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ORIGINAL PAPER

On the convergence of CQ algorithm with variable steps for the split equality problem

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Abstract We investigate CQ algorithm for the split equality problem in Hilbert spaces. In such an algorithm, the selection of the step requires prior information on the matrix norms, which is not always possible in practice. In this paper, we propose a new way to select the step so that the implementation of the algorithm does not need any prior information of the matrix norms. In Hilbert spaces, we establish the weak convergence of the proposed method to a solution of the problem under weaker conditions than usual. Preliminary numerical experiments show that the efficiency of the proposed algorithm when it applies the variable step-size.

Keywords Split feasibility problem · CQ algorithm · Variable-step · Projection

Mathematics Subject Classifications (2010) 47J25 · 47J20 · 49N45 · 65J15

1 Introduction

In this paper, we are concerned with the split feasibility problem (SFP), which requires to find a point $\hat{x} \in \mathbb{R}^n$ satisfying the property:

$$\hat{x} \in C \quad \text{and} \quad A\hat{x} \in Q, \tag{1}$$

where C and Q are nonempty closed convex subset of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is an $m \times n$ matrix (i.e., a linear operator from $\mathbb{R}^n$ into $\mathbb{R}^m$) [5]. The SFP has

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been proved very useful in dealing with a variety of signal processing and image recovery [7].

Various algorithms have been invented to solve the SFP (1) (see [1, 2, 4, 12, 13] and reference therein). In particular, Byrne introduced his CQ algorithm:

$$x_{k+1} = P_C(x_k - \tau A^\top (I - P_Q)Ax_k), \tag{2}$$

where $A^\top$ is the transpose of A and the step τ is a fixed real number in $\left(0, \frac{2}{\|A\|^2}\right)$. The CQ algorithm (2) has been now widely studied since it is more easily performed. However, to implement the CQ algorithm, one has to compute or estimate the value of $\|A\|$, which is not always possible in practice. To overcome this drawback, many authors have conducted worthwhile works on the CQ algorithm so that the choice of the step does not depend on the matrix morms (see for instance [7, 10–14]). Among these works, Yang [14] suggested, instead of the constant-step, a novel variable-step:

$$\tau_k = \frac{\varrho_k}{\|A^\top (I - P_Q)Ax_k\|}, \tag{3}$$

where (ϱ_k) is a sequence of positive real numbers such that

$$\sum_{k=0}^{\infty} \varrho_k = \infty, \quad \sum_{k=0}^{\infty} \varrho_k^2 < \infty. \tag{4}$$

With this choice of the step-sizes, the computation of $\|A\|$ is avoided, and thus one need not know a prior any information of $\|A\|$. Yang proved the convergence of the modified algorithm to a solution of the SFP provided that (i) Q is a bounded subset; and (ii) A is a matrix with full column rank.

Let us now consider the split equality problem (SEP) [8] that consists of finding a pair $(\hat{x}, \hat{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\hat{x} \in C, \ \hat{y} \in Q, \text{ and } A\hat{x} = B\hat{y}, \tag{5}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, A is a $p \times n$ matrix, and B is a $p \times m$ matrix, respectively. It is clear that the SEP includes the SFP as special cases. Indeed, when B is the identity matrix, the SEP is then reduced to the SFP. Recently, Byrne and Moudafi [3] extended CQ algorithm to solve the SEP: choose an arbitrary initial guess x_1, and calculate:

$$\left[\begin{array}{l} x_{k+1} = P_C(x_k - \tau A^\top (Ax_k - By_k)) \\ y_{k+1} = P_Q(y_k - \tau B^\top (By_k - Ax_k)), \end{array}\right. \tag{6}$$

where the step τ is a positive real number. Then the sequence generated by (6) converges to a solution of the SEP if such a solution exists and the step τ is properly chosen.

It is worth noting that in the procedure (6) the step τ is the constant-step whose choice is mainly relying on the norms of matrices A and B. Thus, a similar question of CQ algorithm (6) also arises: Does there exist a way to select the step in CQ algorithm (6) that dose not depend on the matrices norms?

It is the purpose of this paper to answer the above question affirmatively. Motivated by the step choice (3), we can propose a new method for selecting the step in a way that the implementation of CQ algorithm (6) does not need any prior information of matrix norms. We then establish the convergence of the proposed method but without boundedness on Q nor full column rank of the matrix involved.

2 Preliminary

In this section, we assume that H is a Hilbert space and $C \subseteq H$ is a nonempty closed convex subset.

Definition 1 A sequence $(z_k) \subseteq H$ is said to be quasi-Fejér monotone with respect to C if there exists $N \in \mathbb{N}$ such that for any $k \geq N$

$$\|z_{k+1} - z\|^2 \leq \|z_k - z\|^2 + \epsilon_k, \ \forall z \in C$$

where (ϵ_k) is a positive real sequences satisfying $\sum_k \epsilon_k < \infty$.

Lemma 1 [6] *Let $(z_k) \subseteq H$ be quasi-Fejér monotone with respect to C. Then*

(i) (z_k) *is bounded;*
(ii) $(\|z_k - z\|)$ *is convergent for any $z \in C$;*
(iii) (z_k) *is weakly convergent provided all weak cluster points of (z_k) belong to C.*

Lemma 2 [9] *Let (ϵ_k) and (s_k) be nonnegative real sequences. If*

$$s_{k+1} \leq s_k + \epsilon_k, \sum_{k=1}^{\infty} \epsilon_k < \infty,$$

then the limit of (s_k) exists.

Denote by P_C the projection from H onto C; that is,

$$P_C x = \arg\min_{y \in C} \|x - y\|, \quad x \in H.$$

The projection operator has the following properties.

Lemma 3 *Let P_C be the projection operator onto C. Then for any $x, y \in H$,*

(i) P_C *is nonexpansive, i.e.,*

$$\|P_C x - P_C y\| \leq \|x - y\|;$$

(ii) P_C *is firmly nonexpansive, i.e.,*

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle;$$

(iii) $I - P_C$ *is firmly nonexpansive.*

3 The proposed algorithms

In this paper, we shall consider problem (5) in Hilbert spaces, that is, we shall study the following problem: find a pair $(\hat{x}, \hat{y}) \in H_1 \times H_2$ such that

$$\hat{x} \in C, \ \hat{y} \in Q, \ \text{and } A\hat{x} = B\hat{y}, \tag{7}$$

where $C \subseteq H_1$ and $Q \subseteq H_2$ are nonempty closed convex subsets, $A : H_1 \to H_3$ and and $B : H_2 \to H_3$ are two linear bounded operators, and $H_i, i = 1, 2, 3$ are three Hilbert spaces.

Let us now introduce our iterative scheme to solve the SEP. Choose an arbitrary initial guess x_1. Given (x_k, y_k), if $Ax_k = By_k$, stop; otherwise compute:

$$\left[\begin{array}{l} x_{k+1} = P_C\big(x_k - \tau_k A^*(Ax_k - By_k)\big) \\ y_{k+1} = P_Q\big(y_k - \tau_k B^*(By_k - Ax_k)\big), \end{array}\right. \tag{8}$$

where A^* denotes the adjoint operator of A and τ_k is defined as

$$\tau_k := \frac{\varrho_k}{\max(\|A^*(Ax_k - By_k)\|, \|B^*(By_k - Ax_k)\|)}. \tag{9}$$

Remark 1 It is worth noting that

$$\max(\|A^*(Ax_k - By_k)\|, \|B^*(By_k - Ax_k)\|) = 0 \Leftrightarrow Ax_k = By_k, \tag{10}$$

which indicated that $\max(\|A^*(Ax_k - By_k)\|, \|B^*(By_k - Ax_k)\|) > 0$ if $Ax_k \neq By_k$, and thus τ_k is well defined. To show (10), it suffices to show the "$\Rightarrow$" part since the "$\Leftarrow$" part is trivial. Indeed, assume $\max(\|A^*(Ax_k - By_k)\|, \|B^*(By_k - Ax_k)\|) = 0$. It then follows that

$$\begin{aligned} \|Ax_k - By_k\|^2 &= \langle Ax_k - By_k, Ax_k - By_k \rangle \\ &= \langle Ax_k - By_k, Ax_k \rangle - \langle Ax_k - By_k, By_k \rangle \\ &= \langle A^*(Ax_k - By_k), x_k \rangle - \langle B^*(Ax_k - By_k), y_k \rangle \\ &= 0, \end{aligned}$$

that is, $Ax_k = By_k$, and hence (10) follows immediately.

Remark 2 It is not hard to check that if the iteration above terminates within finite steps, then the current iteration must be a solution of the problem. So without loss of generality, we may assume that the algorithm generates an infinite iterative sequence.

In what follows, we denote by $\mathbb{S}$ the solution set of the SEP (7), namely

$$\mathbb{S} = \{(x, y) : x \in C, y \in Q, Ax = By\}.$$

Let $z = (x, y)$ be an element in the product space $H_1 \times H_2$, then its norm is given by $\|z\| = \sqrt{\|x\|^2 + \|y\|^2}$. Let us now establish the convergence results of the proposed algorithm.

Lemma 4 *Let $z_k := (x_k, y_k) \in H_1 \times H_2$ be the sequence generated by (8)–(9). If the SEP (7) is consistent, namely $\mathbb{S} \neq \emptyset$, then*

$$\|z_{k+1} - z^*\|^2 \leq \|z_k - z^*\|^2 - 2\tau_k \|Ax_k - By_k\|^2 + 2\varrho_k^2 \tag{11}$$

holds for each $k \in \mathbb{N}$ and for all $z^* \in \mathbb{S}$.

Proof Taking $z^* = (x^*, y^*) \in \mathbb{S}$, we have that

$$\begin{aligned} \|x_{k+1} - z\|^2 &= \|P_C(x_k - \tau_k A^*(Ax_k - By_k)) - x^*\|^2 \\ &\leq \|(x_k - x^*) - \tau_k A^*(Ax_k - By_k)\|^2 \\ &= \|x_k - x^*\|^2 - 2\tau_k \langle A(x_k - x^*), Ax_k - By_k\rangle \\ &\quad + \tau_k^2 \|A^*(Ax_k - By_k)\|^2 \\ &\leq \|x_k - x^*\|^2 - 2\tau_k \langle A(x_k - x^*), Ax_k - By_k\rangle + \rho_k^2, \end{aligned}$$

and also that

$$\begin{aligned} \|y_{k+1} - y^*\|^2 &= \|P_Q(y_k + \tau_k B^*(Ax_k - By_k)) - y^*\|^2 \\ &\leq \|(y_k - y^*) + \tau_k B^*(Ax_k - By_k)\|^2 \\ &= \|y_k - y^*\|^2 + 2\tau_k \langle B(y_k - y^*), Ax_k - By_k\rangle \\ &\quad + \tau_k^2 \|B^*(Ax_k - By_k)\|^2 \\ &\leq \|y_k - y^*\|^2 + 2\tau_k \langle B(y_k - y^*), Ax_k - By_k\rangle + \rho_k^2. \end{aligned}$$

Note that $Ax^* = By^*$. Then adding up the last two inequalities immediately yields the desired inequality. □

Theorem 1 *Let $z_k := (x_k, y_k) \in H_1 \times H_2$ be the sequence generated by (8)–(9). If the SEP (7) is consistent and the sequence (ϱ_k) satisfies condition (4), then the sequence $z_k := (x_k, y_k)$ converges to an element in $\mathbb{S}$.*

Proof Let $z^* = (x^*, y^*) \in \mathbb{S}$ be fixed. By Lemma 4, we have

$$\|z_{k+1} - z^*\|^2 \leq \|z_k - z^*\|^2 + 2\varrho_k^2.$$

From Definition 1, we see that the sequence (z_k) is quasi-Fejér monotone with respect to $\mathbb{S}$. Thus, by Lemma 1, we conclude that the sequence $(\|z_k - z^*\|)$ is convergent, and in particular, (z_k) is bounded. To complete the proof, we next divide our proof into three steps.

Step 1. show that $\liminf_k \|Ax_k - By_k\| = 0$. From (11), it follows that

$$\tau_k \|Ax_k - By_k\|^2 \leq \frac{1}{2}(\|z_k - z\|^2 - \|z_{k+1} - z\|^2) + \varrho_k^2,$$

which immediately implies that

$$\sum_{j=1}^{k} \tau_j \|Ax_j - By_j\|^2 \leq \frac{1}{2}\|z_1 - z^*\|^2 + \sum_{j=1}^{k} \varrho_j^2.$$

Taking the limit by letting $k \to \infty$ in the last formula and having in mind that $\sum_k \varrho_k^2 < \infty$, we have

$$\sum_{k=1}^{\infty} \tau_k \|Ax_k - By_k\|^2 < \infty. \tag{12}$$

On the other hand, we see that

$$\|A^*(Ax_k - By_k)\| \le \|A\|\|Ax_k - By_k\|,$$
$$\|B^*(By_k - Ax_k)\| \le \|B\|\|Ax_k - By_k\|,$$

which implies that

$$\tau_k \ge \frac{\varrho_k}{\max(\|A\|, \|B\|)\|Ax_k - By_k\|}. \tag{13}$$

Combining (13) and (12), we have

$$\sum_{k=1}^{\infty} \varrho_k \|Ax_k - By_k\| < \infty. \tag{14}$$

This together with the assumption $\sum_k \varrho_k = \infty$ particularly implies that

$$\liminf_{k\to\infty} \|Ax_k - By_k\| = 0.$$

Step 2. show $\lim_k \|Ax_k - By_k\| = 0$. Actually, we have that

$$\begin{aligned} \|A(x_{k+1} - x_k)\| &\le \|A\|\|P_C(x_k - \tau_k A^*(Ax_k - By_k)) - x_k\| \\ &\le \tau_k \|A\|\|A^*(Ax_k - By_k)\| \\ &\le \varrho_k \|A\|, \end{aligned}$$

and also that

$$\begin{aligned} \|B(y_{k+1} - y_k)\| &\le \|B\|\|P_Q(y_k + \tau_k B^*(Ax_k - By_k)) - y_k\| \\ &\le \tau_k \|B\|\|B^*(Ax_k - By_k)\| \\ &\le \varrho_k \|B\|. \end{aligned}$$

Let $a_k = Ax_k - By_k$. By the last two inequalities, we have

$$\begin{aligned} \|a_{k+1} - a_k\| &= \|(Ax_{k+1} - By_{k+1}) - (Ax_k - By_k)\| \\ &\le \|A(x_k - x_{k+1})\| + \|B(y_{k+1} - y_k)\| \\ &\le \varrho_k(\|A\| + \|B\|) = \delta\varrho_k, \end{aligned}$$

where we define $\delta := \|A\| + \|B\|$. Hence, we have

$$\begin{aligned} \|a_{k+1}\|^2 &= \|a_k\|^2 + 2\langle a_k, a_{k+1} - a_k\rangle + \|a_{k+1} - a_k\|^2 \\ &\le \|a_k\|^2 + 2\|a_k\|\|a_{k+1} - a_k\| + \|a_{k+1} - a_k\|^2 \\ &\le \|a_k\|^2 + 2\delta\varrho_k\|a_k\| + \delta^2\varrho_k^2. \end{aligned}$$

Setting $\eta_k = 2\delta\varrho_k\|a_k\| + \delta^2\varrho_k^2$, we have

$$\|a_{k+1}\|^2 \le \|a_k\|^2 + \eta_k. \tag{15}$$

It is clear that $\sum_k \eta_k < \infty$ due to (14) and (4). We can therefore apply Lemma 2 to (15) to get the existence of the $\lim_k \|a_k\|$. Hence, $\lim_k \|Ax_k - By_k\| = 0$, since we have shown that $\liminf_k \|Ax_k - By_k\| = 0$.

Step 3. show that every weak cluster point of (z_k) is in the set $\mathbb{S}$. Suppose that a subsequence $(z_{k_j}) = (x_{k_j}, y_{k_j})$ of (z_k) weakly converges to a point $\hat{z} = (\hat{x}, \hat{y})$. It is readily seen that $\hat{x} \in C$ and $\hat{y} \in Q$; moreover

$$\begin{aligned}\|A\hat{x} - B\hat{y}\| &\leq \liminf_{j\to\infty} \|Ax_{k_j} - By_{k_j}\| \\ &= \lim_{k\to\infty} \|Ax_k - By_k\| = 0,\end{aligned}$$

where we have used the weak lower semi-continuity of the norm, that is, $A\hat{x} = B\hat{y}$. Thus, we conclude that $\hat{z} = (\hat{x}, \hat{y}) \in \mathbb{S}$.

In summary, we have shown that the sequence (z_k) is quasi-Fejér monotone with respect to $\mathbb{S}$ and all weak cluster points of (z_k) belong to $\mathbb{S}$. Consequently, the results follow immediately from Lemma 1.

□

Remark 3 In the theorem above, there are not any requirements of the boundedness of Q and the full column rank of A as used in Yang's result [14].

Remark 4 It is clear our choice of the step does not need any information on the values of $\|A\|$ and $\|B\|$.

4 A demonstration example

For simplicity, we denote algorithms (6) and (8)–(9) by Algorithm 2 and Algorithm 1, respectively. We now conduct an experiment to verify convergence of iterative sequence generated by Algorithm 2 and reveal its efficiency through comparing the performance of Algorithm 1. In the experiment, we consider the case whenever

$$\begin{aligned}C &= \{x \in \mathbb{R}^n : \|x - d\| \leq r\}, \\ Q &= \{y \in \mathbb{R}^m : l \leq y \leq u\},\end{aligned}$$

where $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times m}$, $a_{ij} \in [0, 1]$, $b_{ij} \in [0, 1]$, $d_i \in [0, 10]$, $i = 1, 2, \cdots, n$, $r \in [40, 60]$, $l_j \in [10, 40]$ and $u_j \in [50, 100]$, $j = 1, 2, \cdots, m$ are all generated randomly. The restoration accuracy is measured by means of the mean squared error

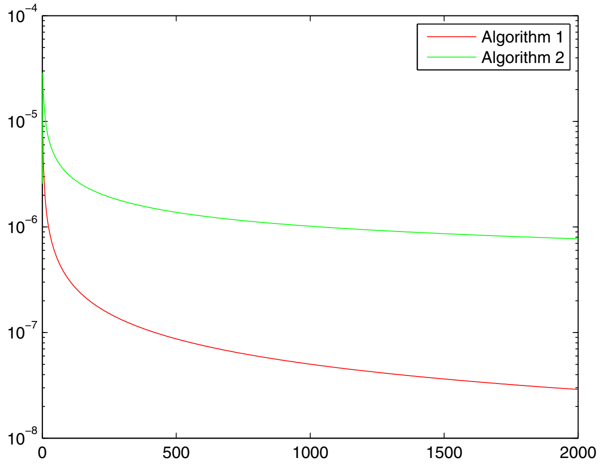
$$\begin{aligned}\mathrm{MSE}(\|x^{k+1} - x^k\|) &= \frac{1}{n}\|x^{k+1} - x^k\|, \\ \mathrm{MSE}(\|y^{k+1} - y^k\|) &= \frac{1}{m}\|y^{k+1} - y^k\|.\end{aligned}$$

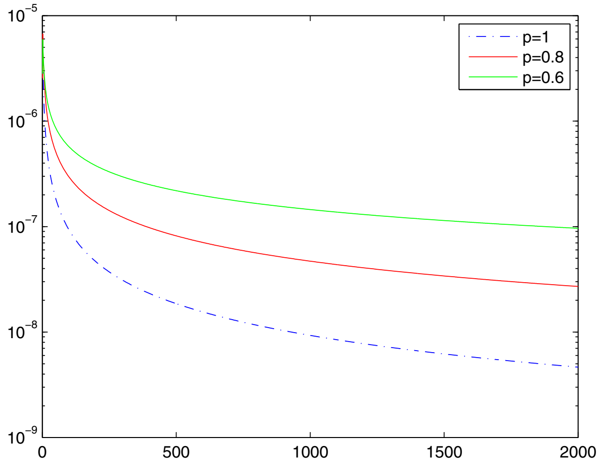
We first compare the efficiency of Algorithms 1 and 2. The parameter is set as $\tau = (rand(1, 1) + 1)/\max(1, norm(A)^2)$ by Algorithm 2 and $\tau_k = 1/(k + 1)$ in

Algorithm 1. As shown in top figure, the convergence of these two algorithms is justified. It is readily seen that Algorithm 1 converges faster than by Algorithm 2 does. This supports in partial the advantage of variable step-size over the constant step-size for the considered problem. We next compare the efficiency of the parameter p in Algorithm 1, whenever we set

$$\tau_k = \frac{1}{(k+1)^p}, k \in (1/2, 1]$$

in Algorithm 1. As shown in bottom figure, it seems that the convergence Algorithm 1 goes faster when p is bigger.





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