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A new numerical approach for solving a class of singular two-point boundary value problems

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Abstract In this paper, we consider the following class of singular two-point boundary value problem posed on the interval $x\epsilon(0, 1]$

$$(g(x)y')' = g(x)f(x, y),$$
$$y'(0) = 0, \mu y(1) + \sigma y'(1) = B.$$

A recursive scheme is developed, and its convergence properties are studied. Further, the error estimation of the method is discussed. The proposed scheme is based on the integral equation formalism and optimal homotopy analysis method in which a recursive scheme is established without any undetermined coefficients. The original differential equation is transformed into an equivalent integral equation to remove the singularity. The integral equation is then made free of undetermined coefficients by imposing the boundary conditions on it. Finally, the integral equation without any undetermined coefficients is efficiently treated by using optimal homotopy analysis method for finding the numerical solution. The optimal control-convergence parameter involved in the components of the series solution is obtained by minimizing the squared residual error equation. The present method is applied to obtain numerical solution of singular boundary value problems arising in various physical models, and numerical results show the advantages of our method over the existing methods.

Keywords Singular boundary value problem · Optimal homotopy analysis method · Undetermined coefficients · Oxygen-diffusion problem · Thermal-explosion problem · Shallow membrane cap problem

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1 Introduction

Nonlinear singular two-point boundary value problems (BVP) arise in a wide variety of problems [1–15] such as reaction-diffusion process, chemical kinetics, physiological process, heat transfer process, nuclear physics, astrophysics, quantum mechanics, thermal-explosion theory, electro hydrodynamics, shallow membrane caps theory, fluid mechanics, and elasticity. Since it may be impossible or difficult to obtain the closed-form solutions to nonlinear singular BVP, these problems must be tackled by analytical approximation or numerical methods. Perturbation method [16] is one of the well-known analytical methods for solving nonlinear problems; however, this method cannot be applied to solve the singular boundary value problems under consideration as it does require small/large physical parameters, the so-called perturbation quantities, in the equations or boundary conditions. To overcome the restriction of the perturbation methods, some nonperturbation methods are developed, including the variational iteration method (VIM) [17], the homotopy perturbation method (HPM) [18], the Adomian decomposition method (ADM) [19], and the homotopy analysis method (HAM) [20–25]. The first three methods can be used to obtain approximate series solution of the problem; however, the convergence of approximation series is not guaranteed. On the other hand, the HAM provides a convenient way to ensure the convergence of solution series via the so-called convergence-control parameter.

The aim of the present paper is to introduce a novel approach based on a combination of integral equation formalism and optimal homotopy analysis method (OHAM) for the numerical solution of a class of nonlinear singular two-point boundary value problems. OHAM is basically a modified version of HAM and has been proposed by S. Liao [26] and V. Marinca et al. [27]. The OHAM has certain advantages over other nonperturbation methods: (1) it is a general method, (ii) can improve the computational efficiency of the method, (iii) greatly accelerates the convergence of the series solution, and (iv) provides fast convergent series solution.

In this work, we consider the following class of singular two-point boundary value problems:

$$(g(x)y')' = g(x)f(x,y) \tag{1}$$

with boundary conditions

$$y'(0) = 0, \mu y(1) + \sigma y'(1) = B. \tag{2}$$

Here, $\mu > 0$, $\sigma \geq 0$ and B is finite constant. The following conditions have been imposed on the functions $g(x)$ and $f(x,y)$:

$C1$: $f(x,y)$ is continuous function for all $(x,y) \in ([0,1] \times R)$

$C2$: $\frac{\partial f(x,y)}{\partial y}$ exists and is continuous for all $(x,y) \in ([0,1] \times R)$

$C3 : \frac{\partial f(x,y)}{\partial y} \geq 0$

$C4 : g(x) \geq 0, g(0) = 0$

$C5 : g(x) \in C^1(0,1]$

$C6 : g(x) \in L^1(0,1]$

$C7 : \int_0^1 \frac{1}{g(\eta)} \int_0^\eta g(t)dtd\eta < \infty.$

The existence and uniqueness of the solution to the problem (1) with boundary conditions (2) have been established in [28].

In general, singular boundary value problems (1)–(2) are difficult to solve because of a singularity at the boundary point $x = 0$. Numerous approximate solution techniques have been developed in the literature to handle the singular problem (1) with $g(x) = x^{\alpha}$ or $x^{\alpha}s(x)$, $\alpha > 0$ and boundary conditions $y(0) = A$ (or $y'(0) = A_1$), $y(1) = C$ (or $\mu y(1) + \sigma y'(1) = B$). To mention a few, in [29], the authors have developed a direct method based on B-spline for a class of non-linear singular BVP (1)–(2) with $g(x) = x^{\alpha}$, $\alpha \geq 1$. Jain and Iyenger [30] described a spline finite difference method of order two for solving the singular problem (1) with $g(x) = x^{\alpha}$, $\alpha > 0$ and boundary conditions $y(0) = A$ (or $y'(0) = 0$), $y(1) = B$. In [31], the author presented a method based on cubic spline for solving a class of singular BVP (1) with $g(x) = x^{\alpha}$, $\alpha \geq 1$ and subject to the boundary conditions $y'(0) = 0$, $y(1) = B$. Further, a novel approach, based on a combination of a modified decomposition method and B-spline collocation, for solving (1) with $g(x) = x^{\alpha}$, $\alpha \geq 1$ and boundary conditions $y'(0) = 0$ (or $y(0) = A$) and $\mu y(1) + \sigma y'(1) = B$ is presented in [32]. Pandey and Singh [33] presented a second-order three-point finite difference method (FDM) for the solution of singular BVP (1)–(2) with $g(x) = x^{\alpha}s(x)$, where $s(x)$ is a general class of non negative function. Furthermore, the variational iteration method (VIM) for solving singular differential equation (1) with $g(x) = x^{\alpha}$, $\alpha \geq 1$ and boundary conditions $y(0) = A$ (or $y'(0) = A_1$), $y(1) = C$ (or $\mu y(1) + \sigma y'(1) = B$) is presented in [34]. Roul and Warbhe [35] have considered the application of HPM for solving the problems (1)–(2).

In this paper, we introduce a new efficient and accurate recursive algorithm for solving the singular BVP given in (1) and (2). The first step of the algorithm converts the singular BVP (1)–(2) into an equivalent integral equation to overcome the singular behaviour at the origin. We note that the resulting integral equation contains an undetermined coefficient. In the second step, the boundary condition at the right end point of the domain of the problem is employed in the integral equation to eliminate the undetermined coefficient. In the last step, the resulting integral equation without any undetermined coefficient is treated by employing OHAM to establish a recursive scheme for obtaining the approximate solution of the singular boundary value problems (1)–(2). In this method, we have introduced an approach based on minimization of the squared residual error to find out the optimal convergence-control parameter involved in the components of the series solution. It is worth pointing out that although some recursive schemes based on modified ADM [32], VIM [34], and ADM [36] have been proposed for solving singular boundary value problems, however, these methods require the computation of undetermined coefficients. This would lead to an increase an computational cost, because a sequence of transcendental equations would have to be solved for that.

Besides the numerical design, we establish the convergence result and error estimate for the proposed method. The method is tested on four nonlinear singular boundary value problems arising in various physical models of engineering and science. We also compare the results obtained by the present technique with those obtained by the various approximatation methods, such as B-spline [29], mixed ADM B-spline [32], cubic spline [31], FDM [33], spline-FDM [30], and VIM [34].

The organization of the remainder of this paper is as follows. In Section 2, a short description of the basic principles of standard homotopy-analysis method is presented. In Section 3, we construct a new recursive scheme based on OHAM to obtain numerical solution of singular boundary value problems considered in the paper. Section 4 is devoted to the convergence analysis and error estimation of the method. The method is illustrated by numerical examples in Section 5. Finally, we summarize our work and present our conclusion in Section 6.

2 Review of classical homotopy analysis method

In this section, we shall give a brief outline of the classical homotopy analysis method. The homotopy analysis method (HAM) was developed and improved by S. Liao [20–25] for solving a wide class of functional equations. This method provides a convenient way to control or adjust the convergence region and the rate of convergence.
To illustrate the basic ideas of the method, let us consider the following nonlinear differential equation of the form

$$N[y(x)] = 0, \tag{3}$$

where N represents the general differential operator, $y(x)$ is an unknown function and x is an independent variable. For simplicity, we ignore all boundary or initial conditions. According to homotopy analysis method, we construct a homotopy $u(x, p)$: $\Omega \times [0, 1] \longrightarrow R$, for (3) which satisfies the following relation

$$\hat{H}(u(x), h, H(x), p) = (1 - p)L[u(x, p) - y_0(x)] - phH(x)N[u(x, p)], \tag{4}$$

where $p \in [0, 1]$ is an embedding parameter, $h \neq 0$ is a constant (called the convergence-control parameter), $H(x) \neq 0$ is an auxiliary function, $y_0(x)$ is an initial approximation of the exact solution $y(x)$ of (3).

Imposing the homotopy (4) to be zero, we have the so called zero-order deformation equation

$$(1 - p)L[u(x, p) - y_0(x)] - phH(x)N[u(x, p)] = 0. \tag{5}$$

When $p = 0$, the zero order deformation (5) becomes

$$u(x, 0) = y_0(x), \tag{6}$$

and when $p = 1$, the zero order deformation (5) reduces to

$$N[u(x, 1)] = 0, \tag{7}$$

$$u(x, 1) = y(x). \tag{8}$$

Hence, $u(x, 1)$ is the solution of nonlinear problem (3). As the parameter p varies through 0 to 1, the solution $u(x, p)$ varies continuously from $y_0(x)$ to $y(x)$. Variation of this kind is called deformation in topology. We now expand the function $u(x, p)$

in Taylor series with respect to the parameter p as follows:

$$u(x, p) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) p^m, \tag{9}$$

where

$$y_m(x) = \frac{1}{m!} \frac{\partial^m u(x, p)}{\partial p^m}.$$

Assume that the auxiliary linear operator L, the auxiliary parameter h, initial approximation $y_0(x)$, and the auxiliary function $H(x)$ are properly chosen so that the series in (9) converges at $p = 1$. Then, at $p = 1$, the series (9) becomes

$$u(x, 1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \tag{10}$$

Now using (8), we have

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \tag{11}$$

The solution components $y_n(x)$ can be obtained from the higher-order deformation equation as described below.
Define the vector $\overrightarrow{y_n}$={$y_0(x)$, $y_1(x)$, ..., $y_n(x)$}.
Inserting (9) into (5), yields

$$(1-p)L\left[\sum_{m=1}^{\infty} y_m(x) p^m\right] = phH(x)N[u(x, p)] \tag{12}$$

Differentiating (12) m times w.r.t. the embedding parameter p, we obtain

$$L[y_m(x) - \chi_m y_{m-1}(x)] = \left.\frac{hH(x)\partial^{m-1}N[u(x, p)]}{(m-1)!\partial p^{m-1}}\right|_{p=0} \tag{13}$$

Therefore, we have the following m-th order deformation equation:

$$L\left[y_m(x) - \chi_m y_{m-1}(x)\right] = hH(x)R_m\left(\overrightarrow{y_{m-1}}(x)\right), \tag{14}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

and

$$R_m(\overrightarrow{y_{m-1}}(x)) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1}N[\sum_{m=1}^{\infty} y_m(x)p^m]}{\partial p^{m-1}}\right|_{p=0} \tag{15}$$

with the initial condition

$$y_m(0) = 0, m > 1. \tag{16}$$

For any given nonlinear operator N, the term $R_m\left(\overrightarrow{y_{m-1}}(x)\right)$ in (14) can be easily expressed by (15). It is worth pointing out that the higher-order deformation (14) is

governed by the same linear operator L. The solution components y_n are computed by means of solving the (14) recursively.
The m-th order approximation solution of $y(x)$ is given by

$$\Phi_m(x) = y_0(x) + \sum_{i=1}^{m} y_i(x). \tag{17}$$

3 A new recursive scheme based on OHAM

In this section, we derive a new recursive scheme based on OHAM for solving nonlinear singular problems (1) with Neumann and Robin boundary conditions (2). To derive the method, we set $z(x) = g(x)y'$ in (1), then integrating (1) from 0 to x, we get

$$z(x) = z(0) + \int_0^x g(t)f(t, y)dt. \tag{18}$$

Now applying the boundary condition at $x = 0$ in (18), it follows that

$$y'(x) = \frac{1}{g(x)}\int_0^x g(t)f(t, y)dt. \tag{19}$$

Again integrating (19) from x to 1, we obtain

$$y(x) = y(1) - \int_x^1 \frac{1}{g(\eta)}\left(\int_0^\eta g(t)f(t, y)dt\right)d\eta. \tag{20}$$

We set y(1)=C^*, where C^* is not known. To determine the value of C^* in (20), we impose the boundary condition at $x = 1$, namely $\mu y(1) + \sigma y'(1) = B$. With the help of the boundary condition, we obtain

$$y(1) = C^* = \frac{B}{\mu} - \frac{\sigma}{\mu g(1)}\int_0^1 g(t)f(t, y)dt. \tag{21}$$

Insert (21) into (20) to get

$$y(x) = \frac{B}{\mu} - \frac{\sigma}{\mu g(1)}\int_0^1 g(t)f(t, y)dt - \int_x^1 \frac{1}{g(\eta)}\left(\int_0^\eta g(t)f(t, y)dt\right)d\eta. \tag{22}$$

Now interchanging the order of integration in (22), we get

$$\begin{aligned} y(x) = {} & \frac{B}{\mu} - \frac{\sigma}{\mu g(1)}\int_0^1 g(t)f(t, y)dt - \int_0^x \left(\int_x^1 \frac{1}{g(\eta)}d\eta\right) g(t)f(t, y)dt \\ & - \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)}d\eta\right) g(t)f(t, y)dt. \end{aligned} \tag{23}$$

The OHAM is extended to solving integral (23) derived from the original (1) with boundary conditions (2).

For this, we consider the (23) as

$$N(y) = y(x) - \frac{B}{\mu} + \frac{\sigma}{\mu g(1)}\int_0^1 g(t)f(t,y)dt + \int_0^x \left(\int_x^1 \frac{1}{g(\eta)}d\eta\right) g(t)f(t,y)dt + \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)}d\eta\right) g(t)f(t,y)dt = 0. \tag{24}$$

Using (24) in (15), we obtain

$$R_m\left(\vec{y_{m-1}}(x)\right) = \frac{1}{(m-1)!}\left(\frac{\partial^{m-1}N(y)}{\partial p^{m-1}}\right)_{p=0}.$$

or

$$R_m(\vec{y_{m-1}}(x)) = y_{m-1}(x) - (1-\chi_m)F(x) + \frac{\sigma}{\mu g(1)}\int_0^1 g(t)T_{m-1}(t,y)dt + \int_0^x \left(\int_x^1 \frac{1}{g(\eta)}d\eta\right) g(t)T_{m-1}(t,y)dt + \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)}d\eta\right) g(t)T_{m-1}(t,y)dt, \tag{25}$$

where

$$F(x) = \frac{B}{\mu} \tag{26}$$

and

$$T_{m-1}(x,y) = \frac{1}{(m-1)!}\left(\frac{\partial^{m-1}f(x,\sum_{i=1}^{\infty} y_i(x)p^i)}{\partial p^{m-1}}\right)_{p=0}. \tag{27}$$

Inserting (25) into (14), we obtain

$$L\Big(y_m(x) - \chi_m y_{m-1}(x)\Big) = hH(x)\Big[y_{m-1}(x) - (1-\chi_m)F(x) + \frac{\sigma}{\mu g(1)}\int_0^1 g(t)T_{m-1}(t,y)dt + \int_0^x \left(\int_x^1 \frac{1}{g(\eta)}d\eta\right) g(t)T_{m-1}(t,y)dt + \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)}d\eta\right) g(t)T_{m-1}(t,y)dt\Big]. \tag{28}$$

We take an initial guess $y_0(x) = F(x)$, auxiliary function $H(x) = 1$, and an auxiliary linear operator $L(u) = u$. In view of these assumptions, from (28), we have the solution components $y_n(x)$, $n \geq 0$ as follows:

$$y_0(x) = F(x),$$

$$y_1(x) = h\Bigg[\frac{\sigma}{\mu g(1)}\int_0^1 g(t)T_0(t,y)dt + \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_0(t,y)dt + \int_x^1\left(\int_t^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_0(t,y)dt\Bigg],$$

$$y_2(x) = (1+h)y_1 + h\Bigg[\frac{\sigma}{\mu g(1)}\int_0^1 g(t)T_1(t,y)dt + \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_1(t,y)dt + \int_x^1\left(\int_t^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_1(t,y)dt\Bigg],$$

$$y_3(x) = (1+h)y_2 + h\Bigg[\frac{\sigma}{\mu g(1)}\int_0^1 g(t)T_2(t,y)dt + \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_2(t,y)dt + \int_x^1\left(\int_t^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_2(t,y)dt\Bigg],$$

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Hence, the present method can be defined by the recurrence relation

$$y_0(x) = F(x),$$

$$y_1(x) = h\Bigg[\frac{\sigma}{\mu g(1)}\int_0^1 g(t)T_0(t,y)dt + \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_0(t,y)dt + \int_x^1\left(\int_t^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_0(t,y)dt\Bigg],$$

$$y_i(x) = (1+h)y_{i-1} + h\Bigg[\frac{\sigma}{\mu g(1)}\int_0^1 g(t)T_{i-1}(t,y)dt + \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_{i-1}(t,y)dt + \int_x^1\left(\int_t^1 \frac{1}{g(\eta)}d\eta\right)g(t)T_{i-1}(t,y)dt\Bigg], \quad i \geq 2 \tag{29}$$

The solution components y_n, $n \geq 1$, defined in (29), contain convergence control parameter h which provides us with a simple way to adjust and control the convergence region of the series solution. By properly choosing this parameter, the present method provides rapidly convergent successive approximations of the exact solutions. In the present approach, the optimum value of the convergence-control parameter h is obtained by minimizing the squared residual of governing equation:

$$R_n(h) = \int_0^1 (N[\Phi_n(x,h)])^2 dx. \tag{30}$$

At the minimum, we have

$$\frac{\partial R_n(h)}{\partial h} = 0.$$

i.e.,

$$\frac{\partial}{\partial h}\int_0^1 (N[\Phi_n(x,h)])^2 dx = 0. \tag{31}$$

Since, the analytical integration may not be always possible to perform especially for transcendental function or it is time consuming particularly for large n, hence the (31) can be discretized in the form:

$$\frac{\partial}{\partial h}\sum_{i=1}^{M^*}(N[\Phi_n(x_i,h)])^2 = 0 \tag{32}$$

Here, M^* is the number of discrete points of the interval [0, 1]. Having computed the optimal value of h, we can proceed to obtain the components y_n of the solution $y(x)$. Hence, the m-term truncated approximate series solution of the singular BVP (1) with boundary condition (2) can be obtained as

$$\phi_m(x) = y_0 + y_1 + y_2 + y_3 + ... + y_m. \tag{33}$$

It is worth pointing out that the recursive scheme defined in (29) does not require the computation of unknown constants for solving the singular boundary value problems (1)–(2). Hence, there is a huge gain in efficiency since the computationally expensive unknown constants evaluation is not performed.

4 Convergence of the method

In this section, we establish the convergence of method defined in (29) for the solution of singular two-point boundary value problems (1) and (2).

Theorem 1 *Let the recursive scheme defined by* (29)*. If the series* $y(x) = \sum_{m=0}^{\infty} y_m(x)$ *is convergent, then it must be a solution of the integral* (23).

Proof Assume that the series $y(x) = \sum_{m=0}^{\infty} y_m(x)$ is convergent. Then, we have

$\lim_{m\to\infty} y_m(x) = 0$.

By means of the operator L, we can write,

$\sum_{m=1}^{n} L[y_m(x) - \chi_m y_{m-1}(x)] = \sum_{m=1}^{n} [y_m(x) - \chi_m y_{m-1}(x)] = y_n(x)$.

Taking limit on both sides, yields

$\sum_{m=1}^{\infty} L[y_m(x) - \chi_m y_{m-1}(x)] = \lim_{n\to\infty} y_n(x) = 0$.

Thereforem from (14), we obtain

$\sum_{m=1}^{\infty} L[y_m(x) - \chi_m y_{m-1}(x)] = \sum_{m=1}^{\infty} [hH(x)R(\vec{y}_{m-1}(x))] = 0$.

Since $h \neq 0$, $H(x) \neq 0$, we have

$$\sum_{m=1}^{\infty} R_m(y_{\vec{m-1}}(x)) = 0. \tag{34}$$

Now using (25) in (34), we obtain

$$0 = \sum_{m=1}^{\infty} \Bigg[y_{m-1}(x) - (1-\chi_m)F(x) + \frac{\sigma}{\mu g(1)} \int_0^1 g(t) T_{m-1}(t, y) dt +$$
$$\int_0^x \left(\int_x^1 \frac{1}{g(\eta)} d\eta \right) g(t) T_{m-1}(t, y) dt + \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)} d\eta \right) g(t) T_{m-1}(t, y) dt \Bigg].$$

$$= y(x) - F(x) + \frac{\sigma}{\mu g(1)} \int_0^1 g(t) \sum_{m=1}^{\infty} T_{m-1}(t, y) dt +$$
$$\int_0^x \left(\int_x^1 \frac{1}{g(\eta)} d\eta \right) g(t) \sum_{m=1}^{\infty} T_{m-1}(t, y) dt + \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)} d\eta \right) g(t) \sum_{m=1}^{\infty} T_{m-1}(t, y) dt.$$

$$= y(x) - F(x) + \frac{\sigma}{\mu g(1)} \int_0^1 g(t) f(t, y) dt +$$
$$\int_0^x \left(\int_x^1 \frac{1}{g(\eta)} d\eta \right) g(t) f(t, y) dt + \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)} d\eta \right) g(t) f(t, y) dt.$$

$$y(x) = F(x) - \frac{\sigma}{\mu g(1)} \int_0^1 g(t) f(t, y) dt -$$
$$\int_0^x \left(\int_x^1 \frac{1}{g(\eta)} d\eta \right) g(t) f(t, y) dt - \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)} d\eta \right) g(t) f(t, y) dt.$$

Therefore, $y(x) = \sum_{m=0}^{\infty} y_m(x)$ be is the exact solution of the integral (23). We next discuss the existence of unique solution of (23). For this purpose, we write the (23) in the following form:

$$y(x) = C + \int_0^1 k(x, t) g(t) f(t, y) dt,$$

where $k(x, t)$ is the kernel and is given by

$$k(x, t) = \begin{cases} \int_x^1 \frac{1}{g(\eta)} d\eta + \frac{\sigma}{\mu g(1)}, & t \leq x \\ \int_t^1 \frac{1}{g(\eta)} d\eta + \frac{\sigma}{\mu g(1)}, & t > x \end{cases} \tag{35}$$

and $C = \frac{B}{\mu}$.

Let X=$C[0,1]$ be the Banach space with the norm

$\parallel z \parallel = \max_{x \in [0,1]} \mid z(x) \mid$. □

Lemma 1 *Let g satisfy $C5$ and $C7$. Then there exists a constant M such that*

$$\max_{x \epsilon [0,1]} \mid \int_0^1 k(x, t) g(t) dt \mid = M.$$

Proof The proof of the Lemma is quite straightforward. In view of $C5$ and $C7$ and the kernel $K(x,t)$ as defined in (35), we must have $\max\limits_{x\epsilon[0,1]} \mid \int_0^1 k(x,t)g(t)dt \mid = M < \infty$. □

Theorem 2 *Let the right-hand side function $f(x,y)$ of the singular differential equation (1) satisfies the Lipschitz condition, that is $\| \ f(x,y_1) - f(x,y_2) \ \| \leq \sigma \ \| \ y_1 - y_2 \ \|$, $\forall y_1, y_2 \epsilon X$, where σ is Lipschitz constant. If we assume that $\beta = \sigma M < 1$, then the (23) has at most one solution.*

Proof To prove this assertion, let us assume that there are two solutions y_1 and y_2 of the (23).

$$\|y_1 - y_2\| = \left\| \int_0^1 k(x,t)g(t)f(t,y_1)dt - \int_0^1 k(x,t)g(t)f(t,y_2)dt \right\|.$$

$$= \left\| \int_0^1 k(x,t)g(t)[f(t,y_1) - f(t,y_2)]dt \right\|.$$

$$\leq \max_{t\in[0,1]} \mid f(t,y_1) - f(t,y_2) \mid \max_{t\in[0,1]} \mid \int_0^1 k(x,t)g(t)dt \mid .$$

In view of Lemma-1, we have

$$\| \ y_1 - y_2 \ \| \leq M \max_{t\in(0,1]} \mid f(t,y_1) - f(t,y_2) \mid .$$

Using Lipschitz continuity of $f(x,y)$, we have

$$\| \ y_1 - y_2 \ \| \leq M \max_{t\in(0,1]} \sigma \mid y_1 - y_2 \mid .$$

$$= M\sigma \ \| \ y_1 - y_2 \ \| .$$

Setting $\beta = \sigma M$ in the above inequality, we have

$$\| \ y_1 - y_2 \ \| \leq \beta \ \| \ y_1 - y_2 \ \| .$$

Since $\beta < 1$, the equality $y_1 = y_2$ must occur. This means that the (23) has a unique solution. □

Theorem 3 *Let X=C[0,1] be the Banach space with the norm $\| \ z \ \| = \max\limits_{x\in(0,1]} \mid z(x) \mid$. The series solution $\sum\limits_{m=0}^{\infty} y_m$ in which the functions $y_m(x)$ are determined by means of relation (29), is uniformly convergent in the intervals [0,1], if there exist $\lambda_n \in (0,1)$ such that $\| \ y_{m+1} \ \| < \lambda_n \ \| \ y_m \ \|$, $\forall m \geq m_0$, for some $m_0 \in N$.*

Proof Let Φ_n be the n-th partial sum of the series $\sum\limits_{m=0}^{\infty} y_m(x)$, that is $\Phi_n = \sum\limits_{m=0}^{n} y_m(x)$.

We complete the proof by showing that:

(i) $\| \Phi_{n+1} - \Phi_n \| \leq \lambda^{n+1-m_0} \| y_{m_0} \|$.

(ii) The sequence Φ_n is a Cauchy sequence in X=C[0,1].

$\| \Phi_{n+1} - \Phi_n \| = \| y_{n+1} \leq \lambda \| y_n \| \leq \lambda^2 \| y_{n-1} \| \leq \cdots\cdots\cdots \leq \lambda^{n+1-m_0} \| y_{m_0} \|$.

For every $m, n \in N$, $m_0 \leq m \leq n$, we have

$$\begin{aligned}
\| \Phi_n - \Phi_m \| &= \|(\Phi_n - \Phi_{n-1}) + (\Phi_{n-1} - \Phi_{n-2}) + \cdots\cdots\cdots + (\Phi_{m+1} - \Phi_m)\|. \\
&\leq \| \Phi_n - \Phi_{n-1} \| + \| \Phi_{n-1} - \Phi_{n-2} \| + \cdots\cdots + \| \Phi_{m+1} - \Phi_m \| \\
&\leq \lambda^{n-m_0} \| y_{m_0} \| + \lambda^{n-m_0-1} \| y_{m_0} \| + \cdots\cdots + \lambda^{m+2-m_0} \\
&\quad \| y_{m_0} \| + \lambda^{m+1-m_0} \| y_{m_0} \| \\
&\leq \| y_{m_0} \| \lambda^{m+1-m_0} (1 + \lambda + \lambda^2 + \cdots\cdots + \lambda^{n-1-m})
\end{aligned}$$

$$\leq \| y_{m_0} \| \lambda^{m+1-m_0} \left(\frac{1 - \lambda^{n-m}}{1 - \lambda} \right). \tag{36}$$

Since $0 < \lambda < 1$, $1 - \lambda^{n-m} < 1$, we have from (36)

$$\| \Phi_n - \Phi_m \| \leq \| y_{m_0} \| \lambda^{m+1-m_0} \left(\frac{1}{1 - \lambda} \right). \tag{37}$$

Taking limit as $n, m \to \infty$, we obtain

$$\lim_{n,m \to \infty} \| \Phi_n - \Phi_m \| = 0.$$

Therefore, Φ_n is a Cauchy sequence in the Banach space X. This implies that the series solution $\sum_{i=0}^{\infty} y_i$ is convergent.

This completes the proof of the Theorem. □

Theorem 4 *Let $y(x)$ be the exact solution of the (23). Let Φ_m be the m-th partial sum of the series solution $\sum_{m=0}^{\infty} y_m(x)$. The error of the m-term truncated approximate series solution of the singular BVP (1)-(2) can be estimated as follows:*

$$\sup_{x \in [0,1]} | y - \sum_{i=0}^{m} y_i(x) | \leq \| y_{m_0} \| \left(\frac{\lambda_n^{m+1-m_0}}{1 - \lambda_n} \right).$$

Proof With the help of the estimation of $(\Phi_n - \Phi_m)$ defined in (37), we have $\forall m, n \in N$ with $m_0 < m \leq n$,

$$\| \sum_{i=0}^{n} y_i(x) - \sum_{i=0}^{m} y_i(x) \| = \| \Phi_n - \Phi_m \| \leq \| y_{m_0} \| \left(\frac{\lambda_n^{m+1-m_0}}{1 - \lambda_n} \right).$$

Taking $n \to \infty$, we get

$$\| y(x) - \sum_{i=0}^{m} y_i(x) \| \leq \| y_{m_0} \| \left(\frac{\lambda_n^{m+1-m_0}}{1-\lambda_n} \right),$$

or

$$\sup_{x\in[0,1]} | y - \sum_{i=0}^{m} y_i(x) | \leq \| y_{m_0} \| \left(\frac{\lambda_n^{m+1-m_0}}{1-\lambda_n} \right).$$

□

5 Numerical illustration

In this section, we illustrate the applicability, reliability, and accuracy of the present recursive scheme by solving several nonlinear singular boundary value problems arising in various physical problems of engineering and science. Numerical results are compared with that of [29–34]. All the numerical computations were done using symbolic computation software package Mathematica. To measure the accuracy of the present method against the exact solution, we determine the maximum absolute error, as defined by

$$E_n(x) = \max_{x\in[0,1]} | \Phi_n(x) - Y(x) | .$$

Here, $Y(x)$ is the exact solution of the problem and $\Phi_n(x)$ is the truncated n-term approximate series solution.

Example 1 Consider the nonlinear singular two-point boundary value problem arising in the theory of thermal explosions

$$(xy'(x))' = -xe^{y(x)} \tag{38}$$

$$y'(0) = 0, \; y(1) = 0.$$

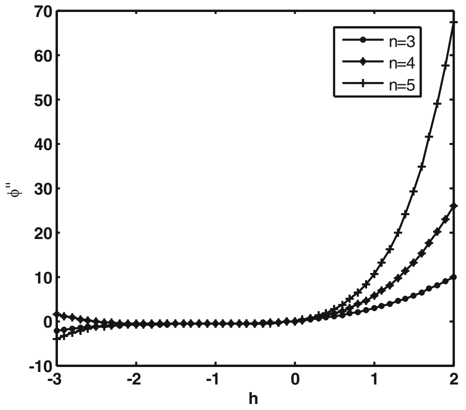
The exact solution is given by $y(x) = 2ln(\frac{A+1}{Ax^2+1})$ with $A = 3 - 2\sqrt{2}$.

This problem is corresponding to (1) and (2) with $g(x) = x$, μ=1, σ=0, $B = 0$ and $f(x, y) = -e^{y(x)}$. We solve the problem (38) using the method defined in (29). The fourth-order approximate solution is as follows:

$$\begin{aligned}\Phi_4(x,h) = \; & h(-1+x^2) + h^2(-1.21875 + 1.125x^2 + 0.09375x^4) \\ & + h^3(-0.677083 + 0.578125x^2 + 0.09375x^4 \\ & + 0.00520833x^6) + h^4(-0.144165 + 0.114258x^2 \\ & + 0.0268555x^4 + 0.00292969x^6 + 0.00012207x^8).\end{aligned} \tag{39}$$

The valid region for the values of h can be identified by means of the so-called h-curve. We plot the h-curve of $\Phi''(0)$ for different values of n in Fig. 1. The flat part (horizontal line) of the curve represents the valid region of h. It is clear from the figure that the valid region of h is $-2 < h < 0$.

Fig. 1 h-curve region of $\phi''(0)$ for example 1

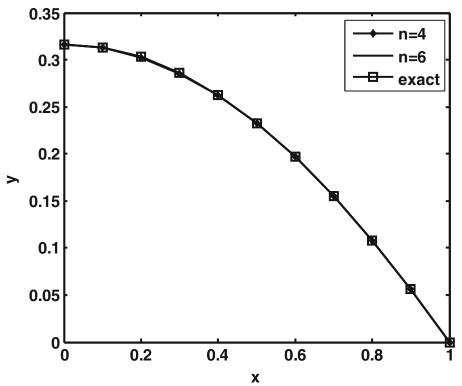


Further, the optimal value of h can be obtained by using the method defined in Section 3. With the help of (32), we find $h = -1.17752$ for n=4. Inserting this value of h in (39), we obtain the following fourth-order optimal approximate solution:

$$\Phi_4(x) = 0.315967 - 0.341885x^2 + 0.0285548x^4 - 0.00287122x^6 + 0.000234688x^8. \tag{40}$$

Figure 2 shows a comparison between the exact solution and the approximate solutions of the problem (38) for n=4,6. It is obvious from the figure that the present method with few solution components approximates the exact solution very well. In addition, we plot the absolute error for n=4,6,7 in Fig. 3, which shows that the error decreases as the number components in the series solution increases. Table 1 compares the result of the maximum absolute error obtained by using the present method with other existing numerical methods such as B-spline method [29], spline-FDM

Fig. 2 Comparison of approximate solution and exact solution of example-1



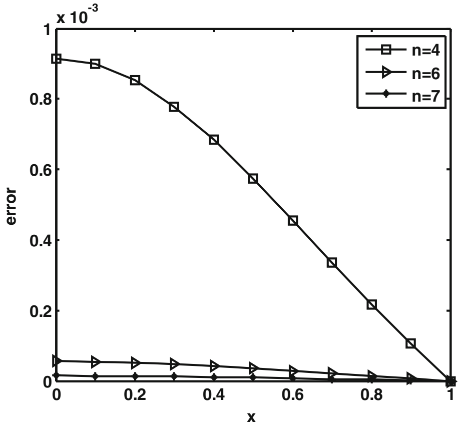


Fig. 3 Numerical results of absolute error of example-1

[30], cubic spline method [31] and mixed B-spline ADM [32]. Comparison shows that our method with few solution components provides better result than these methods. The optimal values of h for n=9, 10, 11, 12 are -1.23655, -1.24143, -1.24782, -1.25162, respectively.

Example 2 Consider the nonlinear singular BVP arising in the study of steady-state oxygen diffusion in a spherical cell

$$(x^2y'(x))' = x^2\frac{\eta y(x)}{y(x)+k}, \tag{41}$$

$$y'(0) = 0,\ 5y(1) + y'(1) = 5,$$

where η and k are positive constants involving the reaction rate and the Michaelis constant, we take $\eta = 0.76129$ and $k = 0.03119$.
This problem has a singular point at $x = 0$ and corresponds to (1)–(2) with $f(x, y) = \frac{\eta y(x)}{y(x)+k}$, $B = 5$, $\mu = 5$ and $\sigma = 1$. The exact solution of the problem is not known.

Table 1 Maximum absolute error of Example 1

n	Present method	N	Method in [29]	N	Method in [30]	N	Method in [31]	N	Method in [32]
9	1.31×10^{-6}	20	3.16×10^{-5}	8	4.7×10^{-4}	20	3.10×10^{-5}	10	8.06×10^{-6}
10	4.27×10^{-7}	40	7.87×10^{-6}	16	3.1×10^{-5}	50	5.04×10^{-6}	20	2.00×10^{-6}
11	1.21×10^{-7}	60	3.50×10^{-6}	32	1.4×10^{-5}				
12	3.98×10^{-8}	90	1.55×10^{-6}	64	4.0×10^{-6}				

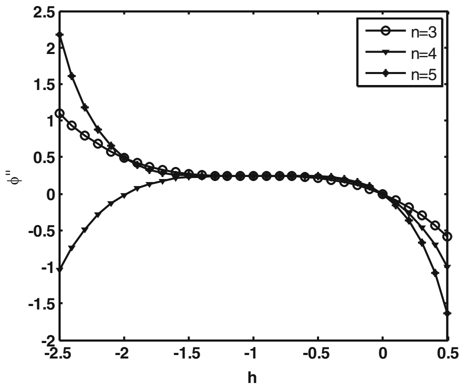


Fig. 4 h-curve region of $\phi''(0)$ for example 2

Using the method defined by (29), we obtain the truncated 6 terms approximate series solution of the problem (40) as given below:

$$\begin{aligned}\Phi_6(x,h) = {} & 1 + h(1.03357 - 0.738264x^2) + h^2(2.59491 - 1.85848x^2 \\ & + 0.00366342x^4) + h^3(3.47207 - 2.49085x^2 + 0.00780413x^4 \\ & + 0.000152952x^6) + h^4(2.61212 - 1.87648x^2 + 0.00753015x^4 \\ & + 0.000289215x^6 + 7.72465 \times 10^{-6}x^8) + h^5(1.0477 \\ & - 0.753483x^2 + 0.00352919x^4 + 0.000194136x^6 \\ & + 0.0000104179x^8 + 2.30924 \times 10^{-7}x^{10}) + h^6(0.175039 \\ & - 0.126001x^2 + 0.000655492x^4 + 0.0000452032x^6 \\ & + 3.65302 \times 10^{-6}x^8 + 1.62597 \times 10^{-7}x^{10} + 3.06728 \times 10^{-9}x^{12}).\end{aligned} \tag{42}$$

The h-curve of $\Phi''(0)$ is displayed in Fig. 4, which shows that the solution series is convergent if $-1.2 < h < -0.2$. The optimal value of h can be obtained by minimizing the sum of the square of the residual error as defined in (32). We find $h = -0.672127$ for $n = 6$. Substituting this value in (42), yields

$$\begin{aligned}\Phi_6(x) = {} & 0.82884 + 0.121725x^2 + 0.000398457x^4 - 9.88018 \times 10^{-6}x^6 + \\ & 4.84241 \times 10^{-7}x^8 - 1.6685 \times 10^{-8}x^{10} + 2.82789 \times 10^{-10}x^{12}.\end{aligned} \tag{43}$$

Figure 5 compares our solution with the B-spline solution [29], mixed ADM B-spline solution [32] and FDM solution [33]. As it is clearly seen in the figure, a close agreement exists between the results obtained by these four methods.

Example 3 Consider the nonlinear singular boundary value problem describing the equilibrium of the isothermal gas sphere

$$\begin{aligned}(x^2y'(x))' &= -x^2y^5(x), \\ y'(0) = 0,\ y(1) &= \sqrt{\tfrac{3}{4}}.\end{aligned} \tag{44}$$

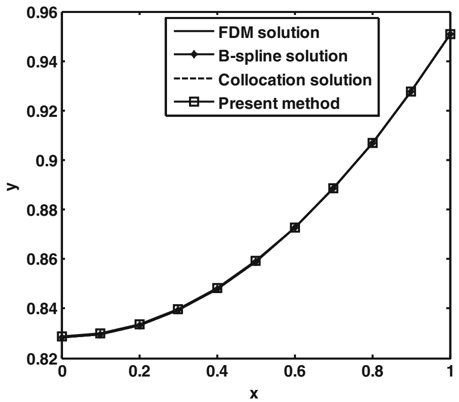


Fig. 5 Approximate solution of example-2

The exact solution is given by $y(x) = \sqrt{\frac{3}{3+x^2}}$.
This problem has a singular point at $x = 0$ and corresponds to (1)–(2) with $f(x, y) = -y^5(x)$, $\mu = 1$, $\sigma = 0$ and $B = \sqrt{\frac{3}{4}}$.

Using the method defined by (29), we obtain the truncated 4 terms approximate series solution of the problem (44) as given by

$$\begin{aligned}\Phi_4(x, h) = &\frac{1}{16777216}\sqrt{3}(8388608 + 3145728h(-1 + x^2)\\&+ 73728h^2(-43 + 34x^2 + 9x^4) + 1536h^3(-1001\\&+ 623x^2 + 333x^4 + 45x^6) + 3h^4(-98839 + 48484x^2\\&+ 39150x^4 + 10260x^6 + 945x^8)).\end{aligned} \tag{45}$$

We plot the h-curve of $\Phi''(0)$ for different values of n in Fig. 6. It is clear from the figures that the valid region of h is $-2 < h < 0$. With the help of (32), we find

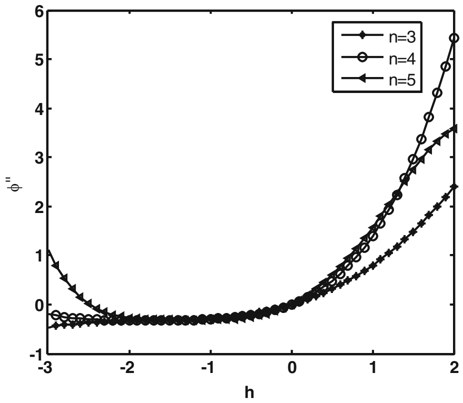


Fig. 6 h-curve region of $\phi''(0)$ for example 3

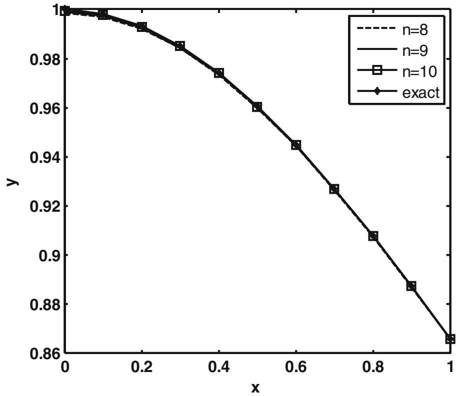


Fig. 7 Comparison of approximate solution and exact solution of example-3

$h = -1.2693$ for n=4. Substituting the optimum value of h in (45), we obtain the following equation:

$$\Phi_4(x) = 0.996076 - 0.158321x^2 + 0.0338558x^4 - 0.00634443x^6 + 0.000759719x^8. \tag{46}$$

We plot the approximate solution of problem (44) for different values of n with the exact solution in Fig. 7. The figure shows that fairly good agreement exist between the approximate and exact solution. Numerical results of absolute errors for $n = 6, 8, 10$ are displayed in Fig. 8.

Example 4 Consider the nonlinear singular BVP arising in the study of the distribution of radial stress on a rotationally symmetric shallow membrane cap

$$(x^3y'(x))' = x^3\left(\frac{1}{2} - \frac{1}{8y^2(x)}\right), \quad 0 < x \le 1 \tag{47}$$

$$y'(0) = 0,\ y(1) = 1.$$

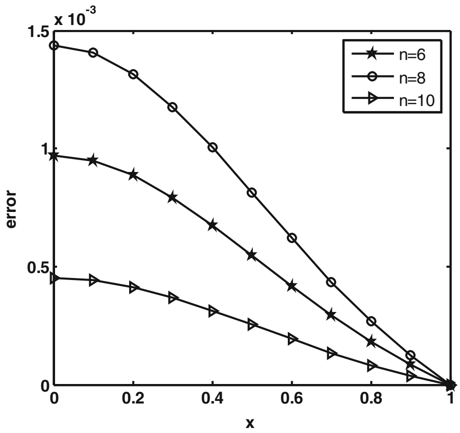


Fig. 8 Numerical results of absolute error of example-3

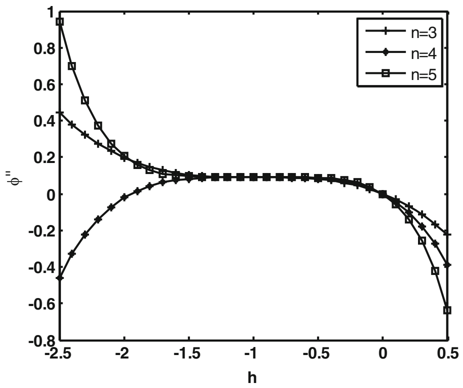


Fig. 9 h-curve region of $\phi''(0)$ for example 4

This problem has a singular point at $x = 0$. The problem (46) corresponds to (1)–(2) with $g(x) = x^3$, $f(x, y) = \left(\frac{1}{2} - \frac{1}{8y^2(x)}\right)$, $B = 1$, $\mu = 1$ and $\sigma = 0$. The exact solution of the problem is not known.

We solve the problem (47) using the method defined in (29). The fourth-order approximate solution is as follows:

$$\Phi_4(x, h) = \frac{1}{503316480}(503316480h(-1 + x^2) + 1474560h^2(98 - 99x^2 + x^4) + 1280h^3(76747 - 78222x^2 + 1452x^4 + 23x^6) + h^4(25016697 - 25695910x^2 + 657700x^4 + 21320x^6 + 193x^8)). \quad (48)$$

The h-curve of $\Phi''(0)$ is displayed in Fig. 9, which shows that the solution series is convergent if $-1.5 < h < -0.1$. The optimal value of h can be obtained by minimizing the sum of the square of the residual error as defined in (32). We find $h = -1.01676$ for $n = 4$. Substituting this value in (48), we obtain:

$$\Phi_4(x) = 0.954135 + 0.0453366x^2 + 0.000543855x^4 - 0.0000162116x^6 + 4.09817 \times 10^{-7}x^8. \quad (49)$$

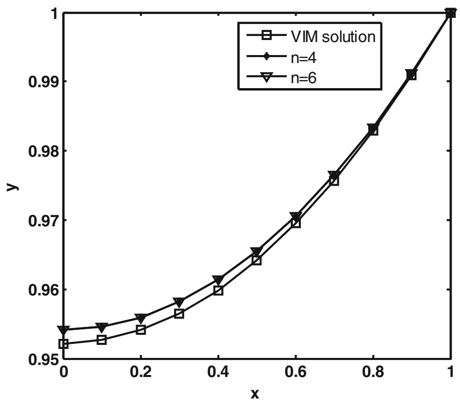


Fig. 10 Approximate solution of example-4

The approximate solutions of (47) obtained using the present method for $n = 4, 6$ and VIM [34] are depicted in Fig. 10. As it is clearly seen in the figure, a close agreement exists between the results obtained by these two methods.

6 Conclusion

In this article, we proposed a new efficient recursive algorithm, based on a combination of integral equation formalism and optimal homotopy analysis method, for the regular singular two-point boundary value problem with Neumann and Robin boundary conditions.

There are three major steps occurring in this algorithm:

◇ The original singular differential equation is transformed into an integral equation to overcome the singularity at the origin.

◇ Boundary condition is imposed to eliminate the undetermined coefficients from the resulting integral equation.

◇ The integral equation without undetermined coefficients is treated by using OHAM to obtain a recursive scheme for the solution of singular BVP considered in the paper.

It is to be noted that the optimal convergence-control parameter involved in the OHAM has been computed by minimizing the squared residual error. We have proved the convergence of the proposed scheme and provided an error estimate of the truncated series solutions. The uniqueness of the solution of the problem has also been discussed in the paper. Four physical model problems have been solved to demonstrate the efficiency and accuracy of the present method. It has been observed that the computed results are in excellent agreement with the exact solutions. Further, our results were compared with those obtained by the methods given in [29–34]. The numerical results indicate that our proposed algorithm is more accurate in comparison to those of [29–34]. Unlike modified ADM, VIM, HPM, or ADM, the proposed approach contains an adjustable parameter, which can be used to control the covergence of series solution. Another advantage of our method over existing recursive schemes using Adomian decomposition method [36], Variational iteration method [34] and modified Adomian decomposition method [32] is that it does not demand the calculation of undetermined coefficient. Besides, it can easily be implemented in symbolic computation softwares like Mathematica and Maple. Moreover, unlike the discretized based numerical methods such as finite difference method, finite element method, or spline method, this method does not require any discretization or linearization of variables. It may be concluded that our proposed algorithm is reliable, effective, and highly accurate for solving singular two-point boundary value problems.

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References

1. Keller, J.B., Electrohydrodynamics, I.: The equilibrium of a charged gas in a container. J. Rational Mech. Anal. **5**, 715–724 (1956)
2. Chambre, P.L.: On the solution of the Poisson-Boltzmann equation with the application to the theory of thermal explosions. J. Chem. Phys. **20**, 1795–1797 (1952)
3. Parter, S.V.: Numerical methods for generalised axially symmetric potentials. SIAM J., Ser. B **2**, 500–516 (1965)
4. Ames, W.F.: Nonlinear ordinary differential equations in transport process. Academia Press, New York (1968)
5. Thomas, L.: The calculation of atomic fields. In: Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge University Press, vol. 23, pp. 542–548 (1927)
6. Fermi, E.: Un metodo statistio per la determinazione di alcune priorieta dell'atome. Rendicondi Accademia Nazionale de Lincei **32**, 602–607 (1927)
7. McElwain, D.L.S.: A re-examination of oxygen diffusion in a spherical cell with Michaelis–Menten oxygen uptake kinetics. J. Theoret. Biol. **71**, 255–263 (1978)
8. Lin, H.S.: Oxygen diffusion in a spherical cell with nonlinear oxygen uptake kinetics. J. Theoretical. Biol. **60**, 449–457 (1976)
9. Flesch, U.: The distribution of heat sources in the human head: a theoretical consideration. J. Theoret. Biol. **54**, 285–287 (1975)
10. Gray, B.F.: The distribution of heat sources in the human head: a theoretical consideration. J. Theoret. Biol. **82**, 473–476 (1980)
11. Adam, J.A.: A simplified mathematical model of tumor growth. Math. Biosci. **81**, 224–229 (1986)
12. Adam, J.A.: A mathematical model of tumor growth II: effect of geometry and spatial nonuniformity on stability. Math. Biosci. **86**, 183–211 (1987)
13. Adam, J.A., Maggelakis, S.A.: Mathematical model of tumor growth IV: effect of necrotic core. Math. Biosci. **97**, 121–136 (1989)
14. Burton, A.C.: Rate of growth of solid tumor as a problem of diffusion. Growth **30**, 157–176 (1966)
15. Greenspan, H.P.: Models for the growth of solid tumor as a problem by diffusion. Stud. Appl. Math. **52**, 317–340 (1972)
16. Nayfeh, A.H.: Perturbation methods. Wiley, New York (2000)
17. He, J.H.: Variational iteration method: a kind of nonlinear analytic technique: some examples. Int. J. Nonlinear Mech. **34**, 699–708 (1999)
18. He, J.H.: Approximate analytical solution for seepage flow with fractional derivative with porous media. Comput. Meth. Appl Mech. **167**, 57–68 (1998)
19. Adomian, G.: Solving frontier problems of physics: the decomposition method. Kluwer Academic, Dordrecht (1994)
20. Liao, S.J.: The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Thesis. Sanghai Jiao Tong University, Shangai (1992)
21. Liao, S.J.: A kind of approximate solution technique which does not depend upon small parameters: a special example. Int. J. Nonlinear Mech. **30**, 371–380 (1995)
22. Liao, S.J., Perturbation, B.eyond.: Introduction to the homotopy analysis method, Chapman and Hall. CRC Press, Boca Raton (2003)
23. Liao, S.J.: On the homotopy analysis method for nonlinear problems. Appl. Math. Comput. **147**, 499–513 (2004)
24. Liao, S.J., Tan, Y.: A general approach to obtain series solutions of nonlinear differential equations. Stud. Appl. Math. **119**, 297–354 (2007)
25. Liao, S.J.: Notes on the homotopy analysis method: some definitions and theorems. Commun. Nonlinear Sci. Numer. Simulat. **14**(4), 983–997 (2009)
26. Liao, S.: An optimal homotopy anlysis approach for strongly nonlinear differential equations. Commun. Nonlinear Sci. Numer. Simulat. **15**, 2003–2016 (2010)
27. Marinca, V., Herisanu, N.: Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer. Int. Commun. Heat Mass Transfer **35**, 1216–1224 (2008)
28. Pandey, R.K., Verma, A.K.: Existance-Uniqueness results for a class of singular boundary value problems arising in physiology. Nonlinear Anal. Real World Appl. **9**, 40–52 (2008)

29. Caglar, H., Caglar, N., Ozer, M.: B-Spline solution of non-linear singular boundary value problems arising in physiology. Chaos, Solitions and Fractals **39**, 1232–1237 (2009)
30. Iyengar, S.R.K., Jain, P.: Spline Finite Differnce Methods for singular two point Boundary Value problem. Numer. Math. **50**, 363–376 (1987)
31. Ravi kant, A.S.V.: Cubic spline polynomial for non-linear singular two-point boundary value problem. Appl. Math. Comput. **189**, 2017–2022 (2007)
32. Khuri, S.A., Sayfy, A.: A novel approach for the solution of a class of singular boundary value problems arising in physiology. Math. Comput. Model. **52**, 626–636 (2010)
33. Pandey, R.K., Singh, A.K.: On the convergence of a finite difference method for a class of singular boundary value problems arising in physiology. J. Comp. Appl. Math **166**, 553–564 (2004)
34. Ravi Kanth, A., Aruna, K.: He's varitional iteration method for treating nonlinear singular boundary value problem. Comput. Math. Appl. **60**(3), 821–829 (2010)
35. Roul, P., Warbhe, U.: New approach for solving a class of singular boundary value problem arising in various physical models. J. Math. Chem. **54**, 1255–1285 (2016)
36. Adomian, G.: Solution of the Thomas-Fermi Equation. Appl. Math. Lett. **11**, 131 (1998)