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An infeasible full-NT step interior point algorithm for CQSCO

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Abstract In this paper, we propose a full Nesterov-Todd (NT) step infeasible interior-point algorithm for convex quadratic symmetric cone optimization based on Euclidean Jordan algebra. The algorithm uses only one feasibility step in each main iteration. The complexity result coincides with the best-known iteration bound for infeasible interior-point methods.

Keywords Convex quadratic symmetric cone optimization · Interior-point method · Infeasible method · Euclidean Jordan algebra · Polynomial complexity

Mathematics Subject Classification (2010) 90C51

1 Introduction

Let $(\mathcal{J}, \circ, \langle \cdot, \cdot \rangle)$, denoted by $\mathcal{J}$, be a Euclidean Jordan algebra and $\mathcal{K}$ be the corresponding symmetric cone. Consider the convex quadratic symmetric cone optimization, denoted by CQSCO, given in the standard form

$$\begin{array}{ll} \min & \langle c, x\rangle + \frac{1}{2}\langle x, \mathcal{H}(x)\rangle \\ s.t. & \mathcal{A}(x) = b \\ & x \in \mathcal{K}, \end{array} \qquad (P)$$

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and its Lagrangian dual problem

$$\begin{aligned} &\max\ b^T y - \frac{1}{2}\langle x, \mathcal{H}(x)\rangle \\ &s.t.\ \mathcal{A}^T y - \mathcal{H}(x) + s = c \qquad\qquad (D) \\ &\qquad s \in \mathcal{K}, \end{aligned}$$

where $c \in \mathcal{J}$ and $b \in R^m$ are given data, $\mathcal{A} : \mathcal{J} \to R^m$ is a given linear map, $\mathcal{A}^T$ is the adjoint of $\mathcal{A}$. $\langle x, s\rangle = \mathrm{tr}(x \circ s)$ stands for the trace inner product in $\mathcal{J}$, and $\mathcal{H}$ is a given self-adjoint monotone, i.e., $\mathcal{H}$ is positive semidefinite linear operator with respect to $\langle\cdot,\cdot\rangle$ on $\mathcal{J}$, i.e., for any $x, s \in \mathcal{J}$, $\langle \mathcal{H}(x), s\rangle = \langle x, \mathcal{H}(s)\rangle$ and $\langle \mathcal{H}(x), x\rangle \geq 0$. Moreover, assume that a_i is the ith row of $\mathcal{A}$, then $\mathcal{A}(x) = b$ means that $\langle a_i, x\rangle = b_i, i = 1, \ldots, m$, while $\mathcal{A}^T y - \mathcal{H}(x) + s = c$ means that $\sum_{i=1}^m y_i a_i - \mathcal{H}(x) + s = c$. Throughout the paper, we assume that the linear map $\mathcal{A}$ is surjective, which implies that $\mathcal{A}\mathcal{A}^T$ is nonsingular.

CQSCO is a generalization of symmetric cone optimization (SCO), which contains linear optimization (LO), second-order cone optimization (SOCO) and semidefinite optimization (SDO) as special cases. Furthermore, CQSCO also includes convex quadratic optimization (CQO) and convex quadratic semidefinite optimization (CQSDO) as special cases. By using Euclidean Jordan algebras, several interior-point methods (IPMs) have been developed for CQSCO [1, 11, 13, 23, 24]. Until now, various algorithms based on different starting points have been introduced. Based on selecting the starting point, are called feasible IPMs or infeasible IPMs (IIPMs). Because finding the feasible starting point is not an easy work, IIPMs have attracted more attention. The primal-dual IIPM was first proposed by Lustig [14]. The first theoretical result on primal-dual IIPMs was obtained by Kojima et al. [12]. They showed that their algorithm is globally convergent. Roos [18] designed an IIPM for LO based on using the perturbed problems. Furthermore, his algorithm begins with infeasible starting point for original primal-dual problems and applies one feasibility step and a few-at most three-centering steps in each main iteration. Kheirfam and Mahdavi-Amiri [5] and Kheirfam [6] extended this algorithm to linear complementarity problem over symmetric cones (SCLCP) and horizontal linear complementarity problems (HLCP), respectively. Some variants of Roos' algorithm studied for SCO by Kheirfam [7, 8]. In [5–8, 18], the authors proposed algorithms with one feasibility step and several centering steps to get an optimal solution of underlying problems. Recently, Roos [19] investigated a new IIPM for LO by improving the full-Newton step IIPM so that the centering steps not be needed. Kheirfam [9, 10] extended this algorithm to $P_*(\kappa)$-SCLCP and HLCP.

Motivated by the recent developments on IIPMs, i.e., [9, 10, 19], we present an infeasible algorithm for CQSCO under the framework of Euclidean Jordan algebras. The purpose of the paper is mainly theoretical, and the symmetrization of the search directions is based on the NT-scaling scheme. The algorithm uses only one feasibility step at each iteration. We derive the complexity bound for the algorithm, and the result shows that it enjoys the best-known iteration bound for IIPMs.

The outline of the paper is as follows. In Section 2, we briefly recall some basic information on Euclidean Jordan algebras that are needed in this paper. In Section 3, we define the perturbed problems and their corresponding central path. In addition, we provide the search directions and also present the algorithm. The complexity analysis is discussed in Section 4. Some numerical results are reported in Section 5. Section 6 ends the paper with a conclusion.

2 Preliminaries

In this section, we present some basic results on Euclidean Jordan algebras that are needed in the subsequent sections. Our presentation is mainly adapted from [2].

Recall that a Euclidean Jordan algebra $\mathcal{J}$ is a finite dimensional inner product space over the field of real numbers endowed with a bilinear map $\circ : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ satisfying the following properties for all $x, y, z \in \mathcal{J}$:

(i) $x \circ y = y \circ x$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$;
(iii) $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$, where $\langle x, y\rangle = \mathrm{tr}(x \circ y)$.

The Jordan algebra $\mathcal{J}$ has an identity element, if there exists an element $e \in \mathcal{J}$ such that $e \circ x = x \circ e = x$ for all $x \in \mathcal{J}$. Recall that an idempotent c is a nonzero element of $\mathcal{J}$ such that $c^2 = c$. An idempotent c is said to be primitive if it is not the sum of two other nonzero idempotents. A set of primitive idempotents $\{c_1, c_2, \dots, c_r\}$ is called a Jordan frame if $c_i \circ c_j = 0$, for any $i \neq j \in \{1, 2, \dots, r\}$, and $\sum_{i=1}^r c_i = e$. For any $x \in \mathcal{J}$, let r be the smallest positive integer such that $\{e, x, \dots, x^r\}$ is linearly dependent; r is called the degree of x and is denoted by $\deg(x)$. Moreover, we define the rank of $\mathcal{J}$ as $r := \max\{\deg(x) : x \in \mathcal{J}\}$. For an element $x \in \mathcal{J}$, let $L : \mathcal{J} \to \mathcal{J}$ be the linear map defined by $L(x)y := x \circ y$, for all $y \in \mathcal{J}$. Furthermore, the linear map $P(x) := 2L(x)^2 - L(x^2)$, where $L(x)^2 := L(x)L(x)$, is called the quadratic representation of x in $\mathcal{J}$. For any $x \in \mathcal{J}$, there exists a Jordan frame $\{c_1, \dots, c_r\}$ and real numbers $\lambda_1(x), \dots, \lambda_r(x)$ such that $x = \sum_{i=1}^r \lambda_i(x)c_i$. The numbers $\lambda_i(x)$, $i = 1, \dots, r$ are uniquely determined by x, and they are called the eigenvalues of x. We denote $\lambda_{\min}(x)(\lambda_{\max}(x))$ be the minimal (maximal) eigenvalue of x. Furthermore, $\mathrm{tr}(x) := \lambda_1(x) + \cdots + \lambda_r(x)$. Note that since $e = c_1 + \cdots + c_r$, the eigenvalues of e are all equal to 1, it follows that $\mathrm{tr}(e) = r$. Recall that for a Euclidean Jordan algebra $\mathcal{J}$, its cone of squares is the set $\mathcal{K} := \{x^2 : x \in \mathcal{J}\}$. A cone is symmetric if and only if it is the cone of squares of some Euclidean Jordan algebra. Recall that two elements x and y in $\mathcal{J}$ operators commute if $L(x)L(y) = L(y)L(x)$, and are called similar, denoted as $x \sim y$, if x and y share the same set of eigenvalues. An element $x \in \mathcal{J}$ is said to be invertible if there exists an element y such that $x \circ y = y \circ x = e$. The element y is called the inverse of x and it is unique. It is denoted by x^{-1}. The Frobenius norm $\|\cdot\|_F$ induced by the inner product $\langle\cdot,\cdot\rangle$ and the spectral norm are given by $\|x\|_F := \sqrt{\langle x, x\rangle} = \sqrt{\mathrm{tr}(x^2)} = \sqrt{\sum_{i=1}^r \lambda_i^2(x)}$ and $\|x\|_\infty := \max_i |\lambda_i(x)|$. We associate with a proper cone $\mathcal{K}$ the partial order defined by $x \succeq_{\mathcal{K}} y \Leftrightarrow x - y \in \mathcal{K}$, and we define an associated strictly partial order by $x \succ_{\mathcal{K}} y \Leftrightarrow x - y \in \mathrm{int}\mathcal{K}$, where $\mathrm{int}\mathcal{K}$ denotes the interior of $\mathcal{K}$. Furthermore, $x \in \mathcal{K} \Leftrightarrow \lambda_i(x) \geq 0$, and $x \in \mathrm{int}\mathcal{K} \Leftrightarrow \lambda_i(x) > 0$ for each $i = 1, \dots, r$.

Lemma 1 (Lemma 3.2 in [3]) *Let $x, s \in int\mathcal{K}$. Then, there exists a unique $w \in int\mathcal{K}$ such that $x = P(w)s$. Moreover,*

$$w = P\left(x^{\frac{1}{2}}\right)\left(P\left(x^{\frac{1}{2}}\right)s\right)^{-\frac{1}{2}}\left[= P\left(s^{-\frac{1}{2}}\right)\left(P(s^{\frac{1}{2}})x\right)^{\frac{1}{2}}\right].$$

This unique w is called the NT-scaling point of x and s [16].

Lemma 2 (Lemma 2.9 in [17]) *Given $x \in int\mathcal{K}$, we have*

$$\|x - x^{-1}\|_F \leq \frac{\|x^2 - e\|_F}{\lambda_{\min}(x)}.$$

Lemma 3 (Lemma 30 in [21]) *Let $x, s \in int\mathcal{K}$. Then*

$$\|P(x)^{\frac{1}{2}}s - e\|_F \leq \|x \circ s - e\|_F.$$

Lemma 4 (Theorem 4 in [22], Lemma 30 in [21]) *Let $x, s \in int\mathcal{K}$. Then*

$$\lambda_{\min}\left(P(x)^{\frac{1}{2}}s\right) \geq \lambda_{\min}(x \circ s), \ \ \lambda_{\max}\left(P(x)^{\frac{1}{2}}s\right) \leq \lambda_{\max}(x \circ s).$$

3 An infeasible algorithm

In this section, we present an infeasible interior-point algorithm for solving the CQSCO problem. As usual for IIPMs, we suppose that an optimal solution exists and let the algorithm start with the following initial infeasible point

$$\left(x^0, y^0, s^0\right) = \left(\rho_p e, 0, \rho_d e\right), \tag{1}$$

where

$$\|x^*\|_\infty \leq \rho_p, \quad \max\{\|s^*\|_\infty, \|\rho_p \mathcal{H}e + c\|_F\} \leq \rho_d \tag{2}$$

for some optimal solution (x^*, y^*, s^*). The initial values of the primal and dual residual vectors present as follows:

$$r_p^0 := b - \mathcal{A}\left(x^0\right), \ \ r_d^0 := c - \mathcal{A}^T y^0 - s^0 + \mathcal{H}\left(x^0\right). \tag{3}$$

Our aim is to show that, under this assumption, our algorithm finds an ϵ solution.

3.1 The perturbed problems

Let (x^0, y^0, s^0) be the starting point. For any ν with $0 < \nu \leq 1$, we consider the perturbed problem (P_ν), defined by

$$\begin{aligned} \min \ & \left\langle c - \nu r_d^0, x\right\rangle + \frac{1}{2}\langle x, \mathcal{H}(x)\rangle \\ s.t. \ & \mathcal{A}(x) = b - \nu r_p^0 \qquad\qquad (P_\nu) \\ & x \in \mathcal{K}, \end{aligned}$$

and its dual problem (D_ν), which is given by

$$\begin{aligned} \max \ & \left(b - \nu r_p^0\right)^T y - \frac{1}{2}\langle x, \mathcal{H}(x)\rangle \\ s.t.\ & \mathcal{A}^T y - \mathcal{H}(x) + s = c - \nu r_d^0 \qquad\qquad (D_\nu) \\ & s \in \mathcal{K}. \end{aligned}$$

It is obvious that $(x, y, s) = (x^0, y^0, s^0)$ is a strictly feasible solution of (P_ν) and (D_ν) when $\nu = 1$. We conclude that (P_ν) and (D_ν) satisfy the interior-point condition (IPC) for $\nu = 1$. More generally, we have the following lemma, whose proof is similar to the proof of Theorem 3.1 in [5] (see also Theorem 5.13 in [18]).

Lemma 5 *The original problems, (P) and (D), are feasible if and only if for each* ν *satisfying* $0 < \nu \leq 1$*, the perturbed problems* (P_ν) *and* (D_ν) *satisfy the IPC.*

3.2 The central path of the perturbed problems

Let (P) and (D) be feasible and $0 < \nu \leq 1$. Then Lemma 5 implies that the perturbed problems (P_ν) and (D_ν) satisfy the IPC, and hence their central paths exist [4]. This means that the system

$$b - \mathcal{A}(x) = \nu r_p^0, \quad x \in \mathcal{K}, \tag{4}$$

$$c - \mathcal{A}^T y + \mathcal{H}(x) - s = \nu r_d^0, \quad s \in \mathcal{K}, \tag{5}$$

$$x \circ s = \mu e, \tag{6}$$

has a unique solution for every $\mu > 0$, as the μ-centers of the perturbed problem pair (P_ν) and (D_ν). Note that, since $x^0 \circ s^0 = \mu^0 e$ with $\mu^0 = \rho_p \rho_d$, (x^0, y^0, s^0) is the μ^0-center of the perturbed problems (P_ν) and (D_ν) for $\nu = 1$. In the sequel, the parameters μ and ν always satisfy the relation $\mu = \nu\mu^0$.

Assuming that x, y, and s satisfy the conditions (4)–(6) for some $\mu > 0$ and $\nu > 0$. Our aim is to obtain search directions Δx, Δy, and Δs that satisfying the conditions (4)–(6) with ν replaced by $\nu^+ := (1 - \theta)\nu$, except that we target the μ^+-centers of (P_{ν^+}) and (D_{ν^+}), i.e.,

$$\begin{aligned} \mathcal{A}(x + \Delta x) &= \nu^+ r_p^0, \\ \mathcal{A}^T(y + \Delta y) - \mathcal{H}(x + \Delta x) + s + \Delta s &= \nu^+ r_d^0, \\ (x + \Delta x) \circ (s + \Delta s) &= \mu^+ e. \end{aligned} \tag{7}$$

So, assuming that (x, y, s) is feasible for the system (4)–(6) and neglecting the quadratic term $\Delta x \circ \Delta s$, it follows that Δx, Δy, and Δs should satisfy

$$\begin{aligned} \mathcal{A}(\Delta x) &= \theta \nu r_p^0, \\ \mathcal{A}^T \Delta y - \mathcal{H}(\Delta x) + \Delta s &= \theta \nu r_d^0, \\ s \circ \Delta x + x \circ \Delta s &= (1 - \theta)\mu e - x \circ s. \end{aligned} \tag{8}$$

Due to the fact that x and s do not operator commute in general, the system (8) does not always have a unique solution. It is well known that this difficulty can be solved by applying a scaling scheme. It goes as follows. Let $x, s, u \in \mathrm{int}\mathcal{K}$, then $x \circ s = \mu e$ if and only if $P(u)x \circ P(u)^{-1}s = \mu e$ (Lemma 28 in [21]). Here, we consider $u = w^{-\frac{1}{2}}$, where w is the NT-scaling point of x and s as defined in Lemma 1. Now replacing (6) by $P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}s = \mu e$, and then applying Newton's method, we obtain the following system

$$\begin{aligned} \mathcal{A}(\Delta x) &= \theta \nu r_p^0, \\ \mathcal{A}^T \Delta y - \mathcal{H}(\Delta x) + \Delta s &= \theta \nu r_d^0, \\ P(w)^{-\frac{1}{2}} \Delta x \circ P(w)^{\frac{1}{2}} s + P(w)^{-\frac{1}{2}} x \circ P(w)^{\frac{1}{2}} \Delta s &= \\ & (1-\theta)\mu e - P(w)^{-\frac{1}{2}} x \circ P(w)^{\frac{1}{2}} s. \end{aligned} \quad (9)$$

The new iterates are obtained by taking a full step as follows

$$x^+ := x + \Delta x, \quad y^+ := y + \Delta y, \quad s^+ := s + \Delta s.$$

Introducing the notations

$$v := \frac{P(w)^{-\frac{1}{2}} x}{\sqrt{\mu}} \left[= \frac{P(w)^{\frac{1}{2}} s}{\sqrt{\mu}} \right], \quad d_x := \frac{P(w)^{-\frac{1}{2}} \Delta x}{\sqrt{\mu}}, \quad d_s := \frac{P(w)^{\frac{1}{2}} \Delta s}{\sqrt{\mu}}, \quad (10)$$

one can easily check that the system (9) which defines the search direction $(\Delta x, \Delta y, \Delta s)$ can be rewritten as follows:

$$\begin{aligned} \bar{\mathcal{A}}(d_x) &= \theta \nu r_p^0, \\ \bar{\mathcal{A}}^T \frac{\Delta y}{\mu} - \bar{\mathcal{H}}(d_x) + d_s &= \frac{1}{\sqrt{\mu}} \theta \nu P(w)^{\frac{1}{2}} r_d^0, \\ d_x + d_s &= (1-\theta) v^{-1} - v, \end{aligned} \quad (11)$$

where $\bar{\mathcal{A}} = \sqrt{\mu} \mathcal{A} P(w)^{\frac{1}{2}}$ and $\bar{\mathcal{H}} = P(w)^{\frac{1}{2}} \mathcal{H} P(w)^{\frac{1}{2}}$. If triple (x, y, s) is feasible for the perturbed problem pair (P_ν) and (D_ν), and $\mu = \nu \rho_p \rho_d$, then we measure proximity to the μ-center of this perturbed problem pair by the quantity

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2} \| v - v^{-1} \|_F, \quad (12)$$

where v is defined in (10). Moreover, it follows that

$$\delta(v) = 0 \Leftrightarrow v = v^{-1} \Leftrightarrow v = e,$$

which this means that (x, y, s) is on the central path of the problem pair. As an immediate consequence we have the following result.

Lemma 6 (cf. Lemma II.62 in [20]) *With $\delta := \delta(v)$, one has $\frac{1}{\rho(\delta)} \leq \lambda_i(v) \leq \rho(\delta)$, for each $i = 1, \ldots, r$, where $\rho(\delta) := \delta + \sqrt{1 + \delta^2}$.*

3.3 An iteration of our algorithm

In Section 3.1, we established that if $\nu = 1$ and $\mu = \mu^0$, then (x^0, y^0, s^0) is the μ-center of the perturbed problem pair (P_ν) and (D_ν). The initial iterates are given as in (1). So, initially, we have $\delta(x, s; \mu) = \delta(x^0, s^0; \mu^0) = 0$. In what follows, we assume that at the start of each iteration, just before the μ-update, $\delta(x, s; \mu) \leq \tau$ where τ is a positive threshold value. This certainly holds at the start of the first iteration. In more details, suppose that for some $\mu \in (0, \mu^0]$ we have x, y, and s satisfying the feasibility conditions (4) and (5) for $\nu = \frac{\mu}{\mu^0}$ and such that $\delta(x, s; \mu) \leq \tau$. Reducing μ to $\mu^+ = (1-\theta)\mu$, with $\theta \in (0, 1)$, we find new iterates x^+, y^+, and s^+ that satisfy (4) and (5), with ν replaced by $\nu^+ = (1-\theta)\nu$, and such that $\delta(x^+, s^+; \mu^+) \leq \tau$.

Note that

$$b - \mathcal{A}(x^+) = b - \mathcal{A}(x + \Delta x) = \nu r_p^0 - \mathcal{A}(\Delta x) = \nu r_p^0 - \theta \nu r_p^0 = (1-\theta)\nu r_p^0,$$

and similarly,

$$\begin{aligned} c - \mathcal{A}^T y^+ + \mathcal{H}(x^+) - s^+ &= c - \mathcal{A}^T(y + \Delta y) + \mathcal{H}(x + \Delta x) - (s + \Delta s) \\ &= \nu r_d^0 - \mathcal{A}^T \Delta y + \mathcal{H}(\Delta x) - \Delta s = \nu r_d^0 - \theta \nu r_d^0 \\ &= (1-\theta)\nu r_d^0. \end{aligned}$$

From the above equations and Lemma 10 (below), it follows that after each iteration the residual vectors and the duality gap are reduced by a factor $1-\theta$. The algorithm stops if the norms of the residuals and the duality gap are less than the accuracy parameter ϵ. At this stage, an ϵ-approximate optimal solution of CQSCO has been found. A formal description of our algorithm is given in Fig. 1.

4 Analysis of the algorithm

Let x, y, and s denote the iterates at the start of an iteration, and assume $\delta(x, s; \mu) \leq \tau$.

4.1 Upper bound for $\delta(v^+)$

As we established in previous sections, the full-NT step generates new iterates x^+, y^+, and s^+ that satisfy the feasibility conditions for (P_{ν^+}) and (D_{ν^+}). A crucial

```
primal − dual Infeasible IPM
Input : accuracy parameter ε > 0;
        update parameter θ, 0 < θ < 1;
        parameters ρ_p > 0, ρ_d > 0.
begin
    x := ρ_p e; y := 0; s := ρ_d e; μ := μ^0 = ρ_p ρ_d; ν = 1;
    while max(tr(x ∘ s), ||r_p||_F, ||r_d||_F) > ε do
    begin
        (x, y, s) := (x, y, s) + (Δx, Δy, Δs);
          μ := (1 − θ)μ; ν := (1 − θ)ν;
    end
end
```

Fig. 1 The algorithm

element in the analysis is to show that after each full-NT step, we have $x^+, s^+ \in$ int$\mathcal{K}$, and such that $\delta(x^+, s^+; \mu^+) \leq \tau$.

To this end, we consider the scaled search directions d_x and d_s and the variance vector v as defined in (10). Using (10), we have

$$x^+ = x + \Delta x = \sqrt{\mu} P(w)^{\frac{1}{2}}(v + d_x), \quad s^+ = s + \Delta s = \sqrt{\mu} P(w)^{-\frac{1}{2}}(v + d_s).$$

Since $P(w)^{\frac{1}{2}}$ and its inverse $P(w)^{-\frac{1}{2}}$ are automorphisms of int $\mathcal{K}$, then x^+ and s^+ belong to int$\mathcal{K}$ if and only if $v + d_x$ and $v + d_s$ belong to int$\mathcal{K}$, respectively. In this case, by using the third equation of (11), we may write

$$(v + d_x) \circ (v + d_s) = v \circ v + v \circ (d_x + d_s) + d_x \circ d_s = (1 - \theta)e + d_x \circ d_s. \quad (13)$$

The proof of the following lemma is similar to the proof of lemma 3.2 in [5], and is therefore omitted.

Lemma 7 *The iterate* (x^+, y^+, s^+) *is strictly feasible if*

$$(1 - \theta)e + d_x \circ d_s \in int\mathcal{K}.$$

Corollary 1 *The iterate* (x^+, y^+, s^+) *is strictly feasible if*

$$\|\lambda(d_x \circ d_s)\|_\infty < 1 - \theta.$$

Proof By Lemma 7, the iterates (x^+, y^+, s^+) are strictly feasible if $(1-\theta)e + d_x \circ d_s \in$ int$\mathcal{K}$. $\|\lambda(d_x \circ d_s)\|_\infty < 1 - \theta$ implies that $-1 + \theta < \lambda_i(d_x \circ d_s) < 1 - \theta$ for each $i = 1, \ldots, r$. Therefore, $\lambda_i((1-\theta)e + d_x \circ d_s) = 1 - \theta + \lambda_i(d_x \circ d_s) > 0$. The last inequalities mean that $(1 - \theta)e + d_x \circ d_s \in$ int$\mathcal{K}$, and the corollary follows. □

In the sequel, we use the notation $\bar{\omega}(v) := \frac{1}{2}\left(\|d_x\|_F^2 + \|d_s\|_F^2\right)$ and assume that $\bar{\omega}(v) < 1 - \theta$. One has

$$\begin{aligned} \|\lambda(d_x \circ d_s)\|_\infty &\leq \|d_x \circ d_s\|_F \leq \|d_x\|_F \|d_s\|_F \\ &\leq \frac{1}{2}\left(\|d_x\|_F^2 + \|d_s\|_F^2\right) = \bar{\omega}(v). \end{aligned} \quad (14)$$

It follows that $\|\lambda(d_x \circ d_s)\|_\infty < 1 - \theta$. Hence $\bar{\omega}(v) < 1 - \theta$ implies that the iterates (x^+, y^+, s^+) are strictly feasible, by Corollary 1. We proceed by driving an upper bound for $\delta(x^+, s^+; \mu^+)$. By definition (12), we have

$$\delta\left(x^+, s^+; \mu^+\right) = \frac{1}{2}\left\|v^+ - \left(v^+\right)^{-1}\right\|_F,$$

where $v^+ = \frac{P(w^+)^{-\frac{1}{2}} x^+}{\sqrt{\mu^+}} \left[= \frac{P(w^+)^{\frac{1}{2}} s^+}{\sqrt{\mu^+}}\right]$. In what follows, we denote $\delta(x^+, s^+; \mu^+)$ shortly by $\delta(v^+)$.

Lemma 8 *If* $\bar{\omega}(v) < 1 - \theta$, *then*

$$\delta\left(v^+\right) \leq \frac{\bar{\omega}(v)}{2\sqrt{(1-\theta)(1-\theta-\bar{\omega}(v))}}.$$

Proof Using Lemma 2, $\sqrt{1-\theta}\ v^+ \sim \left[P(v+d_x)^{\frac{1}{2}}(v+d_s)\right]^{\frac{1}{2}}$ (see the proof of lemma 3.3 in [5]), Lemma 3, Lemma 4, (13), and (14), we have

$$\begin{aligned}
2\delta\left(v^+\right) &= \left\|v^+ - \left(v^+\right)^{-1}\right\|_F \le \frac{\|\left(v^+\right)^2 - e\|_F}{\lambda_{\min}\left(v^+\right)} \\
&= \frac{\left\|P\left(\frac{v+d_x}{\sqrt{1-\theta}}\right)^{\frac{1}{2}}\left(\frac{v+d_s}{\sqrt{1-\theta}}\right) - e\right\|_F}{\lambda_{\min}\left(P\left(\frac{v+d_x}{\sqrt{1-\theta}}\right)^{\frac{1}{2}}\left(\frac{v+d_s}{\sqrt{1-\theta}}\right)\right)^{\frac{1}{2}}} \le \frac{\left\|\left(\frac{v+d_x}{\sqrt{1-\theta}}\right)\circ\left(\frac{v+d_s}{\sqrt{1-\theta}}\right) - e\right\|_F}{\lambda_{\min}\left(\left(\frac{v+d_x}{\sqrt{1-\theta}}\right)\circ\left(\frac{v+d_s}{\sqrt{1-\theta}}\right)\right)^{\frac{1}{2}}} \\
&= \frac{\left\|\frac{d_x\circ d_s}{1-\theta}\right\|_F}{\lambda_{\min}\left(1+\frac{d_x\circ d_s}{1-\theta}\right)^{\frac{1}{2}}} \le \frac{\left\|\frac{d_x\circ d_s}{1-\theta}\right\|_F}{\left(1-\left\|\lambda\left(\frac{d_x\circ d_s}{1-\theta}\right)\right\|_\infty\right)^{\frac{1}{2}}} \\
&\le \frac{\bar{\omega}(v)}{\sqrt{(1-\theta)(1-\theta-\bar{\omega}(v))}}.
\end{aligned}$$

This proves the lemma. □

4.2 Upper bound for $\bar{\omega}(v)$

In this section, we obtain an upper bound $\bar{\omega}(v)$ which will enable us to find a default value for θ, $0 < \theta < 1$. For the moment, let us define

$$\bar{r}_p := \theta v r_p^0, \quad \bar{r}_d := \theta v r_d^0, \quad \bar{r} := (1-\theta)v^{-1} - v. \tag{15}$$

With $\eta := -\frac{\Delta y}{\mu}$, the system (11) (by eliminating d_s) reduces to

$$\begin{aligned}
\bar{\mathcal{A}}(d_x) &= \bar{r}_p, \\
\bar{\mathcal{A}}^T\eta + (I+\bar{\mathcal{H}})(d_x) &= \bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{\frac{1}{2}}\bar{r}_d.
\end{aligned} \tag{16}$$

Multiplying both sides of the second equation in (16) from the left with $\bar{\mathcal{A}}(I+\bar{\mathcal{H}})^{-1}$ and using the first equation of (16), it follows that

$$\bar{\mathcal{A}}\left(I+\bar{\mathcal{H}}\right)^{-1}\bar{\mathcal{A}}^T\eta + \bar{r}_p = \bar{\mathcal{A}}\left(I+\bar{\mathcal{H}}\right)^{-1}\left(\bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{\frac{1}{2}}\bar{r}_d\right).$$

Therefore,

$$\eta = \left(\bar{\mathcal{A}}(I+\bar{\mathcal{H}})^{-1}\bar{\mathcal{A}}^T\right)^{-1}\left[\bar{\mathcal{A}}\left(I+\bar{\mathcal{H}}\right)^{-1}\left(\bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{\frac{1}{2}}\bar{r}_d\right) - \bar{r}_p\right]. \tag{17}$$

Substitution into the second equation of (16) gives

$$d_x = \left(I-\bar{P}\right)\left(I+\bar{\mathcal{H}}\right)^{-1}\left(\bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{\frac{1}{2}}\bar{r}_d\right) + \left(I+\bar{\mathcal{H}}\right)^{-1}\bar{\mathcal{A}}^T\left(\bar{\mathcal{A}}\left(I+\bar{\mathcal{H}}\right)^{-1}\bar{\mathcal{A}}^T\right)^{-1}\bar{r}_p,$$

where $\bar{P} := (I + \bar{\mathcal{H}})^{-1}\bar{\mathcal{A}}^T\left(\bar{\mathcal{A}}(I + \bar{\mathcal{H}})^{-1}\bar{\mathcal{A}}^T\right)^{-1}\bar{\mathcal{A}}$. Let (x^*, y^*, s^*) be the optimal solution satisfying (2). It follows that $\mathcal{A}(x^*) = b$, $\mathcal{A}^T y^* - \mathcal{H}(x^*) + s^* = c$, $0 \preceq_{\mathcal{K}} x^* \preceq_{\mathcal{K}} \rho_p e$ and $0 \preceq_{\mathcal{K}} s^* \preceq_{\mathcal{K}} \rho_d e$. Therefore, it follows that

$$0 \preceq_{\mathcal{K}} x^0 - x^* \preceq_{\mathcal{K}} \rho_p e, \;\; 0 \preceq_{\mathcal{K}} s^0 - s^* \preceq_{\mathcal{K}} \rho_d e. \tag{18}$$

Substituting $\bar{r}_p = \theta\nu r_p^0 = \theta\nu(b - \mathcal{A}x^0) = \theta\nu\mathcal{A}(x^* - x^0)$ and $\bar{r}_d = \theta\nu r_d^0$ into the expression for d_x, we obtain

$$d_x = (I - \bar{P})(I + \bar{\mathcal{H}})^{-1}\left(\bar{r} - \frac{\theta\nu}{\sqrt{\mu}}P(w)^{\frac{1}{2}}r_d^0\right) + \frac{\theta\nu}{\sqrt{\mu}}\bar{P}P(w)^{-\frac{1}{2}}\left(x^* - x^0\right).$$

To proceed, we further simplify the above expression by defining

$$u^x := \frac{\theta\nu}{\sqrt{\mu}}\bar{P}P(w)^{-\frac{1}{2}}\left(x^* - x^0\right), \;\; u^s := \frac{\theta\nu}{\sqrt{\mu}}(I - \bar{P})(I + \bar{\mathcal{H}})^{-1}P(w)^{\frac{1}{2}}r_d^0,$$
$$\bar{r}_1 := (I - \bar{P})(I + \bar{\mathcal{H}})^{-1}\bar{r}.$$

Then we may write

$$\begin{aligned} \|d_x\|_F^2 &= \|\bar{r}_1 - u^s + u^x\|_F^2 = \|\bar{r}_1 - u^s\|_F^2 + \|u^x\|_F^2 \\ &= \|\bar{r}_1\|_F^2 + \|u^s\|_F^2 - 2\langle\bar{r}_1, u^s\rangle + \|u^x\|_F^2 \\ &\le \|\bar{r}_1\|_F^2 + \|u^s\|_F^2 + 2\|\bar{r}_1\|_F\|u^s\|_F + \|u^x\|_F^2 \\ &\le 2\|\bar{r}\|_F^2 + 2\|u^s\|_F^2 + \|u^x\|_F^2. \end{aligned}$$

For $\|d_s\|_F$, we have

$$\begin{aligned} \|d_s\|_F^2 &= \|\bar{r} - d_x\|_F^2 = \|\bar{r}\|_F^2 + \|d_x\|_F^2 - 2\langle\bar{r}, d_x\rangle \\ &\le 2\|\bar{r}\|_F^2 + 2\|d_x\|_F^2 \le 6\|\bar{r}\|_F^2 + 4\|u^s\|_F^2 + 2\|u^x\|_F^2. \end{aligned}$$

Therefore, we may write

$$\bar{\omega}(v) = \frac{1}{2}\left(\|d_x\|_F^2 + \|d_s\|_F^2\right) = 4\|\bar{r}\|_F^2 + \frac{3}{2}\left(2\|u^s\|_F^2 + \|u^x\|_F^2\right). \tag{19}$$

Due to the definitions u^x and u^s, we have

$$2\|u^s\|_F^2 + \|u^x\|_F^2 \le \frac{\theta^2\nu^2}{\mu}\left(2\left\|P(w)^{\frac{1}{2}}r_d^0\right\|_F^2 + \left\|P(w)^{-\frac{1}{2}}\left(x^* - x^0\right)\right\|_F^2\right). \tag{20}$$

We now obtain the upper bounds for $\|P(w)^{-\frac{1}{2}}(x^* - x^0)\|_F^2$ and $\|P(w)^{\frac{1}{2}}r_d^0\|_F^2$. Using that $P(w)^{\frac{1}{2}}$ is self-adjoint with respect to the inner product and $P(w)^{-1}e = w^{-2}$, we have

$$\begin{aligned} \|P(w)^{-\frac{1}{2}}(x^* - x^0)\|_F^2 &= \left\langle P(w)^{-\frac{1}{2}}(x^0 - x^*), P(w)^{-\frac{1}{2}}(x^0 - x^*)\right\rangle \\ &= \left\langle P(w)^{-1}(x^0 - x^*), x^0 - x^*\right\rangle \\ &= \left\langle P(w)^{-1}(x^0 - x^*), \rho_p e\right\rangle - \left\langle P(w)^{-1}(x^0 - x^*), \rho_p e - (x^0 - x^*)\right\rangle \\ &\le \left\langle P(w)^{-1}(x^0 - x^*), \rho_p e\right\rangle = \rho_p\left\langle P(w)^{-1}e, x^0 - x^*\right\rangle \\ &= \rho_p\left\langle P(w)^{-1}e, \rho_p e\right\rangle - \rho_p\left\langle P(w)^{-1}e, \rho_p e - (x^0 - x^*)\right\rangle \\ &\le \rho_p^2 \,\mathrm{tr}(w^{-2}). \end{aligned}$$

Since $x^0 = \rho_p e$, $y^0 = 0$, $s^0 = \rho_d e$ and $\rho_d \geq \|\rho_p \mathcal{H} e + c\|_F$, it follows that $2\rho_d e \succeq r_d^0 \succeq 0$. Therefore, we have

$$\begin{aligned}
\|P(w)^{\frac{1}{2}} r_d^0\|_F^2 &= \left\langle P(w)^{\frac{1}{2}} r_d^0, P(w)^{\frac{1}{2}} r_d^0 \right\rangle = \left\langle P(w) r_d^0, r_d^0 \right\rangle \\
&= \left\langle P(w) r_d^0, 2\rho_d e \right\rangle - \left\langle P(w) r_d^0, 2\rho_d e - r_d^0 \right\rangle \\
&\leq \left\langle P(w) r_d^0, 2\rho_d e \right\rangle = 2\rho_d \left\langle P(w) e, r_d^0 \right\rangle \\
&= 2\rho_d \left\langle P(w) e, 2\rho_d e \right\rangle - 2\rho_d \left\langle P(w) e, 2\rho_d e - r_d^0 \right\rangle \\
&\leq 4\rho_d^2 \left\langle P(w) e, e \right\rangle = 4\rho_d^2 \,\mathrm{tr}(w^2).
\end{aligned}$$

Substituting of the two last inequalities into (20) gives

$$\begin{aligned}
2\|u^s\|_F^2 + \|u^x\|_F^2 &\leq \frac{\theta^2 \nu^2}{\mu} \left(8\rho_d^2 \,\mathrm{tr}\left(w^2\right) + \rho_p^2 \mathrm{tr}\left(w^{-2}\right) \right) \\
&\leq \frac{\theta^2 \nu^2}{\mu} \left(\frac{8\rho_d^2 \,\mathrm{tr}\left(x^2\right)}{\mu \lambda_{\min}(v)^2} + \frac{\rho_p^2 \,\mathrm{tr}\left(s^2\right)}{\mu \lambda_{\min}(v)^2} \right) \\
&\leq \theta^2 \rho(\delta)^2 \left(\frac{8 \,\mathrm{tr}\left(x^2\right)}{\rho_p^2} + \frac{\mathrm{tr}\left(s^2\right)}{\rho_d^2} \right), \qquad (21)
\end{aligned}$$

where the last inequality follows by Lemma 6 and $\mu = \nu \rho_p \rho_d$. Using the orthogonality of v^{-1} and $v^{-1} - v$ with respect to the trace inner product, the triangle inequality and $\|v\|_F^2 = \mathrm{tr}(e) = r$, we get

$$\|\bar{r}\|_F^2 = \|(1-\theta)(v^{-1} - v) - \theta v\|_F^2 \leq 4(1-\theta)^2 \delta(v)^2 + \theta^2 r. \qquad (22)$$

Substitution of (21) and (22) into (19) yields that

$$\bar{\omega}(v) \leq 16(1-\theta)^2 \delta(v)^2 + 4\theta^2 r + \frac{3\theta^2 \rho(\delta)^2}{2} \left(\frac{8 \,\mathrm{tr}\left(x^2\right)}{\rho_p^2} + \frac{\mathrm{tr}\left(s^2\right)}{\rho_d^2} \right). \qquad (23)$$

To continue, we need upper bounds for $\mathrm{tr}(x)$ and $\mathrm{tr}(s)$, which is contained in the following lemma.

Lemma 9 *Let (x, y, s) be feasible for the perturbed problem pair (P_ν) and (D_ν). With (x^0, y^0, s^0) as defined in (1) and ρ_p and ρ_d as defined in (2), we have*

$$tr(x) \leq r\rho_p \left(2 + \rho(\delta)^2\right), \quad tr(s) \leq r\rho_d \left(2 + \rho(\delta)^2\right).$$

Proof Let (x^*, y^*, s^*) be the optimal solution satisfying (2). Then from the feasibility conditions of the perturbed problem pair (P_ν) and (D_ν), it is easily seen that

$$\mathcal{A}(x - \nu x^0 - (1-\nu) x^*) = 0,$$

$$\mathcal{A}^T (y - \nu y^0 - (1-\nu) y^*) - \mathcal{H}(x - \nu x^0 - (1-\nu) x^*) + (s - \nu s^0 - (1-\nu) s^*) = 0.$$

Since $\mathcal{H}$ is self-adjoint positive semidefinite linear operator, we have

$$\begin{aligned}
0 &\le \left\langle \mathcal{H}(x-\nu x^0-(1-\nu)x^*), x-\nu x^0-(1-\nu)x^* \right\rangle \\
&= \left\langle \mathcal{H}(x-\nu x^0-(1-\nu)x^*), x-\nu x^0-(1-\nu)x^* \right\rangle \\
&\quad + \sum_{i=1}^{m} -(y_i-\nu y_i^0-(1-\nu)y_i^*)\langle a_i, x-\nu x^0-(1-\nu)x^*\rangle \\
&= \left\langle \mathcal{H}(x-\nu x^0-(1-\nu)x^*), x-\nu x^0-(1-\nu)x^* \right\rangle \\
&\quad +\langle -\mathcal{A}^T(y-\nu y^0-(1-\nu)y^*), x-\nu x^0-(1-\nu)x^*\rangle \\
&= \left\langle s-\nu s^0-(1-\nu)s^*, x-\nu x^0-(1-\nu)x^* \right\rangle \\
&= \langle x,s\rangle - \nu\big(\langle x^0,s\rangle+\langle x,s^0\rangle\big)-(1-\nu)\big(\langle x^*,s\rangle+\langle x,s^*\rangle\big) \\
&\quad +\nu^2\langle x^0,s^0\rangle+\nu(1-\nu)\big(\langle x^*,s^0\rangle+\langle x^0,s^*\rangle\big)+(1-\nu)^2\langle x^*,s^*\rangle. \qquad (24)
\end{aligned}$$

Since (x^0, y^0, s^0) is as defined in (1), we have

$$\begin{aligned}
\langle x^0,s^0\rangle = \rho_p\rho_d\mathrm{tr}(e) &= \rho_p\rho_d r, \quad \langle x^0,s\rangle+\langle x,s^0\rangle=\rho_p\,\mathrm{tr}(s)+\rho_d\mathrm{tr}(x), \\
\langle x^*,s^0\rangle+\langle x^0,s^*\rangle &= \rho_d\,\mathrm{tr}(x^*)+\rho_p\mathrm{tr}(s^*)=2\rho_p\rho_d r.
\end{aligned}$$

Furthermore, using (10) and Lemma 6, we get

$$\langle x,s\rangle = \mu\left\langle \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}}, \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}}\right\rangle = \mu\langle v,v\rangle=\mu\|v\|_F^2\le\mu\rho(\delta)^2 r.$$

Substituting these into (24), also using $\langle x^*, s^*\rangle = 0$ and $\mu = \nu\rho_p\rho_d$, we obtain

$$\rho_p\mathrm{tr}(s)+\rho_d\mathrm{tr}(x)\le r\rho_p\rho_d(2+\rho(\delta)^2).$$

The required results follow from the last inequality and the fact that $x \succeq_{\mathcal{K}} 0$ and $s \succeq_{\mathcal{K}} 0$. This completes the proof. □

By substituting the results of Lemma 9 into (23), we derive an upper bound for $\bar{\omega}(v)$ as follows

$$\bar{\omega}(v)\le 16(1-\theta)^2\delta(v)^2+4\theta^2 r+\frac{27}{2}r^2\theta^2\rho(\delta)^2(2+\rho(\delta)^2)^2. \qquad (25)$$

4.3 The effect on the duality gap

The following lemma shows that, in each iteration of the algorithm, the duality gap is reduced by the factor $1-\theta$.

Lemma 10 *If the iterate* (x^+, y^+, s^+) *is strictly feasible, then*

$$\langle x^+, s^+\rangle \le 2(1-\theta)\mu r.$$

Proof Since

$$\begin{aligned}\langle x^+, s^+\rangle &= \left\langle \sqrt{\mu^+}P\left(w^+\right)^{\frac{1}{2}} v^+, \sqrt{\mu^+}P\left(w^+\right)^{-\frac{1}{2}} v^+\right\rangle = \mu^+\langle v^+, v^+\rangle \\ &= (1-\theta)\mu\|v^+\|_F^2 = (1-\theta)\mu\sum_{i=1}^{r}\lambda_i\left(v^+\right)^2 \\ &\leq (1-\theta)\mu r\lambda_{\max}\left(v^+\right)^2. \end{aligned} \tag{26}$$

On the other hand, by Lemma 4 and (13), we have

$$\begin{aligned}\lambda_{\max}(v^+)^2 &= \lambda_{\max}\left(P\left(\frac{v+d_x}{\sqrt{1-\theta}}\right)^{\frac{1}{2}}\left(\frac{v+d_s}{\sqrt{1-\theta}}\right)\right) \\ &\leq \lambda_{\max}\left(\left(\frac{v+d_x}{\sqrt{1-\theta}}\right)\circ\left(\frac{v+d_s}{\sqrt{1-\theta}}\right)\right) \\ &= \frac{1}{1-\theta}\lambda_{\max}\big((v+d_x)\circ(v+d_s)\big) \\ &= \frac{1}{1-\theta}\lambda_{\max}\big((1-\theta)e + d_x\circ d_s)\big) \\ &\leq 1+\frac{1}{1-\theta}\|\lambda(d_x\circ d_s)\|_\infty < 2,\end{aligned}$$

where the last inequality follows by Corollary 1. Substituting this bound into (26), we obtain the inequality in the lemma. □

4.4 Values for θ and τ

Our aim is to find a positive number τ such that if $\delta(v) \leq \tau$ holds, then $\delta(v^+) \leq \tau$. By Lemma 8, this will hold if

$$\bar{\omega}(v) < 1-\theta, \tag{27}$$

$$\frac{\bar{\omega}(v)}{2\sqrt{(1-\theta)(1-\theta-\bar{\omega}(v))}} \leq \tau. \tag{28}$$

Using (25), the inequality (27) holds if

$$16(1-\theta)^2\delta(v)^2 + 4\theta^2 r + \frac{27}{2}r^2\theta^2\rho(\delta)^2(2+\rho(\delta)^2)^2 < 1-\theta.$$

Assuming $\delta(v) \leq \tau$, we therefore need to find τ such that the above inequality holds, with θ as large as possible. One easily verifies that the left-hand side expression in the above inequality is monotonically increasing with respect to $\delta(v)$. Hence, it suffices if

$$16(1-\theta)^2\tau^2 + 4\theta^2 r + \frac{27}{2}r^2\theta^2\rho(\tau)^2(2+\rho(\tau)^2)^2 < 1-\theta. \tag{29}$$

In the rest of this section, we show that (29) and (28) hold if θ and τ are taken as follows:

$$\theta = \frac{1}{53r}, \quad \tau = \frac{1}{16}. \tag{30}$$

At this case, defining

$$g(r,\tau)=16\Big(1-\frac{1}{53r}\Big)^2\tau^2+\frac{4}{2809r}+\frac{27}{5618}\rho(\tau)^2(2+\rho(\tau)^2)^2,\quad r\geq 1,$$

we get $g(r,\tau)-(1-\theta)\leq -0.8661<0$. This means that the inequality (29) holds, i.e., the iterate (x^+,y^+,s^+) is strictly feasible. Note that $g(r,\tau)$ provides an upper bound for $\bar{\omega}(v)$, by (25). From Lemma 8, it follows that

$$\delta(v^+)\leq\frac{\bar{\omega}(v)}{2\sqrt{(1-\theta)(1-\theta-\bar{\omega}(v))}}\leq\frac{g(r,\tau)}{2\sqrt{(1-\theta)(1-\theta-g(r,\tau))}}\leq 0.0624<\tau.$$

Therefore, the algorithm is well-defined in the sense that the property $\delta(x,s;\mu)\leq\tau$ is maintained in all iterations.

4.5 Complexity analysis

We have found that if at the start of an iteration the iterates (x,y,s) satisfying $\delta(x,s;\mu)\leq\tau$ and τ and θ are defined as in (30), then after the full step, the new iterate (x^+,y^+,s^+) is strictly feasible and $\delta(x^+,s^+;\mu^+)\leq\tau$. This establishes the algorithm to be well-defined. At each iteration, both the values of $\operatorname{tr}(x\circ s)$ and the norm of the residual vectors are reduced by the factor $1-\theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta}\log\frac{\max\left(\operatorname{tr}(x^0\circ s^0),\|r_p^0\|_F,\|r_d^0\|_F\right)}{\epsilon}.$$

Thus, we may state the main result of our work.

Theorem 1 *If (P) and (D) have an optimal solution (x^*,y^*,s^*) such that $\|x^*\|_\infty\leq\rho_p$ and $\|s^*\|_\infty\leq\rho_d$ for $\rho_p>0$ and $\rho_d>0$, then after at most*

$$53r\log\frac{\max\left(tr\left(x^0\circ s^0\right),\|r_p^0\|_F,\|r_d^0\|_F\right)}{\epsilon}.$$

iterations the algorithm finds an ϵ-solution of (P) and (D).

5 Numerical results

To test the method, we use a number of test problems from the test set given in [15]. We use $(x_0,y_0,s_0)=(3e,0,e)$, $b=3Ae$ and $c=-3He+e$ as the starting values. The results are listed by using MATLAB version 7.8.0.347 (R2009a) on a PC with 2GB RAM under Windows XP to solve the selected test problems. We set $\epsilon=10^{-1}$, and we take the set of parameters $\tau=\frac{1}{16}$ and $\theta=\frac{1}{53n}$ for the proposed algorithm. In Table 1 'Iter.'=number of iterations, 'time'=CPU time (s), 'dual-gap'=$\frac{x^Ts}{n}$ and

Table 1 Numerical results

Name	n	m	Iter.	Time	Dual-gap	Opt.
primal1	325	85	158,208	1610.568951	3.0768e-04	−645.0018
primal2	649	96	339,722	13,230.334885	1.5408e-04	−1.2930e+003
primal3	745	111	395,421	84,226.285055	1.3423e-04	−1.4850e+03
primal4	1489	75	844,964	172,892.618194	6.7159e-05	−2.9731e+03
primalc1	230	9	107,747	262.684171	4.3476-04	−372.3328
cvxqp1_{m}	1000	500	546,370	41,091.772428	9.9999e-05	−2.0267e+07
cvxqp3_{m}	1000	750	546,370	45,551.455874	9.9999e-05	−2.0267e+07
cvxqp1_s	100	50	42,430	56.955248	9.9995e-04	−2.0423e+05
cvxqp2_s	100	25	42,430	48.678467	9.9995e-04	−2.0423e+05
cvxqp3_{s}	100	75	42,430	63.679049	9.9995e-04	−2.0423e+05
gouldqp2	699	349	368,645	10,259.584279	1.4306e-04	1.0470e+03
gouldqp3	699	349	368,645	10,162.697975	1.4306e-04	−4.3617+e03
dual1	85	1	35,333	178.191453	0.0012	−50883
dual2	96	1	40,525	259.038709	0.0010	−34605
dual3	111	1	47,712	416.832747	9.008e-004	−42876
dual4	75	1	30,678	124.237973	0.0013	−25632
dualc1	9	215	2668	27.547797	0.0111	−21009411
dualc2	7	229	1982	18.185493	0.0143	−6.2784e+006
dualc5	8	278	2322	32.318771	0.0125	−909408
dualc8	8	503	2322	48.30303788	0.0125	−7.7594e+007
tame	2	1	432	0.206462	0.0500	6
hs21	2	1	432	0.500003	0.0500	−3.3923
qsc205	203	205	93,754	538.460147	4.9261e-04	−410.9995
ksip	20	1001	6778	127.193175	0.0050	43.8102
genhs28	10	8	3021	0.673330	0.0100	−294.1681

'Opt.'$= c^T x + \frac{1}{2} x^T H x$ provided for the selected problems. The results are given in Table 1.

From the above table, we see that the proposed algorithm converges for the selected problems. Although, in theory, the convergence is not guaranteed for bigger θ values, we performed a MATLAB experiment for $\theta = 0.5$ and $\epsilon = 10^{-8}$. Results are given in Table 2.

It can be seen from tables that for $\theta = \frac{1}{53n}$ and $\theta = \frac{1}{2}$ the algorithm gives the same optimal values, whereas it requires smaller number of iteration and consumes less CPU time for $\theta = \frac{1}{2}$.

Table 2 Results of numerical tests

Name	n	m	Iter.	Time	Dual-gap	Opt.
primal1	85	325	37	2.919720	2.1828e-011	−6.4500e+002
primal2	649	96	38	8.089346	1.0914e-011	−1.2930e+003
primal3	745	111	38	13.847444	1.0914e-011	−1.4850e+003
primal4	1489	75	39	57.604279	5.4570e-012	−2.9730e+003
primalc1	230	9	37	1.186898	2.1828e-011	−3.7233e+002
cvxqp1$_m$	1000	500	39	22.799327	5.4570e-012	−2.0267e+007
cvxqp1$_s$	100	50	35	0.428151	8.7311e-011	−2.0422e+005
cvxqp3$_{\mathrm{m}}$	1000	750	39	24.309886	5.4570e-012	−2.0267e+007
dual1	85	1	35	1.365417	8.7311e-011	−50883
dual2	96	1	35	1.738733	8.7311e-011	−34605
dual3	111	1	35	2.190599	8.7311e-011	−42876
dual4	75	1	35	1.101488	8.7311e-011	−25632
qpcstair	467	356	38	4.809967	1.0914e-011	−2.1722e+004
aug3d	3873	1000	41	868.188402	1.3642e-012	−1.7460e+003
aug3dc	3873	1000	41	802.639363	1.3642e-012	−7.0453e+003
aug3dcqp	3873	1000	41	846.513214	1.3642e-012	−7.0453e+003
aug3dqp	3873	1000	41	804.434460	1.3642e-012	−1.7460e+003
mosarqp2	900	600	38	14.95639	1.0914e-011	−2.3107e+004
mosarqp1	2500	700	40	228.949336	2.7285e-012	−1.2505e+004
qscsd1	760	77	38	9.841303	1.0914e-011	−1.2187e+004
yao	2002	200	40	132.507548	2.7285e-012	−4.0040e+003
$cvxqp3_s$	100	75	35	0.791434	8.7311e-011	−2.0423e+005
$gouldqp2$	699	349	38	7.308872	1.0917e-011	1.0470e+003
$gouldqp2$	699	349	38	7.782762	1.0914e-011	−4.3617e+003
hs21	2	1	30	0.090634	2.7940e-009	−3.3925

6 Conclusions

In this paper, we presented an infeasible interior-point method for convex quadratic symmetric cone optimization based on full-NT step. Our algorithm is used only one feasibility step in each iteration. The obtained complexity bound coincides with the best-known iteration bound for IIPMs. We provided results for $\theta = \frac{1}{53n}$ and $\theta = 0.5$ in Tables 1 and 2. Results of numerical tests are given in Section 5.

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