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ORIGINAL PAPER

Non-polynomial spline method for the solution of two-dimensional linear wave equations with a nonlinear source term

Homa Zadvan[1] **· Jalil Rashidinia**[1]

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Abstract In this paper, two classes of methods are developed for the solution of two space dimensional wave equations with a nonlinear source term. We have used non-polynomial cubic spline function approximations in both space directions. The methods involve some parameters, by suitable choices of the parameters, a new high accuracy three time level scheme of order $O(h^4 + k^4 + \tau^2 + \tau^2h^2 + \tau^2k^2)$ has been obtained. Stability analysis of the methods have been carried out. The results of some test problems are included to demonstrate the practical usefulness of the proposed methods. The numerical results for the solution of two dimensional sine-Gordon equation are compared with those already available in literature.

Keywords Non-polynomial spline approximation · Two-dimensional wave equation · Stability analysis · Sine-Gordon equation

Mathematics Subject Classification (2010) 65D07 · 65M12 · 65M22

1 Introduction

We consider the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + F(u), \qquad (x, y) \in R, \quad t > 0, \tag{1}$$

✉ Jalil Rashidinia
jal.rashidinia@iaustb.ac.ir

Homa Zadvan
homa.zadvan@iauctb.ac.ir

[1] Department of Mathematics, Central Tehran Branch, Islamic Azad University, 13185.768, Tehran, Iran

where

$$R = \{(x, y); L_{0x} < x < L_{1x}, L_{0y} < y < L_{1y}\}$$

subjected to the initial conditions

$$u(x, y, 0) = \phi(x, y), \quad u_t(x, y, 0) = \psi(x, y). \quad L_{0x} \leq x \leq L_{1x}, L_{0y} \leq y \leq L_{1y}, \tag{2}$$

and with the following boundary conditions

$$\begin{cases} u(L_{0x}, y, t) = f_0(y, t), & u(L_{1x}, y, t) = f_1(y, t), \\ u(x, L_{0y}, t) = g_0(x, t), & u(x, L_{1y}, t) = g_1(x, t), \quad t > 0, \end{cases} \tag{3}$$

where $u = u(x, y, t)$ is a real valued sufficiently differentiable function and c^2 is a known constant(often representing wave speed).

Further, we assume that $F : \Re \to \Re$ is the nonlinear smooth function with the following properties [1]

$$i) F(0) = 0, \quad ii) uF(u) \leq 0, \quad iii) \mid F'(u) \mid \leq \upsilon(1 + \mid u \mid^{p-1}),$$

for some $\upsilon > 0$, where $p > 1$. Also let $\phi(x, y)$ and $\psi(x, y)$ are sufficiently differentiable functions of as higher order as possible.

Numerical techniques for the solution of hyperbolic equations have been discussed and developed in the literature. Several finite difference schemes have been presented for one-dimensional linear hyperbolic equations [2–4]. Ragget and Wilson [5] used cubic spline and finite difference approximations for such equations. The numerical solution of one-dimensional linear hyperbolic equations obtained by Rashidinia et al. [6] and Ding et al. [7] based on non-polynomial cubic spline approximation in space and finite difference approximation in time direction. Liu et al. [8] used quartic spline approximation in space direction. Mohanty and Gopal [9, 10] have studied one-dimensional nonlinear wave equation based on cubic spline and compact finite difference methods. Rashidinia and Mohammadi proposed methods using tension spline approximation for the solution of nonlinear Klein-Gordon and nonlinear sine-Gordon equations [11, 12]. The solution of one-dimensional hyperbolic telegraph equation have been approximated by applying radial basis functions (RBF) [13], by cubic B-spline collocation method [14] and by quartic B-spline collocation method [15].

The numerical solution of two-dimensional linear hyperbolic equations has been proposed in [16–18] by using unconditionally stable implicit difference schemes which have second-order accuracy in both space and time. Also for such equations, Dehghan et al. applied compact finite difference approximations of fourth-order for spatial derivatives and collocation method for time component [19]. Meshless methods have been presented for the solution of two-dimensional linear hyperbolic problems, so that Dehghan et al. [20] used collocation points and approximated solution by using thin plate splines (TPS) radial basis functions (RBF). Also in [21] a combination of a mesh free boundary knot method and analog equation method is proposed for such hyperbolic problems. Piperno presented two local time stepping algorithms using discontinuous Galerkin time domain (DGTD) methods for wave problems [22]. Shi and Li [23] discussed the semi-discrete finite element method with rectangular mesh for nonlinear hyperbolic equations. Chabassier and Joly applied finite elements for a class of nonlinear second-order wave equations. They develop a family of three-point schemes that conserve discrete energies in time [24]. Chawla et al. presented linearly implicit one step schemes, for the time integration of second-order nonlinear hyperbolic equations in one and also two space dimensions [25, 26]. The numerical solution of the nonlinear sine-Gordon equation in two space variables is proposed in [27], by a method arises from a two-step one parameter method for solution of second order ordinary differential equations. This method which is second order in time and first order other wise, applied explicitly. Also, this equation is solved in [28], by a method using collocation points and approximating the solution employing thin plate spline (TPS) radial basis functions (RBF). A numerical technique based on polynomial differential quadrature method is proposed in [29] for the solution of two-dimensional sine-Gordon equation. This method reduce the problem into a system of second-order linear differential equations. Then the obtained system changed into a system of ordinary differential equations.

In this paper, we have developed a new implicit three time level method of order two in time and order four in both spaces, for the solution of two space dimensional wave equations with nonlinear source term. We have used non-polynomial cubic spline function approximations in both x and y spatial directions and finite difference approximation in temporal direction. The methods involve some parameters, by suitable choices of the parameters two schemes of orders $O(h^2+k^2+\tau^2+\tau^2h^2+\tau^2k^2)$ and $O(h^4+k^4+\tau^2+\tau^2h^2+\tau^2k^2)$ can be obtained. Stability analysis of the presented methods have been given. Some test examples are provided to demonstrate the viability and practical usefulness of our methods. The errors of the proposed methods for the solution of two dimensional nonlinear sine-Gordon equation are compared with the results given in [27–29].

2 Non-polynomial spline functions

The solution domain, $\Omega = \{(x, y, t); L_{0x} \leq x \leq L_{1x}, L_{0y} \leq y \leq L_{1y}, t > 0\}$, is divided to $(N+1)*(N+1)*J$ mesh. The grid points are (x_l, y_m, t_j), where $x_l = L_{0x} + lh; h = \frac{L_{1x}-L_{0x}}{N+1}, l = 0, 1, ..., N+1, y_m = L_{0y} + mk; k = \frac{L_{1y}-L_{0y}}{N+1}$, $m = 0, ..., N+1$ and $t_j = j\tau; 0 < j \leq J$, N and J are positive integers.

We let $s_{1m}(x)$ be the non-polynomial spline function which interpolates $u(x, y_m)$ in each segment $[(x_l, y_m), (x_{l+1}, y_m)]$ and is defined by

$$s_{1m}(x) = a_{1l} + b_{1l}(x - x_l) + c_{1l} sin\lambda_1(x - x_l) + d_{1l} cos\lambda_1(x - x_l), l = 1, ..., N, \quad (4)$$

where a_{1l}, b_{1l}, c_{1l} and d_{1l} are unknown coefficients and λ_1 is arbitrary parameter. Also, let $s_{2l}(y)$ be the non-polynomial spline function interpolating $u(x_l, y)$ in each segment $[(x_l, y_m), (x_l, y_{m+1})]$ and is defined by

$$s_{2l}(y) = a_{2m} + b_{2m}(y - y_m) + c_{2m} sin\lambda_2(y - y_m) + d_{2m} cos\lambda_2(y - y_m), m = 1, ..., N, \quad (5)$$

where a_{2m}, b_{2m}, c_{2m} and d_{2m} are unknown coefficients and λ_2 is arbitrary parameter. To derive explicit expressions for determining coefficients, we first denote

$$s_{1m}(x_l) = u_{l,m}, \quad s_{1m}(x_{l+1}) = u_{l+1,m},$$
$$s''_{1m}(x_l) = M_{1l,m}, \quad s''_{1m}(x_{l+1}) = M_{1l+1,m}, \tag{6}$$
$$s_{2l}(y_m) = u_{l,m}, \quad s_{2l}(y_{m+1}) = u_{l,m+1},$$
$$s''_{2l}(y_m) = M_{2l,m}, \quad s''_{2l}(y_{m+1}) = M_{2l,m+1}. \tag{7}$$

From (4) and (6) and after algebraic manipulations, we derive

$$a_{1l} = u_{l,m} + \frac{1}{\lambda_1^2} M_{1l,m},$$

$$b_{1l} = \frac{1}{h}(u_{l+1,m} - u_{l,m}) + \frac{1}{\omega_1 \lambda_1}(M_{1l+1,m} - M_{1l,m}),$$

$$c_{1l} = \frac{1}{\lambda_1^2}(cot(\omega_1) M_{1l,m} - csc(\omega_1) M_{1l+1,m}), \quad d_{1l} = \frac{-1}{\lambda_1^2} M_{1l,m},$$

where $\omega_1 = \lambda_1 h$.

Also from (5) and (7), we get

$$a_{2m} = u_{l,m} + \frac{1}{\lambda_2^2} M_{2l,m},$$

$$b_{2m} = \frac{1}{k}(u_{l+1,m} - u_{l,m}) + \frac{1}{\omega_2 \lambda_2}(M_{2l+1,m} - M_{2l,m}),$$

$$c_{2m} = \frac{1}{\lambda_2^2}(cot(\omega_2) M_{2l,m} - csc(\omega_2) M_{2l+1,m}), \quad d_{2m} = \frac{-1}{\lambda_2^2} M_{2l,m},$$

where $\omega_2 = \lambda_2 k$.

From the continuity of the first derivatives of spline functions $s_{1m}(x)$ and $s_{2l}(y)$ at (x_l, y_m), we can obtain the following useful consistency relations

$$u_{l-1,m} - 2u_{l,m} + u_{l+1,m} = h^2(\alpha_1 M_{1l-1,m} + 2\beta_1 M_{1l,m} + \alpha_1 M_{1l+1,m}), \tag{8}$$

$$u_{l,m-1} - 2u_{l,m} + u_{l,m+1} = k^2(\alpha_2 M_{2l,m-1} + 2\beta_2 M_{2l,m} + \alpha_2 M_{2l,m+1}), \tag{9}$$

where

$$\alpha_1 = \frac{1}{\omega_1^2}(\omega_1 csc(\omega_1) - 1), \qquad \beta_1 = \frac{1}{\omega_1^2}(1 - \omega_1 cot(\omega_1)),$$
$$\alpha_2 = \frac{1}{\omega_2^2}(\omega_2 csc(\omega_2) - 1), \qquad \beta_2 = \frac{1}{\omega_2^2}(1 - \omega_2 cot(\omega_2)).$$

It can be shown that the above non-polynomial splines defined in (4) and (5) reduce to standard cubic splines when the parameters tend to zero. So that when $\lambda_1 \to 0$, that $\omega_1 \to 0$, then $(\alpha_1, \beta_1) \to (1/6, 1/3)$ and also when $\lambda_2 \to 0$, then $(\alpha_2, \beta_2) \to (1/6, 1/3)$ and the consistency relations of tension splines defined in (8) and (9) reduce to the following ordinary cubic spline relations, respectively

$$(u_{l-1,m} - 2u_{l,m} + u_{l+1,m}) = \frac{h^2}{6}(M_{1l-1,m} + 4M_{1l,m} + M_{1l+1,m}). \tag{10}$$

$$(u_{l,m-1} - 2u_{l,m} + u_{l,m+1}) = \frac{k^2}{6}(M_{2l,m-1} + 4M_{2l,m} + M_{2l,m+1}). \tag{11}$$

3 Spline numerical methods

We next develop an approximation for (1) in which we use the non-polynomial cubic spline approximation in both spatial directions and finite difference approximation for the temporal direction. At the grid point (x_l, y_m, t_j), (1) may be discretized by

$$(u_{tt})_{l,m}^{j} = c^2((u_{xx})_{l,m}^{j} + (u_{yy})_{l,m}^{j}) + F_{l,m}^{j}, \tag{12}$$

where $F_{l,m}^{j} = F(u_{l,m}^{j})$.

For the time derivative, we use the following finite difference approximation

$$(\bar{u}_{tt})_{l,m}^{j} = \frac{u_{l,m}^{j-1} - 2u_{l,m}^{j} + u_{l,m}^{j+1}}{\tau^2} = (u_{tt})_{l,m}^{j} + O(\tau^2), \tag{13}$$

and the non-polynomial cubic spline function approximations for x and y space derivatives are as follows

$$(\bar{u}_{xx})_{l,m}^{j} = M_{1l,m}^{j} + O(h^2), \tag{14}$$

$$(\bar{u}_{yy})_{l,m}^{j} = M_{2l,m}^{j} + O(k^2). \tag{15}$$

By using (13)–(15) and after neglecting the truncation errors, (12) may be written as follows

$$\frac{u_{l,m}^{j-1} - 2u_{l,m}^{j} + u_{l,m}^{j+1}}{\tau^2} = c^2(M_{1l,m}^{j} + M_{2l,m}^{j}) + F_{l,m}^{j}.$$

or

$$M_{1l,m}^{j} + M_{2l,m}^{j} = \frac{u_{l,m}^{j-1} - 2u_{l,m}^{j} + u_{l,m}^{j+1}}{c^2\tau^2} - \frac{1}{c^2}F_{l,m}^{j}. \tag{16}$$

Then we can conclude that

$$M_{1l\pm1,m}^{j} + M_{2l\pm1,m}^{j} = \frac{u_{l\pm1,m}^{j-1} - 2u_{l\pm1,m}^{j} + u_{l\pm1,m}^{j+1}}{c^2\tau^2} - \frac{1}{c^2}F_{l\pm1,m}^{j}, \tag{17}$$

and

$$M_{1l,m\pm1}^{j} + M_{2l,m\pm1}^{j} = \frac{u_{l,m\pm1}^{j-1} - 2u_{l,m\pm1}^{j} + u_{l,m\pm1}^{j+1}}{c^2\tau^2} - \frac{1}{c^2}F_{l,m\pm1}^{j}. \tag{18}$$

Consistency relation (8) in the j-th time level can be written as

$$(u_{l-1,m}^{j} - 2u_{l,m}^{j} + u_{l+1,m}^{j}) = h^2(\alpha_1 M_{1l-1,m}^{j} + 2\beta_1 M_{1l,m}^{j} + \alpha_1 M_{1l+1,m}^{j}). \tag{19}$$

Then we have

$$(u_{l-1,m\pm1}^{j} - 2u_{l,m\pm1}^{j} + u_{l+1,m\pm1}^{j}) = h^2(\alpha_1 M_{1l-1,m\pm1}^{j} + 2\beta_1 M_{1l,m\pm1}^{j} + \alpha_1 M_{1l+1,m\pm1}^{j}). \tag{20}$$

Similarly consistency relation (9) in the j-th time level can be written as

$$(u_{l,m-1}^{j} - 2u_{l,m}^{j} + u_{l,m+1}^{j}) = k^2(\alpha_2 M_{2l,m-1}^{j} + 2\beta_2 M_{2l,m}^{j} + \alpha_2 M_{2l,m+1}^{j}). \tag{21}$$

Then we have

$$(u^j_{l\pm1,m-1}-2u^j_{l\pm1,m}+u^j_{l\pm1,m+1})=k^2(\alpha_2 M^j_{2l\pm1,m-1}+2\beta_2 M^j_{2l\pm1,m}+\alpha_2 M^j_{2l\pm1,m+1}). \tag{22}$$

At first we multiply (19) by $2k^2\beta_2$ and (20) by $k^2\alpha_2$ and (21) by $2h^2\beta_1$ and (22) by $h^2\alpha_1$, then adding these equations with each other we obtain

$$\begin{aligned}
&2k^2\beta_2(u^j_{l-1,m}-2u^j_{l,m}+u^j_{l+1,m})+k^2\alpha_2(u^j_{l-1,m-1}-2u^j_{l,m-1}+u^j_{l+1,m-1})\\
&+k^2\alpha_2(u^j_{l-1,m+1}-2u^j_{l,m+1}+u^j_{l+1,m+1})+2h^2\beta_1(u^j_{l,m-1}-2u^j_{l,m}+u^j_{l,m+1})\\
&+h^2\alpha_1(u^j_{l-1,m-1}-2u^j_{l-1,m}+u^j_{l-1,m+1})+h^2\alpha_1(u^j_{l+1,m-1}-2u^j_{l+1,m}+u^j_{l+1,m+1})\\
=\ &2h^2k^2\alpha_1\beta_2(M^j_{1l-1,m}+M^j_{2l-1,m}+M^j_{1l+1,m}+M^j_{2l+1,m})\\
&+2h^2k^2\alpha_2\beta_1(M_{1l,m-1}+M_{2l,m-1}+M_{1l,m+1}+M_{2l,m+1})\\
&+h^2k^2\alpha_1\alpha_2(M_{1l-1,m-1}+M_{2l-1,m-1}+M_{1l+1,m-1}+M_{2l+1,m-1})\\
&+h^2k^2\alpha_1\alpha_2(M_{1l-1,m+1}+M_{2l-1,m+1}+M_{1l+1,m+1}+M_{2l+1,m+1})\\
&+4h^2k^2\beta_1\beta_2(M_{1l,m}+M_{2l,m}).
\end{aligned}$$

By substituting $M^j_{1l,m}+M^j_{2l,m}$ and $M^j_{1l\pm1,m}+M^j_{2l\pm1,m}$ and $M^j_{1l,m\pm1}+M^j_{2l,m\pm1}$ with their equivalent expressions in (16)–(18) and after simplification we derive

$$\begin{aligned}
&p_1u^{j+1}_{l-1,m-1}+p_3u^{j+1}_{l,m-1}+p_1u^{j+1}_{l+1,m-1}+p_2u^{j+1}_{l-1,m}+p_4u^{j+1}_{l,m}+p_2u^{j+1}_{l+1,m}\\
&+p_1u^{j+1}_{l-1,m+1}+p_3u^{j+1}_{l,m+1}+p_1u^{j+1}_{l+1,m+1}\\
=\ &-p_1u^{j-1}_{l-1,m-1}-p_3u^{j-1}_{l,m-1}-p_1u^{j-1}_{l+1,m-1}-p_2u^{j-1}_{l-1,m}-p_4u^{j-1}_{l,m}-p_2u^{j-1}_{l+1,m}\\
&-p_1u^{j-1}_{l-1,m+1}-p_3u^{j-1}_{l,m+1}-p_1u^{j-1}_{l+1,m+1}\\
&+p_5u^j_{l-1,m-1}+p_7u^j_{l,m-1}+p_5u^j_{l+1,m-1}+p_6u^j_{l-1,m}+p_8u^j_{l,m}+p_6u^j_{l+1,m}\\
&+p_5u^j_{l-1,m+1}+p_7u^j_{l,m+1}+p_5u^j_{l+1,m+1}+\\
&+\tau^2(p_1F^j_{l-1,m-1}+p_3F^j_{l,m-1}+p_1F^j_{l+1,m-1}+p_2F^j_{l-1,m}+p_4F^j_{l,m}+p_2F^j_{l+1,m}\\
&+p_1F^j_{l-1,m+1}+p_3F^j_{l,m+1}+p_1F^j_{l+1,m+1}),
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
&p_1=\alpha_1\alpha_2,\quad p_2=2\alpha_1\beta_2,\quad p_3=2\alpha_2\beta_1,\quad p_4=4\beta_1\beta_2,\\
&p_5=c^2\tau^2(\alpha_2/h^2+\alpha_1/k^2)+2\alpha_1\alpha_2,\\
&p_6=2c^2\tau^2(\beta_2/h^2-\alpha_1/k^2)+4\alpha_1\beta_2,\\
&p_7=2c^2\tau^2(\beta_1/k^2-\alpha_2/h^2)+4\alpha_2\beta_1,\\
&p_8=-4c^2\tau^2(\beta_2/h^2+\beta_1/k^2)+8\beta_1\beta_2.
\end{aligned} \tag{24}$$

The scheme (23) is an implicit three time level scheme. For the solution u at first time level, that is at $t=\tau$, we use an explicit scheme of $O(\tau^3)$.

Since the initial values of u and u_t are known explicitly at $t = 0$, this implies all their successive partial derivatives are known at t = 0.

By the help of Taylor expansion, a third-order approximation to u at $t = \tau$ can be written as

$$u_{l,m}^1 = u_{l,m}^0 + \tau(u_t)_{l,m}^0 + \frac{\tau^2}{2}(u_{tt})_{l,m}^0 + \frac{\tau^3}{6}(u_{ttt})_{l,m}^0 + O(\tau^4). \tag{25}$$

From initial values in (2), we have

$$u_{l,m}^0 = \phi_{l,m}, \quad (u_t)_{l,m}^0 = \psi_{l,m}. \tag{26}$$

Using (12), we find

$$(u_{tt})_{l,m}^0 = [c^2(u_{xx} + u_{yy}) + F(u)]_{l,m}^0, \tag{27}$$

$$(u_{ttt})_{l,m}^0 = [c^2(u_{txx} + u_{tyy}) + F_u(u).u_t]_{l,m}^0. \tag{28}$$

Thus, by using (25)-(28), we may obtain the approximate solution u at $t = \tau$ as follows

$$u_{l,m}^1 = \phi_{l,m} + \tau\psi_{l,m} + \frac{\tau^2}{2}[c^2((\phi_{xx} + \phi_{yy})_{l,m}) + F(\phi_{l,m})]$$

$$+ \frac{\tau^3}{6}[c^2((\psi_{xx} + \psi_{yy})_{l,m}) + F_u(\phi_{l,m}).\psi_{l,m}]. \tag{29}$$

Then in each time level, we may obtain the solution from the following matrix form of scheme (23)

$$AU^{j+1} = -AU^{j-1} + BU^j + \tau^2 AF(U^j), \; j = 1, ..., J. \tag{30}$$

where A and B are block tridiagonal matrices of order N^2 and U is the solution vector. Each diagonal block of A is a tridiagonal N-order matrix in the form $tri[p_2, p_4, p_2]$ and each upper diagonal block is equal to each lower diagonal block of A and equals to $tri[p_1, p_3, p_1]$. Similarly each diagonal block of B is a tridiagonal N-order matrix in the form $tri[p_6, p_8, p_6]$ and each off diagonal block of B is equal to $tri[p_5, p_7, p_5]$, where the triangular matrix $T = tri[a, b, a]$ is defined by

$$T = (t_{ij}) = \begin{cases} b & i = j \\ a & \mid i - j \mid = 1 \\ 0 & otherwise \end{cases}$$

4 Truncation error

From (12), we have

$$F_{l,m}^j = (u_{tt})_{l,m}^j - c^2((u_{xx})_{l,m}^j + (u_{yy})_{l,m}^j) \tag{31}$$

Then we may have

$$F_{l+\eta,m+\gamma}^{j} = (u_{tt})_{l+\eta,m+\gamma}^{j} - c^{2}((u_{xx})_{l+\eta,m+\gamma}^{j} + (u_{yy})_{l+\eta,m+\gamma}^{j}), \tag{32}$$

where $\eta, \gamma = 0, \pm 1$. Substituting (31) and (32) in the scheme (23) and then expanding both side of the derived equation, in Taylor series in terms of $u(x_l, y_m, t_j)$ and it's partial derivatives we obtain the truncation error as follows

$$\begin{aligned} T_{l,m}^{j} = \{ & 2(\alpha_2+\beta_2)(2\alpha_1+2\beta_1-1)\frac{\partial^2}{\partial x^2} + 2(\alpha_1+\beta_1)(2\alpha_2+2\beta_2-1)\frac{\partial^2}{\partial y^2} \\ & +(h^2\alpha_1(2\alpha_2+2\beta_2-1)+k^2\alpha_2(2\alpha_1+2\beta_1-1))\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} \\ & +(2h^2(\alpha_2+\beta_2)(\alpha_1-\frac{1}{12}))\frac{\partial^4}{\partial x^4} + (2k^2(\alpha_1+\beta_1)(\alpha_2-\frac{1}{12}))\frac{\partial^4}{\partial y^4} \\ & +\frac{1}{3}\tau^2(\alpha_1+\beta_1)(\alpha_2+\beta_2)\frac{\partial^4}{\partial t^4} \\ & +(h^2k^2\alpha_2(\alpha_1-\frac{1}{12})+\frac{1}{6}h^4\alpha_1(\beta_2-\frac{1}{2}))\frac{\partial^4}{\partial x^4}\frac{\partial^2}{\partial y^2} \\ & +(h^2k^2\alpha_1(\alpha_2-\frac{1}{12})+\frac{1}{6}k^4\alpha_2(\beta_1-\frac{1}{2}))\frac{\partial^2}{\partial x^2}\frac{\partial^4}{\partial y^4} \\ & +\frac{1}{6}h^2\tau^2\alpha_1(\alpha_2+\beta_2)\frac{\partial^2}{\partial x^2}\frac{\partial^4}{\partial t^4} + \frac{1}{6}k^2\tau^2\alpha_2(\alpha_1+\beta_1)\frac{\partial^2}{\partial y^2}\frac{\partial^4}{\partial t^4} \\ & +\frac{1}{6}h^4(\alpha_2+\beta_2)(\alpha_1-\frac{1}{30})\frac{\partial^6}{\partial x^6} + \frac{1}{6}k^4(\alpha_1+\beta_1)(\alpha_2-\frac{1}{30})\frac{\partial^6}{\partial y^6} \\ & +\frac{1}{90}\tau^4(\alpha_1+\beta_1)(\alpha_2+\beta_2)\frac{\partial^4}{\partial t^4} + \ldots\}u_{l,m}^{j} \end{aligned} \tag{33}$$

By choosing suitable values of parameters $\alpha_1, \alpha_2, \beta_1$, and β_2, we achieve the following various classes of methods

(i) If we choose

$$\alpha_1+\beta_1 = \frac{1}{2}, \quad \alpha_2+\beta_2 = \frac{1}{2}, \quad \alpha_1 \neq \frac{1}{12}, \quad \alpha_2 \neq \frac{1}{12}. \tag{34}$$

we obtain various schemes of order $O(h^2+k^2+\tau^2+\tau^2h^2+\tau^2k^2)$. In particular, we can choose $\alpha_1 = \alpha_2 = \frac{1}{6}$ and $\beta_1 = \beta_2 = \frac{1}{3}$.

(ii) If we choose

$$\alpha_1+\beta_1 = \frac{1}{2}, \quad \alpha_2+\beta_2 = \frac{1}{2}, \quad \alpha_1 = \frac{1}{12}, \quad \alpha_2 = \frac{1}{12}. \tag{35}$$

we obtain a new scheme of order $O(h^4+k^4+\tau^2+\tau^2h^2+\tau^2k^2)$.

5 Stability analysis

Now we will discuss the stability of the proposed schemes. First, we consider the homogenous part of scheme (23) and assume $u_{l,m}^{j-1} = v_{l,m}^{j}$, then by setting $Y_{l,m}^{j} = (u_{l,m}^{j}, v_{l,m}^{j})^T$ we have

$$\begin{aligned}
& C_1Y_{l-1,m-1}^{j+1} + C_3Y_{l,m-1}^{j+1} + C_1Y_{l+1,m-1}^{j+1} + C_2Y_{l-1,m}^{j+1} + C_4Y_{l,m}^{j+1} + C_2Y_{l+1,m}^{j+1} \\
& +C_1Y_{l-1,m+1}^{j+1} + C_3Y_{l,m+1}^{j+1} + C_1Y_{l+1,m+1}^{j+1} \\
= \; & D_1Y_{l-1,m-1}^{j} + D_3Y_{l,m-1}^{j} + D_1Y_{l+1,m-1}^{j} + D_2Y_{l-1,m}^{j} + D_4Y_{l,m}^{j} + D_2Y_{l+1,m}^{j} \\
& +D_1Y_{l-1,m+1}^{j} + D_3Y_{l,m+1}^{j} + D_1Y_{l+1,m+1}^{j}, \qquad (36)
\end{aligned}$$

where

$$C_1 = \begin{pmatrix} p_1 & 0 \\ 0 & 1 \end{pmatrix}, \; C_2 = \begin{pmatrix} p_2 & 0 \\ 0 & 1 \end{pmatrix}, \; C_3 = \begin{pmatrix} p_3 & 0 \\ 0 & 1 \end{pmatrix}, C_4 = \begin{pmatrix} p_4 & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} p_5 & p_1 \\ 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} p_6 & p_2 \\ 1 & 0 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} p_7 & p_3 \\ 1 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} p_8 & p_4 \\ 1 & 0 \end{pmatrix}.$$

Let $\bar{Y}_{l,m}^{j}$ be the numerical value of $Y_{l,m}^{j}$ then $\varepsilon_{l,m}^{j} = Y_{l,m}^{j} - \bar{Y}_{l,m}^{j}$ is the error vector at the $j-th$ time level. From (36), we have

$$\begin{aligned}
& C_1\varepsilon_{l-1,m-1}^{j+1} + C_3\varepsilon_{l,m-1}^{j+1} + C_1\varepsilon_{l+1,m-1}^{j+1} + C_2\varepsilon_{l-1,m}^{j+1} + C_4\varepsilon_{l,m}^{j+1} + C_2\varepsilon_{l+1,m}^{j+1} \\
& +C_1\varepsilon_{l-1,m+1}^{j+1} + C_3\varepsilon_{l,m+1}^{j+1} + C_1\varepsilon_{l+1,m+1}^{j+1} \\
= \; & D_1\varepsilon_{l-1,m-1}^{j} + D_3\varepsilon_{l,m-1}^{j} + D_1\varepsilon_{l+1,m-1}^{j} + D_2\varepsilon_{l-1,m}^{j} + D_4\varepsilon_{l,m}^{j} + D_2\varepsilon_{l+1,m}^{j} \\
& +D_1\varepsilon_{l-1,m+1}^{j} + D_3\varepsilon_{l,m+1}^{j} + D_1\varepsilon_{l+1,m+1}^{j}. \qquad (37)
\end{aligned}$$

We may assume that the solution of (37) at the grid point (x_l, y_m, t_j) is of the form

$$\varepsilon_{l,m}^{j} = \xi^j e^{i(\theta_1 l + \theta_2 m)}, \qquad (38)$$

where $i = \sqrt{-1}$, θ_1, θ_2 are real phase angles and ξ is in general complex. Substituting (38) into (37) and using Euler identity $e^{i\theta} = cos(\theta) + isin(\theta)$, after simplification we find

$$\xi \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} = \begin{pmatrix} Q_3 & Q_1 \\ Q_2 & 0 \end{pmatrix}, \qquad (39)$$

where

$$Q_1 = 4(\alpha_1 cos\theta_1 + \beta_1)(\alpha_2 cos\theta_2 + \beta_2),$$

$$Q_2 = (2cos\theta_1 + 1)(2cos\theta_2 + 1),$$

$$\begin{aligned}
Q_3 = \; & 4(\alpha_2 cos\theta_2 + \beta_2)(\frac{c^2\tau^2}{h^2}(cos\theta_1 - 1) + \alpha_1 cos\theta_1 + \beta_1) \\
& + 4(\alpha_1 cos\theta_1 + \beta_1)(\frac{c^2\tau^2}{k^2}(cos\theta_2 - 1) + \alpha_2 cos\theta_2 + \beta_2).
\end{aligned}$$

Then we get the amplification matrix of the difference scheme (37) as follows

$$G = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_3 & Q_1 \\ Q_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 - 2c^2\tau^2(\frac{sin^2(\frac{\theta_1}{2})}{h^2(\alpha_1 cos\theta_1+\beta_1)} + \frac{sin^2(\frac{\theta_2}{2})}{k^2(\alpha_2 cos\theta_2+\beta_2)}) & -1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues λ of matrix G satisfy the equation

$$\lambda^2 - 2b\lambda + 1 = 0, \tag{40}$$

where

$$b = 1 - c^2\tau^2 \left(\frac{sin^2(\frac{\theta_1}{2})}{h^2(\alpha_1 cos\theta_1 + \beta_1)} + \frac{sin^2(\frac{\theta_2}{2})}{k^2(\alpha_2 cos\theta_2 + \beta_2)} \right).$$

Using the transformation $\lambda = \frac{1+z}{1-z}$, (40) takes the form

$$(2 + 2b)z^2 + (2 - 2b) = 0. \tag{41}$$

By the above transformation the unit circle can be mapped to the left half of the plane, so the stability criterion $| \lambda |< 1$ will be satisfied, when $| b |< 1$. Thus for stability, by choosing $\alpha_1 + \beta_1 = \frac{1}{2}$ and $\alpha_2 + \beta_2 = \frac{1}{2}$,with $\alpha_1 < \frac{1}{4}$ and $\alpha_2 < \frac{1}{4}$, we must have the following restrictions for the time step

$$\tau \leq h\sqrt{\frac{1 - 4\alpha_1}{2c^2}}, \quad \tau \leq k\sqrt{\frac{1 - 4\alpha_2}{2c^2}}. \tag{42}$$

6 Numerical results

In this section, we applied method (i) and method (ii) presented in (34) and (35) to the following test problems, with the known exact solutions. To demonstrate the applicability of our methods, the computed solutions are compared with the exact solutions at grid points. The computed errors with different norms are tabulated in Tables 1, 2, 3, and 4. The surface plots and pseudo color plots of the estimated solutions at different time levels are given in Figs. 1, 2, 3, and 4. In the first example, our computed errors for the solution of two-dimensional sine-Gordon equation are compared with the results in existing methods given in references [27–29].

Example 1 Consider the two dimensional sine-Gordon equation

$$u_{tt} = u_{xx} + u_{yy} - sinu, \quad -7 < x, y < 7, t > 0, \tag{43}$$

subject to the initial conditions

$$u(x, y, 0) = 4tan^{-1}(exp(x + y)),$$

$$u_t(x, y, 0) = -2sech(x + y), \quad -7 \leq x, y \leq 7$$

The exact solution of this problem is

$$u(x, y, t) = 4tan^{-1}(exp(x + y - t)).$$

The boundary conditions are derived from the exact solution.

Table 1 Errors in the numerical solution of Example 1

	t	Method (i)	Method (ii)	[27]	[28]	[29]
L_∞-errors	1	7.82(−3)	1.34(−4)	3.50(−2)	6.70(−2)	2.7(−3)
	3	2.39(−2)	2.90(−4)	4.31(−2)	8.34(−2)	2.0(−3)
	5	3.18(−2)	2.93(−4)	4.04(−2)	1.01(−1)	3.3(−3)
	7	3.96(−2)	3.40(−4)	3.53(−2)	1.52(−1)	5.9(−3)
RMS-errors	1	2.18(−3)	2.61(−5)	–	5.00(−3)	5.0(−4)
	3	5.48(−3)	5.25(−5)	–	1.03(−2)	5.0(−5)
	5	8.22(−3)	6.36(−5)	–	1.45(−2)	7.0(−4)
	7	8.40(−3)	6.42(−5)	–	1.87(−2)	1.1(−3)

We applied our methods to solve this problem in the domain $-7 \leq x, y \leq 7$ with $h = k = 0.25$, and step size in time direction is $\tau = 0.01$. The L_∞ and RMS-errors in solutions at $t = 1, 3, 5, 7$ are compared with the results in references [27, 28], and [29], with step size in time direction $\tau = 0.001$, which are tabulated in Table 1. In [27, 28], the spatial step sizes are as the same as ours and are equal to $h = k = 0.25$. In reference [29], the number of divides in space directions is $N = 31$. The errors in the estimated solution of our methods are quite accurate because in the above mentioned references the number of levels to achieve such errors are tenfold that of our levels. The computed errors in Table 1 show that in our methods with larger time step and much fewer iterations, we obtain the results with considerable accuracy. The surface plots and pseudo color plots for numerical solutions of $sin(u/2)$ at different time levels are shown in Fig. 1.

Example 2 Consider two-dimensional Klein-Gordon equation

$$u_{tt} = \frac{5}{4}(u_{xx} + u_{yy}) - u - \frac{3}{2}u^3, \quad 0 < x, y < 1, t > 0, \tag{44}$$

Table 2 Errors in the numerical solution of Example 2 with $\mu = 0.1$

		Method (i)			Method (ii)		
(h,k)	τ	L_2-error	L_∞-error	order	L_2-error	L_∞-error	Order
t = 1:							
$(\frac{1}{8}, \frac{1}{8})$	0.02	4.442(−4)	1.239(−4)		2.628(−6)	8.182(−7)	
$(\frac{1}{16}, \frac{1}{16})$	0.02	2.170(−4)	3.033(−5)	2.03	3.336(−7)	5.193(−8)	3.98
$(\frac{1}{32}, \frac{1}{32})$	0.01	1.092(−4)	7.709(−6)	1.98	4.231(−8)	3.294(−9)	3.98
$(\frac{1}{64}, \frac{1}{64})$	0.005	5.485(−5)	1.928(−6)	1.99	5.471(−9)	2.114(−10)	3.96
t = 2:							
$(\frac{1}{8}, \frac{1}{8})$	0.02	1.033(−3)	2.650(−4)		6.076(−6)	1.611(−6)	
$(\frac{1}{16}, \frac{1}{16})$	0.02	5.178(−4)	6.782(−5)	2.03	7.670(−7)	1.047(−7)	3.94
$(\frac{1}{32}, \frac{1}{32})$	0.01	2.610(−4)	1.690(−5)	2.00	9.729(−8)	6.619(−9)	3.98
$(\frac{1}{64}, \frac{1}{64})$	0.005	1.309(−4)	4.139(−6)	2.03	1.258(−8)	4.250(−10)	3.96

Table 3 Errors in the numerical solution of Example 3 with $a = .1, b = 1, \mu = .3$

		Method (i)			Method (ii)		
(h,k)	τ	L_2-error	L_∞-error	order	L_2-error	L_∞-error	Order
t = 1:							
$(\frac{1}{8}, \frac{1}{8})$	0.5	3.554(−4)	7.865(−5)		3.148(−4)	6.513(−5)	
$(\frac{1}{16}, \frac{(1}{16})$	0.2	9.037(−5)	9.248(−6)	3.08	7.948(−5)	8.004(−6)	3.02
$(\frac{1}{32}, \frac{1}{32})$	0.05	1.280(−5)	6.598(−7)	3.08	8.599(−6)	4.359(−6)	4.19
$(\frac{1}{64}, \frac{1}{64})$	0.01	2.650(−6)	6.814(−8)	3.28	6.596(−7)	1.673(−8)	4.70
t=2:							
$(\frac{1}{8}, \frac{1}{8})$	0.5	1.065(−3)	2.096(−4)		9.616(−4)	1.882(−4)	
$(\frac{1}{16}, \frac{1}{16})$	0.2	3.053(−4)	3.059(−5)	2.77	2.705(−4)	2.658(−5)	3.00
$(\frac{1}{32}, \frac{1}{32})$	0.05	4.668(−5)	2.338(−6)	3.71	3.143(−5)	1.553(−6)	4.09
$(\frac{1}{64}, \frac{1}{64})$	0.01	9.894(−6)	2.461(−7)	3.24	2.463(−6)	6.097(−8)	4.67

subject to the initial conditions

$$u(x, y, 0) = Btan(K(x + y)),$$

$$u_t(x, y, 0) = B\mu K sec^2(K(x + y)), \quad 0 \le x, y \le 1$$

$$B = \sqrt{2/3}, K = \sqrt{1/(5 - 2\mu^2)},$$

The exact solution of this problem is

$$u(x, y, t) = Btan(K(x + y) + \mu t).$$

The boundary conditions are derived from the exact solution. We applied our methods to solve this problem with different values of space steps h and k for $\mu = 0.1$. The computed solutions are compared with exact solutions at grid points. The L_2 and L_∞ errors and also the orders of convergence are tabulated in Table 2. The surface plots and pseudo color plots of numerical solutions at different time levels are shown in Fig. 2.

Example 3 Consider the two-dimensional Klein-Gordon equation

Table 4 Errors in the numerical solution of Example 4 with $\mu = 0.1$

(h,k)	t	L_2-error	L_∞-error	$RMS - error$
Method (i)				
(.04,.04)	1	6.750(−4)	2.233(−4)	1.386(−5)
	2	7.018(−4)	2.231(−4)	1.432(−5)
	4	1.049(−3)	2.571(−4)	2.128(−5)
	6	1.343(−3)	3.948(−4)	2.741(−5)
	8	1.880(−3)	5.794(−4)	3.847(−5)
Method (ii)				
(.04,.04)	1	2.825(−4)	4.714(−5)	5.766(−6)
	2	2.962(−4)	4.755(−5)	6.046(−6)
	4	4.567(−4)	4.769(−5)	9.321(−6)
	6	5.873(−4)	8.244(−5)	1.199(−5)
	8	8.532(−4)	1.244(−4)	1.741(−5)

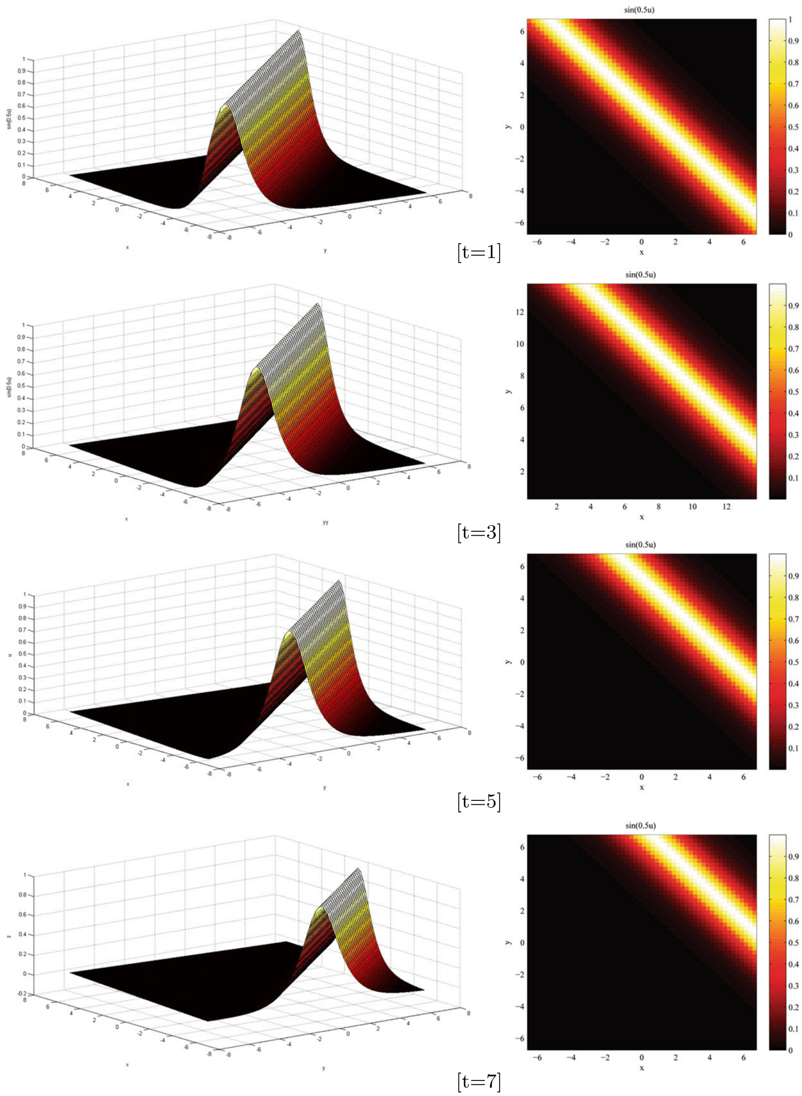

Fig. 1 Surface plots (*left column*) and pseudo color plots (*right column*) for numerical solutions of sin(u/2) at t = 1,3,5,7 with h = k = 0.25, for Example 1

$$u_{tt} = \frac{a^2}{2}(u_{xx} + u_{yy}) - au + bu^3, \quad 0 < x, y < 1, t > 0, \tag{45}$$

where $a, b \in \Re^+$ subject to initial conditions

$$u(x, y, 0) = Btanh(K(x + y)),$$

$$u_t(x, y, 0) = -B\mu Ksech^2(K(x + y - \mu t)), \quad 0 \le x, y \le 1,$$

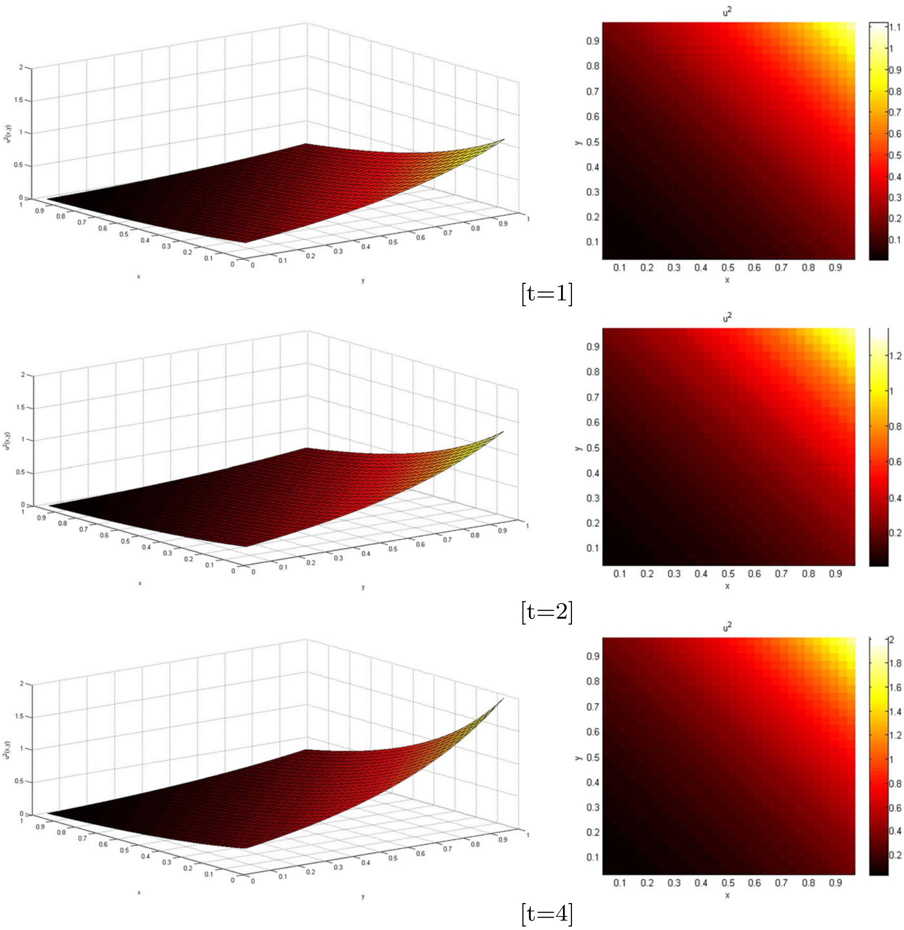

Fig. 2 Surface plots (*left column*) and pseudo color plots (*right column*) of numerical solutions of $u^2(x, y, t)$ at t = 1,2,4 with h = k = 1/32 for Example 2

$$B = \sqrt{\frac{a}{b}}, K = \sqrt{\frac{a}{2(\mu^2 - a^2)}},$$

where $\mu^2 - a^2 > 0$. The kink solution of this problem is

$$u(x, y, t) = Btanh(K(x + y - \mu t)).$$

The boundary conditions are derived from the exact solution. We applied our methods to solve this problem for $a = 0.1, b = 1, \mu = 0.3$ and with different values of space steps h and k. The computed solutions are compared with exact solutions at grid points. The L_2 and L_∞ errors and also the orders of convergence are tabulated in Table 3. The surface plots and pseudo color plots of numerical solutions at different time levels are shown in Fig. 3.

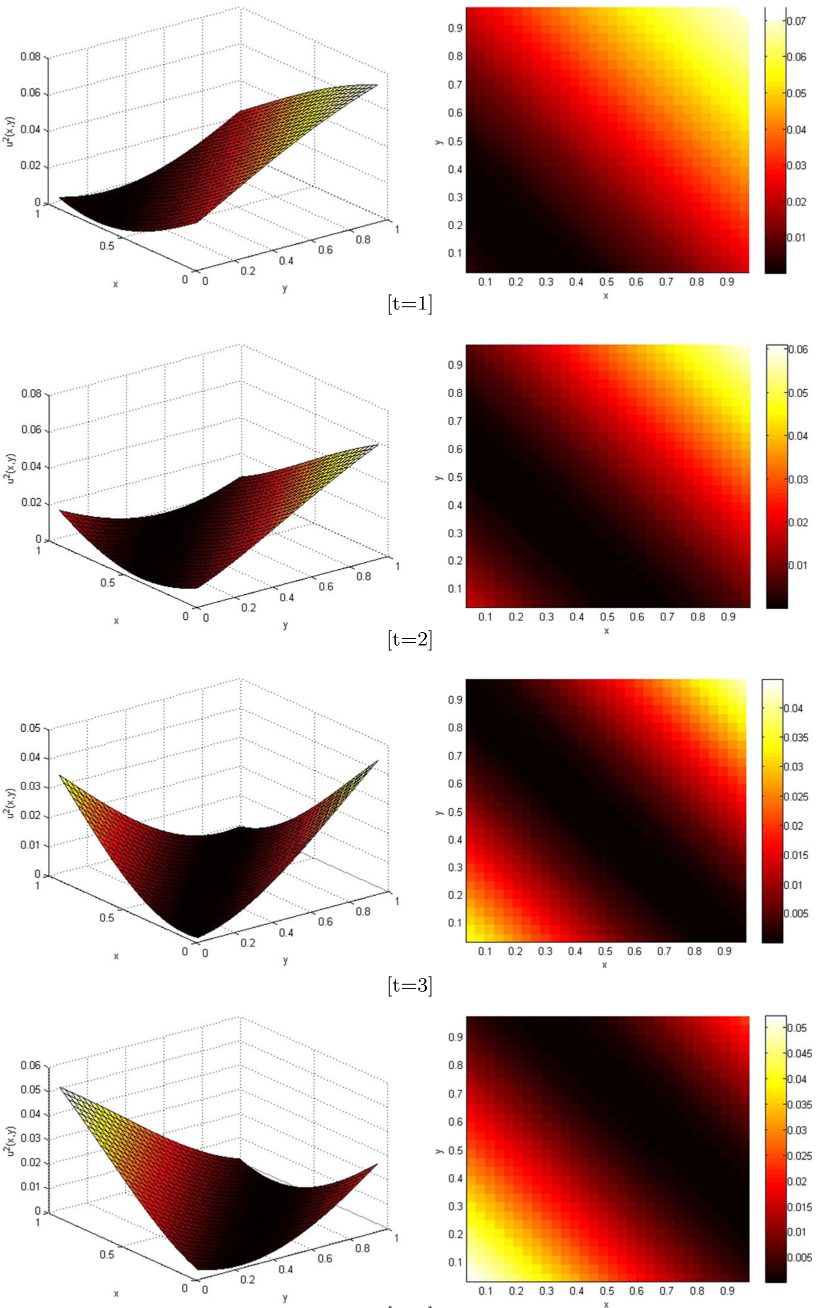

Fig. 3 Surface plots (*left column*) and pseudo color plots (*right column*) for numerical solutions of $u^2(x, y, t)$ at $t = 1, 2, 3, 4$ with $h = k = 1/32$ for Example 3

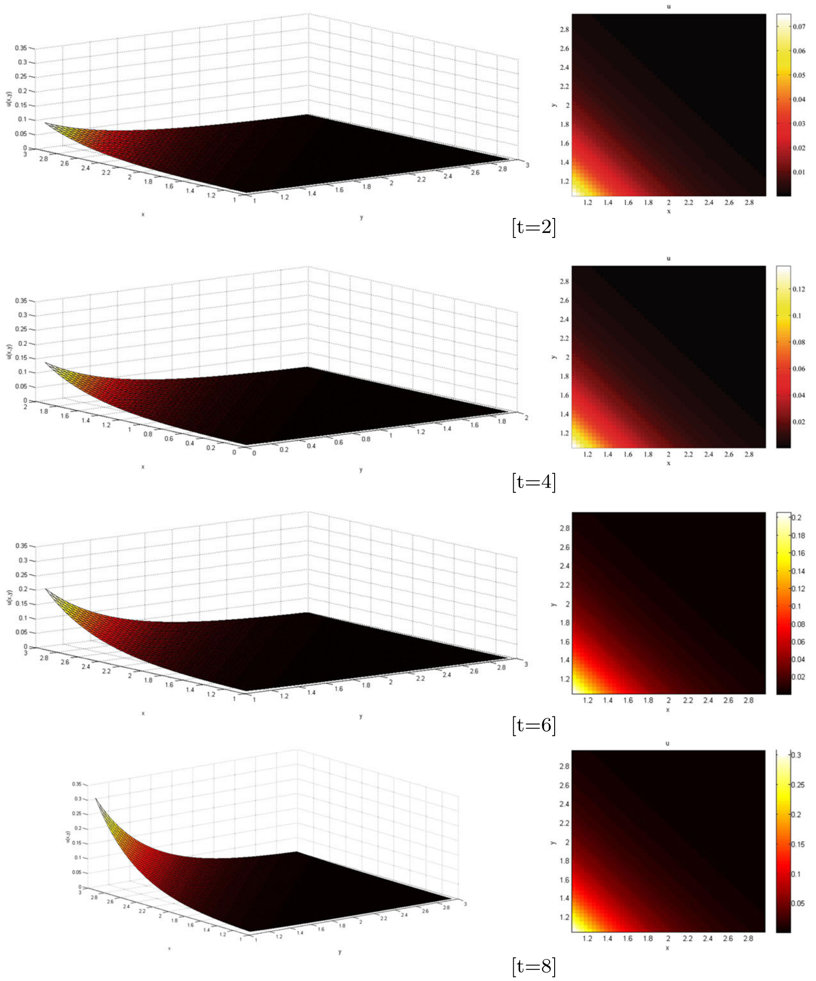

Fig. 4 Surface plots(*left column*) and pseudo color plots(*right column*) of the numerical solutions at $t = 2, 4, 6, 8$ for Example 4

Example 4 Consider the two-dimensional equation

$$u_{tt} = \frac{1}{2}(u_{xx} + u_{yy}) - 2e^{u} + 2e^{-u}, \quad 1 < x, y < 3, \quad t > 0, \tag{46}$$

subject to initial conditions

$$u(x, y, 0) = ln(coth^2(K(x + y))),$$

$$u_t(x, y, 0) = 2\mu K(tanh(x + y) - coth(x + y)), \quad 1 \leq x, y \leq 3,$$

where $K = \frac{1}{\sqrt{1-\mu^2}}$, with $\mu^2 < 1$.

The exact solution of this problem is

$$u(x, y, t) = ln(coth^2(K(x + y - \mu t))).$$

The boundary conditions are derived from the exact solution. We applied our methods to solve this problem for $h = k = 0.04$, with $\mu = 0.1$. The computed solutions are compared with exact solutions at grid points. The L_2 , L_∞, and RMS-errors in solutions at $t = 1, 2, 3, 4$ are tabulated in Table 4. The surface plots and pseudo color plots of numerical solutions at different time levels are shown in Fig. 4.

7 Conclusion

In this paper, we developed two implicit three time level methods for the solution of two dimensional wave equations. We applied non-polynomial cubic spline function approximations for both second-order spatial derivatives. The methods involve some parameters, by choosing the parameters adopted in cubic spline, we may obtain the scheme of order $O(h^2 + k^2 + \tau^2 + \tau^2 h^2 + \tau^2 k^2)$. Also by appropriate choices of the parameters, we can increase the order of accuracy to new scheme of order $O(h^4 + k^4 + \tau^2 + \tau^2 h^2 + \tau^2 k^2)$. In the first example, our computed errors in comparison with results in three existing methods show that our methods with larger time step and fewer iterations give compatible or better approximations. The other numerical results given in the previous section justify that our methods are accurate and easy in application.

References

1. Strauss, W.: Nonlinear wave equations. CBMS 73, AMS (1989)
2. Mohanty, R.K.: An unconditionally stable difference scheme for the one space dimensional linear hyperbolic equation. Appl. Math. Lett. **17**, 101–105 (2004)
3. Gao, F., Chi, C.: Unconditionally stable difference schemes for a one-space dimensional linear hyperbolic equation. Appl. Math. Comput. **187**, 1272–1276 (2007)
4. Mohebbi, A., Dehghan, M.: High order compact solution of the one-space dimensional linear hyperbolic equation. Numer. Methods. Partial. Differ. Equ. **24**, 1222–1235 (2008)
5. Raggett, G.F., Wilson, P.D.: A fully implicit finite difference approximation to the one-dimensional wave equation using a cubic spline technique. J. Inst. Math. Appl. **14**, 75–77 (1974)
6. Rashidinia, J., Jalilian, R., Kazemi, V.: Spline methods for the solution of hyperbolic equations. Appl. Math. Comput. **190**, 882–886 (2007)
7. Ding, H., Zhang, Y.: A new unconditionally stable compact difference scheme of for the 1D linear hyperbolic equation. Appl. Math. Comput. **207**, 236–241 (2009)
8. Liu, H.W., Liu, L.B.: An unconditionally stable spline difference scheme of $O(k^2 + h^4)$ for solving the second order 1D linear hyperbolic equation. Math. Comput. Model. **49**, 1985–1993 (2009)
9. Mohanty, R.K., Gopal, V.: High accuracy cubic spline finite difference approximation for the solution of one-space dimensional non-linear wave equations. Appl. Math. Comput. **218**(8), 4234–4244 (2011)
10. Mohanty, R.K., Gopal, V.: An off-step discretization for the solution of 1D mildly nonlinear wave equations with variable coefficients. J. Adv. Res. Sci. Comput. **04**(02), 1–13 (2012)
11. Rashidinia, J., Mohammadi, R.: Tension spline approach for the numerical solution of nonlinear Klein-Gordon equation. Comput. Phys. Comm. **181**(1), 78–91 (2010)
12. Rashidinia, J., Mohammadi, R.: Tension spline solution of nonlinear sine-Gordon equation. Numer. Algor. **56**(1), 129–142 (2011)
13. Dehghan, M., Mohebbi, A.: High order implicit collocation method for the solution of two-dimensional linear hyperbolic equation. Numer. Methods Partial Differ. Equ. **25**, 232–243 (2009)

14. Mittal, R.C., Bhatia, R.: Numerical solution of second order one dimensional hyperbolic telegraph equation by cubic B-spline collocation method. Appl. Math. Comput. **220**, 496–506 (2013)
15. Dosti, M., Nazemi, A.: Quartic B-spline collocation method for solving one-dimensional hyperbolic telegraph equation. J. Inf. Comput. Sci. **7**(2), 083–090 (2012)
16. Mohanty, R.K., Jain, M.K.: An unconditionally stable alternating direction implicit scheme for the two space dimensional linear hyperbolic equation. Numer. Methods. Partial Differ. Equ. **17**(6), 684–688 (2001)
17. Mohanty, R.K.: An operator splitting method for an unconditionally stable difference scheme for a linear hyperbolic equation with variable coefficients in two space dimensions. Appl. Math. Comput. **152**, 799–806 (2004)
18. Ding, H.F., Zhang, Y.X.: A new fourth order compact finite difference scheme for the two-dimensional second order hyperbolic equations. J. Comput. Appl. Math **230**, 626–632 (2009)
19. Dehghan, M., Mohebbi, A.: High order implicit collocation method for the solution of two-dimensional linear hyperbolic equation. Numer. Methods. Partial Differ. Equ. **25**, 232–243 (2009)
20. Dehghan, M., Shokri, A.: A meshless method for numerical solution of a linear hyperbolic equation with variable coefficients in two space dimensions. Numer. Methods Partial Differ. Equ. **25**, 494–506 (2009)
21. Dehghan, M., Salehi, R.: A method based on meshless approach for the numerical solution of the two space dimensional hyperbolic telegraph equation. Math. Method. Appl. Sci. **35**(10), 1220–1233 (2012)
22. Piperno, S.: Symplectic local time-stepping in non-dissipative dgtd methods applied to wave propagation problems. ESAIM: Math. Modelling. Numer. Analysis **5**, 815–841 (2006)
23. Shi, D.Y., Li, Z.Y.: Superconvergence analysis of the finite element method for nonlinear hyperbolic equations with nonlinear boundary condition. Applied Mathematics - A Journal of the Chinese Universities **4**, 455–462 (2008)
24. Chabassier, J., Joly, P.: Energy preserving schemes for nonlinear hamiltonian systems of wave equations: application to the vibrating piano string. Computer Methods. Appl. Mechanic. Engineering **45**, 2779–2795 (2010)
25. Chawla, M.M., Al-Zanaidi, M.A.: A linearly implicit one-step time integration scheme for nonlinear hyperbolic equations. Intern. J. Computer Math. **76**, 349–361 (2001)
26. Chawla, M.M., Al-Zanaidi, M.A.: A linearly implicit one-step time integration scheme for nonlinear hyperbolic equations in two space dimensions. Intern. J. Computer Math. **80**(3), 357–365 (2003)
27. Djidjeli, K., Price, W.G., Twizell, E.H.: Numerical solutions of a damped sine-Gordon equation in two space variables. J. Eng. Math. **29**, 347–369 (1995)
28. Dehghan, M., Shokri, A.: A numerical method for solution of the two dimensional sine-Gordon equation using the radial basis functions. Math. Comput. Simulation **79**, 700–715 (2008)
29. Jiwari, R., Pandit, S., Mittal, R.C.: Numerical simulation of two dimensional sine-Gordon solitons by differential quadrature method. Comput. Phys. Commun. **183**, 600–616 (2012)