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ORIGINAL PAPER

An improvement to double-step Newton method and its multi-step version for solving system of nonlinear equations and its applications

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Abstract In this work, we have improved the order of the double-step Newton method from four to five using the same number of evaluation of two functions and two first order Fréchet derivatives for each iteration. The multi-step version requires one more function evaluation for each step. The multi-step version converges with order $3r + 5$, $r \geq 1$. Numerical experiments are done comparing the new methods with some existing methods. Our methods are also tested on Chandrasekhar's problem and the 2-D Bratu problem to illustrate the applications.

Keywords System of nonlinear equation · Newton's method · Order of convergence · Multi-step method · Fréchet derivatives

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1 Introduction

The construction of iterative methods for approximating the solution of systems of nonlinear equations is an important and interesting task in numerical analysis and

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applied scientific branches. In order to solve many application problems, one need to find an approximate solution of system of nonlinear equations $F(x) = 0$, where $F(x) = (f_1(x), f_2(x), ..., f_n(x))^T$, $x = (x_1, x_2, ..., x_n)^T$, $f_i : \mathbb{R}^n \to \mathbb{R}, \forall i = 1, 2, \ldots, n$, and $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map and D is an open and convex set, where we assume that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ is a zero of the systems and $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}\right)^T$ is an initial guess sufficiently close to α. One of the basic procedure for solving system of nonlinear equations is the classical one-step second order Newton method ($2NR$) [14]. It is defined by

$$x^{(k+1)} = G_{2NR}(x^{(k)}) = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}),\ k = 0, 1, 2, ... \tag{1}$$

where $[F'(x^{(k)})]^{-1}$ is the inverse of the first Fréchet derivative $F'(x^{(k)})$ of the function $F(x^{(k)})$. It is straightforward to see that this method requires the evaluation of one function, one first derivative, and one matrix inversion per iteration. Traub [16] suggested that multi-step iterative methods are better way to improve the order of convergence free from second derivatives, such modifications of Newton's method have been proposed in the literature; for example see [1, 4, 6, 11, 15] and references therein. The double-step third and fourth order Newton's methods have been proposed in the recent literature, see [9–11, 15].

Traub [16] proposed a two-step variant of Newton's method ($3TM$) having convergence order three by evaluating two functions, one Fréchet derivative, and its inverse for

$$x^{(k+1)} = G_{3TM}(x^{(k)}) = G_{2NR}(x^{(k)}) - [F'(x^{(k)}))]^{-1} F(G_{2NR}(x^{(k)})). \tag{2}$$

The double-step fourth order Newton method ($4NR$) is given by

$$x^{(k+1)} = G_{4NR}(x^{(k)}) = G_{2NR}(x^{(k)}) - [F'(G_{2NR}(x^{(k)}))]^{-1} F(G_{2NR}(x^{(k)})), \tag{3}$$

which was recently rediscovered by Noor et al. [11] using the variational iteration technique, where two functions, two Fréchet derivatives, and their inverse were evaluated. Recently, Abad et al. [1] combined the Newton and Traub methods to obtain a three-step fourth order method ($4ACT$), where two functions, two Fréchet derivatives, and their inverse were evaluated

$$x^{(k+1)} = G_{4ACT}(x^{(k)}) = G_{2NR}(x^{(k)}) - [F'(G_{3TM}(x^{(k)}))]^{-1} F(G_{2NR}(x^{(k)})). \tag{4}$$

Again in [1], a different combination to get a three-step fifth order method ($5ACT$), where three functions, two Fréchet derivatives, and their inverse were evaluated for

$$x^{(k+1)} = G_{5ACT}(x^{(k)}) = G_{3TM}(x^{(k)}) - [F'(G_{2NR}(x^{(k)}))]^{-1} F(G_{3TM}(x^{(k)})). \tag{5}$$

In this paper, we have proposed a two-step fifth order method which is an improvement over the double-step Newton method, which uses two functions and two Fréchet derivative evaluations and only one inverse. A multi-step version with order $3r + 5$, $r \geq 1$ for solving a system of nonlinear equations is also suggested which uses one more additional functional evaluation only for each step. The rest of this paper is organized as follows. In Section 2, we present new algorithms one having fifth order and the other a multi-step version having order $3r+5$, $r \geq 1$. In Section 3, we study the convergence analysis of the new methods. In Section 4, numerical examples and their results are discussed comparing with some existing methods. In Section 5, two application problems are solved using the present method and some existing methods. A brief conclusion is given in Section 6.

2 Development of the methods

We propose the following two-step method ($5MBJ$):

$$\begin{aligned} x^{(k+1)} &= G_{5MBJ}(x^{(k)}) = G_{2NR}(x^{(k)}) - H_1(x^{(k)})[F'(x^{(k)})]^{-1}F(G_{2NR}(x^{(k)})), \\ H_1(x^{(k)}) &= 2I - \tau(x^{(k)}) + \frac{5}{4}(\tau(x^{(k)}) - I)^2, \\ \tau(x^{(k)}) &= [F'(x^{(k)})]^{-1}F'\left(G_{2NR}(x^{(k)})\right), \end{aligned} \tag{6}$$

where I is the $n \times n$ identity matrix. This method uses two functions and two Fréchet derivative evaluations and only one inverse. We will show that this method is fifth order. We further improve the $5MBJ$ method by an additional function evaluation to get the multi-step version called $(3r+5)MBJ$ ($r \geq 1$) method and it is given by

$$\begin{aligned} x^{(k+1)} &= G_{(3r+5)MBJ}(x^{(k)}) = \mu_r(x^{(k)}), \\ \mu_j(x^{(k)}) &= \mu_{j-1}(x^{(k)}) - H_2(x^{(k)})[F'(x^{(k)})]^{-1}F(\mu_{j-1}(x^{(k)})), \\ H_2(x^{(k)}) &= 2I - \tau(x^{(k)}) + \frac{3}{2}(\tau(x^{(k)}) - I)^2, \\ \mu_0(x^{(k)}) &= G_{5MBJ}(x^{(k)}), \quad j = 1, 2, ..., r, \ r \geq 1. \end{aligned} \tag{7}$$

This multi-step version has order $3r+5$, $r \geq 1$. The case $r = 0$ is the $5MBJ$ method given in (6).

3 Convergence analysis

The main theorem is going to be demonstrated by means of the n-dimensional Taylor expansion of the functions involved. In the following, we use certain notations and results found in [6]:

Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently Fréchet differentiable in D. Suppose the qth derivative of F at $u \in \mathbb{R}^n$, $q \geq 1$, is the q-linear function $F^{(q)}(u) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $F^{(q)}(u)(v_1, \ldots, v_q) \in R^n$. Given $\alpha + h \in \mathbb{R}^n$, which lies in a neighborhood of a solution α of the nonlinear system $F(x) = 0$, Taylor's expansion can be applied (assuming Fréchet derivatives $F'(\alpha)$ are nonsingular) to obtain

$$F(\alpha + h) = F'(\alpha)\left[h + \sum_{q=2}^{p-1} C_q h^q\right] + O(h^p) \tag{8}$$

where $C_q = (1/q!)[F'(\alpha)]^{-1}F^{(q)}(\alpha)$, $q \geq 2$. It is noted that $C_q h^q \in \mathbb{R}^n$ since $F^{(q)}(\alpha) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $[F'(\alpha)]^{-1} \in \mathcal{L}(\mathbb{R}^n)$. Also, we can expand $F'(\alpha + h)$ in Taylor series

$$F'(\alpha + h) = F'(\alpha)\left[I + \sum_{q=2}^{p-1} qC_q h^{q-1}\right] + O(h^p) \tag{9}$$

where I is the identity matrix. It is also noted that $qC_q h^{q-1} \in \mathcal{L}(\mathbb{R}^n)$. Denote $e^{(k)} = x^{(k)} - \alpha$, so the error at the $(k+1)$th iteration is $e^{(k+1)} = Le^{(k)^p} + O(e^{(k)^{p+1}})$, where L is a p-linear function $L \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ called the *error equation* and p is the *order of convergence*. Observe that $e^{(k)^p}$ is $(e^{(k)}, e^{(k)}, \cdots, e^{(k)})$.

In order to prove the convergence order for the method (6), we need to recall some important definitions and results from the theory of point of attraction.

Definition 3.01 (**Point of Attraction**) [13] Let $G : D \subset \mathbb{R}^n \to \mathbb{R}^n$. Then, α is a point of attraction of the iteration

$$x^{(k+1)} = G(x^{(k)}),\ k = 0, 1, ... \tag{10}$$

if there is an open neighborhood S of α defined by

$$S(\alpha) = \{x \in \mathbb{R}^n \,\big|\, \|x - \alpha\| < \delta\},\ \delta > 0,$$

such that $S \subset D$ and, for any $x^{(0)} \in S$, the iterates $\{x^{(k)}\}$ defined by (10) all lie in D and converge to α.

Theorem 3.01 (**Ostrowski Theorem**) [14] *Assume that $G : D \subset \mathbb{R}^n \to \mathbb{R}^n$ has a fixed point $\alpha \in int(D)$ and $G(x)$ is Fréchet differentiable on α. If*

$$\rho(G'(\alpha)) = \sigma < 1 \tag{11}$$

then α is a point of attraction for $x^{(k+1)} = G(x^{(k)})$.

We state below a general result which has been proved in Babajee et al. [4], showing that α is a point of attraction for a general iteration function $G(x) = P(x) - Q(x)R(x)$.

Theorem 3.02 (**Babajee et al. Theorem**) [4] *Let $F : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently Fréchet differentiable at each point of an open convex neighborhood D of $\alpha \in D$, which is a solution of the system $F(x) = 0$. Suppose that $P, Q, R : D \subset \mathbb{R}^n \to \mathbb{R}^n$ are sufficiently Fréchet differentiable functionals (depending on F) at each point in D with $P(\alpha) = \alpha$, $Q(\alpha) \neq 0$, $R(\alpha) = 0$.*

Then, there exists a ball

$$S = \overline{S}(\alpha, \delta) = \Big\{\|\alpha - x\| \leq \delta\Big\} \subset S_0,\ \delta > 0,$$

on which the mapping

$$G : S \to \mathbb{R}^n, \quad G(x) = P(x) - Q(x)R(x),\ \forall x \in\ S$$

is well-defined; moreover, G is Fréchet differentiable at α, thus

$$G'(\alpha) = P'(\alpha) - Q(\alpha)R'(\alpha).$$

Theorem 3.03 *Let $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently Fréchet differentiable at each point of an open convex neighborhood D of $\alpha \in \mathbb{R}^n$, that is a solution of the system $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular in α, and $x^{(0)}$ close enough to α. Then, the sequence $\{x^{(k)}\}_{k\geq 0}$ obtained using the iterative expression (6) converges locally to α with order 5, where the error equation obtained is*

$$\begin{aligned} e^{(k+1)} &= G_{5MBJ}(x^{(k)}) - \alpha = L_1 {e^{(k)}}^5 + O({e^{(k)}}^6), \\ L_1 &= 14C_2^4 + \frac{1}{2}C_2C_3C_2 + 12C_2^2C_3 - \frac{27}{2}C_3C_2^2 \end{aligned} \tag{12}$$

Proof We first show that α is a point of attraction using Theorem 3.02. In this case,

$$P(x) = G_{2NR}(x), \quad Q(x) = H_1(x)[F'(x)]^{-1}, \quad R(x) = F(G_{2NR}(x)).$$

Now, since $F(\alpha) = 0$, we have

$$\begin{aligned} G_{2NR}(\alpha) &= \alpha - [F'(\alpha)]^{-1}F(\alpha) = \alpha, \\ \tau(\alpha) &= F'(\alpha)^{-1}F'(G_{2NR}(\alpha)) = [F'(\alpha)]^{-1}F'(\alpha) = I, \quad H_1(\alpha) = I, \\ P(\alpha) &= G_{2NR}(\alpha), \quad P'(\alpha) = G'_{2NR}(\alpha) = 0, \\ Q(\alpha) &= H_1(\alpha)[F'(\alpha)]^{-1} = I[F'(\alpha)]^{-1} = [F'(\alpha)]^{-1} \neq 0, \\ R(\alpha) &= F(G_{2NR}(\alpha)) = F(\alpha) = 0, \\ R'(\alpha) &= F'(G_{2NR}(\alpha))G'_{2NR}(\alpha) = 0, \end{aligned}$$

$$G'(\alpha) = P'(\alpha) - Q(\alpha)R'(\alpha) = 0,$$

so that $\rho(G'(\alpha)) = 0 < 1$ and by Ostrowski's theorem, α is a point of attraction of 6.

From (8) and (9), we obtain

$$F(x^{(k)}) = F'(\alpha)\left[e^{(k)} + C_2{e^{(k)}}^2 + C_3{e^{(k)}}^3 + C_4{e^{(k)}}^4 + C_5{e^{(k)}}^5\right] + O({e^{(k)}}^6), \tag{13}$$

and

$$F'(x^{(k)}) = F'(\alpha)\left[I + 2C_2e^{(k)} + 3C_3{e^{(k)}}^2 + 4C_4{e^{(k)}}^3 + 5C_5{e^{(k)}}^4\right] + O({e^{(k)}}^5). \tag{14}$$

We have

$$[F'(x^{(k)})]^{-1} = \left[I + X_1e^{(k)} + X_2{e^{(k)}}^2 + X_3{e^{(k)}}^3 + X_4{e^{(k)}}^4\right][F'(\alpha)]^{-1} + O({e^{(k)}}^5), \tag{15}$$

where $X_1 = -2C_2$, $X_2 = 4C_2^2 - 3C_3$, $X_3 = -8C_2^3 + 6C_2C_3 + 6C_3C_2 - 4C_4$, and $X_4 = -5C_5 + 9C_3^2 + 8C_2C_4 + 8C_4C_2 + 16C_2^4 - 12C_2^2C_3 - 12C_3C_2^2 - 12C_2C_3C_2$. Then,

$$\begin{aligned} [F'(x^{(k)})]^{-1}F(x^{(k)}) = {} & e^{(k)} - C_2{e^{(k)}}^2 + 2(C_2^2 - C_3){e^{(k)}}^3 \\ & +\Big(-3C_4 - 4C_2^3 + 4C_2C_3 + 3C_3C_2\Big){e^{(k)}}^4 \\ & +\Big(6C_3^2 + 8C_2^4 - 8C_2^2C_3 - 6C_2C_3C_2 - 6C_3C_2^2 \\ & +6C_2C_4 + 4C_4C_2 - 4C_5\Big){e^{(k)}}^5 + O({e^{(k)}}^6). \end{aligned} \tag{16}$$

Also, we have

$$\begin{aligned} G_{2NR}(x^{(k)}) = {} & \alpha + C_2{e^{(k)}}^2 + 2\Big(-C_2^2 + C_3\Big){e^{(k)}}^3 + \Big(3C_4 + 4C_2^3 \\ & -4C_2C_3 - 3C_3C_2\Big){e^{(k)}}^4 + \Big(-6C_3^2 - 8C_2^4 + 8C_2^2C_3 \\ & +6C_2C_3C_2 + 6C_3C_2^2 - 6C_2C_4 - 4C_4C_2 + 4C_5\Big){e^{(k)}}^5. \end{aligned} \tag{17}$$

Expanding $F(G_{2NR}(x^{(k)}))$ and $F'(G_{2NR}(x^{(k)}))$ about α in Taylor's series respectively given below

$$\begin{aligned} F(G_{2NR}(x^{(k)})) = {} & F'(\alpha)\Big[(G_{2NR}(x^{(k)}) - \alpha) + C_2(G_{2NR}(x^{(k)}) - \alpha)^2 \\ & +C_3(G_{2NR}(x^{(k)}) - \alpha)^3 + ...\Big] \\ = {} & F'(\alpha)\Big[C_2{e^{(k)}}^2 + 2(-C_2^2 + C_3){e^{(k)}}^3 \\ & +\Big(3C_4 + 5C_2^3 - 4C_2C_3 - 3C_3C_2\Big){e^{(k)}}^4 \\ & +\Big(-6C_3^2 - 12C_2^4 + 10C_2^2C_3 + 8C_2C_3C_2 \\ & +6C_3C_2^2 - 6C_2C_4 \\ & -4C_4C_2 + 4C_5\Big){e^{(k)}}^5\Big], \end{aligned} \tag{18}$$

$$\begin{aligned} F'(G_{2NR}(x^{(k)})) = {} & F'(\alpha)\Big[I + 2C_2(G_{2NR}(x^{(k)}) - \alpha) \\ & +3C_3(G_{2NR}(x^{(k)}) - \alpha)^2 + ...\Big] \\ = {} & F'(\alpha)\Big[I + P_1{e^{(k)}}^2 + P_2{e^{(k)}}^3 + P_3{e^{(k)}}^4\Big] + O({e^{(k)}}^5), \end{aligned} \tag{19}$$

where

$$P_1 = 2C_2^2,\ P_2 = 4C_2C_3 - 4C_2^3,\ and\ P_3 = 8C_2^4 + 6C_2C_4 - 8C_2^2C_3 + 3C_3C_2^2 - 6C_2C_3C_2.$$

Using (15) and (19), we have

$$\begin{aligned}[F'(x^{(k)})]^{-1}F'(G_{2NR}(x^{(k)})) \;=\;& I-2C_2e^{(k)}+\Big(6C_2^2-3C_3\Big)e^{(k)^2}\\ &+\Big(10C_2C_3+6C_3C_2-16C_2^3-4C_4\Big)e^{(k)^3}\\ &+\Big(-5C_5+9C_3^2+40C_2^4+14C_2C_4+8C_4C_2\\ &-28C_2^2C_3-15C_3C_2^2-18C_2C_3C_2\Big)e^{(k)^4}\\ &+O(e^{(k)^5}).\end{aligned} \tag{20}$$

Then

$$\begin{aligned}H_1(x^{(k)}) \;=\;& 2I-\tau(x^{(k)})+\frac{5}{4}(\tau(x^{(k)})-I)^2\\ =\;& I+2C_2e^{(k)}-\Big(C_2^2-3C_3\Big)e^{(k)^2}+\Big(-\frac{5}{2}C_2C_3+\frac{3}{2}C_3C_2\\ &-14C_2^3+4C_4\Big)e^{(k)^3}\\ &+O(e^{(k)^4}).\end{aligned} \tag{21}$$

Using (15) and (18), we have

$$\begin{aligned}[F'(x^{(k)})]^{-1}F(G_{2NR}(x^{(k)})) \;=\;& C_2e^{(k)^2}+\Big(2C_3-4C_2^2\Big)e^{(k)^3}\\ &+\Big(13C_2^3-8C_2C_3-6C_3C_2+3C_4\Big)e^{(k)^4}\\ &+\Big(-12C_3^2-38C_2^4+26C_2^2C_3+20C_2C_3C_2\\ &+18C_3C_2^2-12C_2C_4-8C_4C_2+4C_5\Big)e^{(k)^5}\\ &+O(e^{(k)^6}).\end{aligned} \tag{22}$$

Then

$$\begin{aligned}&H_1(x^{(k)})[F'(x^{(k)})]^{-1}F(G_{2NR}(x^{(k)}))\\ &=C_2e^{(k)^2}+(2C_3-2C_2^2)e^{(k)^3}+(3C_4+4C_2^3-4C_2C_3-3C_3C_2)e^{(k)^4}\\ &+\Big(-6C_3^2+4C_5-6C_2C_4-4C_4C_2-22C_2^4+\frac{11}{2}C_2C_3C_2+8C_2^2C_3\\ &+\frac{39}{2}C_3C_2^2-12C_2^2C_3\Big)e^{(k)^5}+O(e^{(k)^6}).\end{aligned} \tag{23}$$

Using (17) and (23) in (6), we have

$$e^{(k+1)}=(14C_2^4+\frac{1}{2}C_2C_3C_2+12C_2^2C_3-\frac{27}{2}C_3C_2^2)e^{(k)^5}+O(e^{(k)^6}), \tag{24}$$

which proves fifth order convergence. □

Theorem 3.04 *Let $F:D\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be sufficiently Fréchet differentiable at each point of an open convex neighborhood D of $\alpha\in\mathbb{R}^n$ that is a solution of the system $F(x)=0$. Let us suppose that $x\in S=\overline{S}(\alpha,\delta)$ and $F'(x)$ is continuous* *and nonsingular in* α*, and* $x^{(0)}$ *is close enough to* α*. Then* α *is a point of attraction of the sequence* $\{x^{(k)}\}$ *obtained using the iterative expression (7). Furthermore, the sequence converges locally to* α *with order* $3r+5$*, where* r *is a positive integer and* $r \geq 1$.

Proof In this case,

$$P(x) = \mu_{j-1}(x), \quad Q(x) = H_2(x)F(x)^{-1}, \quad R(x) = F(\mu_{j-1}(x)),\ j = 1, ..., r.$$

We can show by induction that

$$\mu_{j-1}(\alpha) = \alpha, \quad \mu'_{j-1}(\alpha) = 0, \quad \forall\, j = 1, ..., r$$

so that

$$\begin{aligned}
&P(\alpha) = \mu_{j-1}(\alpha) = \alpha,\ H_2(\alpha) = I,\ Q(\alpha) = I[F'(\alpha)]^{-1} = [F'(\alpha)]^{-1} \neq 0,\\
&R(\alpha) = F(\mu_{j-1}(\alpha)) = F(\alpha) = 0,\\
&P'(\alpha) = \mu'_{j-1}(\alpha) = 0,\ R'(\alpha) = F'(\mu_{j-1}(\alpha))\mu'_{j-1}(\alpha) = 0,\\
&G'(\alpha) = P'(\alpha) - Q(\alpha)R'(\alpha) = 0.
\end{aligned}$$

So $\rho(G'(\alpha)) = 0 < 1$ and by Ostrowski's theorem, α is a point of attraction of (7). A Taylor expansion of $F(\mu_{j-1}(x^{(k)}))$ about α yields

$$F(\mu_{j-1}(x^{(k)})) = F'(\alpha)\left[(\mu_{j-1}(x^{(k)}) - \alpha) + C_2(\mu_{j-1}(x^{(k)}) - \alpha)^2 + ...\right] \tag{25}$$

Also, let

$$\begin{aligned}
H_2(x^{(k)}) &= 2I - \tau(x^{(k)}) + \frac{3}{2}(\tau(x^{(k)}) - I)^2 \\
&= I + 2C_2e^{(k)} + 3C_3{e^{(k)}}^2 + \left(-C_2C_3 - 20C_2^3 + 3C_3C_2 + 4C_4\right){e^{(k)}}^3 + ...
\end{aligned} \tag{26}$$

Using (15) and (26), we have

$$H_2(x^{(k)})[F'(x^{(k)})]^{-1} = \left[I + L_2\, {e^{(k)}}^3 + ...\right][F'(\alpha)]^{-1}, \quad L_2 = -20C_2^3 - C_2C_3 + 3C_3C_2 \tag{27}$$

Using (27) and (25), we have

$$\begin{aligned}
H_2(x^{(k)})[F'(x^{(k)})]^{-1}F(\mu_{j-1}(x^{(k)})) &= \left(I + L_2\, {e^{(k)}}^3 + ...\right) \times \Big((\mu_{j-1}(x^{(k)}) - \alpha) \\
&\quad + C_2(\mu_{j-1}(x^{(k)}) - \alpha)^2 + ...\Big) \\
&= \mu_{j-1}(x^{(k)}) - \alpha + L_2\, {e^{(k)}}^3(\mu_{j-1}(x^{(k)}) - \alpha) \\
&\quad + C_2(\mu_{j-1}(x^{(k)}) - \alpha)^2 + ...
\end{aligned} \tag{28}$$

Using (28) in (7), we obtain

$$\begin{aligned}
\mu_j(x^{(k)}) - \alpha &= \mu_{j-1}(x^{(k)}) - \alpha - \Big(\mu_{j-1}(x^{(k)}) - \alpha + L_2\, {e^{(k)}}^3(\mu_{j-1}(x^{(k)}) - \alpha) \\
&\quad + C_2(\mu_{j-1}(x^{(k)}) - \alpha)^2 + ...\Big) \\
&= -L_2\, {e^{(k)}}^3(\mu_{j-1}(x^{(k)}) - \alpha) + ...
\end{aligned} \tag{29}$$

As we know that $\mu_0(x^{(k)}) - \alpha = O(e^{(k)^5})$ and from (29), for $j = 1, 2, ...$

$$\begin{aligned}
\mu_1(x^{(k)}) - \alpha &= -L_2(e^{(k)^{(3)}})\Big(\mu_0(x^{(k)}) - \alpha\Big) + ... \\
&= -L_2 L_1 e^{(k)^8} + ... \\
\mu_2(x^{(k)}) - \alpha &= -L_2(e^{(k)^{(3)}})\Big(\mu_1(x^{(k)}) - \alpha\Big) + ... \\
&= -L_2(-L_2 L_1) e^{(k)^{11}} + ... \\
&= L_2^2 L_1 e^{(k)^{11}} + ...
\end{aligned}$$

Proceeding by induction, we have

$$\mu_r(x^{(k)}) - \alpha = (-L_2)^r L_1 \ (e^{(k)^{(3r+5)}}) + O(e^{(k)^{(3r+6)}}), r \geq 1. \tag{30}$$

□

Remark : Multi-step version $(3r+5)MBJ$ ($r \geq 0$) methods are constructed from $4+r$ evaluation of F and F' together. Only one inverse evaluation of Fréchet derivatives F' at $(x^{(k)})$ is used for the proposed method (7).

4 Numerical examples

The numerical experiments have been carried out using the MATLAB software for the examples given below. The approximate solutions are calculated correctly to 1000 digits by using variable precision arithmetic. We use the following stopping criterion for the iteration scheme:

$$err_{min} = \|x^{(k+1)} - x^{(k)}\|_2 < 10^{-100}. \tag{31}$$

We have used the approximated computational order of convergence p_c given by

$$p_c \approx \frac{\log(\|x^{(k+1)} - x^{(k)}\|_2 / \|x^{(k)} - x^{(k-1)}\|_2)}{\log(\|x^{(k)} - x^{(k-1)}\|_2 / \|x^{(k-1)} - x^{(k-2)}\|_2)}. \tag{32}$$

Let M be the number of iterations required for reaching the minimum residual (err_{min}).

Test Problem 1 (TP1) We consider the following nonlinear system:

$$F(x_1, x_2) = (x_1 + exp(x_2) - cos(x_2), \quad 3x_1 - x_2 - sin(x_2)).$$

The Jacobian matrix is given by $F'(x) = \begin{pmatrix} 1 & exp(x_2) + sin(x_2) \\ 3 & -1 - cos(x_2) \end{pmatrix}$. The starting vector is $x^{(0)} = (1.5, 2)^T$ and the exact solution is $\alpha = (0, 0)^T$.

Test Problem 2 (TP2) We consider the following nonlinear system:

$$\begin{cases} x_2x_3 + x_4(x_2 + x_3) = 0, \\ x_1x_3 + x_4(x_1 + x_3) = 0, \\ x_1x_2 + x_4(x_1 + x_2) = 0, \\ x_1x_2 + x_1x_3 + x_2x_3 = 1. \end{cases} \tag{33}$$

We solve this system using the initial approximation $x^{(0)} = (0.5, 0.5, 0.5, -0.2)^T$. The solution of this system is $\alpha \approx (0.577350, 0.577350, 0.577350, -0.288675)^T$. The Jacobian matrix which has 12 non-zero elements is given by

$$F'(x) = \begin{pmatrix} 0 & x_3 + x_4 & x_2 + x_4 & x_2 + x_3 \\ x_3 + x_4 & 0 & x_1 + x_4 & x_1 + x_3 \\ x_2 + x_4 & x_1 + x_4 & 0 & x_1 + x_2 \\ x_2 + x_3 & x_1 + x_3 & x_1 + x_2 & 0 \end{pmatrix}. \tag{34}$$

Test Problem 3 (TP3) We consider the following nonlinear system:

$$\begin{cases} x_i x_{i+1} - 1 = 0, \quad i = 1, 2, 3, ...15, \\ x_{15}x_1 - 1 = 0. \end{cases} \tag{35}$$

The solution is $\alpha = (1, 1, 1, ..., 1)^T$. We choose the starting vector $x^{(0)} = (1.5, 1.5, 1.5, ..., 1.5)^T$. The Jacobian matrix has 30 non-zero elements and it is given by

$$F'(x) = \begin{pmatrix} x_2 & x_1 & 0 & 0 & ... & 0 & 0 \\ 0 & x_3 & x_2 & 0 & ... & 0 & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & 0 & 0 & ... & x_{15} & x_{14} \\ x_{15} & 0 & 0 & 0 & ... & 0 & x_1 \end{pmatrix}.$$

Table 1 shows the results for the test problems (TP1, TP2, TP3), from which we conclude that $8MBJ$ and $11MBJ$ methods are the most efficient methods out of the methods compared with least number of iterations and residual error.

Table 1 Comparison of different methods for system of nonlinear equations

Methods	TP1			TP2			TP3		
	M	err_{min}	p_c	M	err_{min}	p_c	M	err_{min}	p_c
$2NR$	10	1.0385e−103	1.99	8	3.9287e−145	2.00	9	8.9692e−179	1.99
$3TM$	7	9.6505e−104	2.99	6	8.8773e−236	3.01	6	5.7903e−142	2.99
$4NR$	6	5.3845e−207	3.99	5	2.9883e−291	4.03	5	8.9692e−179	3.99
$4ACT$	6	2.8073e−309	3.99	5	3.8694e−283	4.03	5	2.0352e−203	3.99
$5ACT$	6	0	4.99	4	5.7140e−121	5.12	5	0	4.99
$5MBJ$	6	1.0954e−315	4.99	4	5.0835e−102	5.15	5	1.3994e−304	4.99
$8MBJ$	5	0	7.80	4	0	8.80	4	3.6805e−226	7.53
$11MBJ$	4	4.5504e−104	10.89	4	0	12.00	4	0	10.51

5 Applications

5.1 Chandrasekhar's equation

Consider the quadratic integral equation related to Chandrasekhar's work [5, 8]

$$x(s) = f(s) + \lambda x(s) \int_0^1 k(s,t)x(t)dt, \tag{36}$$

which arise in the study of the radiative transfer theory, the transport of neutrons and the kinetic theory of the gases. Equation 36 is also studied by Argyros [2, 3] and along with some conditions for the kernel $k(s,t)$ in [7]. We consider the maximum norm for the kernel $k(s,t)$ as a continuous function in $s, t \in [0,1]$ such that $0 < k(s,t) < 1$ and $k(s,t)+k(t,s) = 1$. Moreover, we assume that $f(s) \in C[0,1]$ is a given function and λ is a real constant. Note that finding a solution for (36) is equivalent to solving the equation $F(x) = 0$, where $F : C[0,1] \to C[0,1]$ and

$$F(x)(s) = x(s) - f(s) - \lambda x(s) \int_0^1 k(s,t)x(t)dt, \; x \in C[0,1], \; s \in [0,1]. \tag{37}$$

In particular, we consider $f(s) = 1$, $\lambda = 1/4$, and $k(s,t) = \frac{s}{s+t}$ in above equation, we have

$$F(x)(s) = x(s) - 1 - \frac{x(s)}{4} \int_0^1 \frac{s}{s+t} x(t)dt, \; x \in C[0,1], \; s \in [0,1]. \tag{38}$$

Finally, we approximate numerically a solution for $F(x) = 0$, where $F(x)$ is given in (38) by means of a discretization procedure. We solve the integral (38) by the Gauss-Legendre quadrature formula:

$$\int_0^1 f(t)dt \approx \frac{1}{2} \sum_{j=1}^{m} \beta_j f(t_j), \tag{39}$$

where β_j are the weights and t_j are the knots tabulated in Table 2 for $m = 8$. Denote x_i for the approximations of $x(t_i), i = 1, 2, ...8$, we obtain the following nonlinear system:

$$x_i \approx 1 + \frac{1}{8} x_i \sum_{j=1}^{8} a_{ij} x_j, \; where \; a_{ij} = \frac{t_i \beta_j}{8(t_i + t_j)}, \; i = 1, ...8. \tag{40}$$

For this application, we use the following stopping criterion

$$err_{min} = \|x^{(k+1)} - x^{(k)}\|_2 < 10^{-5},$$

the initial approximation assumed is $x^{(0)} = \{1, 1, ..., 1\}^t$ for obtaining the solution of this problem given by $x^* = \{1.02171973146..., 1.07318638173..., 1.12572489365..., 1.16975331216..., 1.20307175130..., 1.22649087463..., 1.24152460059..., 1.24944851669...\}^t$. Table 3 compares the iteration numbers and their errors for this application. The results show that the proposed method $5MBJ$ is better than $2NR$ and some other methods.

Table 2 Weights and knots for the Gauss-Legendre formula ($m = 8$)

j	t_j	β_j
1	0.0198550717512...	0.101228536290...
2	0.101666761293...	0.222381034453...
3	0.237233795041...	0.313706645877...
4	0.408282678752...	0.362683783378...
5	0.591717321247...	0.362683783378...
6	0.762766204958...	0.31370664587...
7	0.898333238706...	0.222381034453...
8	0.980144928248...	0.101228536290...

5.2 The 2-D Bratu problem

We consider the solution of the Bratu problem in two dimensions

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \lambda exp(U) = 0, \;\; x, y \in D = [0, 1] \times [0, 1] \tag{41}$$

subject to the boundary conditions

$$U(x, y) = 0, \;\; x, y \in D. \tag{42}$$

The 2-D planar Bratu problem has two known, bifurcated, exact solutions for values of $\lambda < \lambda_c$, one solution for $\lambda = \lambda_c$, and no solutions for $\lambda > \lambda_c$. The exact solution to (41) is known and can be presented here as

$$U(x, y) = 2 \ln \left[\frac{\cosh\left(\frac{\theta}{4}\right) \cosh\left((x - \frac{1}{2})(y - \frac{1}{2})\theta\right)}{\cosh\left((x - \frac{1}{2})(\frac{\theta}{2})\right) \cosh\left((y - \frac{1}{2})(\frac{\theta}{2})\right)} \right], \tag{43}$$

where θ is a constant to be determined, which satisfies the boundary conditions and is carefully chosen and assumed to be the solution of the differential (41). The following procedure found in [12], for how to obtain the critical value of λ. Substituting (43) in (41), simplifying and collocating at the point $x = \dfrac{1}{2}$ and $y = \dfrac{1}{2}$ because it is the midpoint of the interval. Another point could be chosen, but low-order

Table 3 Comparison of iteration and errors for Chandrasekhar's equation

M	$2NR$	$3TM$	$4NR$	$4ACT$	$5ACT$	$5MBJ$
1	4.9e-001	5.1e-001	5.1e-001	5.1e-001	5.1e-001	5.1e-001
2	1.6e-002	9.8e-004	1.5e-005	1.4e-005	1.7e-006	5.9e-006
3	1.5e-005	5.3e-012	3.8e-016	2.2e-016	–	–
4	1.2e-011	–	–	–	–	–

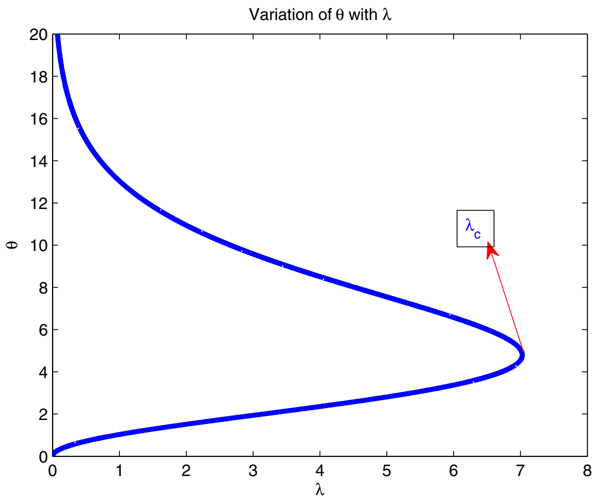

Fig. 1 Variation of θ for different values of λ

approximations are likely to be better if the collocation points are distributed somewhat evenly throughout the region. Then, we have

$$\theta^2 = \lambda \cosh^2\left(\frac{\theta}{4}\right). \tag{44}$$

Differentiating (44) with respect to θ and setting $\dfrac{d\lambda}{d\theta} = 0$, the critical value λ_c satisfies

$$\theta = \frac{1}{4}\lambda_c \cosh\left(\frac{\theta}{4}\right) \sinh\left(\frac{\theta}{4}\right). \tag{45}$$

By eliminating λ from (44) and (45), we have the value of θ_c for the critical λ_c satisfying

$$\frac{\theta_c}{4} = \coth\left(\frac{\theta_c}{4}\right) \tag{46}$$

and $\theta_c = 4.798714561$. We then get $\lambda_c = 7.027661438$ from (45).

Table 4 Comparison of number of λ's for the 2-D Bratu problem for $N = 10$

Method	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$\overline{M_\lambda}$
$2NR$	0	101	520	79	3.96
$3TM$	18	633	49	0	3.04
$4NR$	101	599	0	0	2.85
$4ACT$	101	599	0	0	2.85
$5ACT$	200	500	0	0	2.71
$5MBJ$	121	579	0	0	2.82
$8MBJ$	514	186	0	0	2.26

Table 5 Comparison of number of λs for the 2-D Bratu problem for $N = 20$

Method	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$\overline{M_\lambda}$
$2NR$	1	212	487	0	3.69
$3TM$	39	661	0	0	2.94
$4NR$	213	487	0	0	2.69
$4ACT$	213	487	0	0	2.69
$5ACT$	419	281	0	0	2.40
$5MBJ$	217	483	0	0	2.69
$8MBJ$	700	0	0	0	2

Figure 1 illustrates this critical value of λ_c. The differential (41) is usually discretized by using the finite-difference five-point formula with the step size h, the resulting nonlinear equations are

$$F(U_{i,j}) = -(4U_{i,j} - \lambda h^2 exp(U_{i,j})) + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} \quad (47)$$

where $U_{i,j}$ is U at (x_i, y_j), $x_i = ih$, $y_j = jh$, $i, j = 1, 2, ...N$. Equation 47 represents a set of $N \times N$ nonlinear equations in $U_{i,j}$ which are then solved by using iterative methods. We use $N = 10$ and $N = 20$ to test 700 λs in the interval (0, 7] (interval width = 0.01). For each λ, we let M_λ be the minimum number of iterations for which $\|U_{i,j}^{(k+1)} - U_{i,j}^{(k)}\|_2 < 1e - 11$, where the approximation $U_{i,j}^{(k)}$ is calculated correctly to 14 decimal places. Let $\overline{M_\lambda}$ be the mean of iteration number for the 700 λs.

Tables 4 and 5 give the results for the 2-D Bratu problem, where M represents number of iterations for convergence. It can be observed from the Table 5 that the proposed method $8MBJ$ is convergent for all the grid points in two iterations. Also the method $8MBJ$ is the most efficient method among the compared methods for the cases $N = 10$ and $N = 20$ because it has the lowest mean iteration number.

6 Conclusion

In this work, we have proposed a two-step fifth order method which is an improvement to the double-step Newton method and also proposed a multi-step version of the two-step fifth order method. The main advantages of the proposed schemes are the following: (i) they do not use second order Fréchet derivative and (ii) evaluate only one inverse of first order Fréchet derivative. Also, we have verified that the root α is a point of attraction for the proposed schemes in the sense of Ostrowski [14]. The proposed new methods and their theoretical results are validated through examples whose results are tabulated. The performance of our methods are compared with Newton's method and some existing methods. For practical applications, the new methods are verified on Chandrasekhar's equation and the 2-D Bratu problem which gives encouraging results compared to some existing methods.

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