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Galerkin finite element method for nonlinear fractional Schrödinger equations

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Abstract In this paper, a class of nonlinear Riesz space-fractional Schrödinger equations are considered. Based on the standard Galerkin finite element method in space and Crank-Nicolson difference method in time, the semi-discrete and fully discrete systems are constructed. By Brouwer fixed point theorem and fractional Gagliardo-Nirenberg inequality, we prove the fully discrete system is uniquely solvable. Moreover, we focus on a rigorous analysis and consideration of the conservation and convergence properties for the semi-discrete and fully discrete systems. Finally, a linearized iterative finite element algorithm is introduced and some numerical examples are given to confirm the theoretical results.

Keywords Nonlinear fractional Schrödinger equation · Finite element method · Crank-Nicolson scheme · Conservation · Unique solvability · Convergence

1 Introduction

Fractional calculus is considered as the generalization of the classical (or integer order) calculus with a history of at least 300 years. It has turned out that derivatives and integrals of non-integer order are very suitable for the description in many phenomena. The growing number of fractional derivative applications in various fields

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of science and engineering indicate that there are significant demands for the mathematical models of real objects. As the result of the non-local property of the fractional derivatives, fractional differential equations are well used to describe the phenomena in physics, chemistry, biology, engineering, and even economics [21, 24, 25, 28]. However, the analytical solutions to fractional differential equations are very difficult to derive explicitly, although many considerable works have been carried on the theoretical analysis [11, 13, 36]. Therefore, there have been growing interests recently in developing numerical methods for solving fractional differential equations. Until now, various numerical methods are given for solving fractional differential equations such as finite difference methods [8, 19, 22, 41], spectral methods [18, 20], collocation methods [38], finite element methods [5, 9], etc.

As is well known, the nonlinear Schrödinger equations (NLSs) play an important role in quantum mechanics. During the past few decades, there are various numerical methods in the numerical analysis and scientific computing for NLSs [2, 14, 15, 34]. In recent years, as the generalization of the standard nonlinear Schrödinger equation, there have been growing interests in the analysis and computing for the numerical solutions to nonlinear fractional Schrödinger equations (FSEs). For the time-fractional Schrödinger equations, Mohebbi et al. [23] employed a meshless technique based on collocation methods and radial basis functions, Khan et al. [16] derived approximating solutions by homotopy analysis methods, and Wei et al. [35] gave discrete solution via a rigorous analysis of implicit fully discrete local discontinuous Galerkin method. For the space-fractional Schrödinger equations, some fully or linearly implicit difference methods were introduced and discrete conservation properties were analyzed in [30, 31, 33]. Two-dimensional problems were considered and a fourth-order ADI scheme was presented in [40]. Some other classes of methods, including HSS-like iteration method [26] and differential transform method [3], have also been studied. To the best of our knowledge, however, the finite element method, which is an important approach to solve partial differential equations, has not been considered for such equations. Compared with finite difference methods, it has the advantage of being able to utilize nonuniform meshes.

In this article, we are concerned with the Galerkin finite element method for the following nonlinear fractional Schrödinger equation with the Riesz space fractional derivative ($1 < \alpha \leq 2$)

$$iu_t - (-\triangle)^{\frac{\alpha}{2}} u + f(u) = 0, \quad -\infty < x < +\infty, \; 0 < t \leq T, \tag{1}$$

with the initial and Dirichlet boundary conditions given by

$$u(x, 0) = u_0(x), \quad -\infty < x < +\infty, \tag{2}$$

$$\lim_{|x|\to\infty} u(x, t) = 0, \;\; 0 < t \leq T, \tag{3}$$

where $i^2 = -1$, $f : C \to C$ is locally Lipschitz and $u_0(x)$ is a given smooth function. The Riesz fractional derivative is given as [37]

$$\frac{\partial^\alpha}{\partial |x|^\alpha} u(x, t) = -(-\triangle)^{\frac{\alpha}{2}} u(x, t) = -\frac{1}{2cos(\frac{\alpha}{2}\pi)}[{}_x D_L^\alpha u(x, t) + {}_x D_R^\alpha u(x, t)], \tag{4}$$

where ${}_xD_L^{\alpha}u(x,t)$ and ${}_xD_R^{\alpha}u(x,t)$ are the left and right Riemann-Liouville fractional derivatives defined as follows

$$
{}_xD_L^{\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(2-\alpha)}\left(\frac{d}{dx}\right)^2 \int_{-\infty}^{x}(x-s)^{1-\alpha}u(s,t)ds, & 1<\alpha<2, \\ \frac{d^2}{dx^2}u(x,t), & \alpha=2, \end{cases}
$$

$$
{}_xD_R^{\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(2-\alpha)}\left(-\frac{d}{dx}\right)^2 \int_{x}^{+\infty}(s-x)^{1-\alpha}u(s,t)ds, & 1<\alpha<2, \\ \frac{d^2}{dx^2}u(x,t), & \alpha=2. \end{cases}
$$

The Riesz fractional derivative can also be given as the following equivalent Fourier form [7]

$$
-(-\triangle)^{\alpha/2}u(x,t) := -\mathscr{F}^{-1}(|\xi|^{\alpha}\tilde{u}(\xi,t)), \tag{5}
$$

where $\mathscr{F}$ is the Fourier transform and $\tilde{u}$ is the fourier transform of u.

Generally speaking, we always assume $Im(f(v),v)=0$, where $(\cdot,\cdot)$ be the inner product on $L^2(\mathscr{R})$. In this paper, we are interested in a particular case

$$
f(u)=\beta|u|^2u, \quad \beta\in\mathscr{R}. \tag{6}
$$

For this case, Guo et al. [11] proved the existence and uniqueness of the global smooth solution to the period boundary value problem, and derived fractional mass and energy conservation

$$
\|u\|=\|u_0\|, \quad E(t)=E(0), \tag{7}
$$

where

$$
E(t)=\|(-\triangle)^{\frac{\alpha}{4}}u\|^2-\frac{\beta}{2}\|u\|_{L^4}^4. \tag{8}
$$

Here, $\|\cdot\|$ denotes the L^2-norm and $\|\cdot\|_{L^4}$ denotes the L^4-norm.

In [2], Akrivis et al. constructed two fully discrete finite element schemes to approximate the solutions of the classical (integer order) nonlinear Schrödinger equations and gave detailed analysis of unique solvability, conservation, and convergence properties. The main objective of this paper is to extent and develop FEMs to solve the nonlinear space-fractional Schrödinger equations subject to initial-boundary conditions. We construct semi-discrete scheme and fully discrete one which not only satisfy mass conservation but also satisfy energy conservation in some sense. Meanwhile, by virtue of the Brouwer fixed point theorem and fractional Gagliardo-Nirenberg inequality, the existence and uniqueness of the solution to the fully discrete scheme are proved rigorously. Moreover, through the application of some useful lemmas, we give L^2-norm convergence results of semi-discrete scheme and fully discrete one. In order that we can implement the proposed scheme efficiently, we introduce a linearized iterative finite element algorithm, based on which, we give some numerical examples to examine the theoretical results.

The remainder of this paper is arranged as follows. In Section 2, the definitions and properties of fractional derivative spaces and fractional Sobolev space are introduced. In Section 3, we obtain a semi-discrete variational scheme for nonlinear FSE (1)–(3), whose solution keeps mass conservation and energy conservation, and then give a L^2-norm error estimate with respect to the semi-discrete solution of $u(t)$. In Section 4, we construct an implicit Galerkin finite element fully discrete system based on the standard Galerkin finite element method in space and Crank-Nicolson difference method in time. The unique solvablility, mass conservation, and energy conservation properties of fully discrete system are studied, and then L^2-norm convergence result is derived. In Section 6, a linearized iterative finite element algorithm is proposed. In Section 7, some numerical examples are reported to confirm our theoretical analysis. The final section is the summary of the paper. Throughout the paper, we use C and C_i many times to denote positive constants which may be different in different situations.

2 Preliminaries

In this section, we recall some definitions and lemmas we will use thereafter. For the cases of multi-dimensional spaces, these results can be similarly generalized [6].

Definition 1 (Left fractional derivative space [10, 27]). For $\mu > 0$, we define the semi-norm

$$|u|_{J_L^\mu(\mathscr{R})} := \|{}_xD_L^\mu u\|, \tag{9}$$

and norm

$$\|u\|_{J_L^\mu(\mathscr{R})} := \left(\|u\|^2 + |u|^2_{J_L^\mu(\mathscr{R})}\right)^{\frac{1}{2}}, \tag{10}$$

and let $J_{L,0}^\mu(\mathscr{R})$ denote the closure of $C_0^\infty(\mathscr{R})$ with respect to $\|\cdot\|_{J_L^\mu(\mathscr{R})}$.

Definition 2 (Right fractional derivative space [10, 27]). For $\mu > 0$, we define the semi-norm

$$|u|_{J_R^\mu(\mathscr{R})} := \|{}_xD_R^\mu u\|, \tag{11}$$

and norm

$$\|u\|_{J_R^\mu(\mathscr{R})} := \left(\|u\|^2 + |u|^2_{J_R^\mu(\mathscr{R})}\right)^{\frac{1}{2}}, \tag{12}$$

and let $J_{R,0}^\mu(\mathscr{R})$ denote the closure of $C_0^\infty(\mathscr{R})$ with respect to $\|\cdot\|_{J_R^\mu(\mathscr{R})}$.

Definition 3 (Symmetric fractional derivative space [10, 27]). For $\mu > 0$ and $\mu \neq n - \frac{1}{2}, n \in N$, we define the semi-norm

$$|u|_{J_S^\mu(\mathscr{R})} := \left|({}_xD_L^\mu u, {}_xD_R^\mu u)\right|^{\frac{1}{2}}, \tag{13}$$

and norm

$$\|u\|_{J_S^{\mu}(\mathscr{R})} := \left(\|u\|^2 + |u|^2_{J_S^{\mu}(\mathscr{R})}\right)^{\frac{1}{2}}, \tag{14}$$

and let $J_{S,0}^{\mu}(\mathscr{R})$ denote the closure of $C_0^{\infty}(\mathscr{R})$ with respect to $\|\cdot\|_{J_S^{\mu}(\mathscr{R})}$.

Analogously, we define above spaces and respective semi-norms and norms in bounded domain $\Omega \subset \mathscr{R}$. The following definition gives the semi-norm and norm of fractional Sobolev space.

Definition 4 (Fractional Sobolev space [10, 27]) For $\mu > 0$, we define the semi-norm

$$|u|_{H^{\mu}(\mathscr{R})} := \left\| |\xi|^{\mu}\tilde{u}(\xi)\right\|_{L^2(\mathscr{R})}, \tag{15}$$

and norm

$$\|u\|_{H^{\mu}(\mathscr{R})} := \left(\|u\|^2 + |u|^2_{H^{\mu}(\mathscr{R})}\right)^{\frac{1}{2}}, \tag{16}$$

and let $H_0^{\mu}(\mathscr{R})$ denote the closure of $C_0^{\infty}(\mathscr{R})$ with respect to $\|\cdot\|_{H^{\mu}(\mathscr{R})}$, where ξ and $\tilde{u}$ are the Fourier transform parameter and the Fourier transform of u, respectively.

For bounded domain Ω, we define the fractional Sobolev space $H^{\mu}(\Omega)$ as follows [17]

$$H^{\mu}(\Omega) = \{v \in L^2(\Omega) : \exists \tilde{v} \in H^{\mu}(\mathscr{R}), \tilde{v}|_{\Omega} = v\} \tag{17}$$

with the semi-norm

$$|v|_{H^{\mu}(\Omega)} = \inf_{\tilde{v}\in H^{\mu}(\mathscr{R}), \tilde{v}|_{\Omega}=v} |\tilde{v}|_{H^{\mu}(\mathscr{R})}, \tag{18}$$

and norm

$$\|v\|_{H^{\mu}(\Omega)} = \inf_{\tilde{v}\in H^{\mu}(\mathscr{R}), \tilde{v}|_{\Omega}=v} \|\tilde{v}\|_{H^{\mu}(\mathscr{R})}. \tag{19}$$

As [10, 27], we have the following lemmas about their equivalent properties.

Lemma 1 *If $\mu > 0$ and $\mu \neq n - \frac{1}{2}$, $n \in N$, then $J_L^{\mu}(\mathscr{R})$, $J_R^{\mu}(\mathscr{R})$, $J_S^{\mu}(\mathscr{R})$ and $H^{\mu}(\mathscr{R})$ are equal with equivalent norms and semi-norms, and $J_{L,0}^{\mu}(\mathscr{R})$, $J_{R,0}^{\mu}(\mathscr{R})$, $J_{S,0}^{\mu}(\mathscr{R})$ and $H_0^{\mu}(\mathscr{R})$ are equal with equivalent norms and semi-norms.*

Lemma 2 *If $\mu > 0$ and $\mu \neq n - \frac{1}{2}$, $n \in N$, then $J_L^{\mu}(\Omega)$, $J_R^{\mu}(\Omega)$, $J_S^{\mu}(\Omega)$, and $H^{\mu}(\Omega)$ are equal with equivalent norms and semi-norms, and $J_{L,0}^{\mu}(\Omega)$, $J_{R,0}^{\mu}(\Omega)$, $J_{S,0}^{\mu}(\Omega)$, and $H_0^{\mu}(\Omega)$ are equal with equivalent norms and semi-norms.*

Lemma 3 [29]. *For $\mu > 0$, we have*

$$({}_xD_L^{\mu}u, {}_xD_R^{\mu}u) = cos(\mu\pi)\|{}_xD_L^{\mu}u\|^2, \quad ({}_xD_L^{\mu}u, {}_xD_R^{\mu}u) = cos(\mu\pi)\|{}_xD_R^{\mu}u\|^2. \tag{20}$$

3 Semi-discrete variational scheme

In this section, we mainly discuss the conservation and convergence of semi-discrete variational formulation. Assume that the solution of the system (1)–(3) is negligibly small outside of the interval $\Omega = (a, b)$, i.e., $u|_{x\in\mathscr{R}\setminus\Omega} = 0$. Denote S_h to be a family of partitions of Ω with grid parameter h, and associated with S_h define the finite-dimensional subspace $X_h := \{v \in C(\bar{\Omega}) \cap H_0^{\alpha/2}(\Omega) : v|_K \in P_{m-1}, \ \forall K \in S_h\}$, where P_{m-1} denotes the space of polynomials with the order no more than $m-1$. In order to obtain the variational formulation of (1)–(3), we introduce the following lemma first.

Lemma 4 [39]. *For* $1 < \alpha \leq 2$, *if* $u, v \in J_L^{\alpha}(\Omega)$*(or* $J_R^{\alpha}(\Omega)$*),* $u|_{\partial\Omega} = 0$, $v|_{\partial\Omega} = 0$, *then*

$$({}_xD_L^{\alpha}u, v) = ({}_xD_L^{\alpha/2}u, {}_xD_R^{\alpha/2}v), \quad ({}_xD_R^{\alpha}u, v) = ({}_xD_R^{\alpha/2}u, {}_xD_L^{\alpha/2}v).$$

By above lemma, we know

$$\big((-\triangle)^{\frac{\alpha}{2}}u, v\big) = \frac{1}{2cos(\frac{\alpha}{2}\pi)}[({}_xD_L^{\alpha/2}u, {}_xD_R^{\alpha/2}v) + ({}_xD_R^{\alpha/2}u, {}_xD_L^{\alpha/2}v)]. \tag{21}$$

Hence, we get the following variational formulation of problem (1)–(3)

$$i(u_t, v) - B(u, v) + (f(u), v) = 0, \quad \forall v \in H_0^{\frac{\alpha}{2}}(\Omega), \tag{22}$$

with the initial condition given by

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{23}$$

where

$$B(u, v) := \frac{1}{2cos(\frac{\alpha}{2}\pi)}\big[({}_xD_L^{\alpha/2}u, {}_xD_R^{\alpha/2}v) + ({}_xD_R^{\alpha/2}u, {}_xD_L^{\alpha/2}v)\big].$$

For convenience, we define the following semi-norm and norm

$$|u|_{\frac{\alpha}{2}} := B(u, u)^{\frac{1}{2}}, \quad \|u\|_{\frac{\alpha}{2}} := (\|u\|^2 + |u|^2_{\frac{\alpha}{2}})^{\frac{1}{2}}. \tag{24}$$

It is easy to note by Lemmas 2 and 3 that $|u|_{\frac{\alpha}{2}}$ and $\|u\|_{\frac{\alpha}{2}}$ are equivalent with the semi-norms and norms of $J_L^{\mu}(\Omega)$, $J_R^{\mu}(\Omega)$, $H^{\mu}(\Omega)$ and $J_S^{\mu}(\Omega)$. Therefore, the fractional item $B(\cdot, \cdot)$ has the following properties: there exist positive constants C_1, C_2 such that for $u, v \in H_0^{\frac{\alpha}{2}}(\Omega)$, $1 < \alpha \leq 2$

$$B(u, v) \leq C_1\|u\|_{\frac{\alpha}{2}}\|v\|_{\frac{\alpha}{2}}, \tag{25}$$

$$B(u, u) \geq C_2\|u\|^2_{\frac{\alpha}{2}}. \tag{26}$$

Next, in order to get the desired conclusions by finite element method, one may define first the semi-discrete approximation of $u(t)$ in the customary way. Define $u_h : [0, T] \to X_h$ as the semi-discrete approximation of the exact solution $u(t)$, satisfying

$$i(\partial_t u_h, v_h) - B(u_h, v_h) + (f(u_h), v_h) = 0, \quad \forall v_h \in X_h, \tag{27}$$

with the discrete initial condition

$$u_h(0) = u_h^0, \tag{28}$$

where $u_h^0 \in X_h$ is an approximation of u^0, such that

$$\|u^0 - u_h^0\| \le ch^m. \tag{29}$$

Now, we have following conservation results of semi-discrete scheme (27)–(29).

Theorem 1 *If f is defined as (6), then the semi-discrete scheme (27)–(28) is conservative in the sense*

$$Q_h(t) = Q_h(0), \quad 0 \le t \le T, \tag{30}$$
$$E_h(t) = E_h(0), \quad 0 \le t \le T, \tag{31}$$

where

$$Q_h(t) := \|u_h(t)\|^2, \quad E_h(t) := |u_h(t)|^2_{\frac{\alpha}{2}} - \frac{\beta}{2}\|u_h(t)\|^4_{L^4(\Omega)}$$

are the mass and energy, respectively.

Proof Let $v_h = u_h$ in (27), we have

$$i(\partial_t u_h, u_h) - B(u_h, u_h) + (f(u_h), u_h) = 0.$$

Since $Im\{(f(u_h), u_h)\} = 0$, then take the imaginary part of above equation to arrive at

$$Re\{(\partial_t u_h, u_h)\} = 0.$$

Using relation

$$\frac{d}{dt}\|u_h\|^2 = (\partial_t u_h, u_h) + (u_h, \partial_t u_h) = 2Re\{(\partial_t u_h, u_h)\} = 0,$$

we obtain (30) immediately.

Similarly, let $v_h = \partial_t u_h$ in (27), we have

$$i(\partial_t u_h, \partial_t u_h) - B(u_h, \partial_t u_h) + (f(u_h), \partial_t u_h) = 0.$$

Taking the real parts in the above equation, we immediately conclude

$$Re\{B(u_h, \partial_t u_h)\} = Re\{(f(u_h), \partial_t u_h)\}. \tag{32}$$

Notice that

$$
\begin{aligned}
&\frac{d}{dt}B(u_h, u_h)\\
&= \frac{1}{2cos(\frac{\alpha}{2}\pi)}\frac{d}{dt}\left\{({}_xD_L^{\alpha/2}u_h, {}_xD_R^{\alpha/2}u_h) + ({}_xD_R^{\alpha/2}u_h, {}_xD_L^{\alpha/2}u_h)\right\}\\
&= \frac{1}{2cos(\frac{\alpha}{2}\pi)}\Big\{({}_xD_L^{\alpha/2}(\partial_t u_h), {}_xD_R^{\alpha/2}u_h) + ({}_xD_L^{\alpha/2}u_h, {}_xD_R^{\alpha/2}(\partial_t u_h))\\
&\qquad +({}_xD_R^{\alpha/2}(\partial_t u_h), {}_xD_L^{\alpha/2}u_h) + ({}_xD_R^{\alpha/2}u_h, {}_xD_L^{\alpha/2}(\partial_t u_h))\Big\}\\
&= \frac{1}{2cos(\frac{\alpha}{2}\pi)}\left\{2Re({}_xD_L^{\alpha/2}u_h, {}_xD_R^{\alpha/2}(\partial_t u_h)) + 2Re({}_xD_R^{\alpha/2}u_h, {}_xD_L^{\alpha/2}(\partial_t u_h))\right\}\\
&= 2Re\{B(u_h, \partial_t u_h)\},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\beta}{2}\frac{d}{dt}\|u_h\|^4_{L^4(\Omega)} &= \frac{\beta}{2}\Big(\int_\Omega \partial_t u_h\cdot\bar{u}_h\cdot u_h\cdot\bar{u}_h d\Omega + \int_\Omega u_h\cdot\partial_t\bar{u}_h\cdot u_h\cdot\bar{u}_h d\Omega\\
&\qquad + \int_\Omega u_h\cdot\bar{u}_h\cdot\partial_t u_h\cdot\bar{u}_h d\Omega + \int_\Omega u_h\cdot\bar{u}_h\cdot u_h\cdot\partial_t\bar{u}_h d\Omega\Big)\\
&= \beta\left(\int_\Omega |u_h|^2\cdot\bar{u}_h\cdot\partial_t u_h d\Omega + \int_\Omega |u_h|^2\cdot u_h\cdot\partial_t\bar{u}_h d\Omega\right)\\
&= 2Re\{(f(u_h), \partial_t u_h)\}.
\end{aligned}
$$

Therefore, by (32), we have

$$
\frac{d}{dt}B(u_h, u_h) = \frac{\beta}{2}\frac{d}{dt}\|u_h\|^4_{L^4(\Omega)},
$$

which implies (31). □

Next, our task is to derive the semi-discrete approximating property by using the technique in [2]. To this end, we need introduce the following inverse inequality.

Lemma 5 [2]. *For any discrete function $v_h \in X_h$, the inequality*

$$
|v_h|_\infty \le Ch^{-\frac{1}{2}}\|v_h\| \tag{33}
$$

holds.

As in the case of the classical nonlinear Schrödinger equation (see [2]), we suppose that, if u is the solution of (1)–(3), there holds

$$
\lim_{h\to 0}\sup_{0\le t\le T}\inf_{v_h\in X_h}\{|u(t) - v_h|_\infty + h^{-\frac{1}{2}}\|u(t) - v_h\|\} = 0. \tag{34}
$$

Define

$$\Upsilon_\delta := \left\{ y \in C : \ |y - u(x,t)| < \delta, \ \exists (x,t) \in \bar{\Omega} \times [0,T] \right\},$$

and function $\varphi_h : [0,T] \to X_h$, such that

$$\begin{cases} i(\partial_t \varphi_h, v_h) - B(\varphi_h, v_h) + (f_\delta(\varphi_h), v_h) = 0, & \forall v_h \in X_h, \ \ t \in [0,T], \\ \varphi_h(0) = u_h^0, \end{cases} \tag{35}$$

where $f_\delta : \mathscr{C} \to \mathscr{C}$ is a global Lipschitz continuous function coincide with f on Υ_δ [2]. In order to find a bound for $\|u(t) - u_h(t)\|$, we shall firstly estimate $\|u(t) - \varphi_h(t)\|$. For achieving this goal, we give the definition of Ritz Projection $P_h : H_0^{\frac{\alpha}{2}}(\Omega) \to X_h$, satisfying

$$B(P_h u, v_h) = B(u, v_h), \ \ v_h \in X_h. \tag{36}$$

By [4], we know there exists a constant $C > 0$ satisfying the following approximating properties: if $u \in H^m(\Omega) \cap H_0^{\frac{\alpha}{2}}(\Omega)$, $\alpha/2 < \eta \le m$

$$\|u - P_h u\| \le C h^\eta \|u\|_\eta, \ \alpha \ne 3/2; \tag{37}$$

$$\|u - P_h u\| \le C h^{\eta - \kappa} \|u\|_\eta, \ \alpha = 3/2, \ 0 < \kappa < 1/2. \tag{38}$$

Split

$$u - \varphi_h = (u - P_h u) + (P_h u - \varphi_h) := \rho + \theta.$$

We firstly discuss the case of $\alpha \ne 3/2$. Let $v = v_h \in X_h$ in (22) and take a subtraction of (22) and (35) to arrive at

$$i(\theta_t, v_h) - B(\theta, v_h) = -i(\rho_t, v_h) + (f_\delta(\varphi_h) - f_\delta(u), v_h), \tag{39}$$

where is due to $B(\rho, v_h) = 0$ and the coincidence of f and f_δ on Υ_δ. Set $v_h = \theta$ in (39), and take imaginary parts to obtain

$$Re\{(\theta_t, \theta)\} = Re\{-(\rho_t, \theta)\} + Im\{(f_\delta(\varphi_h) - f_\delta(u), \theta)\}. \tag{40}$$

Then, by Cauchy-Schwarz inequality and the global Lipschitz property of f_δ, we have

$$Re\{(\theta_t, \theta)\} \le \big(\|\rho_t\| + L(\|\rho\| + \|\theta\|) \big) \|\theta\|,$$

where L is the Lipschitz constant. Hence, we have

$$\frac{d}{dt}\|\theta\| \le \|\rho_t\| + L(\|\rho\| + \|\theta\|). \tag{41}$$

By (37), we know that

$$\|\rho\| \le C h^m \|u\|_m, \quad \|\rho_t\| \le C h^m \|u_t\|_m.$$

Above inequalities and (41) yield

$$\frac{d}{dt}\|\theta\| \le C\big(\|\theta\| + h^m\big), \tag{42}$$

where $C > 0$ is a constant depending on L, u, and u_t. By (42) and Gronwall's Lemma, we conclude

$$\|\theta(t)\| \le C(\theta(0) + h^m). \tag{43}$$

By using Triangle inequality and (29), we obtain

$$\|u(t) - \varphi_h(t)\| \le C(\|u^0 - u_h^0\| + h^m) \le Ch^m, \tag{44}$$

where C is a constant independent of h. By Lemma 5 and Triangle inequality, we get following one for any $v_h \in X_h$

$$\begin{aligned}
|u(t) - \varphi_h(t)|_\infty &\le |\rho|_\infty + |\theta|_\infty \\
&\le |u(t) - v_h(t)|_\infty + |v_h(t) - P_h u(t)|_\infty + |\theta|_\infty \\
&\le |u(t) - v_h(t)|_\infty + h^{-\frac{1}{2}}\|v_h(t) - P_h u(t)\| + h^{-\frac{1}{2}}\|\theta\| \\
&\le |u(t) - v_h(t)|_\infty + h^{-\frac{1}{2}}\|v_h(t) - u(t)\| + h^{-\frac{1}{2}}\|u(t) \\
&\quad - P_h u(t)\| + h^{-\frac{1}{2}}\|\theta\|.
\end{aligned}$$

Therefore, via (34), (37), and (43), we have

$$\lim_{h\to 0} |u(t) - \varphi_h(t)|_\infty = 0, \quad 0 \le t \le T, \tag{45}$$

which shows there exists $h_0 > 0$ such that for $h \le h_0$, $\varphi_h(x,t) \in \Upsilon_\delta$ for $(x,t) \in \bar{\Omega} \times [0, T]$. Equation (45) also implies $\varphi_h = u_h$. Hence, by (44), we get

$$\|u(t) - u_h(t)\| \le Ch^m. \tag{46}$$

By (38), we immediately get the error estimate for the case of $\alpha = 3/2$.

The following theorem follows immediately from what we have discussed above.

Theorem 2 *Suppose that u is the exact solution to (1)–(3), u_h is the solution of semi-discrete scheme (27)–(28) and u_h^0 satisfies (29). Then*

$$\max_{0\le t\le T} \|u(t) - u_h(t)\| \le Ch^m, \ \ \alpha \ne \frac{3}{2}, \tag{47}$$

and

$$\max_{0\le t\le T} \|u(t) - u_h(t)\| \le Ch^{m-\kappa}, \ \ \alpha = 3/2, \ 0 < \kappa < \frac{1}{2} \tag{48}$$

hold.

4 Fully discrete scheme

In this section, we first introduce the fully discrete scheme and then give its rigorous analysis of the conservation, unique solvability, and convergence properties. Assume that $U^n \in X_h$ is the approximation of $u(x,t)$ with $t = t_n$. Let $t = t_{n+\frac{1}{2}}$ in (22).

We discretize (22) by CN scheme in temporal direction, and finite element method in spatial direction. Then, we get the fully discrete scheme of variational formulation, which is to find $U^{n+1} \in X_h$, such that

$$i(\delta_t U^{n+\frac{1}{2}}, v_h) - B(U^{n+\frac{1}{2}}, v_h) + (f(U^{n+\frac{1}{2}}), v_h) = 0, \quad \forall v_h \in X_h,\ 0 \le n \le N-1, \tag{49}$$

with the initial condition

$$U^0 = P_h u^0, \tag{50}$$

where $\delta_t U^{n+1/2} = (U^{n+1} - U^n)/\tau$, $U^{n+1/2} = (U^n + U^{n+1})/2$. Akrivis et al. [2] show that the scheme (49)–(50) hold

$$\|U^n\| = \|U^0\|, \quad 1 \le n \le N,$$

for $\alpha = 2$. However, the energy conservative does not keep all the time. Accordingly, we are interested in another fully discrete scheme, which is to find $U^{n+1} \in X_h$, such that for $0 \le n \le N-1$

$$i(\delta_t U^{n+\frac{1}{2}}, v_h) - B(U^{n+\frac{1}{2}}, v_h) + \frac{\beta}{2}\Big((|U^{n+1}|^2 + |U^n|^2)U^{n+\frac{1}{2}}, v_h\Big) = 0, \quad \forall v_h \in X_h, \tag{51}$$

with the same initial condition given by (50).

4.1 Conservation

The semi-discrete scheme (27)–(28) satisfies two conservation laws given by Theorem 1. We would like to prove that the fully discrete scheme (51) with initial condition (50) is also keep these invariant quantities. This subsection is devoted to considering the conservation properties of the discrete solution $U^n (0 \le n \le N)$.

Lemma 6 *For the fully discrete solution $U^n \in X_h,\ 0 \le n \le N$, we have*

$$Re\big\{B(U^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}})\big\} = \frac{1}{2\tau}\left(|U^{n+1}|^2_{\frac{\alpha}{2}} - |U^n|^2_{\frac{\alpha}{2}}\right). \tag{52}$$

Proof Using the definition of $B(\cdot,\cdot)$, we get

$$\begin{aligned}
&B\left(U^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}}\right)\\
&= \frac{1}{2cos(\frac{\alpha}{2}\pi)}\Big[\Big({}_xD_L^{\alpha/2}U^{n+\frac{1}{2}}, {}_xD_R^{\alpha/2}\delta_t U^{n+\frac{1}{2}}\Big) + \Big({}_xD_R^{\alpha/2}U^{n+\frac{1}{2}}, {}_xD_L^{\alpha/2}\delta_t U^{n+\frac{1}{2}}\Big)\Big]\\
&= \frac{1}{4\tau cos(\frac{\alpha}{2}\pi)}\Big[\big({}_xD_L^{\alpha/2}U^{n+1}, {}_xD_R^{\alpha/2}U^{n+1}\big) - \big({}_xD_L^{\alpha/2}U^{n+1}, {}_xD_R^{\alpha/2}U^{n}\big)\\
&\quad + \big({}_xD_L^{\alpha/2}U^{n}, {}_xD_R^{\alpha/2}U^{n+1}\big) - \big({}_xD_L^{\alpha/2}U^{n}, {}_xD_R^{\alpha/2}U^{n}\big) + \big({}_xD_R^{\alpha/2}U^{n+1}, {}_xD_L^{\alpha/2}U^{n+1}\big)\\
&\quad - \big({}_xD_R^{\alpha/2}U^{n+1}, {}_xD_L^{\alpha/2}U^{n}\big) + \big({}_xD_R^{\alpha/2}U^{n}, {}_xD_L^{\alpha/2}U^{n+1}\big) - \big({}_xD_R^{\alpha/2}U^{n}, {}_xD_L^{\alpha/2}U^{n}\big)\Big]\\
&= \frac{1}{2\tau}\left(|U^{n+1}|^2_{\frac{\alpha}{2}} - |U^n|^2_{\frac{\alpha}{2}}\right) + \frac{1}{2\tau}\Big(B(U^n, U^{n+1}) - B(U^{n+1}, U^n)\Big).
\end{aligned}$$

It is easy to note that $B(U^n, U^{n+1})$ is conjugate with $B(U^{n+1}, U^n)$, hence, we have

$$Re\left\{B\left(U^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}}\right)\right\} = \frac{1}{2\tau}\left(|U^{n+1}|_{\frac{\alpha}{2}}^2 - |U^n|_{\frac{\alpha}{2}}^2\right).$$

This completes the proof. □

Theorem 3 *The fully discrete scheme* (51) *with the initial condition* (50) *is conservative in the sense*

$$Q^n = Q^0, \quad 0 \le n \le N, \tag{53}$$

$$E^n = E^0, \quad 0 \le n \le N, \tag{54}$$

where

$$Q^n := \|U^n\|^2, \quad E^n := |U^n|_{\frac{\alpha}{2}}^2 - \frac{\beta}{2}\|U^n\|_{L^4(\Omega)}^4$$

are the mass and energy, respectively, in the fully discrete sense.

Proof Let $v_h = U^{n+\frac{1}{2}}$ in (51), we have for $0 \le n \le N-1$,

$$i(\delta_t U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}) - B(U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}) + \frac{\beta}{2}\left((|U^{n+1}|^2 + |U^n|^2)U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right) = 0. \tag{55}$$

We note that $Im\left\{\frac{\beta}{2}\left((|U^{n+1}|^2 + |U^n|^2)U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right)\right\} = 0$ and $Im\left\{B(U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}})\right\} = 0$, thus, take the imaginary part of (55) to arrive at

$$(\delta_t U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}) = \|U^{n+1}\|^2 - \|U^n\|^2 = 0, \tag{56}$$

which implies (53). Let $v_h = \delta_t U^{n+\frac{1}{2}}$ in (51), then we have for $0 \le n \le N-1$,

$$i(\delta_t U^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}}) - B(U^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}}) + \frac{\beta}{2}\left((|U^{n+1}|^2 + |U^n|^2)U^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}}\right) = 0. \tag{57}$$

Take the real parts of (57) to arrive at

$$Re\left\{B(U^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}})\right\} = \frac{\beta}{2} Re\left\{\left((|U^{n+1}|^2 + |U^n|^2)U^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}}\right)\right\}. \tag{58}$$

By Lemma 6, we have

$$\frac{1}{2\tau}\left(|U^{n+1}|_{\frac{\alpha}{2}}^2 - |U^n|_{\frac{\alpha}{2}}^2\right) = \frac{\beta}{4\tau}\left(\|U^{n+1}\|_{L^4(\Omega)}^4 - \|U^n\|_{L^4(\Omega)}^4\right). \tag{59}$$

Thus, (54) is valid and the proof is complete. □

Remark 1 It is noted from Theorem 3 that the numerical solution of fully discrete scheme (51) is long-time bounded, i.e., there exists some constant $C > 0$, such that

$$\|U^n\| \le C, \; 0 \le n \le N. \tag{60}$$

4.2 Solvability and uniqueness

In order to ensure well-posedness of the algorithm, in this subsection, we show that the fully discrete scheme (51) with the initial condition (50) is uniquely solvable. Akrivis et al. in [2] use Brouwder fixed point theorem and integer Gagliardo-Nirenberg inequality to prove the unique solvability of classical NLS. To extent the method to the nonlinear FSE, we need use fractional Gagliardo-Nirenberg inequality shown in Lemma 8.

Lemma 7 (Brouwder fixed point theorem [1]). *Let* $(\mathscr{H}, <\cdot,\cdot>)$ *be a finite dimensional inner product space,* $\|\cdot\|$ *be the associated norm, and* $f : \mathscr{H} \to \mathscr{H}$ *be continuous. Assume, moreover, that there exists* $\lambda > 0$ *such that for every* $z \in \mathscr{H}$ *with* $\|z\| = \lambda$ *there holds* $Re(f(z), z) \geq 0$*. Then, there exists a* $z^* \in \mathscr{H}$ *such that* $g(z^*) = 0$ *and* $\|z^*\| \leq \lambda$.

Lemma 8 (Fractional Gagliardo-Nirenberg inequality [12]). *Assume* d *is the dimensional number. Let* $1 \leq p_1, p_2 < \infty$, $0 < \theta < p < \infty$, $0 < s < d$ *and* $1 < p_1 < d/s$. *We have*

$$\|u\|_{L^p} \leq B^{\frac{\theta}{p}} \|(-\Delta)^{s/2}u\|_{L^{p_1}}^{\frac{\theta}{p}} \|u\|_{L^{p_2}}^{\frac{p-\theta}{p}}, \tag{61}$$

with

$$\theta\left(\frac{1}{p_1} - \frac{s}{d}\right) + \frac{p-\theta}{p_2} = 1,$$

and

$$B = 2^{-s}\pi^{-s/2}\frac{\Gamma((d-s)/2)}{\Gamma((d+s)/2)}\left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{s/d}.$$

Lemma 9 [2]. *For* $\forall z_1, z_2 \in \mathbb{C}$, *we have*

$$\left| |2z_1 - z|^2 z_1 - |2z_2 - z|^2 z_2 \right| \leq 4\left(|z_1| + |z_2| + \frac{1}{2}|z|\right)^2 |z_1 - z_2|. \tag{62}$$

Remark 2 By Lemma 8, for $d = 1$, letting $s = \alpha/2$, $p = 4$, $p_1 = p_2 = 2$, we have $\theta = 2/\alpha$ and thus, according to the equivalent quality between norm and norm, we get

$$\|u\|_{L^4}^4 \leq B_1 \|(-\Delta)^{\alpha/4}u\|^{\frac{2}{\alpha}} \|u\|^{4-\frac{2}{\alpha}} \leq C|u|_{\frac{\alpha}{2}}^{\frac{2}{\alpha}} \|u\|^{4-\frac{2}{\alpha}}, \tag{63}$$

where B_1 and C are positive constants dependent of α.

Theorem 4 *The solution of fully discrete finite element scheme* (51) *with initial condition* (50) *is uniquely solvable.*

Proof Rewrite (51) as following one by using the relations $\delta_t U^{n+1/2} = (U^{n+1} - U^n)/\tau$ and $U^{n+1/2} = (U^{n+1} + U^n)/2$

$$i\left(U^{n+\frac{1}{2}}, v_h) - i(U^n, v_h\right) - \frac{1}{2}\tau B(U^{n+\frac{1}{2}}, v_h) + \frac{1}{4}\beta\tau\left((|2U^{n+\frac{1}{2}} - U^n|^2 + |U^n|^2)U^{n+\frac{1}{2}}, v_h\right) = 0. \tag{64}$$

For convenience, we denote $\Phi : X_h \to X_h$, such that

$$(\Phi(\omega), v_h) = \frac{1}{2}B(\omega, v_h) - \frac{\beta}{4}\left((|2\omega - U^n|^2 + |U^n|^2)\omega, v_h\right), \quad \forall \omega \in X_h.$$

Now, we first consider the existence of the solution to the following equation

$$U^{n+\frac{1}{2}} = U^n - i\tau\Phi(U^{n+\frac{1}{2}}). \tag{65}$$

In order to prove the solvability of fully discrete scheme (50)–(51), we just need prove the solution of (65) with initial condition (50) exists. For achieving this result, we let $\mathscr{F} : X_h \to X_h$

$$\mathscr{F}(\omega) = \omega - U^n + i\tau\Phi(\omega). \tag{66}$$

By Lemma 7, we intend to conclude $Re\{(\mathscr{F}(w), w)\} \geq 0$. We note that $Im\{(\Phi(\omega), \omega)\} = 0$ and thus

$$\begin{aligned} Re\{(\mathscr{F}(\omega), \omega)\} &= \|\omega\|^2 - Re\{(U^n, \omega)\} \\ &\geq \|\omega\|^2 - \|U^n\| \cdot \|\omega\| \\ &= \|\omega\|(\|\omega\| - \|U^n\|). \end{aligned}$$

Setting $\|\omega\| = \|U^n\|$, which is a constant by Theorem 4.1, we get

$$Re\{(\mathscr{F}(\omega), \omega)\} \geq 0.$$

By Lemma 7, we complete the proof of the existence of discrete solution.

Next, we proceed to prove the uniqueness of the discrete solution U^{n+1}. Assume that there are two solutions $X, Y \in X_h$ to solve the fully discrete scheme (50)–(51). Then by (65), we get

$$\|X - Y\|^2 = -i\tau(\Phi(X) - \Phi(Y), X - Y). \tag{67}$$

By definition, we have

$$\begin{aligned} (\Phi(X) - \Phi(Y), X - Y) &= \frac{1}{2}B(X, X - Y) - \frac{\beta}{4}\left((|2X - U^n|^2 + |U^n|^2)X, X - Y\right) \\ &\quad - \left(\frac{1}{2}B(Y, X - Y) - \frac{\beta}{4}((|2Y - U^n|^2 + |U^n|^2)Y, X - Y)\right) \\ &= \frac{1}{2}B(X - Y, X - Y) - \frac{\beta}{4}(g(X, Y), X - Y) \\ &\quad - \frac{\beta}{4}(|U^n|^2(X - Y), X - Y), \end{aligned}$$

where

$$g(X,Y) := |2X - U^n|^2 X - |2Y - U^n|^2 Y.$$

It is straight to note that $B(X-Y, X-Y)/2$ and $-\beta(|U^n|^2(X-Y), X-Y)/4$ are real numbers. Hence, by taking the real parts of (67), we have

$$\|X-Y\|^2 = -Im\left\{\frac{\beta\tau}{4}(g(X,Y), X-Y)\right\}. \tag{68}$$

By virtue of *Hölder* inequality, we get

$$\|X-Y\|^2 \le \frac{|\beta|\tau}{4}\|g(X,Y)\|_{L^{\frac{4}{3}}(\Omega)}\|X-Y\|_{L^4(\Omega)}. \tag{69}$$

Then, take the imaginary part of (67) to arrive at

$$|X-Y|^2_{\frac{\alpha}{2}} = \frac{\beta}{2}Re\left\{(g(X,Y), X-Y)\right\} + \frac{\beta}{2}(|U^n|^2(X-Y), X-Y). \tag{70}$$

Similarly, by virtue of *Hölder* inequality, we get

$$|X-Y|^2_{\frac{\alpha}{2}} \le \frac{\beta}{2}\|g(X,Y)\|_{L^{\frac{4}{3}}(\Omega)}\|X-Y\|_{L^4(\Omega)} + \frac{\beta}{2}(|U^n|^2(X-Y), X-Y). \tag{71}$$

According to Lemma 9 and *Hölder* inequality, we conclude

$$\begin{aligned}
\|g(X,Y)\|_{L^{\frac{4}{3}}(\Omega)} &= \left(\int_\Omega |g(X,Y)|^{\frac{4}{3}}dx\right)^{\frac{3}{4}} \\
&\le 4\left(\int_\Omega (|X|+|Y|+\frac{1}{2}|U^n|)^{\frac{8}{3}}|X-Y|^{\frac{4}{3}}dx\right)^{\frac{3}{4}} \\
&\le 4\left(\int_\Omega (|X|+|Y|+\frac{1}{2}|U^n|)^4 dx\right)^{\frac{1}{2}}\left(\int_\Omega |X-Y|^4 dx\right)^{\frac{1}{4}} \\
&\le C\|X,Y,U^n\|^2_{L^4(\Omega)}\|X-Y\|_{L^4(\Omega)},
\end{aligned} \tag{72}$$

where $\|X,Y,U^n\|_{L^4(\Omega)} := \max(\|X\|_{L^4(\Omega)}, \|Y\|_{L^4(\Omega)}, \|U^n\|_{L^4(\Omega)})$ and C is a positive constant. Suppose the initial condition U^0 satisfies

$$|U^0|_{\frac{\alpha}{2}} \le \bar{C}, \tag{73}$$

where $\bar{C} > 0$ is a constant independent of h and τ. In fact, since $U^0 = P_h u^0$ and $u^0 \in H_0^{\frac{\alpha}{2}}(\Omega)$, then taking $v_h = P_h u^0$ in (36), we easily yield (73).

Next, we consider two cases: $\beta \ge 0$ and $\beta < 0$. For $\beta \ge 0$, by (53) and (63), we have

$$\|U^n\|^4_{L^4(\Omega)} \le \tilde{c}|U^n|^{\frac{2}{\alpha}}_{\frac{\alpha}{2}}, \tag{74}$$

where $\tilde{c} > 0$ is a constant dependent of d and α. Then, by (54) and (73), we obtain

$$|U^n|^2_{\frac{\alpha}{2}} - \frac{\tilde{c}\beta}{2}|U^n|^{\frac{2}{\alpha}}_{\frac{\alpha}{2}} - C \leq 0.$$

We note that $|U^n|_{\frac{\alpha}{2}}$ is impossible to be an infinite real number, thus we conclude

$$|U^n|_{\frac{\alpha}{2}} \leq C', \tag{75}$$

where C' is a positive constant. For $\beta < 0$, by (54) and (74), we have

$$\|U^n\|^4_{L^4(\Omega)} \leq \tilde{c}|U^n|^{\frac{2}{\alpha}}_{\frac{\alpha}{2}} \leq C. \tag{76}$$

By the similar analysis as (74)–(76), we conclude $\|X, Y, U^n\|^4_{L^4(\Omega)} \leq C$. Next, use (63) again to arrive at

$$\|X - Y\|^4_{L^4(\Omega)} \leq C|X - Y|^{\frac{2}{\alpha}}_{\frac{\alpha}{2}}\|X - Y\|^{4-\frac{2}{\alpha}}. \tag{77}$$

Then, by (69) and (72), we get

$$\|X - Y\|^{4-\frac{2}{\alpha}} \leq C|\beta|^{2-\frac{1}{\alpha}}\tau^{2-\frac{1}{\alpha}}\|X - Y\|^{4-\frac{2}{\alpha}}_{L^4(\Omega)}. \tag{78}$$

By $H\ddot{o}lder$ inequality and (76), we obtain

$$\int_\Omega |U^n|^2|X - Y|^2 dx \leq \|U^n\|^2_{L^4(\Omega)}\|X - Y\|^2_{L^4(\Omega)} \leq C\|X - Y\|^2_{L^4(\Omega)}.$$

According to (71) and (72), we have

$$|X - Y|^{\frac{2}{\alpha}}_{\frac{\alpha}{2}} \leq C|\beta|^{\frac{1}{\alpha}}\|X - Y\|^{\frac{2}{\alpha}}_{L^4(\Omega)}. \tag{79}$$

Equations (77)–(79) yield

$$\|X - Y\|^4_{L^4(\Omega)} \leq C|\beta|^2\tau^{2-1/\alpha}\|X - Y\|^4_{L^4(\Omega)}. \tag{80}$$

By (80), we can get $X = Y$, which implies the solution of fully discrete scheme is unique.

This completes the proof. □

In the last part, we analyze the error between the finite element approximation given by (50)–(51) and the true solution. By the similar method as the integer order Schrödinger-type equation [2], we obtain a prior error estimate for the approximation given in Theorem 5.

Theorem 5 *Suppose that the exact solution u of the system* (1)–(3) *is sufficiently smooth in bounded domain* Ω. *Let* $\tau = o(h^{1/4})$. *Then there exists a unique discrete solution* U^n *such that*

$$\max_{0 \leq n \leq N} \|u^n - U^n\| \leq C(\tau^2 + h^m), \quad \alpha \neq \frac{3}{2}, \tag{81}$$

and

$$\max_{0\le n\le N}\|u^n-U^n\|\le C(\tau^2+h^{m-\kappa}),\quad \alpha=\frac{3}{2}, 0<\kappa<\frac{1}{2}, \tag{82}$$

where $u^n=u(x,t_n)$ *and* $C>0$ *is a constant independent of* h *and* τ.

Proof First, we suppose that $\alpha\ne 3/2$. Denote $H:\mathscr{C}\times\mathscr{C}\to\mathscr{C}$,

$$H(z_1,z_2)=\frac{\beta}{4}(|z_1|^2+|z_2|^2)(z_1+z_2).$$

For given $\delta>0$, we let

$$\tilde{\Upsilon}_\delta=\big\{(z_1,z_2)\in\mathscr{C}\times\mathscr{C}:\ |z_1-u(x,t)|<\delta, |z_2-u(y,s)|<\delta,\ \exists\,(x,t),(y,s)\in\bar{\Omega}\times[0,T]\big\},$$

and $H_\delta:\mathscr{C}\times\mathscr{C}\to\mathscr{C}$ be a globally Lipschitz continuous function which is equivalent to H on $\tilde{\Upsilon}_\delta$. Define $\Psi^n,\ 0\le n\le N$, in X_h by

$$\begin{cases} i(\delta_t\Psi^{n+\frac{1}{2}},v_h)-B(\Psi^{n+\frac{1}{2}},v_h)+(H_\delta(\Psi^{n+1},\Psi^n),v_h)=0, & \forall v_h\in X_h,\ 0\le n\le N-1,\\ \Psi^0=u_h^0. \end{cases} \tag{83}$$

We note by Theorem 4 that $\Psi^n(0\le n\le N)$ is the unique solution to (83). For convenience, we split

$$\Psi^n-u^n=(\Psi^n-P_hu^n)+(P_hu^n-u^n):=\theta^n+\rho^n. \tag{84}$$

By (83) and (1), we get

$$i(\theta^{n+1}-\theta^n,v_h)-\frac{\tau}{2}B(\theta^{n+1}+\theta^n,v_h)=i\sum_{k=1}^{4}Y_k,\ \forall v_h\in X_h, \tag{85}$$

where

$$\begin{aligned} Y_1 &:= (\rho^n-\rho^{n+1},v_h),\\ Y_2 &:= \Big(\tau u_t^{n+\frac{1}{2}}-(u^{n+1}-u^n),v_h\Big),\\ Y_3 &:= i\left[\tau B(u^{n+\frac{1}{2}},v_h)-\frac{\tau}{2}\Big(B(u^{n+1},v_h)+B(u^n,v_h)\Big)\right],\\ Y_4 &:= i\ \tau\Big(H_\delta(\Psi^{n+1},\Psi^n)-f_\delta(u^{n+\frac{1}{2}}),v_h\Big). \end{aligned}$$

By Cauchy-Schwarz inequality, Taylor's Theorem, and (37), we have

$$\begin{aligned} \|Y_1\| &\le C_1\tau h^m\|v_h\|,\\ \|Y_2\| &\le C_2\tau^3\|v_h\|,\\ \|Y_3\| &\le \tau\|(-\Delta)^{\frac{\alpha}{2}}((u^{n+1}+u^n)/2-u^{n+\frac{1}{2}})\|\cdot\|v_h\|\\ &\le C_3\tau^3\|v_h\|, \end{aligned}$$

where $C_k(k=1,2,3)>0$ are constants independent of h and τ. For Y_4, we estimate it by the following way

$$\begin{aligned}
Y_4 &= \tau(H_\delta(\Psi^{n+1},\Psi^n)-f_\delta(u^{n+\frac{1}{2}}),v_h)\\
&= \tau\bigg(H_\delta(\Psi^{n+1},\Psi^n)-H_\delta(u^{n+1},u^n),v_h\bigg)\\
&\quad +\tau\bigg(H_\delta(u^{n+1},u^n)-\frac{\beta}{2}|u^{n+\frac{1}{2}}|^2(u^n+u^{n+1}),v_h\bigg)\\
&\quad +\tau\bigg(\frac{\beta}{2}|u^{n+\frac{1}{2}}|^2(u^n+u^{n+1})-f_\delta(u^{n+\frac{1}{2}}),v_h\bigg)\\
&:= \sum_{i=1}^{3}\tau(L_i,v_h).
\end{aligned}$$

For $L_i(i=1,2,3)$, we have

$$\begin{aligned}
&\|L_1\|\le \tilde{L}(\|\Psi^n-u^n\|+\|\Psi^{n+1}-u^{n+1}\|)\le C(\tilde{L})(\|\theta^n\|+\|\theta^{n+1}\|+h^m),\\
&\|L_j\|\le C(u,\beta)\tau^2,\quad j=2,3,
\end{aligned}$$

where $C(\tilde{L})>0$ is a constant dependent of Lipschitz constant $\tilde{L}$ and $C(u,\beta)>0$ is constant dependent of u and β. Hence, we obtain

$$\|Y_4\|\le C_4\tau(\|\theta^n\|+\|\theta^{n+1}\|+\tau^2+h^m)\|v_h\|,$$

where $C_4>0$ is constant depending on u, β, and $\tilde{L}$ but independent of h and τ. Let $v_h=(\theta^n+\theta^{n+1})/2$ in (85) and take the imaginary part to arrive at

$$\|\theta^{n+1}\|\le\|\theta^n\|+\hat{C}\tau(\|\theta^n\|+\|\theta^{n+1}\|+\tau^2+h^m). \tag{86}$$

From (86), we get the following inequality when τ is chosen sufficiently small

$$\|\theta^n\|\le C(\tau^2+h^m),$$

where $C=(e^{2\hat{C}T/(1-\hat{C}/2)}-1)/2\hat{C}$. Then, triangle inequality yields

$$\|\Psi^n-u^n\|\le C(\tau^2+h^m). \tag{87}$$

Next, we proceed to get the conclusion (81). By similar deduction as (45), we get the following inequality for any $v_h\in X_h$,

$$\begin{aligned}
|u^n-\Psi^n|_\infty &\le |u^n-v_h|_\infty+|v_h-P_hu^n|_\infty+|P_hu^n-\Psi^n|_\infty\\
&\le |u^n-v_h|_\infty+Ch^{-\frac{1}{2}}(\|v_h-u^n\|+\|u^n-P_hu^n\|)+|\theta^n|_\infty\\
&\le |u^n-v_h|_\infty+Ch^{-\frac{1}{2}}\|v_h-u^n\|+Ch^{m-\frac{1}{2}}+Ch^{-\frac{1}{2}}\tau^2.
\end{aligned}$$

According to (34) and given condition $\tau = o(h^{1/4})$, we can see that for sufficient small $h > 0$,

$$|u^n - \Psi^n|_\infty \le \frac{\delta}{2}, \ \forall\, 0 \le n \le N, \tag{88}$$

which implies $\Psi^n \in \tilde{\Upsilon}_\delta$. Then, by (83), we get $\Psi^n = U^n$. Now, it would be straight to get (81) from (87) by substituting Ψ^n by U^n.

By (38), the stated result for $\alpha = 3/2$ follows analogously. Thus, we complete the proof. □

5 Iterative algorithm

It is noted that the discrete scheme (51) with initial condition (50) is a fully implicit method. In order to solve the discrete problem, we propose a linearized iterative algorithm to compute the solution of the fully discrete finite element scheme in this section.

As [33], we define the following algorithm, that is to find $U^{n+1(s+1)} \in X_h$, such that for $0 \le n \le N-1$, $s \ge 0$,

$$i\left(\frac{U^{n+1(s+1)} - U^n}{\tau}, v_h\right) - \frac{1}{2}B(U^{n+1(s+1)} + U^n, v_h) + (H^{n+1(s)}, v_h) = 0, \tag{89}$$

with the boundary condition

$$U^{n+1(s+1)} = 0, \quad on\ \partial\Omega, \tag{90}$$

where

$$H^{n+1(s)} = \frac{\beta}{4}(|U^{n+1(s)}|^2 + |U^n|^2)(U^{n+1(s)} + U^n), \tag{91}$$

and

$$U^{n+1(0)} = \begin{cases} U^n, & n = 0, \\ 2U^n - U^{n-1}, & n \ge 1. \end{cases} \tag{92}$$

We note that the scheme (89)–(92) is linearized and once we get the solution $U^{n+1(s+1)}$, then the discrete solution of problem (51) is reached if $U^{n+1(s+1)}$ converges. We also notice that the global stiff matrix of (89) does not change as the process of iterative, which can reduce the amount of computation efficiently.

6 Numerical experiments

In this section, we give some numerical examples to support our theoretical analysis proposed in the previous sections. Consider the following nonlinear Riesz space-fractional Schrödinger equation [30]

$$iu_t - (-\triangle)^{\frac{\alpha}{2}}u + 2|u|^2u = 0, \tag{93}$$

with the initial value condition

$$u(x,0) = sech(x) \cdot exp(2ix). \tag{94}$$

When $\alpha = 2$, the problem becomes the classic integer-order Schrödinger equation and by [30, 32], we know the exact solution is given by

$$u(x,t) = sech(x-4t)\cdot exp(i(2x-3t)). \tag{95}$$

Since the initial value $u(x,0)$ exponentially decays to zero with the variable x away from the origin, the wave function can be negligible outside the interval $[a,b]$ for $a \ll 0$ and $b \gg 0$, so that we can set $u(a,t) = u(b,t) = 0$ when a is chosen sufficiently small and b is chosen sufficiently big. In this section, we set $a = -20,\ b = 20$ and the outer iteration tolerance $tol = 10^{-8}$.

According to the fully discrete iterative FEM scheme (89)–(92), comparing with finite element methods to solve traditional differential equations, the mainly difficult parts are the computing of bilinear form $B(U^{n+1(s+1)} + U^n, v_h)$ and nonlinear item $(H^{n+1(s)}, v_h)$. In this paper, we take the Lagrange linear shape functions for example. Let

$$\Omega = (-20,20),\ \bar{\Omega} = \bigcup_{i=1}^{N} e_i,\ e_i = [x_i, x_{i+1}](i = 1\cdots N),\ L^i = (L_1^i, L_2^i),$$

where

$$L_1^i = \begin{cases} \frac{x_{i+1}-x}{h}, & x \in e_i, \\ 0, & else, \end{cases}$$

and

$$L_2^i = \begin{cases} \frac{x-x_i}{h}, & x \in e_i, \\ 0, & else. \end{cases}$$

Since the fractional derivative is a non-local operator, the compute of bilinear form $B(\cdot,\cdot)$ is more complex than the case of integer one. Bu et al. [6] gives detailed statements, thus we do not need state here. For the computing of the nonlinear item $(H^{n+1(s)}, v_h)$, we write the main steps as follows. Let

$$U^{n+1(s)}|_{e_i} = a_s L_1^i + b_s L_2^i,\ \ U^n|_{e_i} = aL_1^i + bL_2^i. \tag{96}$$

It is obvious that

$$\begin{aligned}(H^{n+1(s)}, v_h) &= \left(\frac{1}{2}(|U^{n+1(s)}|^2 + |U^n|^2)(U^{n+1(s)} + U^n), v_h\right) \\ &= \sum_{i=1}^{N}\left(\frac{1}{2}(|U^{n+1(s)}|^2 + |U^n|^2)(U^{n+1(s)} + U^n), v_h\right)_i, \end{aligned} \tag{97}$$

where $(u,v)_i := \int_{x_i}^{x_{i+1}} u\bar{v}dx$. When $x \in e_i$, we have

$$|U^{n+1(s)}|^2 = |a_s L_1^i + b_s L_2^i|^2 = |a_s|^2(L_1^i)^2 + |b_s|^2(L_2^i)^2 + F(a_s,b_s)L_1^i L_2^i, \tag{98}$$

and

$$|U^n|^2 = |aL_1^i + bL_2^i|^2 = |a|^2(L_1^i)^2 + |b|^2(L_2^i)^2 + F(a,b)L_1^i L_2^i, \tag{99}$$

where $F(x, y) := 2Re(x)Re(y) + 2Im(x)Im(y)$. Assume $v_h|_{e_i} = v_1^i L_1^i + v_2^i L_2^i$ and combine (96)–(99) to arrive at

$$(H^{n+1(s)}, v_h)_i = (P_1, P_2)A_1^i(v_1^i, v_2^i)^T + (P_3, P_4)A_2^i(v_1^i, v_2^i)^T, \tag{100}$$

where

$$\begin{aligned}
(P_1, P_2) &:= \Big((|a_s|^2 + |a|^2)(a_s + a), (|b_s|^2 + |b|^2)(b_s + b)\Big),\\
(P_3, P_4) &:= \Big((|a_s|^2 + |a|^2)(b_s + b) + Q(a_s + a), (|b_s|^2 + |b|^2)(a_s + a) + Q(b_s + b)\Big),\\
Q &:= \Big(2Re(a_s)Re(b_s) + 2Im(a_s)Im(b_s)\Big) + \Big(2Re(a)Re(b) + 2Im(a)Im(b)\Big),
\end{aligned}$$

and

$$A_1^i := \begin{pmatrix} ((L_1^i)^3, L_1^i)_i & ((L_1^i)^3, L_2^i)_i \\ ((L_2^i)^3, L_1^i)_i & ((L_2^i)^3, L_2^i)_i \end{pmatrix} = \begin{pmatrix} \frac{h}{5} & \frac{h}{20} \\ \frac{h}{20} & \frac{h}{5} \end{pmatrix},$$

$$A_2^i := \begin{pmatrix} ((L_1^i)^2 L_2^i, L_1^i)_i & ((L_1^i)^2 L_2^i, L_2^i)_i \\ ((L_2^i)^2 L_1^i, L_1^i)_i & ((L_2^i)^2 L_1^i, L_2^i)_i \end{pmatrix} = \begin{pmatrix} \frac{h}{20} & \frac{h}{30} \\ \frac{h}{30} & \frac{h}{20} \end{pmatrix}.$$

The following step is to construct the global stiff matrix similar as the process of classical integer-order problems. Based on above analysis, we obtain the following results.

Firstly, we take $\tau = h = 0.05$, then the figures of the numerical solutions to (93)–(94) for different α are depicted in Figs. 1, 2, 3, and 4. We note that the order α will affect the shape of the soliton. When α becomes smaller, the shape of the soliton will change more quickly. As shown in [30], this property of the fractional Schrödinger equation can be used in physics to modify the shape of wave without the change of the nonlinearity and dispersion effects. From Figs. 1, 2, 3, and 4, when α tends to 2, the numerical solutions of the nonlinear fractional Schrödinger equation are convergent

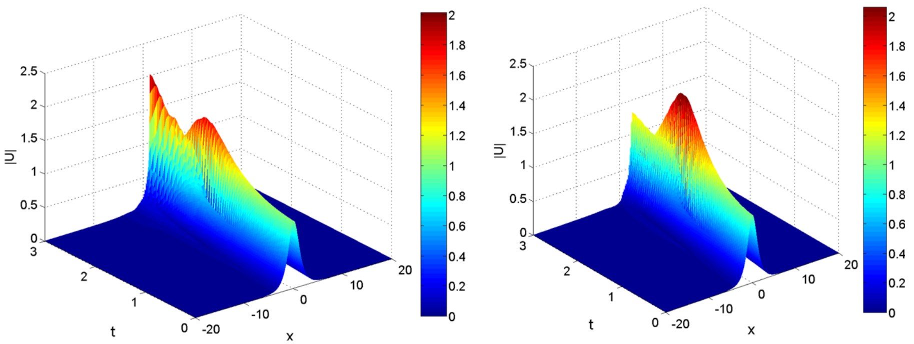

Fig. 1 FEM numerical solutions for $\alpha = 1.1$ (*left*) and $\alpha = 1.3$ (*right*)

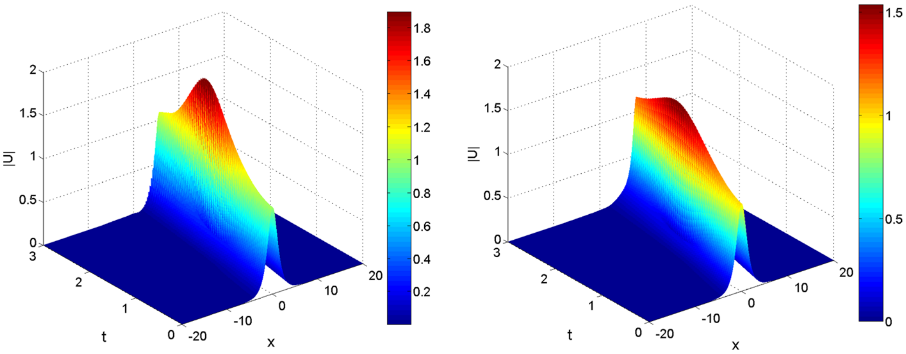

Fig. 2 FEM numerical solutions $\alpha = 1.5$ (*left*) and $\alpha = 1.7$ (*right*)

to the solutions of the usual classical integer one. By the view of Fig. 5, we see that it will not only change the height and width of the solitary wave solution but produce two turning points.

Secondly, numerical accuracy of the proposed iterative finite element scheme is examined. For $\alpha = 2$, the numerical solution of the fractional equation is convergent to the classical non-fractional one and the exact solution is given by (95). As expected, Table 1 shows that the method has second order accuracy for $\alpha = 2$ both in temporal and spatial directions. However, for $1 < \alpha < 2$, there are no way to get the exact solution of problem (93)–(94). Accordingly, we have to get the 'exact' solution by choosing a very fine mesh and a sufficiently small time step. In this section, we select $h = 0.01$ and $\tau = 0.00005$. Table 2 confirms the theoretical accuracy stated in Theorem 5. Here, u is the 'exact' solution, u_h is the numerical one, and $e_h := u - u_h$ is the error function.

Finally, we compute the discrete conservation laws to confirm Theorem 3. Let $\tau = h = 0.05$. Tables 3 and 4 show the values of mass Q^n and energy E^n at different time for different α. Figure 6 depicts the evolution of mass Q^n and energy E^n. By Fig. 6, we note that the mass Q^n is independent of the time and the value of α, as a result, the four curves in the left figure of Fig. 6 overlap with each other. However, the energy E^n is independent of the time but dependent of the value of α. Meanwhile,

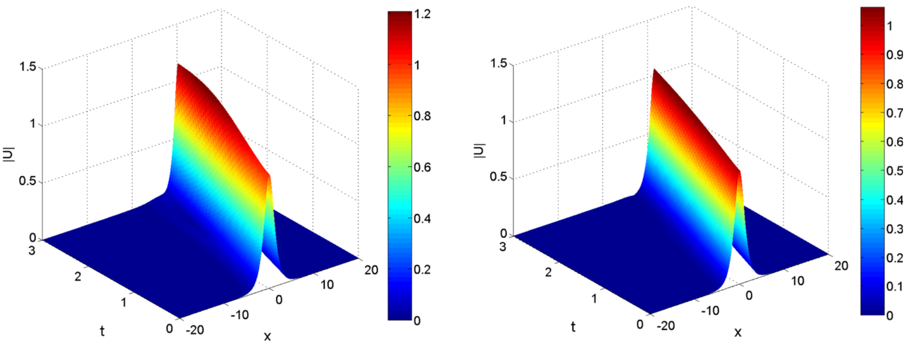

Fig. 3 FEM numerical solutions for $\alpha = 1.9$ (*left*) and $\alpha = 1.99$ (*right*)

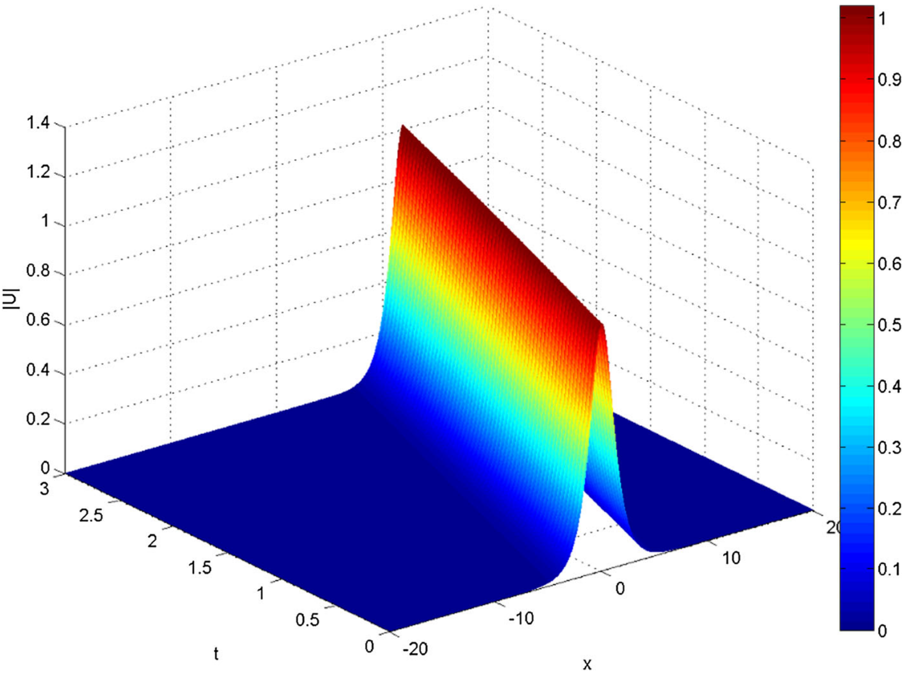

Fig. 4 FEM numerical solutions for $\alpha = 2.0$

by Tables 3 and 4, the different values of mass Q^n and energy E^n for fixed α mainly comes from the choose of the iteration tolerance (10^{-8}), therefore, once we reduce the iteration tolerance, the conservation results will be better. All these show that the scheme (89)–(91) preserves the mass and energy conservation very well.

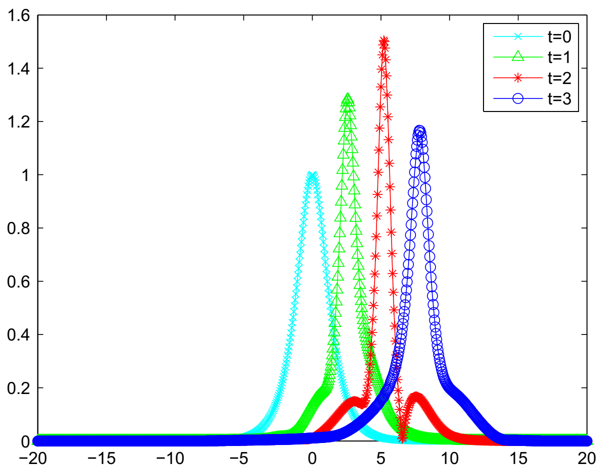

Fig. 5 FEM numerical solutions at the time $t = 0,\ 1,\ 2,\ 3$ for $\alpha = 1.7$

Table 1 The error and the order of convergence using piecewise P^1 elements for $\alpha = 2$ with $\tau = 0.1h$

τ	h	$\|u - u_h\|$	Order
0.02	0.2	2.3491e-01	
0.01	0.1	6.0874e-02	1.9482
0.005	0.05	1.5352e-02	1.9874
0.0025	0.025	3.8464e-03	1.9969
0.00125	0.0125	9.6214e-04	1.9992

Table 2 The error $\|u - u_h\|$ and the order of convergence using piecewise P^1 elements for different α ($\tau = 0.1h$)

α	$h = 0.1$	$h = 0.05$	Order
1.1	1.0378e-01	2.4162e-02	2.1027
1.3	1.6540e-01	3.8015e-02	2.1214
1.5	2.1429e-01	4.5236e-02	2.2441
1.7	2.2121e-01	4.6317e-02	2.2558
1.9	2.2996e-01	5.1283e-02	2.1648
1.99	2.4834e-01	5.7698e-02	2.1057

Table 3 The value of Q^n at different time for different α

T	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.9$	$\alpha = 2.0$
0	1.9963931340652	1.9963931340653	1.9963931340653	1.9963931340653
1/2	1.9963931341950	1.9963931321130	1.9963931349491	1.9963931348856
1	1.9963931347466	1.9963931361416	1.9963931370828	1.9963931357332
3/2	1.9963931381068	1.9963931361416	1.9963931422997	1.9963931366545
2	1.9963931307797	1.9963931250771	1.9963931449086	1.9963931376258
5/2	1.9963931305633	1.9963931266803	1.9963931471321	1.9963931386181
3	1.9963931282832	1.9963931286137	1.9963931519087	1.9963931396030

Table 4 The value of E^n at different time for different α

T	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.9$	$\alpha = 2.0$
0	2.9633848214952	4.4825224320512	6.6138662499513	7.3282437524857
1/2	2.9633848218691	4.4825224284203	6.6138662506749	7.3282437533072
1	2.9633848210085	4.4825224280589	6.6138662501820	7.3282437537592
3/2	2.9633848240941	4.4825224399996	6.6138662458918	7.3282437542151
2	2.9633848271497	4.4825224411663	6.6138662474977	7.3282437546740
5/2	2.9633848278166	4.4825224425385	6.6138662502774	7.3282437551357
3	2.9633848379934	4.4825224419623	6.6138662481048	7.3282437556003

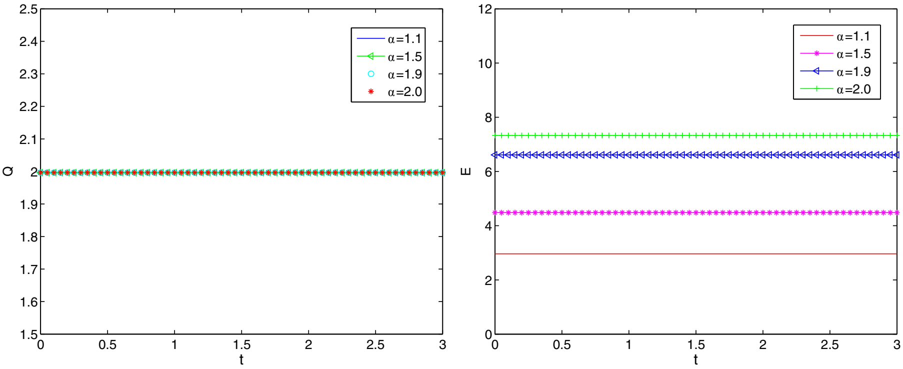

Fig. 6 Evolution of mass Q^n (*left*) and energy E^n (*right*) for different value of α

7 Conclusion

In this paper, Crank-Nicolson scheme in temporal direction and finite element method in spatial direction are used to solve a class of nonlinear Riesz space-fractional Schrödinger equations. We analyze the conservation and convergence properties of the semi-discrete scheme and the fully discrete one. Meanwhile, we give a rigorous analysis of the uniqueness and existence of the fully discrete solution. Also, we propose a linearized iterative finite element algorithm for performing the nonlinear CN finite element scheme. Numerical tests show the scheme is efficient. From the processes of numerical tests, we note that the obtained global rigidity matrix of nonlinear Riesz space-fractional Schrödinger equation by the finite element method is dense which shows the computing of inner product of fractional derivative is more elaborate than integer one.

We remark that it is easy to extend the conservation, convergence results of the semi-discrete and the fully discrete schemes to the cases of multi-dimensional spaces. However, the proof of the unique solvability of discrete solutions in the cases of multi-dimensional spaces is totally different from the one-dimensional case, which is due to the different forms of fractional Gagliardo-Nirenberg inequality in different dimensions. This needs to be further investigated.

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