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# A new block preconditioner for complex symmetric indefinite linear systems

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**Abstract** Using the equivalent block two-by-two real linear systems and relaxing technique, we establish a new block preconditioner for a class of complex symmetric indefinite linear systems. The new preconditioner is much closer to the original block two-by-two coefficient matrix than the Hermitian and skew-Hermitian splitting (HSS) preconditioner. We analyze the spectral properties of the new preconditioned matrix, discuss the eigenvalue distribution and derive an upper bound for the degree of its minimal polynomial. Finally, some numerical examples are provided to show the effectiveness and robustness of our proposed preconditioner.

**Keywords** Block two-by-two matrix · Preconditioning · Complex symmetric linear system · Relaxing technique

**Mathematics Subject Classifications (2010)** 65F10 · 75F50

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## 1 Introduction

We consider the iterative solution of large and sparse complex symmetric linear systems

$$Ax = b, \quad A \in \mathbb{C}^{N\times N} \quad and \quad x, b \in \mathbb{C}^{N}. \tag{1}$$

The nonsingular complex symmetric matrix $A$ can be written as

$$A = W + iT, \tag{2}$$

where the matrices $W, T \in \mathbb{R}^{N\times N}$ are symmetric, and $i = \sqrt{-1}$ denotes the imaginary unit. Many scientific and engineering applications often lead to the problem (1), such as diffuse optical tomography [1], electromagnetic problem [17], and quantum mechanics [34]. For other applications, we refer to [19] and references therein.

A class of splitting iteration methods based on Hermitian and skew-Hermitian splitting (HSS) has been established by Bai et al. [10, 12], and it has been further discussed and generalized by many researchers, see, e.g., [8, 9, 11, 13, 18]. When the matrices $W$ and $T$ are symmetric positive semi-definite and one of them is positive definite, Bai et al. constructed a modified HSS (MHSS) [5] iteration method and the preconditioned MHSS (PMHSS) [6] iteration method. These two methods are unconditionally convergent and the latter shows $h$-independent convergence behavior. Recently, Bai [4] further analyzed algebraic and convergence properties of the PMHSS iteration method for solving complex linear systems, and presented analytical and numerical comparisons among several iteration methods. When the real part of the coefficient $A$ is dominant, Li et al. [27] derived a lopsided PMHSS (LPMHSS) iteration method. In [36], Xu generalized the PMHSS iteration method to solve two classes of complex symmetric indefinite linear systems. Recently, Cao and Ren [21] considered two variants of the PMHSS iteration method. For the problem (1) with symmetric indefinite matrix $W \in \mathbb{R}^{N\times N}$ and symmetric positive definite matrix $T \in \mathbb{R}^{N\times N}$, the MHSS and PMHSS iteration methods converge slowly or stagnate. To overcome this problem, Wu [35] designed the Hermitian normal splitting (HNS) iteration method and simplified HNS (SHNS) iteration method. Recently, Zhang and Dai [38] proposed their preconditioned versions. For other effective splitting iteration methods, we refer to [39, 40].

The problem (1) can be equivalently rewritten as the following real block two-by-two linear systems

$$\mathcal{A}x = \begin{bmatrix} W & -T \\ T & W \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \tag{3}$$

or

$$\mathcal{A}x = \begin{bmatrix} T & -W \\ W & T \end{bmatrix}\begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} g \\ f \end{bmatrix} \equiv b, \tag{4}$$

which can be solved in real arithmetics by some Krylov subspace iteration methods, such as the preconditioned GMRES [32] method. A high-quality preconditioner is very crucial to improve the numerical behavior of some Krylov subspace iteration methods. For the symmetric positive semidefinite matrices $W$ and $T$, a number of block preconditioners have been derived. Bai [3] developed a class of rotated block triangular (RBT) preconditioners based on the PMHSS preconditioning matrix

[7]. To increase the speed-up ratios of the RBT preconditioners, Lang and Ren [26] studied inexact RBT (IRBT) preconditioners. By introducing a new equivalent variant of the problem (3), Yan and Huang [37] presented a class of splitting-based block preconditioners. The SOR-like methods is a class of effective iteration methods for solving the saddle point problems. Bai et al. [15] proposed a generalized SOR (GSOR) method for the saddle point problems, analyzed its convergence, and determined its optimal iteration parameter and the corresponding optimal convergence factor. Bai and Wang [16] developed the method to generalized saddle point problems. Recently, Salkuyeh et al. [33] applied the GSOR method to the linear system (3) and derived the GSOR iteration method for solving complex symmetric linear systems, and Hezari et al. [25] designed its preconditioned version. Liang and Zhang [28] established the symmetric SOR (SSOR) method and its accelerated variant for solving the problem (3).

In this paper, we construct a novel block preconditioner based on the new splitting of the coefficient matrix $\mathcal{A}$. In order to obtain a better approximation to the matrix $\mathcal{A}$, we apply the relaxing technique to modify this new preconditioner. Theoretical analysis shows that all eigenvalues of the new preconditioned matrix are located in the interval (0, 1]. We also investigate the eigenvalue distribution and derive an upper bound of the degree of the minimal polynomial of the new preconditioned matrix.

The framework of the paper is organized as follows. In Section 2, we present the new block preconditioner and describe the detail implementation of the preconditioning process. In Section 3, we give some theoretical analyses about the preconditioned matrix. In Section 4, some numerical examples are tested to show the effectiveness of our proposed preconditioner. Finally, we make some concluding remarks in Section 5.

## 2 A new block splitting preconditioner

In this section, we will construct a new block splitting preconditioner for the equivalent real block two-by-two linear systems (4) with symmetric indefinite matrix $W \in \mathbb{R}^{N\times N}$ and symmetric positive definite matrix $T \in \mathbb{R}^{N\times N}$.

Bai et al. [9] first introduced the positive definite and skew-Hermitian splitting (PSS), established the PSS iteration method for solving non-Hermitian positive definite linear systems, and showed its unconditional convergence. Pan et al. [29] developed the PSS preconditioner for saddle point problems, which can be considered as a deteriorated version of the PSS iteration method. For the linear system (4), it is easy to obtain the following splitting of the coefficient matrix $\mathcal{A}$,

$$\mathcal{A} = \mathcal{J} + \mathcal{K}, \tag{5}$$

where $\mathcal{J} = \begin{bmatrix} T & O \\ O & O \end{bmatrix}$ and $\mathcal{K} = \begin{bmatrix} O & -W \\ W & T \end{bmatrix}$. Analogous to the PSS preconditioner, we set

$$\mathcal{P} = \frac{1}{2\alpha}(\alpha\mathcal{I} + \mathcal{K})(\alpha\mathcal{I} + \mathcal{J}). \tag{6}$$

When the matrix $\mathcal{P}$ is considered as the preconditioner, the pre-factor $\frac{1}{2\alpha}$ has no effect on the preconditioned system. Therefore, we can let

$$\mathcal{P}_1 = \frac{1}{\alpha}(\alpha\mathcal{I}+\mathcal{K})(\alpha\mathcal{I}+\mathcal{J}) = \frac{1}{\alpha}\begin{bmatrix} \alpha I & -W \\ W & \alpha I + T \end{bmatrix}\begin{bmatrix} \alpha I + T & O \\ O & \alpha I \end{bmatrix} = \begin{bmatrix} \alpha I + T & -W \\ W(I+\frac{1}{\alpha}T) & \alpha I + T \end{bmatrix}. \tag{7}$$

From (6) and (7), we have

$$\mathcal{R}_1 = \mathcal{P}_1 - \mathcal{A} = \begin{bmatrix} \alpha I & O \\ \frac{1}{\alpha}WT & \alpha I \end{bmatrix}. \tag{8}$$

As seen from (8), the two diagonal blocks trend to zero, but the nonzero off-diagonal block becomes unbounded as $\alpha$ approaches 0. Therefore, a practical parameter $\alpha$ should be introduced to balance the two parts. For more details, we refer to [24].

Inspired by the idea of the relaxed preconditioner [20, 22, 23], we modify the preconditioner $\mathcal{P}_1$ by

$$\mathcal{P}_2 = \frac{1}{\alpha}\begin{bmatrix} \alpha I & -W \\ W & T \end{bmatrix}\begin{bmatrix} \alpha I + T & O \\ O & \alpha I \end{bmatrix} = \begin{bmatrix} \alpha I + T & -W \\ W(I+\frac{1}{\alpha}T) & T \end{bmatrix}. \tag{9}$$

From (9), it holds that

$$\mathcal{R}_2 = \mathcal{P}_2 - \mathcal{A} = \begin{bmatrix} \alpha I & O \\ \frac{1}{\alpha}WT & O \end{bmatrix}. \tag{10}$$

Comparing the matrix $\mathcal{R}_2$ with the matrix $\mathcal{R}_1$, we know that the preconditioner $\mathcal{P}_2$ is much closer to the coefficient matrix $\mathcal{A}$ than the preconditioner $\mathcal{P}_1$.

Note that the preconditioner $\mathcal{P}_2$ can be constructed based on the new splitting of the coefficient matrix $\mathcal{A}$

$$\mathcal{A} = \mathcal{P}_2 - \mathcal{R}_2 = \begin{bmatrix} \alpha I + T & -W \\ W(I+\frac{1}{\alpha}T) & T \end{bmatrix} - \begin{bmatrix} \alpha I & O \\ \frac{1}{\alpha}WT & O \end{bmatrix}. \tag{11}$$

For the preconditioned Krylov subspace iteration method with the preconditioner $\mathcal{P}_2$, we need to solve the following linear subsystems

$$\frac{1}{\alpha}\begin{bmatrix} \alpha I & -W \\ W & T \end{bmatrix}\begin{bmatrix} \alpha I + T & O \\ O & \alpha I \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \alpha I + T & -W \\ W(I+\frac{1}{\alpha}T) & T \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \end{bmatrix}, \tag{12}$$

where $(z_1^T, z_2^T)^T$ and $(\bar{r}_1^T, \bar{r}_2^T)^T$ are the current and generalized residual vectors, respectively. From (12), we have

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \alpha \begin{bmatrix} \alpha I + T & O \\ O & \alpha I \end{bmatrix}^{-1}\begin{bmatrix} \alpha I & -W \\ W & T \end{bmatrix}^{-1}\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} \alpha I + T & O \\ O & \alpha I \end{bmatrix}^{-1}\begin{bmatrix} I & \frac{1}{\alpha}W \\ O & I \end{bmatrix}\begin{bmatrix} \alpha I & O \\ O & T+\frac{1}{\alpha}W^2 \end{bmatrix}^{-1}\begin{bmatrix} I & O \\ -\frac{1}{\alpha}W & I \end{bmatrix}\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \end{bmatrix} \tag{13}$$

Using the above results, we can describe the implementing process of the preconditioner $\mathcal{P}_2$ in Algorithm 1.

Fortunately, in Algorithm 1, we only need to solve two symmetric positive definite linear subsystems under the assumptions of symmetric indefinite matrix $W$ and symmetric positive definite matrix $T$. Therefore, we can apply the sparse

**Algorithm 1** Implementing process of preconditioner $\mathcal{P}_2$

1. Give a residual vector $\bar{r} = (\bar{r}_1^T, \bar{r}_2^T)^T$;
2. Compute $u_1 = \bar{r}_2 - \frac{1}{\alpha} W \bar{r}_1$;
3. Solve the linear system $(T + \frac{1}{\alpha} W^2) z_2 = u_1$;
4. Compute $u_2 = \bar{r}_1 + W z_2$;
5. Solve the linear system $(\alpha I + T) z_1 = u_2$;
6. Set the generalized residual vector $z = (\bar{z}_1^T, \bar{z}_2^T)^T$.

Cholesky decomposition, the conjugate gradient (CG) method or the preconditioned CG method to solve symmetric positive definite linear systems $(T + \frac{1}{\alpha} W^2) z_2 = u_1$ and $(\alpha I + T) z_1 = u_2$.

For the HSS preconditioner, we can derive the computing process as in Algorithm 2.

**Algorithm 2** Implementing process of HSS preconditioner

1. Give a residual vector $\bar{r} = (\bar{r}_1^T, \bar{r}_2^T)^T$;
2. Solve the linear system $(\alpha I + T) v_1 = \bar{r}_1$;
3. Solve the linear system $(\alpha I + T) v_2 = \bar{r}_2$;
4. Compute $u_1 = v_2 - \frac{1}{\alpha} W v_1$;
5. Solve the linear system $(\alpha I + \frac{1}{\alpha} W^2) z_2 = u_1$;
6. Compute $z_1 = \frac{1}{\alpha}(v_1 + W z_2)$;
7. Set the generalized residual vector $z = (\bar{z}_1^T, \bar{z}_2^T)^T$.

From Algorithm 2, we may see that there are three symmetric positive definite linear subsystems to be solved. Therefore, our preconditioner returns better computing efficiency than the HSS preconditioner.

## 3 Theorectical analysis of the preconditioned matrix

In this section, we investigate the spectral properties of the preconditioned matrix and give an upper bound of the degree of its minimal polynomial.

First, we analyze the eigenvalue distribution of the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$.

**Theorem 1** *Assume that the coefficient matrix $\mathcal{A}$ is nonsingular, $W \in \mathbb{R}^{N\times N}$ is symmetric indefinite and $T \in \mathbb{R}^{N\times N}$ is symmetric positive definite. Let $\alpha$ be a real positive constant. Then for the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$, the following results hold.*

*(1) $\mathcal{P}_2^{-1}\mathcal{A}$ has an eigenvalue 1 with multiplicity at least $N$;*

*(2) the remaining nonunit eigenvalues of $\mathcal{P}_2^{-1}\mathcal{A}$ satisfy the generalized eigenvalue problem $\alpha(U + T)y = \lambda(\alpha I + U)(\alpha I + T)y$ and locate in the interval* $(0, 1)$.

*Proof* From (8)–(10), we have

$$\begin{aligned}
\mathcal{P}_2^{-1}\mathcal{A} &= \mathcal{P}_2^{-1}(\mathcal{P}_2-\mathcal{R}_2)=\mathcal{I}-\mathcal{P}_2^{-1}\mathcal{R}_2\\
&= \mathcal{I}-\alpha\begin{bmatrix}\alpha I+T & O\\ O & \alpha I\end{bmatrix}^{-1}\begin{bmatrix}\alpha I & -W\\ W & T\end{bmatrix}^{-1}\begin{bmatrix}\alpha I & O\\ \frac{1}{\alpha}WT & O\end{bmatrix}\\
&= \mathcal{I}-\alpha\begin{bmatrix}\alpha I+T & O\\ O & \alpha I\end{bmatrix}^{-1}\begin{bmatrix}I & \frac{1}{\alpha}W\\ O & I\end{bmatrix}\begin{bmatrix}\alpha I & O\\ O & T+\frac{1}{\alpha}W^2\end{bmatrix}^{-1}\begin{bmatrix}I & O\\ -\frac{1}{\alpha}W & I\end{bmatrix}\begin{bmatrix}\alpha I & O\\ \frac{1}{\alpha}WT & O\end{bmatrix}\\
&= \mathcal{I}-\begin{bmatrix}\alpha(\alpha I+T)^{-1}+(\alpha I+T)^{-1}W(T+\frac{1}{\alpha}W^2)^{-1}(\frac{1}{\alpha}WT-W) & O\\ (T+\frac{1}{\alpha}W^2)^{-1}(\frac{1}{\alpha}WT-W) & O\end{bmatrix}.
\end{aligned}$$

Setting $U=WT^{-1}W$, we have

$$\begin{aligned}
&\alpha(\alpha I+T)^{-1}+(\alpha I+T)^{-1}W(T+\tfrac{1}{\alpha}W^2)^{-1}(\tfrac{1}{\alpha}WT-W)\\
&=(\alpha I+T)^{-1}(\alpha I+W(T+\tfrac{1}{\alpha}W^2)^{-1}(\tfrac{1}{\alpha}WT-W))\\
&=(\alpha I+T)^{-1}(\alpha I+W(W(W^{-1}TW^{-1}+\tfrac{1}{\alpha}I)W)^{-1}W(\tfrac{1}{\alpha}T-I))\\
&=(\alpha I+T)^{-1}(\alpha I+(\tfrac{1}{\alpha}I+W^{-1}TW^{-1})^{-1}(\tfrac{1}{\alpha}T-I))\\
&=(\alpha I+T)^{-1}(\alpha I+(I+\tfrac{1}{\alpha}W^{-1}TW^{-1})^{-1}WT^{-1}W(\tfrac{1}{\alpha}T-I))\\
&=(\alpha I+T)^{-1}(\alpha I+(I+\tfrac{1}{\alpha}U)^{-1}U(\tfrac{1}{\alpha}T-I))\\
&=(\alpha I+T)^{-1}(\alpha I+(I+\tfrac{1}{\alpha}U)^{-1}U(\tfrac{1}{\alpha}T-I))\\
&=(\alpha I+T)^{-1}(\alpha I+U)^{-1}(\alpha^2 I+UT).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\mathcal{P}_2^{-1}\mathcal{A} &= \begin{bmatrix}I-(\alpha I+T)^{-1}(\alpha I+U)^{-1}(\alpha^2 I+UT) & O\\ (T+\frac{1}{\alpha}W^2)^{-1}(W-\frac{1}{\alpha}WT) & I\end{bmatrix}\\
&= \begin{bmatrix}\alpha(\alpha I+T)^{-1}(\alpha I+U)^{-1}(U+T) & O\\ (T+\frac{1}{\alpha}W^2)^{-1}(W-\frac{1}{\alpha}WT) & I\end{bmatrix}.
\end{aligned} \tag{14}$$

From (14), we know that the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$ has an eigenvalue 1 with multiplicity at least $N$, and the remaining nonunit eigenvalues of $\mathcal{P}_2^{-1}\mathcal{A}$ satisfy the following generalized eigenvalue problem

$$\alpha(U+T)y=\lambda(\alpha I+U)(\alpha I+T)y. \tag{15}$$

Since the matrix $W\in\mathbb{R}^{N\times N}$ is symmetric indefinite and the matrix $T\in\mathbb{R}^{N\times N}$ is symmetric positive definite, then the matrix $U=WT^{-1}W$ is symmetric positive semidefinite, so $U+T$ and $\alpha I+U$ are symmetric positive definite for $\alpha>0$. It is easy to verify that all the eigenvalues of the generalized eigenvalue problem (15) are positive real numbers.

For $\forall y\in\mathbb{R}^N$, $y\neq 0$, let

$$a=\frac{(Ty,y)}{(y,y)},\quad b=\frac{(Uy,y)}{(y,y)}\quad \textit{and}\quad c=\frac{(UTy,y)}{(y,y)}, \tag{16}$$

then $a>0$ and $b\geq 0$.

Consider the following eigenvalue problem

$$TWT^{-1}Wv=\beta v,$$

where $\beta$ is an eigenvalue and $v$ is the corresponding eigenvector. Let $T=LL^T$ be the Cholesky factorization of the matrix $T$ and $z=L^{-1}v$, then we have $L^TWT^{-1}Wz=$

$\beta z$. Since the matrix $L^T W T^{-1} W L$ is symmetric positive semidefinite, then $\beta \geq 0$ and $c \geq 0$.

From (15) and (16), we can deduce $\lambda(\mathcal{P}_2^{-1}\mathcal{A}) = \frac{\alpha(a+b)}{\alpha^2+\alpha(a+b)+c}$. Finally, it follows from $\alpha > 0$, $a > 0$, $b \geq 0$, and $c \geq 0$ that $0 < \lambda(\mathcal{P}_2^{-1}\mathcal{A}) < 1$, i.e., the remaining nonunit eigenvalues of $\mathcal{P}_2^{-1}\mathcal{A}$ are located in (0, 1). □

It is very crucial to choose an optimal parameter $\alpha$ during the implementation of the preconditioner $\mathcal{P}_2$. For the HSS preconditioner, Bai [2] derived the theoretically optimal parameter $\alpha$ for solving the saddle point problems. However, it is very difficult for the preconditioner $\mathcal{P}_2$ to obtain the theoretically optimal parameter $\alpha$. Using the algebraic estimation technique given by Golub and Greif [24], we can get an estimate parameter $\alpha_{est} = \|W^2\|_2/\|T\|_2$. A practical method for determining the parameter $\alpha$ is to find $\alpha$ such that

$$\|\mathcal{R}_2\|_F = \|\mathcal{P}_2 - \mathcal{A}\|_F = \| \begin{bmatrix} \alpha I & O \\ \frac{1}{\alpha} W T & O \end{bmatrix} \|_F = min.$$

Then, we obtain the parameter $\alpha_{pra} = \left(\frac{tr(TW^2T)}{N}\right)^{\frac{1}{4}}$. However, our numerical results show that $\alpha_{est}$ and $\alpha_{pra}$ are not very effective to improve the convergence behavior of the preconditioned GMRES method. Similar to the PMHSS preconditioner [7], the parameter $\alpha$ may be determined by performing numerical experiments.

Bai and Ng [14] established some inexact preconditioners and discussed the finite-step termination properties of the corresponding preconditioned Krylov subspace method with an optimal or Galerkin property. Similar to the proposition 2.1 in [14] and following its proof, we have the following result.

**Theorem 2** *Assume that the conditions of Theorem 1 are satisfied and the preconditioner $\mathcal{P}_2$ is defined in (9). Then $\partial(\mathcal{P}_2^{-1}\mathcal{A}) \leq N + 1$, where $\partial(\mathcal{P}_2^{-1}\mathcal{A})$ denotes the degree of the minimal polynomial of the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$.*

*Proof* From (14), the preconditioned matrix can be expressed as

$$\mathcal{P}_2^{-1}\mathcal{A} = \begin{bmatrix} \Xi_1 & O \\ \Xi_2 & I \end{bmatrix}, \tag{17}$$

where $\Xi_1 = \alpha(\alpha I + T)^{-1}(\alpha I + U)^{-1}(U + T) \in \mathbb{R}^{N\times N}$ and $\Xi_2 = (T + \frac{1}{\alpha}W^2)^{-1}(W - \frac{1}{\alpha}WT) \in \mathbb{R}^{N\times N}$. Suppose that $\beta_i (i = 1, \ldots, N)$ are the eigenvalues of the matrix $\Xi_1$. Then, $\beta_i (i = 1, \ldots, N)$ are also the eigenvalues of the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$. From (17), the characteristic polynomial of the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$ is

$$(\lambda - 1)^N \prod_{i=1}^{N} (\lambda - \beta_i).$$

It is easy to compute

$$(\mathcal{P}_2^{-1}\mathcal{A} - I) \prod_{i=1}^{N} (\mathcal{P}_2^{-1}\mathcal{A} - \beta_i I) = \begin{bmatrix} (\Xi_1 - I) \prod_{i=1}^{N} (\Xi_1 - \beta_i I) & O \\ \Xi_2 \prod_{i=1}^{N} (\Xi_1 - \beta_i I) & O \end{bmatrix}.$$

Using Hamilton-Cayley theorem [30], we obtain $\prod_{i=1}^{N}(\Xi_1 - \beta_i I) = 0$. Therefore, we have $\partial(\mathcal{P}_2^{-1}\mathcal{A}) \leq N + 1$. □

It follows from Proposition 6.1 in [31] that the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}_2^{-1}\mathcal{A})$ is at most $N + 1$.

Next, we discuss the eigenvector of the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$.

**Theorem 3** *Assume that the preconditioner $\mathcal{P}_2$ is given in (9), then there are $N + j$ linear independent eigenvectors of the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$, which are described as follows:*

*(1)* *$N$ eigenvectors* $\begin{bmatrix} 0 \\ v_l \end{bmatrix}$ $(l = 1, 2, \ldots, N)$ *that correspond to the eigenvalue 1;*

*(2)* *$j(0 \leq j \leq N)$ eigenvectors* $\begin{bmatrix} u_l^1 \\ v_l^1 \end{bmatrix}$ $(1 \leq l \leq j)$ *that correspond to the nonunit eigenvalues $\lambda_l$, where $\alpha(U + T)u_l^1 = \lambda(\alpha I + U)(\alpha I + T)u_l^1, u_l^1 \neq 0$ and $v_l^1 = \frac{1}{1-\lambda}(T + \frac{1}{\alpha}W^2)^{-1}(W - \frac{1}{\alpha}WT)u_l^1$.*

*Proof* The proof is similar to that of Theorem 3.2 in [22]; hence, it is omitted. □

## 4 Numerical examples

In this section, we give some numerical examples of complex symmetric linear systems from the references [5, 6, 21] to illustrate the effectiveness and robustness of the preconditioned GMRES(♯) [31, 32] method which is combined with our proposed preconditioner $\mathcal{P}_2$ and the HSS preconditioner. All computations are carried out using double precision float point arithmetic in MATLAB (version R2010b). In our implementations, we choose the initial guess $x_0 = \mathbf{zeros}(2N, 1)$ and set the stopping criterion to be $\frac{\|r_j\|}{\|r_0\|} \leq 1.e - 6$, where $r_j = b - \mathcal{A}x_j$. Note that Its and CPU denote iteration steps and CPU time (in seconds) for computing an approximate solution, respectively. Like the PMHSS preconditioner, we choose the optimal parameter $\alpha$ by performing numerical experiments which minimize the numbers of iteration steps and computing times, see [7] for more details. $\alpha_{exp}$ denotes the optimal iteration parameter in this section. A symbol "–" is used to indicate that the method does not obtain the required stopping criterion before maximum iterations or out of memory. The maximum number of iteration steps allowed is set to 5000 for the GMRES(♯) method, and to 200 for the preconditioned GMRES(♯) method.

*Example 1* In this example, we compare the computing efficiency of the preconditioned GMRES(50) method with HSS preconditioner ($\mathcal{P}_{HSS}$) and our proposed preconditioner ($\mathcal{P}_2$). The complex symmetric linear system is of the form [38]

$$[(T_m \otimes I_m + I_m \otimes T_m - k^2h^2(I_m \otimes I_m)) + i\sigma_2(I_m \otimes I_m)]x = b,$$

**Table 1** Its and CPU for the preconditioned GMRES(50) method in Example 1

| Preconditioner | $k$ | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|---|
| | $m^2$ | $16^2$ | $32^2$ | $64^2$ | $128^2$ | $256^2$ |
| No-prec | Its | – | – | – | – | – |
| | CPU | – | – | – | – | – |
| $\mathcal{P}_{HSS}$ | Its | 6 | 5 | 11 | 26 | 67 |
| ($\alpha = 0.01$) | CPU | 0.008 | 0.021 | 0.167 | 1.887 | 27.848 |
| | $\alpha_{exp}$ | 0.01 | 0.05 | 0.001 | 0.0001 | 0.0001 |
| $\mathcal{P}_2$ | Its | 3 | 2 | 2 | 2 | 2 |
| | CPU | 0.007 | 0.015 | 0.071 | 0.384 | 1.937 |
| | $\alpha_{pra}$ | 0.038 | 0.019 | 0.010 | 0.005 | 0.002 |
| $\mathcal{P}_2$ | Its | 3 | 3 | 3 | 4 | 3 |
| | CPU | 0.007 | 0.023 | 0.087 | 0.459 | 2.369 |
| $\mathcal{P}_2$ | Its | 3 | 3 | 3 | 4 | 4 |
| ($\alpha = 0.01$) | CPU | 0.008 | 0.018 | 0.076 | 0.445 | 2.437 |

where $T_m = tridiag(-1, 2, -1)$ is a tridiagonal matrix with order $m$ and $k$ denotes the wavenumber. We choose the matrices $W = T_m \otimes I_m + I_m \otimes T_m - k^2h^2(I_m \otimes I_m)$ and $T = \sigma_2(I_m \otimes I_m)$, where $\sigma_2 = 0.1$ and $h = \frac{1}{m+1}$. $T$ is symmetric positive definite. We set the right-hand side $b = \mathcal{A} * ones(2m^2, 1)$. Table 1, Figs. 1, 2 and 3 report the numerical results.

From Table 1, we can conclude some observations as follows. Firstly, the unpreconditioned GMRES(50) method does not converge in all cases. Secondly, the two preconditioners can improve the convergence behavior of the GMRES(50) method, but the $\mathcal{P}_2$ preconditioned GMRES(50) method returns better numerical results than the HSS preconditioned GMRES(50) method in terms of Its and CPU time. Thirdly, the iteration steps of the $\mathcal{P}_2$ preconditioned GMRES(50) method are almost

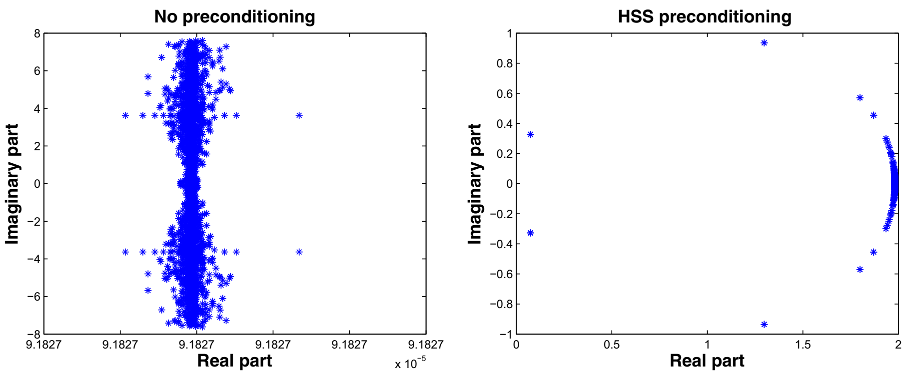


**Fig. 1** The spectral distributions of unpreconditioned matrix (*on the left*) and HSS preconditioned matrix (*on the right*)

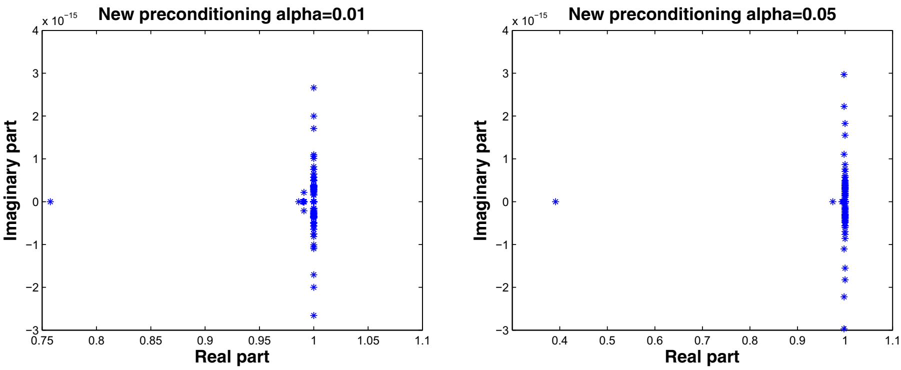


**Fig. 2** The spectral distributions of $\mathcal{P}_2$ preconditioned matrix with $\alpha = 0.01$ (*on the left*) and $\alpha = 0.05$ (*on the right*)

constant. Therefore, the $\mathcal{P}_2$ preconditioned GMRES(50) method demonstrates $h$-independent convergence behavior. The value of parameter $\alpha$ decreases as the grid size $m$ increases. Fourthly, numerical result shows that the practical parameter $\alpha_{pra}$ is effective to improve the convergence behavior of the preconditioned GMRES(50) method compared with the parameter $\alpha_{exp}$. Lastly, for $\alpha = 0.01$, the $\mathcal{P}_2$ preconditioned GMRES(50) method converges faster and requires less CPU times than the HSS preconditioned GMRES(50) method. From Figs. 1 and 2, all the eigenvalues of the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$ (with $\alpha = 0.01, 0.05$) are located in a circle centered at (1,0) with a radius strictly less than 1. We also see that the spectral distribution of the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$ is better than that of the preconditioned matrix $\mathcal{P}_{HSS}^{-1}\mathcal{A}$ and unpreconditioned matrix $\mathcal{A}$, which is consistent with the theoretical results in Theorem 1. From Fig. 3, we can observe that the HSS preconditioner is more sensitive to the parameter $\alpha$ than the $\mathcal{P}_2$ preconditioner. Therefore, our proposed preconditioner is more effective and practical for solving the complex symmetric linear systems, in comparison with the HSS preconditioner.

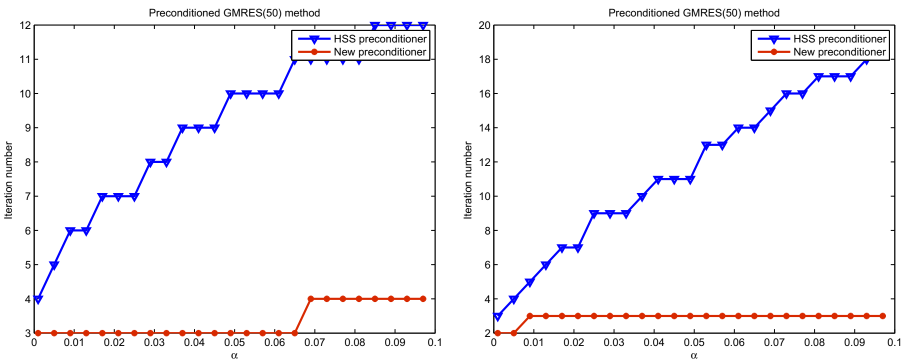


**Fig. 3** Number of iterations versus $\alpha$ with $m = 16 \times 16$ (*on the left*) and $m = 32 \times 32$ (*on the right*)

**Table 2** Its and CPU for the preconditioned GMRES(10) method in Example 2

| Preconditioner | $M$ | $5I$ | $10I$ | $20I$ | $30I$ | $50I$ |
|---|---|---|---|---|---|---|
| No-prec | Its | – | 2864 | 1221 | 744 | 385 |
| | CPU | – | 1.810 | 0.860 | 0.542 | 0.307 |
| $\mathcal{P}_{HSS}$ | Its | 23 | 24 | 28 | 31 | 38 |
| $(\alpha = 0.01)$ | CPU | 0.071 | 0.079 | 0.084 | 0.090 | 0.128 |
| | $\alpha_{exp}$ | 0.2 | 0.1 | 0.09 | 0.19 | 0.19 |
| $\mathcal{P}_2$ | Its | 9 | 9 | 8 | 7 | 6 |
| | CPU | 0.034 | 0.038 | 0.031 | 0.028 | 0.025 |
| | $\alpha_{pra}$ | 0.747 | 0.774 | 0.817 | 0.849 | 0.883 |
| $\mathcal{P}_2$ | Its | 11 | 15 | 18 | 16 | 12 |
| | CPU | 0.040 | 0.067 | 0.071 | 0.062 | 0.043 |
| $\mathcal{P}_2$ | Its | 21 | 18 | 17 | 16 | 18 |
| $(\alpha = 0.01)$ | CPU | 0.057 | 0.051 | 0.049 | 0.048 | 0.050 |

*Example 2* We consider the following complex symmetric linear system [5, 6, 35]

$$[(-\omega^2\mathbf{M} + \mathbf{K}) + i(\omega C_{\mathbf{V}} + C_{\mathbf{H}})]x = b.$$

where $M$ and $K$ are the inertia and stiffness matrices, $C_{\mathbf{V}}$ and $C_{\mathbf{H}}$ are the viscous and hysteretic damping matrices, respectively, $\omega$ is the driving circular frequency, $K = I \otimes V_m + V_m \otimes I$, $V_m = h^{-2}tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$ is a tridiagonal matrix, $h = \frac{1}{m+1}$, $C_{\mathbf{V}} = \frac{1}{2}M$, $C_{\mathbf{H}} = \mu K$ with $\mu$ being a damping coefficient.

We choose the matrices $W = h^2(-\omega^2\mathbf{M} + \mathbf{K})$ and $T = h^2(\omega C_{\mathbf{V}} + C_{\mathbf{H}})$, and set $\omega = 2\pi$ and $\mu = 0.02$. For $M = 5I, 10I, 20I, 30I, 50I$, we can easily show that the matrix $W$ is symmetric indefinite and the matrix $T$ is symmetric positive definite. In this example, we set $m = 32$ and the right-hand side $b = \mathcal{A} * ones(2m^2, 1)$.

As observed from Table 2, we can see that the GMRES(10) method does not converge in case of $M = 5I$. The HSS preconditioned GMRES(10) method and the $\mathcal{P}_2$ preconditioned GMRES(10) method return better convergence behavior than

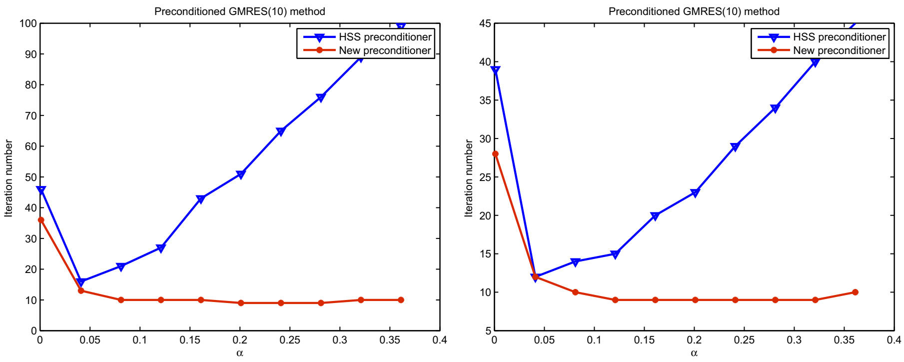


**Fig. 4** Number of iterations versus $\alpha$ with $M = 5I$ (*on the left*) and $M = 10I$ (*on the right*)

the unpreconditioned GMRES(10) method. Furthermore, for the $\mathcal{P}_2$ preconditioned GMRES(10) method, we also observe that the change of iteration steps is relatively stable with the inertia matrix $M$ changing. For the parameter $\alpha = 0.01$, the $\mathcal{P}_2$ preconditioned GMRES(10) method requires less iteration steps and CPU times than the HSS preconditioned GMRES(10) method. Numerical result shows that the practical parameter $\alpha_{pra}$ is not effective to improve the convergence behavior of the preconditioned GMRES(10) method compared with the parameter $\alpha_{exp}$. From Fig. 4, we can obtain the same result as that of Fig. 3.

*Example 3* We consider the following complex symmetric linear system [21]

$$[(K - (3 - \sqrt{3})\omega^2 I) + i(K + (3 + \sqrt{3})\tau^2 I)]x = b,$$

where $K = I \otimes V_m + V_m \otimes I$, $\omega = 10\pi$, $\tau = 2\pi$, $h = \frac{1}{m+1}$, $n = m^2$ and $V_m = h^{-2} tridiag(-1, 2, -1) \in \mathbb{R}^{m \times m}$ is a tridiagonal matrix. We choose the symmetric indefinite matrix $W = K - (3 - \sqrt{3})\omega^2 I$ and the symmetric positive definite matrix $T = K + (3 + \sqrt{3})\tau^2 I$.

From Table 3, for $m \geq 64$ we can see that the GMRES(20) method does not converge. Both the HSS preconditioner and the $\mathcal{P}_2$ preconditioner improve computing efficiency of the GMRES(20) method, and the $\mathcal{P}_2$ preconditioned GMRES(20) method returns better numerical results than the HSS preconditioned GMRES(20) method in all cases. The iteration steps of the $\mathcal{P}_2$ preconditioned GMRES(20) method increase slowly with the grid size $m$ increasing, but the amplitude is relatively stable. Numerical result shows that the practical parameter $\alpha_{pra}$ is not effective to improve the convergence behavior of the preconditioned GMRES(20) method compared with the parameter $\alpha_{exp}$. From Fig. 5, we can obtain the same results as those of Figs. 3 and 4.

**Table 3** Its and CPU for the preconditioned GMRES(20) method in Example 3

| Preconditioner | $m$ | 16 | 32 | 64 | 128 | 256 |
|---|---|---|---|---|---|---|
| No-prec | Its | 105 | 1657 | – | – | – |
| | CPU | 0.041 | 1.220 | – | – | – |
| $\mathcal{P}_{HSS}$ | Its | 24 | 26 | 29 | 77 | 181 |
| $(\alpha = 1)$ | CPU | 0.026 | 0.086 | 0.501 | 6.817 | 90.069 |
| $\mathcal{P}_2$ | Its | 8 | 11 | 11 | 19 | 36 |
| $(\alpha = 1)$ | CPU | 0.009 | 0.040 | 0.199 | 1.608 | 16.009 |
| | $\alpha_{exp}$ | 2.54 | 1.97 | 1.03 | 0.49 | 0.24 |
| $\mathcal{P}_2$ | Its | 6 | 8 | 11 | 14 | 20 |
| | CPU | 0.007 | 0.030 | 0.194 | 1.246 | 9.580 |
| | $\alpha_{pra}$ | 3.10 | 4.63 | 4.97 | 5.07 | 5.09 |
| $\mathcal{P}_2$ | Its | 7 | 11 | 21 | 43 | 89 |
| | CPU | 0.009 | 0.042 | 0.478 | 3.941 | 41.54 |
| $\mathcal{P}_{HSS}$ | Its | 10 | 18 | 30 | 48 | 77 |
| $(\alpha = \alpha_{exp})$ | CPU | 0.013 | 0.063 | 0.508 | 4.370 | 38.491 |

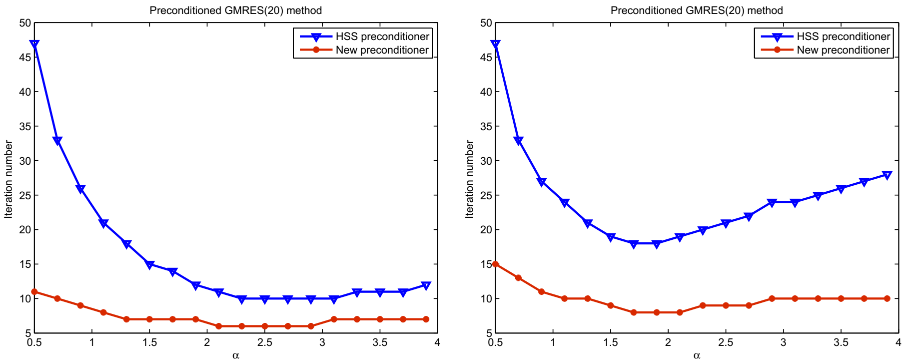


**Fig. 5** Number of iterations versus $\alpha$ with $m = 16$ (*on the left*) and $m = 32$ (*on the right*)

## 5 Conclusion

In this paper, we have developed a new block preconditioner for solving a class of complex symmetric indefinite linear systems. Theoretical properties of the new preconditioner have been studied in detail. Numerical results show that the new preconditioner is more effective than the HSS preconditioner in improving the convergence behavior of the restarted GMRES method. How to select an optimal and practical parameter $\alpha$ should be investigated in the future.

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## References

1. Arridge, S.R.: Optical tomography in medical imaging. Inverse Probl. **15**, R41–R93 (1999)
2. Bai, Z.-Z.: Optimal parameters in the HSS-like methods for saddle-point problems. Numer. Linear Algebra Appl. **16**, 447–479 (2009)
3. Bai, Z.-Z.: Rotated block triangular preconditioning based on PMHSS. Sci. China Math. **56**, 2523–2538 (2013)
4. Bai, Z.-Z.: On preconditioned iteration methods for complex linear systems. J. Eng. Math. **93**, 41–60 (2015)
5. Bai, Z.-Z., Benzi, M., Chen, F.: Modified HSS iteration methods for a class of complex symmetric linear systems. Computing. **87**, 93–111 (2010)
6. Bai, Z.-Z., Benzi, M., Chen, F.: On preconditioned MHSS iteration methods for complex symmetric linear systems. Numer. Algor. **56**, 297–317 (2011)
7. Bai, Z.-Z., Benzi, M., Chen, F., Wang, Z.-Q.: Preconditioned MHSS iteration methods for a class of block two-by-two linear systems with applications to distributed control problems. IMA J. Numer. Anal. **33**, 343–369 (2013)
8. Bai, Z.-Z., Golub, G.H.: Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems. IMA J. Numer. Anal. **27**, 1–23 (2007)

9. Bai, Z.-Z., Golub, G.H., Lu, L.-Z., Yin, J.-F.: Block triangular and skew-Hermitian splitting method for positive definite linear systems. SIAM J. Sci. Comput. **26**, 844–863 (2005)
10. Bai, Z.-Z., Golub, G.H., Ng, M.K.: Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. SIAM J. Matrix Anal. Appl. **24**, 603–626 (2003)
11. Bai, Z.-Z., Golub, G.H., Ng, M.K.: On successive-overrelaxation acceleration of the Hermitian and skew-Hermitian splitting iterations. Numer. Linear Algebra Appl. **14**, 319–335 (2007)
12. Bai, Z.-Z., Golub, G.H., Ng, M.K.: On inexact Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. Linear Algebra Appl. **428**, 413–440 (2008)
13. Bai, Z.-Z., Golub, G.H., Pan, J.-Y.: Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems. Numer. Math. **98**, 1–32 (2004)
14. Bai, Z.-Z., Ng, M.K.: On inexact preconditioners for nonsymmetric matrices. SIAM J. Sci. Comput. **26**, 1710–1724 (2005)
15. Bai, Z.-Z., Parlett, B.N., Wang, Z.-Q.: On generalized successive overrelaxation methods for augmented linear systems. Numer. Math. **102**, 1–38 (2005)
16. Bai, Z.-Z., Wang, Z.-Q.: On parameterized inexact Uzawa methods for generalized saddle point problems. Linear Algebra Appl. **428**, 2900–2932 (2008)
17. Bao, G., Sun, W.-W.: A fast algorithm for the electromagnetic scattering from a large cavity. SIAM J. Sci. Comput. **27**, 553–574 (2005)
18. Benzi, M.: A generalization of the Hermitian and skew-Hermitian splitting iteration. SIAM J. Matrix Anal. Appl. **31**, 360–374 (2009)
19. Benzi, M., Bertaccini, D.: Block preconditioning of real-valued iterative algorithms for complex linear systems. IMA J. Numer. Anal. **28**, 598–618 (2008)
20. Cao, Y., Dong, J.-L., Wang, Y.-M.: A relaxed deteriorated PSS preconditioner for nonsymmetric saddle point problems from the steady Navier–Stokes equation. J. Comput. Appl. Math. **273**, 41–60 (2015)
21. Cao, Y., Ren, Z.-R.: Two variants of the PMHSS iteration method for a class of complex symmetric indefinite linear systems. Appl. Math. Comput. **264**, 61–71 (2015)
22. Cao, Y., Ren, Z.-R., Shi, Q.: A simplified HSS preconditioner for generalized saddle point problems. BIT Numer. Math. **56**, 423–439 (2016)
23. Cao, Y., Yao, L.-Q., Jiang, M.-Q., Niu, Q.: A relaxed HSS preconditioner for saddle point problems from meshfree discretization. J. Comput. Math. **31**, 398–421 (2013)
24. Golub, G.H., Greif, C.: On solving block-structured indefinite linear systems. SIAM J. Sci. Comput. **24**, 2076–2092 (2003)
25. Hezari, D., Edalatpour, V., Salkuyeh, D.K.: Preconditioned GSOR iterative method for a class of complex symmetric system of linear equations. Numer. Linear Algebra Appl. **22**, 761–776 (2015)
26. Lang, C., Ren, Z.-R.: Inexact rotated block triangular preconditioners for a class of block two-by-two matrices. J. Eng. Math. **93**, 87–98 (2015)
27. Li, X., Yang, A.-L., Wu, Y.-J.: Lopsided PMHSS iteration method for a class of complex symmetric linear systems. Numer. Algor. **66**, 555–568 (2014)
28. Liang, Z.-Z., Zhang, G.-F.: On SSOR iteration method for a class of block two-by-two linear systems. Numer. Algor. **71**, 655–671 (2016)
29. Pan, J.-Y., Ng, M.K., Bai, Z.-Z.: New preconditioners for saddle point problems. Appl. Math. Comput. **172**, 762–771 (2006)
30. Parlett, B.N.: The Symmetric Eigenvalue Problem. SIAM, Philadelphia (1998)
31. Saad, Y. Iterative Methods for Sparse Linear Systems, 2nd edn. SIAM, Philadelphia (2003)
32. Saad, Y., Schultz, M.H.: GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems. SIAM J. Sci. Stat. Comput. **7**, 856–869 (1986)
33. Salkuyeh, D.K., Hezari, D., Edalatpour, V.: Generalized successive overrelaxation iterative method for a class of complex symmetric linear system of equations. Int. J. Comput. Math. **92**, 802–815 (2015)
34. van Dijk, W., Toyama, F.M.: Accurate numerical solutions of the time-dependent Schrödinger equation. Phys. Rev. E. **75**, 036707-1–036707-10 (2007)
35. Wu, S.-L.: Several variants of the Hermitian and skew-Hermitian splitting method for a class of complex symmetric linear systems. Numer. Linear Algebra Appl. **22**, 338–356 (2015)
36. Xu, W.-W.: A generalization of preconditioned MHSS iteration method for complex symmetric indefinite linear systems. Appl. Math. Comput. **219**, 10510–10517 (2013)
37. Yan, H.-Y., Huang, Y.-M.: Splitting-based block preconditioning methods for block two-by-two matrices of real square blocks. Appl. Math. Comput. **243**, 825–837 (2014)

38. Zhang, J.-H., Dai, H.: A new splitting preconditioner for the iterative solution of complex symmetric indefinite linear systems. Appl. Math. Lett. **49**, 100–106 (2015)
39. Zhang, G.-F., Zheng, Z.: A parameterized splitting iteration method for complex symmetric linear systems. J.pan J. Indust. Appl. Math. **31**, 265–278 (2014)
40. Zheng, Q.-Q., Ma, C.-F.: Accelerated PMHSS iteration methods for complex symmetric linear systems. Numer. Algor. in press (2016)