{N}$,

$$\begin{aligned} 2\langle x_n - z, \quad & J_E x_n - J_E z_n\rangle \\ &= 2\langle x_n - z, r_n T^*(J_F T x_n - J_F Q_{\mu_n} T x_n)\rangle \\ &= 2r_n \langle T x_n - T z, J_F T x_n - J_F Q_{\mu_n} T x_n\rangle \\ &= 2r_n \langle T x_n - Q_{\mu_n} T x_n + Q_{\mu_n} T x_n - T z, J_F T x_n - J_F Q_{\mu_n} T x_n\rangle \\ &= 2r_n \langle T x_n - Q_{\mu_n} T x_n, J_F T x_n - J_F Q_{\mu_n} T x_n\rangle \\ &\qquad + 2r_n \langle Q_{\mu_n} T x_n - T z, J_F T x_n - J_F Q_{\mu_n} T x_n\rangle \\ &\geq 2r_n \langle T x_n - Q_{\mu_n} T x_n, J_F T x_n - J_F Q_{\mu_n} T x_n\rangle \\ &= r_n \big(\phi_F(T x_n, Q_{\mu_n} T x_n) + \phi_F(Q_{\mu_n} T x_n, T x_n)\big) \\ &\geq r_n \phi_F(T x_n, Q_{\mu_n} T x_n). \end{aligned} \tag{3.1}$$

Then, we have that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n$ for all $n \in \mathbb{N}$. Next, to show that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset D_n$ for all $n \in \mathbb{N}$, let us show that $\langle y_n - z, J_E z_n - J_E y_n\rangle \geq 0$ for all $z \in A^{-1}0$ and $n \in \mathbb{N}$. In fact, we have that for all $z \in A^{-1}0$ and $n \in \mathbb{N}$,

$$\langle y_n - z, J_E z_n - J_E y_n\rangle = \langle J_{\lambda_n} z_n - z, J_E z_n - J_E J_{\lambda_n} z_n\rangle \geq 0. \tag{3.2}$$

Then, we have that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset D_n$ for all $n \in \mathbb{N}$. We shall show that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_n$ for all $n \in \mathbb{N}$. Since $\langle x_1 - z, J_E x_1 - J_E x_1\rangle \geq 0$ for all

$z \in E$, it is obvious that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_1 = E$. Suppose that, for some $k \in \mathbb{N}$, $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_k$. Then, $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_k \cap D_k \cap Q_k$. From $x_{k+1} = \Pi_{C_k \cap D_k \cap Q_k} x_1$, we have that

$$\langle x_{k+1} - z, J_E x_1 - J_E x_{k+1} \rangle \geq 0, \quad \forall z \in C_k \cap D_k \cap Q_k.$$

Since $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_k \cap D_k \cap Q_k$, we have that

$$\langle x_{k+1} - z, J_E x_1 - J_E x_{k+1} \rangle \geq 0, \quad \forall z \in A^{-1}0 \cap T^{-1}(B^{-1}0).$$

Then, we get that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_{k+1}$. By mathematical induction, we have that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset Q_n$ for all $n \in \mathbb{N}$. Thus, we have that

$$A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n \cap D_n \cap Q_n$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $A^{-1}0 \cap T^{-1}(B^{-1}0)$ is a nonempty, closed and convex subset of E, there exists $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$ such that $z_0 = \Pi_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$. We have from $x_{n+1} = \Pi_{C_n \cap D_n \cap Q_n} x_1$ that

$$\phi_E(x_{n+1}, x_1) \leq \phi_E(y, x_1), \quad \forall y \in C_n \cap D_n \cap Q_n.$$

Since $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n \cap D_n \cap Q_n$, we have that

$$\phi_E(x_{n+1}, x_1) \leq \phi_E(z_0, x_1). \tag{3.3}$$

This means that $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty} \phi_E(x_{n+1}, x_n) = 0$. From $x_{n+1} = \Pi_{C_n \cap D_n \cap Q_n} x_1$, we have that $x_{n+1} \in Q_n$ and hence

$$2\langle x_n - x_{n+1}, J_E x_1 - J_E x_n \rangle \geq 0.$$

From this, we have that

$$2\langle x_n - x_1 + x_1 - x_{n+1}, J_E x_1 - J_E x_n \rangle \geq 0.$$

This implies that

$$2\langle x_1 - x_{n+1}, J_E x_1 - J_E x_n \rangle \geq 2\langle x_1 - x_n, J_E x_1 - J_E x_n \rangle$$

and hence

$$\phi_E(x_1, x_n) + \phi_E(x_{n+1}, x_1) - \phi_E(x_{n+1}, x_n) \geq \phi_E(x_1, x_n) + \phi_E(x_n, x_1).$$

Then, we have that

$$\phi_E(x_{n+1}, x_1) \geq \phi_E(x_{n+1}, x_n) + \phi_E(x_n, x_1). \tag{3.4}$$

Therefore, $\{\phi_E(x_n, x_1)\}$ is bounded and nondecreasing. Then, there exists the limit of $\{\phi_E(x_n, x_1)\}$. Using (3.4), we also have that

$$\lim_{n\to\infty} \phi_E(x_{n+1}, x_n) = 0. \tag{3.5}$$

We get from Lemma 2 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Using $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, let us show that $\lim_{n\to\infty}\|Tx_n-Q_{\mu_n}Tx_n\|=0$. We have from $x_{n+1}\in C_n$ that

$$2\langle x_n-x_{n+1}, J_Ex_n-J_Ez_n\rangle \geq r_n\phi_F(Tx_n, Q_{\mu_n}Tx_n). \tag{3.7}$$

Furthermore, we claim that $\{J_Ex_n-J_Ez_n\}$ is bounded. That $\{J_Ex_n-J_Ez_n\}$ is bounded is proved as follows. We first have that $\|J_Ex_n-J_Ez_n\|=\|r_nT^*(J_FTx_n-J_FQ_{\mu_n}Tx_n)\|$. Furthermore, we have that

$$\|J_FTx_n\|=\|Tx_n\|\leq\|T\|\|x_n\|.$$

We also have that, for $z\in T^{-1}(B^{-1}0)$,

$$\begin{aligned}(\|Tz\|-\|Q_{\mu_n}Tx_n\|)^2 &\leq \phi_F(Tz, Q_{\mu_n}Tx_n)\\ &\leq \phi_F(Tz, Tx_n)\leq(\|Tz\|+\|Tx_n\|)^2\\ &\leq \|T\|^2(\|z\|+\|x_n\|)^2.\end{aligned}$$

Using this, we have that

$$\|Q_{\mu_n}Tx_n\|\leq\|T\|(\|z\|+\|x_n\|)+\|Tz\|\leq\|T\|(\|z\|+\|x_n\|)+\|T\|\|z\|.$$

Then, we have that

$$\|J_FQ_{\mu_n}Tx_n\|=\|Q_{\mu_n}Tx_n\|\leq\|T\|(2\|z\|+\|x_n\|).$$

Hence, we have that

$$\begin{aligned}\|J_Ex_n-J_Ez_n\| &= \|r_nT^*(J_FTx_n-J_FQ_{\mu_n}Tx_n)\|\\ &\leq \frac{1}{\|T\|^2}\|T\|\left(\|J_FTx_n\|+\|J_FQ_{\mu_n}Tx_n)\|\right)\\ &\leq \frac{1}{\|T\|^2}\|T\|\left(\|T\|\|x_n\|+\|T\|(2\|z\|+\|x_n\|)\right)\\ &\leq 2(\|x_n\|+\|z\|).\end{aligned}$$

This implies that $\{J_Ex_n-J_Ez_n\}$ is bounded. Since $\{J_Ex_n-J_Ez_n\}$ is bounded and $r_n\geq a>0$, we have from (3.6) and (3.7) that

$$\lim_{n\to\infty}\phi_F(Tx_n, Q_{\mu_n}Tx_n)=0.$$

Therefore, we get from Lemma 2 that

$$\lim_{n\to\infty}\|Tx_n-Q_{\mu_n}Tx_n\|=0. \tag{3.8}$$

Furthermore, since F is uniformly smooth, we have from (3.8) that

$$\lim_{n\to\infty}\|J_FTx_n-J_FQ_{\mu_n}Tx_n\|=0. \tag{3.9}$$

Since $\|J_Ex_n-J_Ez_n\|=\|r_nT^*(J_FTx_n-J_FQ_{\mu_n}Tx_n)\|$ and $\{r_n\}$ is bounded, we get from (3.9) that

$$\lim_{n\to\infty}\|J_Ex_n-J_Ez_n\|=0. \tag{3.10}$$

Since E^* is uniformly smooth, we have from (3.10) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.11}$$

We also have from $x_{n+1} \in D_n$ that

$$2\langle y_n - x_{n+1}, J_E z_n - J_E y_n\rangle \geq 0$$

and hence

$$2\langle y_n - z_n + z_n - x_n + x_n - x_{n+1}, J_E z_n - J_E y_n\rangle \geq 0.$$

This implies that

$$2\langle z_n - x_n + x_n - x_{n+1}, J_E z_n - J_E y_n\rangle \geq 2\langle z_n - y_n, J_E z_n - J_E y_n\rangle$$

and hence

$$2\langle z_n - x_n + x_n - x_{n+1}, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n) + \phi_E(y_n, z_n).$$

Since $\{J_E z_n - J_E y_n\}$ is bounded, we have from (3.6) and (3.11) that

$$\lim_{n\to\infty} \phi_E(z_n, y_n) = 0.$$

Using Lemma 2 and $y_n = J_{\lambda_n} z_n$, we have that

$$\lim_{n\to\infty} \|z_n - J_{\lambda_n} z_n\| = 0. \tag{3.12}$$

Since E is uniformly smooth, we have from (3.12) that

$$\lim_{n\to\infty} \|J_E z_n - J_E J_{\lambda_n} z_n\| = 0. \tag{3.13}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. We have from $\lim_{n\to\infty} \|x_n - z_n\| = 0$ that $\{z_{n_i}\}$ converges weakly to w. We also have from (3.12) that $\{J_{\lambda_{n_i}} z_{n_i}\}$ converges weakly to w. Since J_{λ_n} is the generalized resolvent of A, we have that

$$\frac{J_E z_n - J_E J_{\lambda_n} z_n}{\lambda_n} \in A J_{\lambda_n} z_n, \quad \forall n \in \mathbb{N}.$$

From the monotonicity of A, it follows that

$$0 \leq \left\langle s - J_{\lambda_{n_i}} z_{n_i}, t^* - \frac{J_E z_{n_i} - J_E J_{\lambda_{n_i}} z_{n_i}}{\lambda_{n_i}}\right\rangle$$

for all $(s, t^*) \in A$. Since $\|J_E z_{n_i} - J_E J_{\lambda_{n_i}} z_{n_i}\| \to 0$ and $0 < b \leq \lambda_{n_i}$, we have that $0 \leq \langle s - w, t^* - 0\rangle$ for all $(s, t^*) \in A$. Since A is maximal monotone, we have that $w \in A^{-1}0$. Furthermore, since T is bounded and linear, we also have that $\{T x_{n_i}\}$ converges weakly to Tw. From (3.8), we have that $\{Q_{\mu_{n_i}} T x_{n_i}\}$ converges weakly to Tw. Since Q_{μ_n} is the generalized resolvent of B, we have that

$$\frac{J_F T x_n - J_F Q_{\mu_n} T x_n}{\mu_n} \in B Q_{\mu_n} T x_n, \quad \forall n \in \mathbb{N}.$$

From the monotonicity of B, it follows that

$$0 \leq \left\langle u - Q_{\mu_{n_i}} T x_{n_i}, v^* - \frac{J_F T x_{n_i} - J_F Q_{\mu_{n_i}} T x_{n_i}}{\mu_{n_i}}\right\rangle$$

for all $(u, v^*) \in B$. From $\|J_F T x_{n_i} - J_F Q_{\mu_{n_i}} T x_{n_i}\| \to 0$ and $0 < b \leq \mu_{n_i}$, we have that $0 \leq \langle u - Tw, v^* - 0\rangle$ for all $(u, v^*) \in B$. Since B is maximal monotone, we have that $Tw \in B^{-1}0$. Therefore, $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

Using $z_0 = \Pi_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$, $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$ and (3.3), we have that

$$\begin{aligned}
\phi_E(z_0, x_1) &\leq \phi_E(w, x_1)\\
&= \|w\|^2 - 2\langle w, J_E x_1\rangle + \|x_1\|^2\\
&\leq \liminf_{i\to\infty} \left(\|x_{n_i}\|^2 - 2\langle x_{n_i}, J_E x_1\rangle + \|x_1\|^2\right)\\
&= \liminf_{i\to\infty} \phi_E(x_{n_i}, x_1)\\
&\leq \limsup_{i\to\infty} \phi_E(x_{n_i}, x_1) \leq \phi_E(z_0, x_1).
\end{aligned}$$

From the definition of $\Pi_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$, we get that $z_0 = w$ and

$$\lim_{i\to\infty} \phi_E(x_{n_i}, x_1) = \phi_E(w, x_1) = \phi_E(z_0, x_1).$$

So, we have that $\lim_{i\to\infty} \|x_{n_i}\| = \|z_0\|$. From the Kadec-Klee property of E, we have that $x_{n_i} \to z_0$. Therefore, we have $x_n \to z_0$. This completes the proof. □

4 Applications and a numerical example

In this section, using Theorem 6, we get new strong convergence theorems in Banach spaces. Let E be a Banach space and let f be a proper, lower semicontinuous, and convex function of E into $(-\infty, \infty]$. The subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z^* \in E^* : f(x) + \langle y - x, z^*\rangle \leq f(y),\ \forall y \in E\}$$

for all $x \in E$. From Rockafellar [21], we know that ∂f is a maximal monotone operator. Let C be a nonempty, closed, and convex subset of E and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & x \in C,\\ \infty, & x \notin C. \end{cases}$$

Then i_C is a proper, lower semicontinuous and convex function on E and then the subdifferential ∂i_C of i_C is a maximal monotone operator. Thus we can define the generalized resolvent J_λ of ∂i_C for $\lambda > 0$, i.e.,

$$J_\lambda x = (J + \lambda \partial i_C)^{-1} J x$$

for all $x \in E$. We have that for any $x \in E$ and $u \in C$,

$$
\begin{aligned}
u = \;& J_\lambda x \Longleftrightarrow Jx \in Ju + \lambda \partial i_C u \\
& \Longleftrightarrow \frac{1}{\lambda}(Jx - Ju) \in \partial i_C u \\
& \Longleftrightarrow i_C y \geq \langle y - u, \frac{1}{\lambda}(Jx - Ju)\rangle + i_C u, \ \forall y \in E \\
& \Longleftrightarrow 0 \geq \langle y - u, \frac{1}{\lambda}(Jx - Ju)\rangle, \ \forall y \in C \\
& \Longleftrightarrow \langle y - u, Jx - Ju\rangle \leq 0, \ \forall v \in C \\
& \Longleftrightarrow u = \Pi_C x. \qquad (4.1)
\end{aligned}
$$

Using Theorem 6, we prove a strong convergence theorem for finding minimizers of convex functions in two Banach spaces.

Theorem 7 *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let f and g be proper, lower semicontinuous, and convex functions of E into $(-\infty, \infty]$ and F into $(-\infty, \infty]$ such that $(\partial f)^{-1}0 \neq \emptyset$ and $(\partial g)^{-1}0 \neq \emptyset$, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $(\partial f)^{-1}0 \cap T^{-1}((\partial g)^{-1}0) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
t_n = \arg\min_{y\in F}\{g(y) + \frac{1}{2\mu_n}\|y\|^2 - \frac{1}{\mu_n}\langle y, J_F T x_n\rangle\}, \\
z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F t_n)\big), \\
y_n = \arg\min_{x\in E}\{f(x) + \frac{1}{2\lambda_n}\|x\|^2 - \frac{1}{\lambda_n}\langle x, J_E z_n\rangle\}, \\
C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(T x_n, t_n)\}, \\
D_n = \{z \in E : \langle y_n - z, J_E z_n - J_E y_n\rangle \geq 0\}, \\
Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N},
\end{cases}
$$

where $\{r_n\}, \{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities

$$
0 < a \leq r_n \leq \frac{1}{\|T\|^2} \text{ and } 0 < b \leq \lambda_n, \mu_n, \quad \forall n \in \mathbb{N}.
$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in (\partial f)^{-1}0 \cap T^{-1}((\partial g)^{-1}0)$, where $w_1 = \Pi_{(\partial f)^{-1}0 \cap T^{-1}((\partial g)^{-1}0)} x_1$.

Proof We know from [5] that

$$
t_n = \arg\min_{y\in F}\{g(y) + \frac{1}{2\mu_n}\|y\|^2 - \frac{1}{\mu_n}\langle y, J_F T x_n\rangle\}
$$

is equivalent to

$$
0 \in (\partial g)t_n + \frac{1}{\mu_n} J_F t_n - \frac{1}{\mu_n} J_F T x_n.
$$

From this, we have $J_F T x_n \in J_F t_n + \mu_n(\partial g)t_n$, i.e., $t_n = Q_{\mu_n} T x_n$.

Similarly, we have that

$$y_n = \arg\min_{x\in E}\{f(x) + \frac{1}{2\lambda_n}\|x\|^2 - \frac{1}{\lambda_n}\langle x, J_E z_n\rangle\}$$

is equivalent to $y_n = J_{\lambda_n} z_n$. Using Theorem 6, we get the conclusion. □

Using (4.1) and Theorem 7, we obtain the following result for the split feasibility problem in two Banach spaces.

Theorem 8 *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and F, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $C \cap T^{-1}D \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F \Pi_D T x_n)\big), \\ y_n = \Pi_C z_n, \\ C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(T x_n, \Pi_D T x_n)\}, \\ D_n = \{z \in E : \langle y_n - z, J_E z_n - J_E y_n\rangle \geq 0\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in C \cap T^{-1}D$, where $w_1 = \Pi_{C\cap T^{-1}D} x_1$.

A numerical example Let us give a numerical example which supports our theorem. Let $E = F = \mathbb{R}$, $C = [2, 4] \subset E$ and $D = [2, 6] \subset F$ in Theorem 8. Take a bouded linear operator $T : E \to F$ by $y = Tx = 2x$ for all $x \in E$ and $r_n = \frac{1}{8}$ for all $n \in \mathbb{N}$. Then we have $C \cap T^{-1}D = [2, 3]$. For $x_1 = 5$, we have

$$z_1 = 4,\ y_1 = 4,\ C_1 = (-\infty, 4],\ D_1 = \mathbb{R},\ Q_1 = \mathbb{R}.$$

Then we have $x_2 = \Pi_{C_1\cap D_1\cap Q_1} x_1 = 4$. For $x_2 = 4$, we have

$$z_2 = 3.5,\ y_2 = 3.5,\ C_2 = (-\infty, 3.5],\ D_2 = \mathbb{R},\ Q_2 = (-\infty, 4].$$

Then we have $x_3 = \Pi_{C_2\cap D_2\cap Q_2} x_1 = 3.5$. Furthermore, for $x_3 = 3.5$, we have

$$z_3 = 3.25,\ y_3 = 3.25,\ C_3 = (-\infty, 3.25],\ D_3 = \mathbb{R},\ Q_3 = (-\infty, 3.5].$$

Then we have $x_4 = \Pi_{C_3\cap D_3\cap Q_3} x_1 = 3.25$. Simiarly, we have $x_5 = 3.125$. By such a method, we have $x_n \to 3$, where $3 = \Pi_{C\cap T^{-1}D} x_1$.

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