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ORIGINAL PAPER

Inequalities and asymptotics for the Euler–Mascheroni constant based on DeTemple's result

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Abstract Let $R_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right)$. DeTemple proved the following inequality:

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}$$

for all integers $n \geq 1$, where γ denotes the Euler–Mascheroni constant. In this paper, we give a pair of recurrence relations for determining the constants a_ℓ and b_ℓ such that

$$R_n - \gamma \sim \sum_{\ell=1}^{\infty} \frac{a_\ell}{(n^2 + n + b_\ell)^{2\ell-1}}, \qquad n \to \infty.$$

Based on this expansion, we establish some inequalities for the Euler–Mascheroni constant.

Keywords Psi function · Euler–Mascheroni constant · Asymptotic formula · Inequality

Mathematics Subject Classifications (2010) Primary 33B15; Secondary 11Y60 · 41A60

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1 Introduction

The Euler–Mascheroni constant $\gamma = 0.577215664\ldots$ is defined as the limit of the sequence

$$D_n = H_n - \ln n, \qquad n \in \mathbb{N} := \{1, 2, 3, \ldots\}, \tag{1.1}$$

where H_n denotes the nth harmonic number defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

There has been a significant amount of interest and research on γ as testified by survey papers (cf., [7]) and expository books (cf., [11]), which reveal its essential properties and surprising connections with other areas of mathematics.

Several bounds for $D_n - \gamma$ have been given in the literature [3, 4, 15, 17–20]. For example, the following inequality for $D_n - \gamma$ was established in [15, 20]:

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n}, \qquad n \in \mathbb{N}. \tag{1.2}$$

Alzer [3, Theorem 3] gave the following sharp form of the inequality (1.2):

$$\frac{1}{2\left(n + \frac{1}{2(1-\gamma)} - 1\right)} \le D_n - \gamma < \frac{1}{2\left(n + \frac{1}{6}\right)}, \qquad n \in \mathbb{N}. \tag{1.3}$$

The constants $\frac{1}{2(1-\gamma)} - 1$ and $\frac{1}{6}$ are the best possible.

The editorial comment in [19] said that the inequality (1.3) holds, and equality holds only when $n = 1$. However, the proof of (1.3) was not published in [19]. So, the first published proof is due to Alzer.

The convergence of the sequence D_n to γ is very slow. By changing the logarithmic term in (1.1), DeTemple [8, 9] presented the following inequality:

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \tag{1.4}$$

where

$$R_n = H_n - \ln\left(n + \frac{1}{2}\right). \tag{1.5}$$

Karatsuba [12] improved (1.4) and obtained the following inequality:

$$\frac{1}{24n^2} - \frac{1}{24n^3} + \frac{1}{120n^4} - \frac{1}{126n^6} \le R_n - \gamma \le \frac{1}{24n^2} - \frac{1}{24n^3} + \frac{23}{960n^4}. \tag{1.6}$$

Chen [5] obtained the following sharp form of the inequality (1.4):

$$\frac{1}{24(n+a)^2} \le R_n - \gamma < \frac{1}{24(n+b)^2}, \qquad n \in \mathbb{N} \tag{1.7}$$

with the best possible constants

$$a = \frac{1}{\sqrt{24[-\gamma + 1 - \ln(3/2)]}} - 1 = 0.55106\ldots \quad \text{and} \quad b = \frac{1}{2}.$$

For some $a \in (0, \infty)$, Sîntămărian [16] considered the sequence

$$\left(\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\frac{a+n-1}{a}\right)_{n\in\mathbb{N}}$$

and evaluated the limit $\gamma(a)$ of this sequence, which for $a = 1$ is equal to the Euler constant γ. In particular, Sîntămărian [16] presented some sequences that are stated to converge quickly to $\gamma(a)$, like the sequences

$$\alpha_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + \frac{1}{2a}\right) - \frac{1}{24(a+n-\frac{1}{2})^2} + \frac{7}{960(a+n-\frac{1}{2})^4} - \frac{31}{8064(a+n-\frac{1}{2})^6} + \frac{127}{30720(a+n-\frac{1}{2})^8}$$

and

$$\beta_n = \alpha_n - \frac{511}{67584(a+n-\frac{1}{2})^{10}}.$$

Moreover, Sîntămărian [16, Theorem 3.2] presented the following results:

(i) $\gamma(a) < \alpha_{n+1} < \alpha_n$, for each $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} n^{10}(\alpha_n - \gamma(a)) = \frac{511}{67584}.$$

(ii) $\beta_n < \beta_{n+1} < \gamma(a)$, for each $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} n^{12}(\gamma(a) - \beta_n) = \frac{1414477}{67092480}.$$

(iii) For each $n \in \mathbb{N}$,

$$\frac{511}{67584(a+n)^{10}} < \alpha_n - \gamma(a) < \frac{511}{67584(a+n-\frac{1}{2})^{10}}. \tag{1.8}$$

In this paper, we give a pair of recurrence relations for determining the constants a_ℓ and b_ℓ such that

$$R_n - \gamma \sim \sum_{\ell=1}^{\infty} \frac{a_\ell}{(n^2+n+b_\ell)^{2\ell-1}}, \qquad n \to \infty.$$

Based on this expansion, we establish some inequalities for the Euler–Mascheroni constant.

2 A useful lemma

Euler's gamma function:

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\mathrm{d}t, \qquad x > 0$$

is one of the most important functions in mathematical analysis and has applications in many diverse areas. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is known as the psi (or digamma) function. $\psi(x)$ is connected to the Euler–Mascheroni constant and harmonic numbers through the well known relation (see [10, p. 137, Eq. (5.4.14)])

$$\psi(n+1) = H_n - \gamma, \qquad n \in \mathbb{N}. \tag{2.1}$$

Hence, various approximations of the psi function are used in this relation and interpreted as approximation for the harmonic number H_n or as approximation of the constant γ.

Lemma 2.1 (see [2]) *For $x > \frac{1}{2}$ and $N = 0, 1, 2, \ldots$,*

$$\ln\left(x - \tfrac{1}{2}\right) - \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}} < \psi(x) < \ln\left(x - \tfrac{1}{2}\right) - \sum_{k=1}^{2N+1} \frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}}, \tag{2.2}$$

where

$$B_k(1/2) = -\left(1 - \frac{1}{2^{k-1}}\right) B_k, \quad k = 0, 1, 2, \ldots. \tag{2.3}$$

Here B_n $(n \in \mathbb{N}_0)$ are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi.$$

It follows from (2.2) that for $x > 0$,

$$\frac{1}{24x^2} - \frac{7}{960x^4} + \frac{31}{8064x^6} - \frac{127}{30720x^8} < \psi\left(x + \frac{1}{2}\right) - \ln x < \frac{1}{24x^2} - \frac{7}{960x^4} + \frac{31}{8064x^6} - \frac{127}{30720x^8} + \frac{511}{67584x^{10}}. \tag{2.4}$$

Using (2.4), we find[1]

$$P^2(x) < \frac{1}{576(x+\frac{1}{2})^4} - \frac{7}{11520(x+\frac{1}{2})^6} + \frac{7229}{19353600(x+\frac{1}{2})^8}, \qquad x \geq 2, \tag{2.5}$$

where

$$P(x) = \psi(x+1) - \ln\left(x + \frac{1}{2}\right). \tag{2.6}$$

[1] The inequality (2.5) is proved in the Appendix.

It follows from (2.2) that for $x > 0$,

$$\frac{1}{24x^2} - \frac{7}{960x^4} + \frac{31}{8064x^6} - \frac{127}{30720x^8} + \frac{511}{67584x^{10}} - \frac{1414477}{67092480x^{12}} < \psi\left(x + \frac{1}{2}\right) - \ln x$$
$$< \frac{1}{24x^2} - \frac{7}{960x^4} + \frac{31}{8064x^6} - \frac{127}{30720x^8} + \frac{511}{67584x^{10}} - \frac{1414477}{67092480x^{12}} + \frac{8191}{98304x^{14}}. \tag{2.7}$$

The inequalities (2.5) and (2.7) are used in Section 2.

3 Asymptotic expansions

Theorem 3.1 *As $x \to \infty$, we have*

$$\psi\left(x + \frac{1}{2}\right) \sim \ln x + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(x^2 + \mu_\ell)^{2\ell-1}}, \tag{3.1}$$

where λ_ℓ and μ_ℓ are given by a pair of recurrence relations

$$\lambda_\ell = \left(1 - \frac{1}{2^{4\ell-3}}\right)\frac{B_{4\ell-2}}{4\ell-2} - \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k}, \qquad \ell \geq 2 \tag{3.2}$$

and

$$\mu_\ell = -\frac{1}{(2\ell-1)\lambda_\ell}\left\{\left(1 - \frac{1}{2^{4\ell-1}}\right)\frac{B_{4\ell}}{4\ell} + \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k+1}\binom{2\ell-1}{2\ell-2k+1}\right\}, \quad \ell \geq 2 \tag{3.3}$$

with $\lambda_1 = \frac{1}{24}$ and $\mu_1 = \frac{7}{40}$. Here B_n are the Bernoulli numbers.

Proof Write (3.1) as

$$\psi\left(x + \frac{1}{2}\right) - \ln x \sim \sum_{j=1}^{\infty} \frac{\lambda_j}{x^{4j-2}}\left(1 + \frac{\mu_j}{x^2}\right)^{-2j+1}.$$

Direct computation yields

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{\lambda_j}{x^{4j-2}}\left(1 + \frac{\mu_j}{x^2}\right)^{-2j+1} &\sim \sum_{j=1}^{\infty} \frac{\lambda_j}{x^{4j-2}} \sum_{k=0}^{\infty} \binom{-2j+1}{k} \frac{\mu_j^k}{x^{2k}} \\ &\sim \sum_{j=1}^{\infty} \frac{\lambda_j}{x^{4j-2}} \sum_{k=0}^{\infty} (-1)^k \binom{k+2j-2}{k} \frac{\mu_j^k}{x^{2k}} \\ &\sim \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \lambda_{k+1}\mu_{k+1}^{j-k-1}(-1)^{j-k-1}\binom{j+k-1}{j-k-1}\frac{1}{x^{2j+2k}},\end{aligned}$$

which can be written as

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{x^{4j-2}} \left(1+\frac{\mu_j}{x^2}\right)^{-2j+1} \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+2}{2} \rfloor} \lambda_k \mu_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{x^{2j}}.$$

We then obtain

$$\psi\left(x+\frac{1}{2}\right) - \ln x \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+2}{2} \rfloor} \lambda_k \mu_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{x^{2j}}. \tag{3.4}$$

On the other hand, it follows (see [13, p. 33]) that

$$\psi\left(x+\frac{1}{2}\right) - \ln x \sim -\sum_{j=1}^{\infty} \frac{B_{2j}(1/2)}{2jx^{2j}}, \qquad x \to \infty.$$

Using (2.3), we have

$$\psi\left(x+\frac{1}{2}\right) - \ln x \sim \sum_{j=1}^{\infty} \left(1-\frac{1}{2^{2j-1}}\right) \frac{B_{2j}}{2j} \frac{1}{x^{2j}}. \tag{3.5}$$

Equating coefficients of the term x^{-2j} on the right sides of (3.4) and (3.5), we obtain

$$\left(1-\frac{1}{2^{2j-1}}\right) \frac{B_{2j}}{2j} = \sum_{k=1}^{\lfloor \frac{j+2}{2} \rfloor} \lambda_k \mu_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1}, \qquad j \in \mathbb{N}. \tag{3.6}$$

Setting $j = 2\ell - 1$ and $j = 2\ell$ in (3.6), respectively, yields

$$\left(1-\frac{1}{2^{4\ell-3}}\right) \frac{B_{4\ell-2}}{4\ell-2} = \sum_{k=1}^{\ell} \lambda_k \mu_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \tag{3.7}$$

and

$$\begin{aligned}\left(1-\frac{1}{2^{4\ell-1}}\right) \frac{B_{4\ell}}{4\ell} &= -\sum_{k=1}^{\ell+1} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} \\ &= -\sum_{k=1}^{\ell} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - \lambda_{\ell+1}\mu_{\ell+1}^{-1} \binom{2\ell-1}{-1} \\ &= -\sum_{k=1}^{\ell} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1}.\end{aligned} \tag{3.8}$$

For $\ell = 1$, from (3.7) and (3.8) we obtain

$$\lambda_1 = \frac{1}{24} \quad \text{and} \quad \mu_1 = \frac{7}{40},$$

and for $\ell \geq 2$ we have

$$\left(1-\frac{1}{2^{4\ell-3}}\right)\frac{B_{4\ell-2}}{4\ell-2}=\sum_{k=1}^{\ell-1}\lambda_k\mu_k^{2\ell-2k}\binom{2\ell-2}{2\ell-2k}+\lambda_\ell$$

and

$$\left(1-\frac{1}{2^{4\ell-1}}\right)\frac{B_{4\ell}}{4\ell}=-\sum_{k=1}^{\ell-1}\lambda_k\mu_k^{2\ell-2k+1}\binom{2\ell-1}{2\ell-2k+1}-(2\ell-1)\lambda_\ell\mu_\ell.$$

We then obtain the recurrence relations (3.2) and (3.3). The proof of Theorem 3.1 is complete. □

Here we give explicit numerical values of some first terms of λ_ℓ and μ_ℓ by using the recurrence relations (3.2) and (3.3). This shows how easily we can determine the constants λ_ℓ and μ_ℓ in (3.1).

$$\lambda_1 = \frac{1}{24}, \quad \mu_1 = \frac{7}{40},$$

$$\lambda_2 = \frac{31}{192}B_6 - \lambda_1\mu_1^2 = \frac{31}{192}\cdot\left(\frac{1}{6}\right) - \frac{1}{24}\cdot\left(\frac{7}{40}\right)^2 = \frac{2071}{806400},$$

$$\mu_2 = -\frac{127B_8 + 1024\lambda_1\mu_1^3}{3072\lambda_2} = -\frac{127\cdot\left(-\frac{1}{30}\right)+1024\cdot\left(\frac{1}{24}\right)\cdot\left(\frac{7}{40}\right)^3}{3072\cdot\left(\frac{2071}{806400}\right)} = \frac{42049}{82840},$$

$$\begin{aligned}\lambda_3 &= \frac{511}{5120}B_{10} - \lambda_1\mu_1^4 - 6\lambda_2\mu_2^2\\ &= \frac{511}{5120}\cdot\left(\frac{5}{66}\right) - \frac{1}{24}\cdot\left(\frac{7}{40}\right)^4 - 6\cdot\left(\frac{2071}{806400}\right)\cdot\left(\frac{42049}{82840}\right)^2 = \frac{4971169273}{1399664640000},\end{aligned}$$

$$\begin{aligned}\mu_3 &= -\frac{2047B_{12} + 24576\lambda_1\mu_1^5 + 245760\lambda_2\mu_2^3}{122880\lambda_3}\\ &= -\frac{2047\cdot\left(-\frac{691}{2730}\right)+24576\cdot\left(\frac{1}{24}\right)\cdot\left(\frac{7}{40}\right)^5+245760\cdot\left(\frac{2071}{806400}\right)\cdot\left(\frac{42049}{82840}\right)^3}{122880\cdot\left(\frac{4971169273}{1399664640000}\right)}\\ &= \frac{186936308635618223}{187374306471770600}.\end{aligned}$$

Noting that (2.1) holds, replacing x by $n+\frac{1}{2}$ in (3.1), we obtain the following

Corollary 3.1 *As $n\to\infty$, we have*

$$R_n - \gamma \sim \sum_{\ell=1}^{\infty}\frac{a_\ell}{(n^2+n+b_\ell)^{2\ell-1}}, \tag{3.9}$$

where

$$a_\ell = \lambda_\ell \quad \textit{and} \quad b_\ell = \frac{1}{4} + \mu_\ell. \tag{3.10}$$

The first few constants a_ℓ and b_ℓ are

$$\begin{aligned} a_1 &= \lambda_1 = \frac{1}{24}, \quad b_1 = \frac{1}{4} + \mu_1 = \frac{17}{40}, \\ a_2 &= \lambda_2 = \frac{2071}{806400}, \quad b_2 = \frac{1}{4} + \mu_2 = \frac{62759}{82840}, \\ a_3 &= \lambda_3 = \frac{4971169273}{1399664640000}, \quad b_3 = \frac{1}{4} + \mu_3 = \frac{233779885253560873}{187374306471770600}. \end{aligned}$$

We then obtain

$$\begin{aligned} R_n - \gamma \quad &\sim \frac{1}{24(x^2 + x + \frac{17}{40})} + \frac{2071}{806400\left(x^2 + x + \frac{62759}{82840}\right)^3} \\ &+ \frac{4971169273}{1399664640000\left(n^2 + n + \frac{233779885253560873}{187374306471770600}\right)^5} + \cdots. \end{aligned} \tag{3.11}$$

Remark 3.1 The inequality (2.2) implies the following asymptotic expansion:

$$\psi(x+1) - \ln\left(x + \frac{1}{2}\right) \sim \sum_{k=1}^{\infty} \left(1 - \frac{1}{2^{k-1}}\right) \frac{B_k}{2k(x + \frac{1}{2})^{2k}}, \qquad x \to \infty. \tag{3.12}$$

Noting that (2.1) holds, we obtain from (3.12) that

$$R_n - \gamma \sim \sum_{k=1}^{\infty} \left(1 - \frac{1}{2^{k-1}}\right) \frac{B_k}{2k(n + \frac{1}{2})^{2k}}, \qquad n \to \infty. \tag{3.13}$$

From a computational viewpoint, the formula (3.9) improves the formula (3.13).

4 Inequalities

Theorem 4.1 *For $n \in \mathbb{N}$, we have*

$$\frac{1}{24(n^2 + n + \alpha)} < R_n - \gamma \le \frac{1}{24(n^2 + n + \beta)} \tag{4.1}$$

with the best possible constants

$$\alpha = \frac{17}{40} = 0.425 \quad \textit{and} \quad \beta = \frac{47 - 48\gamma + 48\ln(2/3)}{-24 + 24\gamma - 24\ln(2/3)} = 0.40580406\ldots. \tag{4.2}$$

Proof The inequality (4.1) can be written as

$$\alpha > f(n) \ge \beta, \tag{4.3}$$

where

$$f(x)=\frac{1}{24P(x)}-x^2-x$$

with $P(x)$ is defined by (2.6). Direct computation yields

$$f(1)=0.40580406\ldots,\qquad f(2)=0.41657108\ldots. \tag{4.4}$$

We conclude from the asymptotic expansion of ψ [1, p. 259] that

$$\lim_{x\to\infty} f(x)=\frac{17}{40}.$$

In order to prove Theorem 4.1, it suffices to show that the sequence $\{f(n)\}$ is strictly increasing for $n\geq 1$.

From the well known continued fraction for ψ' (see [6, p. 232]):

$$\psi'(z)=\frac{1}{z}+\frac{1}{2z^2}+\frac{2\pi}{z}\left(\frac{a_1^{(1)}}{z^2}{}_{+}\ \frac{a_2^{(1)}}{1}{}_{+}\ \frac{a_3^{(1)}}{z^2}{}_{+}\ \frac{a_4^{(2)}}{1}{}_{+}\ \ldots\right),\qquad |\arg z|<\frac{\pi}{2},$$

where

$$a_1^{(1)}=\frac{1}{12\pi},\quad a_m^{(1)}=\frac{m^2(m^2-1)}{4(4m^2-1)},\qquad m\geq 2,$$

we find that for $x>0$,

$$\frac{2\pi}{x}\left(\frac{a_1^{(1)}}{x^2}{}_{+}\ \frac{a_2^{(1)}}{1}{}_{+}\ \frac{a_3^{(1)}}{x^2}{}_{+}\ \frac{a_4^{(1)}}{1}\right)<\psi'(x)-\frac{1}{x}-\frac{1}{2x^2}$$
$$<\frac{2\pi}{x}\left(\frac{a_1^{(1)}}{x^2}{}_{+}\ \frac{a_2^{(1)}}{1}{}_{+}\ \frac{a_3^{(1)}}{x^2}{}_{+}\ \frac{a_4^{(1)}}{1}{}_{+}\ \frac{a_5^{(1)}}{x^2}\right),$$

that is

$$\frac{7(15x^2+22)}{30x(21x^4+35x^2+4)}<\psi'(x)-\frac{1}{x}-\frac{1}{2x^2}<\frac{385x^4+1148x^2+300}{210x^3(11x^4+35x^2+14)},\qquad x>0. \tag{4.5}$$

Differentiating $f(x)$ and applying the second inequality in (4.5) and the inequality (2.5), we obtain that for $x\geq 2$,

$$\begin{aligned}-576P^2(x)f'(x) &= 24\left(\psi'(x)-\frac{1}{x^2}\right)-\frac{48}{2x+1}+576(2x+1)P^2(x)\\ &< 24\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{385x^4+1148x^2+300}{210x^3(11x^4+35x^2+14)}\right)-\frac{48}{2x+1}\\ &\quad +576(2x+1)\left(\frac{1}{576(x+\frac{1}{2})^4}-\frac{7}{11520(x+\frac{1}{2})^6}+\frac{7229}{19353600(x+\frac{1}{2})^8}\right)\\ &= -\frac{4P_7(x-2)}{525x^3(11x^4+35x^2+14)(2x+1)^7},\end{aligned}$$

where

$$\begin{aligned}P_7(x) &= 584490+5851566x+11488246x^2+10750439x^3+5783365x^4\\ &\quad +1839404x^5+318934x^6+22781x^7.\end{aligned}$$

So

$$f'(x) > 0, \qquad x \geq 2,$$

and therefore $f(x)$ is strictly increasing for $x \geq 2$. Noting that (4.4) holds, we see that the sequence $\{f(n)\}$ is strictly increasing for $n \geq 1$. The proof of Theorem 4.1 is complete. □

Remark 4.1 The inequality (4.1) is sharper than the inequality (1.7).

Remark 4.2 Direct computation yields

$$\left[R_n - \gamma - \frac{1}{24(n^2+n+\frac{2}{5})}\right]_{n=1} = \frac{283}{288} - \gamma - \ln 3 + \ln 2 = -0.000041\ldots < 0.$$

By using the second inequality in (2.4), we find that for $n \geq 2$,

$$\begin{aligned} R_n - \gamma - \frac{1}{24(n^2+n+\frac{2}{5})} &= \psi(n+1) - \ln\left(n+\frac{1}{2}\right) - \frac{1}{24(n^2+n+\frac{2}{5})} \\ &< \frac{1}{24(n+\frac{1}{2})^2} - \frac{7}{960(n+\frac{1}{2})^4} + \frac{31}{8064(n+\frac{1}{2})^6} - \frac{1}{24(n^2+n+\frac{2}{5})} \\ &= -\frac{18485+41410(n-2)+29282(n-2)^2+8400(n-2)^3+840(n-2)^4}{2520(2n+1)^6(5n^2+5n+2)} < 0. \end{aligned}$$

Noting that the lower bound in (4.1) holds, we have

$$\frac{1}{24(n^2+n+\frac{17}{40})} < R_n - \gamma < \frac{1}{24(n^2+n+\frac{2}{5})}, \qquad n \in \mathbb{N}. \tag{4.6}$$

The upper bound in (4.6) does not contain the Euler–Mascheroni constant.

Theorem 4.2 *For $x \geq 3$, we have*

$$\begin{aligned} \frac{4971169273}{1399664640000\left(x+\frac{1}{2}\right)^{10}} &< \psi\left(x+\frac{1}{2}\right) - \ln x - \frac{1}{24(x^2+\frac{7}{40})} - \frac{2071}{806400\left(x^2+\frac{42049}{82840}\right)^3} \\ &< \frac{4971169273}{1399664640000x^{10}}. \end{aligned} \tag{4.7}$$

Proof In order to prove (4.7), it suffices to show that

$$g(x) > 0 \quad \text{and} \quad h(x) < 0 \quad \text{for} \quad x \geq 3, \tag{4.8}$$

where

$$\begin{aligned} g(x) = \psi\left(x+\frac{1}{2}\right) - \ln x - \frac{1}{24(x^2+\frac{7}{40})} - \frac{2071}{806400\left(x^2+\frac{42049}{82840}\right)^3} \\ - \frac{4971169273}{1399664640000\left(x+\frac{1}{2}\right)^{10}} \end{aligned}$$

and

$$h(x) = \psi\left(x+\frac{1}{2}\right) - \ln x - \frac{1}{24(x^2+\frac{7}{40})} - \frac{2071}{806400\left(x^2+\frac{42049}{82840}\right)^3} - \frac{4971169273}{1399664640000x^{10}}.$$

By using (2.7), we find that

$$g(x) > \frac{1}{24x^2} - \frac{7}{960x^4} + \frac{31}{8064x^6} - \frac{127}{30720x^8} + \frac{511}{67584x^{10}} - \frac{1414477}{67092480x^{12}} - \frac{1}{24(x^2+\frac{7}{40})} - \frac{2071}{806400\left(x^2+\frac{42049}{82840}\right)^3} - \frac{4971169273}{1399664640000\left(x+\frac{1}{2}\right)^{10}}$$
$$= \frac{P_{19}(x-3)}{191054223360000x^{12}(40x^2+7)(82840x^2+42049)^3(2x+1)^{10}}$$

and

$$h(x) < \frac{1}{24x^2} - \frac{7}{960x^4} + \frac{31}{8064x^6} - \frac{127}{30720x^8} + \frac{511}{67584x^{10}} - \frac{1414477}{67092480x^{12}} + \frac{8191}{98304x^{14}} - \frac{1}{24(x^2+\frac{7}{40})} - \frac{2071}{806400\left(x^2+\frac{42049}{82840}\right)^3} - \frac{4971169273}{1399664640000x^{10}}$$
$$= -\frac{P_{10}(x-3)}{34737131520000x^{14}(40x^2+7)(82840x^2+42049)^3},$$

where

$$P_{19}(x) = 691164101510110055828273508054311740027 01 + 50564435864551418435545680292873795026 4204x + \cdots + 790026075639633784990531584000 00x^{19}$$

and

$$P_{10}(x) = 4889457809166407733690019500369 12 + 1908267281276275760947093211291016x + \cdots + 1399466168075905080077049 6000x^{10}$$

are polynomials of 19th, respectively 10th degree, having all coefficients positive. Hence, (4.8) holds true. The proof of Theorem 4.1 is complete. □

Corollary 4.1 *For $n \in \mathbb{N}$, let*

$$\omega_n = R_n - \frac{1}{24(n^2+n+\frac{17}{40})} - \frac{2071}{806400\left(n^2+n+\frac{62759}{82840}\right)^3}. \tag{4.9}$$

Then, we have

$$\frac{4971169273}{1399664640000(n+1)^{10}} < \omega_n - \gamma < \frac{4971169273}{1399664640000\left(n+\frac{1}{2}\right)^{10}}. \tag{4.10}$$

Proof Replacement of x by $n+\frac{1}{2}$ in (4.7) yields

$$\frac{4971169273}{1399664640000(n+1)^{10}} < \psi(n+1) - \ln\left(n+\frac{1}{2}\right) - \frac{1}{24(n^2+n+\frac{17}{40})} - \frac{2071}{806400\left(n^2+n+\frac{62759}{82840}\right)^3} < \frac{4971169273}{1399664640000\left(n+\frac{1}{2}\right)^{10}} \tag{4.11}$$

for $n \geq 3$. Elementary calculations show that the inequality (4.11) is also valid for $n = 1$ and 2. Noting that (2.1) holds, we see that (4.11) can be written as (4.10). The proof is complete. □

Mortici [14] introduced the sequences (ν_n) and (μ_n) by

$$\nu_n = H_n - \ln \frac{n^2+n+\frac{7}{24}}{n+\frac{1}{2}} \tag{4.12}$$

and

$$\mu_n = H_n - \ln \frac{n^3+\frac{3}{2}n^2+\frac{227}{240}n+\frac{107}{480}}{n^2+n+\frac{97}{240}}. \tag{4.13}$$

Moreover, the author offered some numerical computations to prove the superiority of his sequences (ν_n) and (μ_n) over the classical sequence (D_n) and the DeTemple sequence (R_n).

The choice $a = 1$ in (1.8) yields

$$\frac{511}{67584(n+1)^{10}} < \alpha_n - \gamma < \frac{511}{67584(n+\frac{1}{2})^{10}}, \tag{4.14}$$

where

$$\alpha_n = H_n - \ln\left(n+\frac{1}{2}\right) - \frac{1}{24(n+\frac{1}{2})^2} + \frac{7}{960(n+\frac{1}{2})^4} - \frac{31}{8064(n+\frac{1}{2})^6} + \frac{127}{30720(n+\frac{1}{2})^8}. \tag{4.15}$$

We now offer some numerical computations to show the superiority of our sequence (ω_n) over the sequences (ν_n), (μ_n) and and (α_n).

n	$\gamma - \nu_n$	$\mu_n - \gamma$	$\alpha_n - \gamma$	$\omega_n - \gamma$
1	1.0299×10^{-3}	1.3667×10^{-4}	6.4852×10^{-5}	1.4623×10^{-5}
10	5.2564×10^{-7}	1.8859×10^{-9}	4.5283×10^{-13}	2.0856×10^{-13}
100	6.2963×10^{-11}	2.4862×10^{-15}	7.1911×10^{-23}	3.3772×10^{-23}
1000	6.4107×10^{-15}	2.5544×10^{-21}	7.5232×10^{-33}	3.5339×10^{-33}

Appendix: Proof of (2.5)

Using (2.4), we find that for $x \geq 2$,

$$\begin{aligned}
&\left(\psi\left(x+\frac{1}{2}\right)-\ln x\right)^2-\left(\frac{1}{576x^4}-\frac{7}{11520x^6}+\frac{7229}{19353600x^8}\right)\\
&<\left(\frac{1}{24x^2}-\frac{7}{960x^4}+\frac{31}{8064x^6}-\frac{127}{30720x^8}+\frac{511}{67584x^{10}}\right)^2\\
&\quad-\left(\frac{1}{576x^4}-\frac{7}{11520x^6}+\frac{7229}{19353600x^8}\right)\\
&=-\frac{Q(x-2)}{50357757542400x^{20}},
\end{aligned}$$

where

$$\begin{aligned}
Q(x) = {} & 11972450715095+68182955378648x+170375802258662x^2\\
&+247321024184696x^3+231764684292911x^4+146843565663872x^5\\
&+63807592875488x^6+18796851375104x^7+3595428926144x^8\\
&+403437619200x^9+20171880960x^{10}.
\end{aligned}$$

We then obtain (2.5).

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