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Accelerated PMHSS iteration methods for complex symmetric linear systems

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Abstract In this paper, we introduce and analyze an accelerated preconditioning modification of the Hermitian and skew-Hermitian splitting (APMHSS) iteration method for solving a broad class of complex symmetric linear systems. This accelerated PMHSS algorithm involves two iteration parameters α, β and two preconditioned matrices whose special choices can recover the known PMHSS (preconditioned modification of the Hermitian and skew-Hermitian splitting) iteration method which includes the MHSS method, as well as yield new ones. The convergence theory of this class of APMHSS iteration methods is established under suitable conditions. Each iteration of this method requires the solution of two linear systems with real symmetric positive definite coefficient matrices. Theoretical analyses show that the upper bound $\sigma_1(\alpha, \beta)$ of the asymptotic convergence rate of the APMHSS method is smaller than that of the PMHSS iteration method. This implies that the APMHSS method may converge faster than the PMHSS method. Numerical experiments on a few model problems are presented to illustrate the theoretical results and examine the numerical effectiveness of the new method.

Keywords Complex symmetric linear system · PMHSS iteration · Convergence theory · Iterative methods · Accelerated technology

Mathematics Subject Classification (2010) 65F10 · 65F50

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1 Introduction

Let n be a positive integer. We consider the iterative solution of systems of linear equations of the form

$$Ax = b, \quad A \in C^{n\times n} \quad \text{and} \quad x, b \in C^n, \tag{1.1}$$

where $A \in C^{n\times n}$ is a complex symmetric matrix of the form

$$A = W + \mathrm{i}T, \tag{1.2}$$

and $W, T \in R^{n\times n}$ are real, symmetric, and positive semidefinite matrices with at least one of them, e.g., W, being positive definite. Here and in the sequel, we use $\mathrm{i} = \sqrt{-1}$ to denote the imaginary unit.

Complex symmetric linear systems of this kind arise in many important problems in scientific computing and engineering applications. For example, FFT-based solution of certain time-dependent PDEs [17], diffuse optical tomography [1], molecular scattering [32], lattice quantum chromody-namics [26], numerical solutions of the complex Helmholtz equation and numerical computations in eddy current problems. For more examples, we refer to [2, 5, 13, 14, 19, 20, 22] and the references therein.

As a matter of fact, the complex symmetric linear system (1.1) is formally identical to the following block two-by-two systems of linear equations:

$$Cx = \begin{pmatrix} W & -T \\ T & W \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} = g. \tag{1.3}$$

In fact, let $x = y + iz$ and $b = p + iq$, then from (1.1) and (1.2), we can get $(W + iT)(y + iz) = p + iq$, which implies that we can obtain the (1.3). This is the same as in [7], the authors there also transformed the system (1.1) into a real and block two-by-two linear system. Conversely, from the linear (1.3), we can get the complex symmetric linear system (1.1). Moreover, the block two-by-two systems of linear (1.3) can be formally regarded as a special case of the generalized saddle point problem [4, 15, 16]. It frequently arises from finite element discretizations of elliptic partial differential equation (PDE)-constrained optimization problems such as distributed control problems [3, 29, 30, 33, 34] and so on. All in all, the complex symmetric linear system (1.1) is a very important problem in practical application. Hence, there is a strong need for the fast solution of complex symmetric linear systems.

For solving the complex symmetric linear system (1.1) efficiently, van derVorst and Mellissen [37] proposed the conjugate orthogonal conjugate gradient (COCG) method, which is regarded as an extension of the Conjugate Gradient (CG) method [28]. Relatively complicated but robust algorithms such as QMR [25], CSYM [18], and Bi-CGCR [21] are also useful. QMR is derived from the complex symmetric Lanczos algorithm, CSYM is obtained from the idea of QMR and tridiagonalization of A by Householder reflections, and Bi-CGCR is derived from a particular case in Bi-CG [24] for solving non-Hermitian linear systems. In [36] Sogabe and Zhang extended the conjugate residual (CR) method described in [27, 35] to complex symmetric linear systems based on an observation of deriving CG, CR, and COCG. Moreover, based on the Hermitian and skew-Hermitian splitting (HSS)

$$A = H + S$$

of the matrix $A \in C^{n\times n}$, with

$$H = \frac{1}{2}(A + A^*) \text{ and } S = \frac{1}{2}(A - A^*)$$

being the Hermitian and skew-Hermitian parts and A^* being the conjugate transpose of the matrix $A \in C^{n\times n}$, we can apply the HSS iteration method [10] or its preconditioned variant PHSS (i.e., the preconditioned HSS, see [12]) which were proposed by Bai et al. or the generalized PHSS methods [40] to compute an approximate solution of the linear system (1.1). In addition, the convergence properties of the PHSS method can be found in [9]. In [11], Bai et al. further generalized the technique for constructing HSS iteration method for solving large sparse non-Hermitian positive definite system of linear equations to the normal/skew-Hermitian (NS) splitting obtaining a class of normal/skew-Hermitian splitting (NSS) iteration methods. Theoretical analyses shown that the NSS iteration method converges unconditionally to the exact solution of the system of linear (1.1). Moreover, [31, 39] proposed the generalized HSS method and generalized preconditioned HSS method for solving singular linear systems and non-Hermitian positive definite linear systems, respectively.

A potential difficulty with the HSS iteration approach is the need to solve the shifted skew-Hermitian sub-system of linear equations at each iteration step. In some cases its solution is as difficult as that of the original problem, although there are situations where the matrix T is structured in such a way as to make linear systems involving $\alpha I + iT$ easy to solve. In general, however, this will not be the case. Hence, Bai et al. presented a modification of the HSS iteration scheme in [5] and some of its basic properties are studied. Moreover, in [6], the authors proposed a preconditioned variant of the modified HSS (PMHSS) iteration method for solving the complex symmetric systems of linear equations. That PMHSS iteration method has faster convergence than the MHSS method which was illustrated by the numerical implementations in the paper [6].

To further generalize the PMHSS iteration method and accelerate its convergence rate, in this paper we propose a accelerated PMHSS (APMHSS) iteration method for solving the complex symmetric linear system (1.1). We establish the convergence theory for the APMHSS iteration method under the condition that both W and T are symmetric positive semidefinite and, at least, one of them is positive definite. A considerable advantage of the APMHSS iteration method consists in the fact that solution of two linear sub-systems with coefficient matrices both being real and symmetric positive definite, need to be solved at each step. This is just like the MHSS and PMHSS iteration methods.

The organization of the paper is as follows. In Section 2, we establish the APMHSS iteration method for the complex symmetric linear system (1.1). In Section 3, the analysis of the convergence property of this new method is given. Moreover, numerical results are given in Section 4 to show the correctness of our theoretical analysis and the effectiveness of this APMHSS iteration method. Finally, in Section 5 we put forth some conclusions and remarks to end the paper.

The following notations will be used throughout this paper. We denote the identity matrix and the 0-matrix by I and O, respectively. For a vector v, we denote the l_2 norm of v by $\|v\|_2$. And for a matrix B, we denote the conjugate transpose and the inverse of B by B^* and B^{-1}, respectively. Moreover, $\mathrm{sp}(B)$ denotes the spectrum of the matrix B, and the spectral radius of B is denoted by $\rho(B)$.

2 The APMHSS method

Based on the preconditioned MHSS (PMHSS) iteration method for complex symmetric linear system (1.1), in this section we derive a accelerated PMHSS iteration method which has two parameters α and β. To this end, we first introduce the PMHSS iteration method proposed in Bai et al. [6] which is much more efficient than the MHSS iteration method for solving the complex symmetric linear system (1.1). This PMHSS iteration method is algorithmically described in the following.

Method 2.1 (The PMHSS iteration method) Let $x \in C^n$ be an arbitrary initial guess and α be a given positive constant. For $k = 0, 1, 2, \cdots$, until the sequence of iterates $\{x^{(k)}\}_0^\infty$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha V + W)x^{(k+\frac{1}{2})} = (\alpha V - \mathrm{i}T)x^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} = (\alpha V + \mathrm{i}W)x^{(k+\frac{1}{2})} - \mathrm{i}b, \end{cases}$$

where $V \in C^{n\times n}$ is a prescribed symmetric positive definite matrix.

As $W, V \in C^{n\times n}$ are symmetric positive definite, $T \in C^{n\times n}$ is symmetric positive semidefinite, and α is positive, we see that both matrices $\alpha V + W$ and $\alpha V + T$ are symmetric positive definite. Hence, the two linear sub-systems involved in each step of the PMHSS iteration can also be solved effectively using mostly real arithmetic either exactly by a Cholesky factorization or inexactly by some conjugate gradient or multigrid scheme.

Now we can establish the following APMHSS iteration method for solving the complex symmetric linear system (1.1) in an analogous fashion to the PMHSS iteration scheme. More precisely, we have the following algorithmic description of the APMHSS iteration method.

Method 2.2 (The APMHSS iteration method) Let $P \in R^{n\times n}$ be a symmetric positive definite matrix, with $PW = WP$ and $PT = TP$. Given an initial vector $x^{(0)} \in C^n$, and two relaxation factors $\alpha > 0, \beta > 0$. For $k = 0, 1, 2, \cdots$, until the sequence of iterates $\{x^{(k)}\}_0^\infty$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure:

$$\begin{cases} (\alpha P + W)x^{(k+\frac{1}{2})} = (\alpha P - \mathrm{i}T)x^{(k)} + b, \\ (\beta P + T)x^{(k+1)} = (\beta P + \mathrm{i}W)x^{(k+\frac{1}{2})} - \mathrm{i}b. \end{cases} \tag{2.1}$$

When $P \in R^{n\times n}$ is a symmetric positive definite matrix and $\alpha = \beta > 0$, the APMHSS iteration method reduces to the PMHSS iteration method. Moreover, when $P = I \in R^{n\times n}$ and $\alpha = \beta > 0$, then the APMHSS iteration method reduces to the MHSS method. We can suitably choose P and α, β such that the induced APMHSS iteration method possesses faster convergence rate and higher computing efficiency. In addition, the symmetric positive definite matrix P and the positive constants α, β should be judiciously selected so that the sub-systems of linear (2.1) with the coefficient matrices $\alpha P + W$ and $\beta P + T$ can be solved economically and rapidly. This two-parameter generalizations of the PHSS [12] iteration method has been also studied in [8] (a class of AHSS iteration methods for solving the large sparse saddle-point problem studied by Bai and Golub).

After straightforward derivations we can reformulate the APMHSS iteration scheme into the standard form

$$x^{(k+1)} = H(\alpha, \beta)x^{(k)} + M^{-1}(\alpha, \beta)b, \tag{2.2}$$

where

$$H(\alpha, \beta) = (\beta P + T)^{-1}(\beta P + \mathrm{i}W)(\alpha P + W)^{-1}(\alpha P - \mathrm{i}T)$$

and

$$M^{-1}(\alpha, \beta) = (\beta P + T)^{-1}(\beta P - \mathrm{i}\alpha P)(\alpha P + W)^{-1}.$$

Obviously, we can obtain that $H(\alpha, \beta)$ is the iteration matrix of the APMHSS iteration (2.1) or (2.2). Therefore, the APMHSS method (2.1) is convergent if and only if the spectral radius of the iteration matrix $H(\alpha, \beta)$ of the stationary iterative (2.2) is less than one, i.e., $\rho(H(\alpha, \beta)) < 1$. See [27, 38, 41]. In addition, if let

$$N(\alpha, \beta) = M(\alpha, \beta) - A = \frac{(\beta + \mathrm{i}\alpha)}{\alpha^2 + \beta^2}(\beta P + \mathrm{i}W)P^{-1}(\alpha P - \mathrm{i}T),$$

then

$$A = M(\alpha, \beta) - N(\alpha, \beta) \tag{2.3}$$

defines a splitting of the coefficient matrix (1.2) of the complex symmetric linear system (1.1), and the APMHSS iteration method (2.2) can also be induced by the matrix splitting (2.3). Easily, we see that $H(\alpha, \beta) = M^{-1}(\alpha, \beta)N(\alpha, \beta)$ is the iteration matrix of the APMHSS iteration method (2.2).

Moreover, when $P = W \in R^{n\times n}$, i.e., $P = W$, we have

$$x^{(k+1)} = H_1(\alpha, \beta)x^{(k)} + M_1^{-1}(\alpha, \beta)b, \tag{2.4}$$

with

$$H_1(\alpha,\beta) = (\beta W + T)^{-1}(\beta W + \mathrm{i}W)(\alpha W + W)^{-1}(\alpha W - \mathrm{i}T)$$
$$= \frac{\beta + \mathrm{i}}{\alpha + 1}(\beta W + T)^{-1}(\alpha W - \mathrm{i}T)$$

and

$$M_1(\alpha,\beta) = ((\beta W + T)^{-1}(\beta W - \mathrm{i}\alpha W)(\alpha W + W)^{-1})^{-1}$$
$$= \frac{(\alpha+1)(\beta + \mathrm{i}\alpha)}{\alpha^2+\beta^2}(\beta W + T).$$

Then the PMHSS iteration scheme (2.4) is now induced by the matrix splitting $A = M_1(\alpha,\beta) - N_1(\alpha,\beta)$ with

$$N_1(\alpha,\beta) = M_1(\alpha,\beta) - A = \frac{(\beta + \mathrm{i}\alpha)(\beta+\mathrm{i})}{\alpha^2+\beta^2}(\alpha W - \mathrm{i}T),$$

where $\alpha = \beta$.

3 Analysis for the APMHSS method

In this section, we discuss the convergence property of the APMHSS iteration method. Sufficient conditions for the convergence of the APMHSS method are also provided in the following theorems that we will present.

Theorem 3.1 *Let $A = W + iT \in C^{n\times n}$, with $W \in R^{n\times n}$ and $T \in R^{n\times n}$ symmetric positive definite and symmetric positive semidefinite, respectively, and let α, β be two given positive constants. If $0 < \beta \le \alpha$, $\alpha^2 - \beta^2 \le 2\beta\mu_{\min}$ and $P \in R^{n\times n}$ is a symmetric positive definite matrix which satisfies $PW = WP$, $PT = TP$. Then the spectral radius $\rho(H(\alpha,\beta))$ of the APMHSS iteration matrix $H(\alpha,\beta)$ satisfies $\rho(H(\alpha,\beta)) \le \sigma_1(\alpha,\beta)$, where $\mu_{\min}$ is the smallest eigenvalue of the matrix $P^{-1}T$ and*

$$\sigma_1(\alpha,\beta) = \max_{\lambda_j \in sp(P^{-1}W)} \frac{\sqrt{\beta^2+\lambda_j^2}}{\alpha+\lambda_j}.$$

Therefore, it holds that

$$\rho(H(\alpha,\beta)) \le \sigma_1(\alpha,\beta) < 1, \quad \forall\, \alpha \ge \beta > 0.$$

That is, the APMHSS iteration (2.2) converges to the unique solution $\overline{x}$ of the complex symmetric linear system (1.1) for any initial guess $x^{(0)}$.

Proof By the similarity invariance of the matrix spectrum, we have

$$\begin{aligned}\rho(H(\alpha,\beta)) &= \rho((\beta P + \mathrm{i}W)(\alpha P + W)^{-1}(\alpha P - \mathrm{i}T)(\beta P + T)^{-1})\\ &= \rho((\beta I + \mathrm{i}P^{-1}W)(\alpha I + P^{-1}W)^{-1}(\alpha I - \mathrm{i}P^{-1}T)(\beta I + P^{-1}T)^{-1})\\ &\le \|(\beta I + \mathrm{i}P^{-1}W)(\alpha I + P^{-1}W)^{-1}(\alpha I - \mathrm{i}P^{-1}T)(\beta I + P^{-1}T)^{-1}\|_2\\ &\le \|(\beta I + \mathrm{i}P^{-1}W)(\alpha I + P^{-1}W)^{-1}\|_2\|(\alpha I - \mathrm{i}P^{-1}T)(\beta I + P^{-1}T)^{-1}\|_2.\end{aligned}$$

Because $W, T, P \in R^{n\times n}$ are symmetric matrices, and $PW = WP$, $PT = TP$, we can see that $P^{-1}W$, $P^{-1}T$ are also symmetric matrices. Then analogously to Theorem 2.1 in [5], we can obtain

$$\begin{aligned}\rho(H(\alpha,\beta)) &\leq \max_{\lambda_j\in sp(P^{-1}W)}\left|\frac{\beta+\mathrm{i}\lambda_j}{\alpha+\lambda_j}\right|\cdot\max_{\mu_j\in sp(P^{-1}T)}\left|\frac{\alpha+\mathrm{i}\mu_j}{\beta+\mu_j}\right|\\ &= \max_{\lambda_j\in sp(P^{-1}W)}\frac{\sqrt{\beta^2+\lambda_j^2}}{\alpha+\lambda_j}\cdot\max_{\mu_j\in sp(P^{-1}T)}\frac{\sqrt{\alpha^2+\mu_j^2}}{\beta+\mu_j}.\end{aligned}$$

Note that $\mu_j \geq 0$ $(1 \leq j \leq n)$ and $\alpha^2 - \beta^2 \leq 2\beta\mu_{\min}$, we can obtain

$$\begin{aligned}\alpha^2+\mu_j^2 &\leq \beta^2+2\beta\mu_{\min}+\mu_j^2\\ &\leq \beta^2+2\beta\mu_j+\mu_j^2\\ &= (\beta+\mu_j)^2.\end{aligned}$$

It then follows that

$$\frac{\sqrt{\alpha^2+\mu_j^2}}{\beta+\mu_j} \leq 1,\ \ (1\leq j\leq n),$$

which implies

$$\rho(H(\alpha,\beta)) \leq \max_{\lambda_j\in sp(P^{-1}W)}\frac{\sqrt{\beta^2+\lambda_j^2}}{\alpha+\lambda_j} = \sigma_1(\alpha,\beta).$$

Moreover, from the condition that $0 < \beta \leq \alpha$, we can obtain $\rho(H(\alpha,\beta)) \leq \sigma_1(\alpha,\beta) < 1$, therefore the APMHSS iteration converges to the unique solution $\overline{x}$ of the complex symmetric linear system (2.1).

The proof is completed. □

Corollary 3.1 *If the conditions of the Theorem 3.1 is satisfied, then the upper bound $\sigma_1(\alpha,\beta)$ of the asymptotic convergence rate of the APMHSS method is smaller than that of the PMHSS iteration method.*

Proof Let $\beta = \alpha$, then the APMHSS iteration method (2.2) reduces to the PMHSS method studied in [6]. By making use of the above Theorem 3.1 that we have obtained, we can see that the upper bound of the asymptotic convergence rate of the PMHSS method is

$$\sigma(\alpha) = \max_{\lambda_j\in sp(P^{-1}W)}\frac{\sqrt{\alpha^2+\lambda_j^2}}{\alpha+\lambda_j}.$$

Then from the condition that $0 < \beta \leq \alpha$, we have

$$\sigma_1(\alpha,\beta) = \max_{\lambda_j\in sp(P^{-1}W)}\frac{\sqrt{\beta^2+\lambda_j^2}}{\alpha+\lambda_j} \leq \max_{\lambda_j\in sp(P^{-1}W)}\frac{\sqrt{\alpha^2+\lambda_j^2}}{\alpha+\lambda_j} = \sigma(\alpha).$$

The proof is completed. □

It's worth noting that when $0 < \beta < \alpha$, then $\sigma_1(\alpha, \beta) < \sigma(\alpha)$, which implies that the APMHSS method may converge faster than the PMHSS iteration method. And this result will be illustrated by the numerical experiments in Section 4.

Corollary 3.2 *If the conditions of the Theorem 3.1 is satisfied and $P = W$, then $\sigma_1(\alpha, \beta) = \frac{\sqrt{\beta^2+1}}{\alpha+1} < 1$.*

Proof From the proof of Theorem 3.1, we can get the result directly. □

Theorem 3.2 *Let $A = W + iT \in C^{n\times n}$, with $W \in R^{n\times n}$ and $T \in R^{n\times n}$ symmetric positive definite and symmetric positive semidefinite, respectively, and let α, β be two given positive constants. If $0 < \alpha < \beta$, $\beta^2 - \alpha^2 \leq 2\alpha\lambda_{\min}$ and $P \in R^{n\times n}$ is a symmetric positive definite matrix which satisfies $PW = WP$, $PT = TP$. Then the spectral radius $\rho(H(\alpha, \beta))$ of the APMHSS iteration matrix $H(\alpha, \beta)$ satisfies $\rho(H(\alpha, \beta)) \leq \sigma_2(\alpha, \beta)$, where $\lambda_{\min}$ is the smallest eigenvalue of the matrix $P^{-1}W$ and*

$$\sigma_2(\alpha, \beta) = \max_{\mu_j \in sp(P^{-1}T)} \frac{\sqrt{\alpha^2 + \mu_j^2}}{\beta + \mu_j}.$$

Therefore, it holds that

$$\rho(H(\alpha, \beta)) \leq \sigma_2(\alpha, \beta) < 1, \quad \forall \beta \geq \alpha > 0.$$

That is, the APMHSS iteration (2.2) converges to the unique solution $\overline{x}$ of the complex symmetric linear system (1.1) for any initial guess $x^{(0)}$.

Proof Similar to the proof of Theorem 3.1, we can obtain

$$\rho(H(\alpha, \beta)) \leq \max_{\lambda_j \in sp(P^{-1}W)} \frac{\sqrt{\beta^2 + \lambda_j^2}}{\alpha + \lambda_j} \cdot \max_{\mu_j \in sp(P^{-1}T)} \frac{\sqrt{\alpha^2 + \mu_j^2}}{\beta + \mu_j}.$$

Then from $\lambda_j > 0$ $(1 \leq j \leq n)$ and $\beta^2 - \alpha^2 \leq 2\alpha\lambda_{\min}$, we can obtain

$$\beta^2 + \lambda_j^2 \leq (\alpha + \lambda_j)^2.$$

This implies

$$\rho(H(\alpha, \beta)) \leq \max_{\mu_j \in sp(P^{-1}T} \frac{\sqrt{\alpha^2 + \mu_j^2}}{\beta + \mu_j} = \sigma_2(\alpha, \beta).$$

Moreover, from the condition that $0 < \alpha \leq \beta$, we can obtain $\rho(H(\alpha, \beta)) \leq \sigma_2(\alpha, \beta) < 1$, therefore the APMHSS iteration converges to the unique solution $\overline{x}$ of the complex symmetric linear system (2.1).

The proof is completed. □

4 Numerical experiments

In this section, we perform some numerical examples to illustrate the theoretical results and show the effectiveness of the APMHSS iteration method for solving the complex symmetric linear system (2.1) in terms of both iteration count (denoted as IT) and computing time (in seconds, denoted as CPU), and the norm of the residual (denoted as "RES") defined by

$$\mathrm{RES} = \|b - Ax^{(k)}\|_2.$$

In actual computations, the iteration schemes are started from the zero vector and terminated if the current iterations satisfy ERR$\leq 10^{-6}$ or the number of the prescribed iteration steps $k = 500$ are exceeded, where

$$\mathrm{ERR} = \frac{\|b - Ax^{(k)}\|_2}{\|b\|_2}.$$

All experiments are performed in MATLAB (version 7.4.0.336 (R2010b)) with machine precision 10^{-16}, and all experiments are implemented on a personal computer with 2.20 GHz central processing unit (Intel(R) Core(TM) i3-2310M), 2.00G memory and Win7 operating system.

- 4.1 Example descriptions

Example 4.1 (See [2, 5, 6]) The system of linear (1.1) is of the form

$$[(K + \frac{3-\sqrt{3}}{\tau} I) + \mathrm{i}(K + \frac{3+\sqrt{3}}{\tau} I)]x = b, \tag{4.1}$$

Table 1 IT, CPU and RES for MHSS, PMHSS and APMHSS methods for Example 4.1

Method	$m \times m$	16×16	32×32	64×64	128×128	256×256
MHSS	α_{opt}	1.069	0.673	0.440	0.410	0.312
	IT	40	59	73	99	134
	CPU	0.012	0.069	0.581	4.423	35.112
	RES	4.27e-008	2.41e-008	3.42e-008	3.52e-007	2.33e-007
PMHSS	α_{opt}	1.091	1.356	1.350	1.120	1.451
	IT	21	21	21	21	21
	CPU	0.009	0.004	0.245	1.099	6.188
	RES	4.33e-008	2.02e-008	3.04e-008	3.20e-007	6.86e-007
APMHSS	α_{opt}	1.069	1.310	1.350	1.000	1.455
	β_{opt}	0.71	0.980	1.210	0.950	1.235
	IT	19	16	17	15	15
	CPU	0.003	0.002	0.082	0.964	4.655
	RES	4.35e-008	3.02e-008	1.94e-008	4.80e-007	3.45e-007

where τ is the time step-size and K is the five-point centered difference matrix approximating the negative Laplacian operator $L = -\Delta$ with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0, 1]\times[0, 1]$ with the mesh-size $h = \frac{1}{m+1}$. The matrix $K \in R^{n\times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2}\mathrm{tridiag}(-1, 2, -1) \in R^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. We take

$$W = K + \frac{3-\sqrt{3}}{\tau}I, \text{ and } T = K + \frac{3+\sqrt{3}}{\tau}I,$$

and the right-hand side vector b with its jth entry $[b]_j$ being given by

$$[b]_j = \frac{(1-\mathrm{i})j}{\tau(j+1)^2}, \quad j = 1, 2, \cdots, n.$$

In our tests, we take $\tau = h$. Furthermore, we normalize coefficient matrix and right-hand side by multiplying both by h^2. For more details, we refer to [2].

Example 4.2 (See [5, 6, 14]) The system of linear (1.1) is of the form

$$[(-\omega^2 M + K) + \mathrm{i}(\omega C_V + C_H)]x = b, \tag{4.2}$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ω is the driving circular frequency. We take $C_H = \mu K$ with μ a damping coefficient, $M = I$, $C_V = 10I$, and K the five-point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0, 1]\times[0, 1]$ with the mesh-size $h = \frac{1}{m+1}$. The matrix $K \in R^{n\times n}$ possesses the tensor-product form $K = I \otimes B_m + B_m \otimes I$, with $B_m = h^{-2}\mathrm{tridiag}(-1, 2, -1) \in$

Table 2 IT, CPU and RES for MHSS, PMHSS and APMHSS methods for Example 4.2

Method	$m \times m$	16×16	32×32	64×64	128×128	256×256
MHSS	α_{opt}	0.518	0.269	0.052	0.021	0.002
	IT	53	86	90	99	139
	CPU	0.012	0.050	0.434	3.903	35.436
	RES	3.53e-008	1.49e-008	6.48e-008	3.59e-007	4.35e-007
PMHSS	α_{opt}	0.681	0.988	1.20	1.120	0.972
	IT	34	37	38	38	38
	CPU	0.015	0.052	0.449	1.756	10.455
	RES	3.79e-008	1.14e-008	4.54e-008	1.20e-007	2.82e-007
APMHSS	α_{opt}	0.578	0.985	1.100	1.110	0.860
	β_{opt}	0.655	1.230	1.500	1.255	1.125
	IT	29	30	28	20	25
	CPU	0.005	0.025	0.165	0.992	8.565
	RES	3.49e-008	1.02e-008	3.90e-008	2.25e-007	3.75e-007

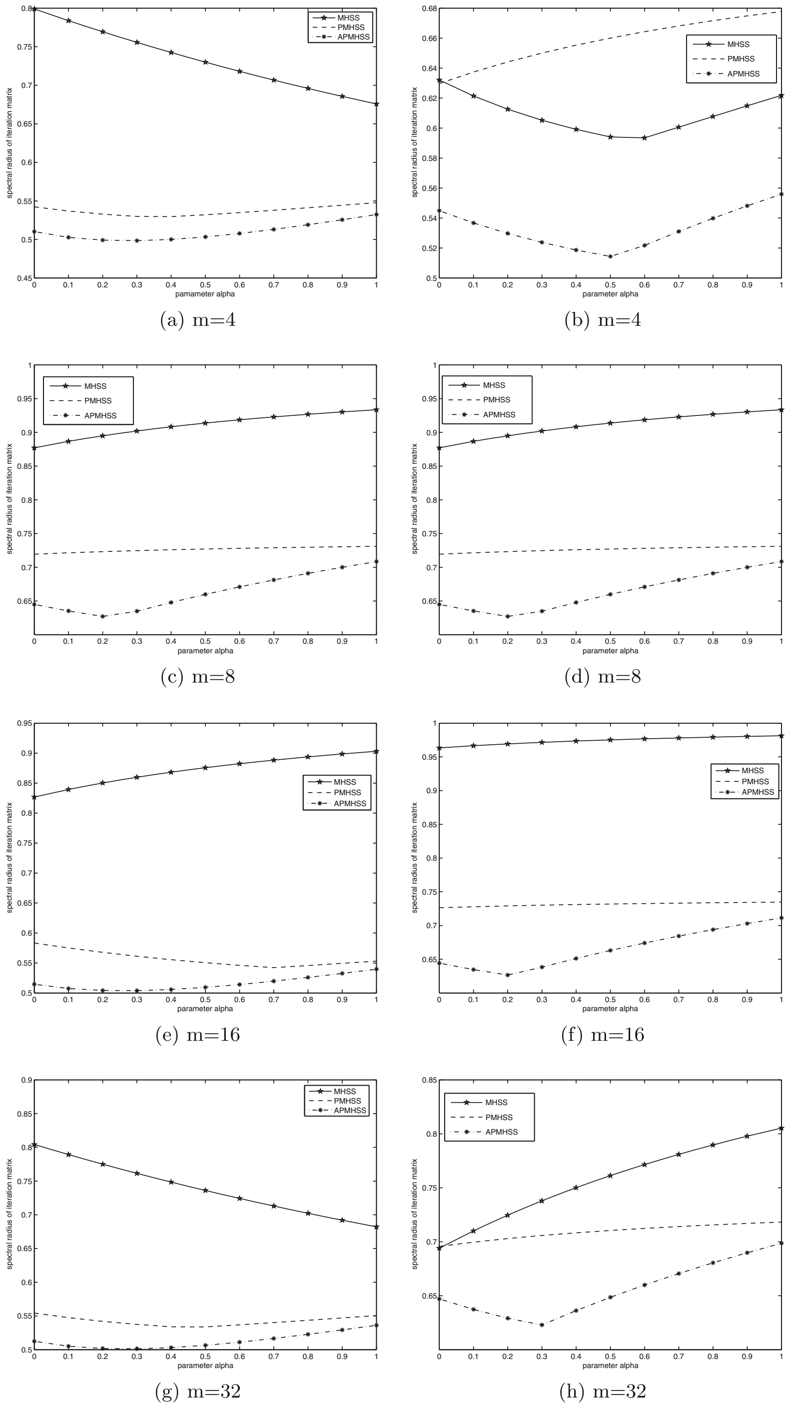
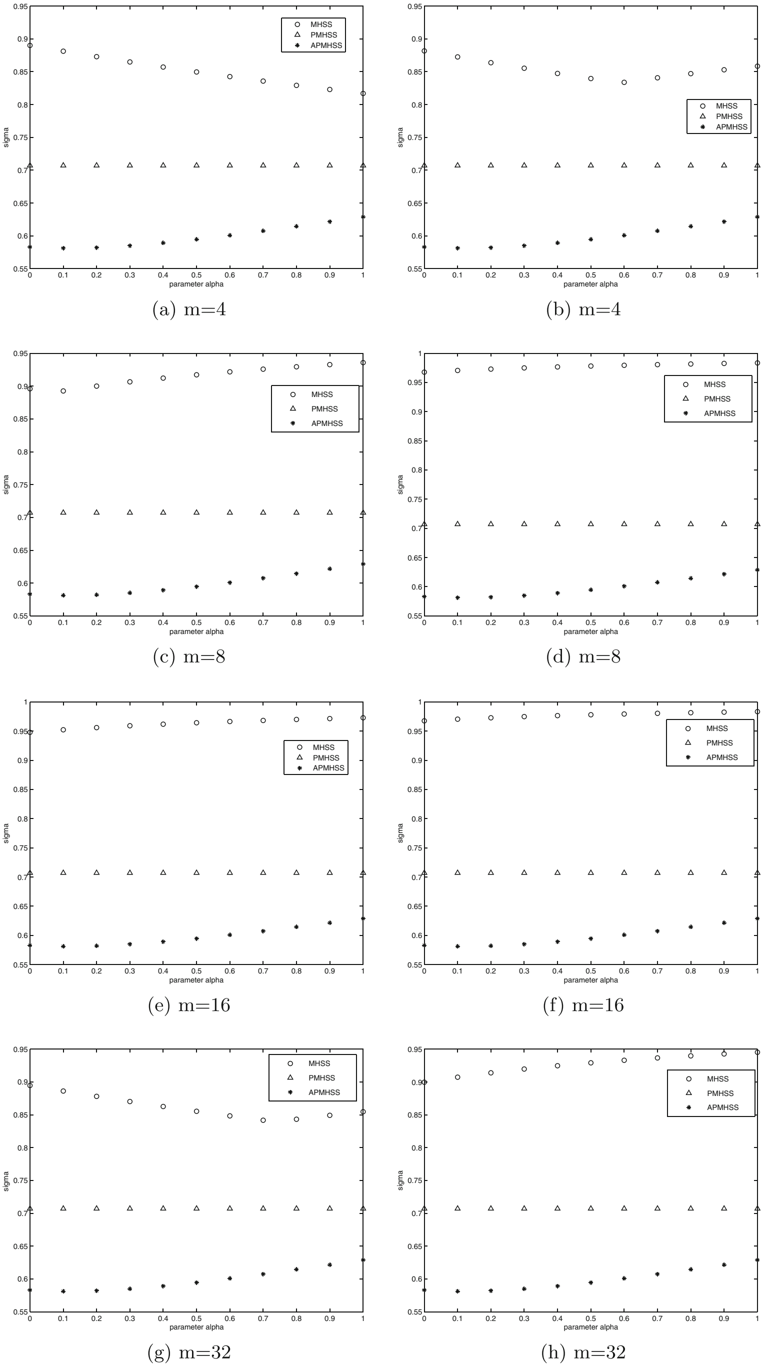


Fig. 1 The spectral radius $\rho(H(\alpha))$ of the iteration matrices; *left*: Example 4.1, *right*: Example 4.2



(a) m=4 (b) m=4

(c) m=8 (d) m=8

(e) m=16 (f) m=16

(g) m=32 (h) m=32

Fig. 2 The upper bound $\sigma(H(\alpha))$ of the spectral radius for the iteration matrices; *left*: Example 4.1, *right*: Example 4.2

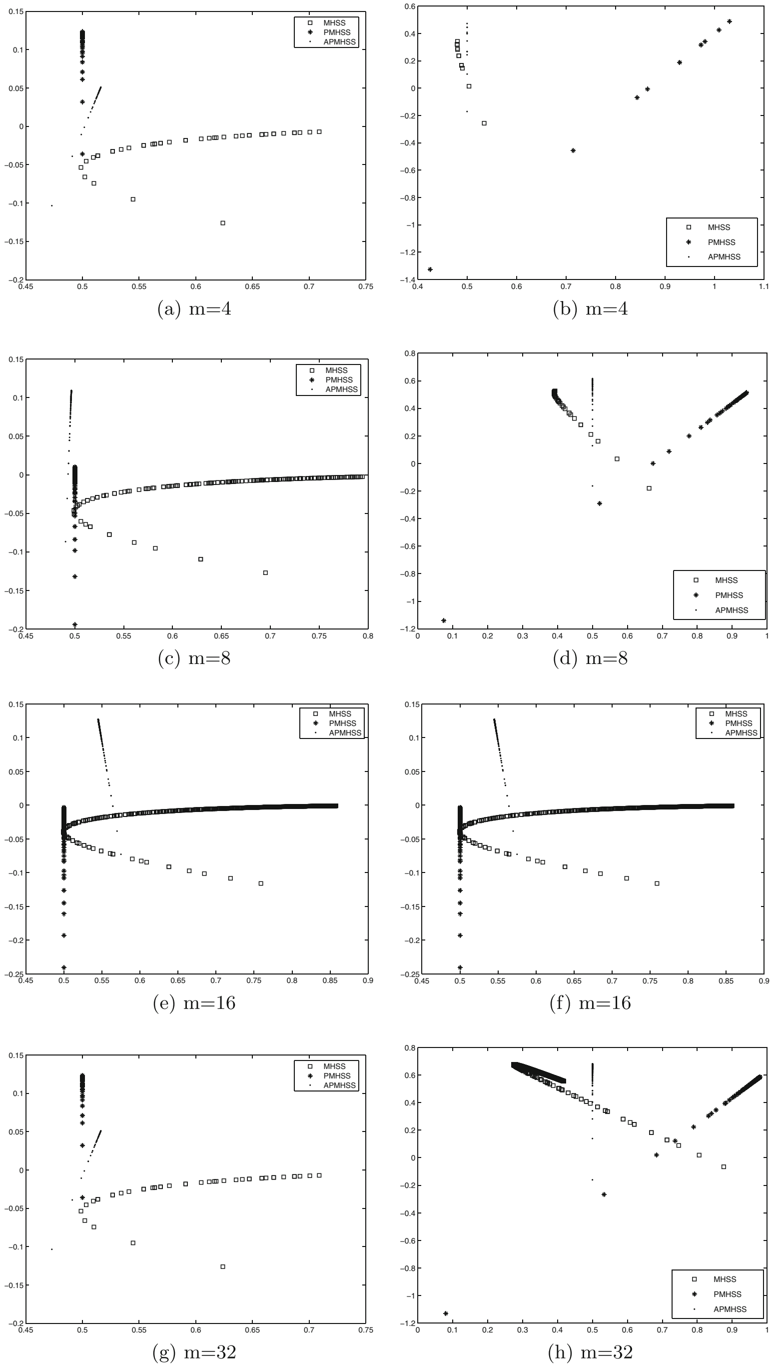


Fig. 3 The eigenvalue distributions of the iteration matrices when $\alpha = \alpha_{opt}$ and $\beta = \beta_{opt}$; *left*: Example 4.1, *right*: Example 4.2

$R^{m\times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. In addition, we set $\omega = \pi$, $\mu = 0.02$, and the right-hand side vector b to be $b = (1 + \mathrm{i})A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. As before, we normalize the system by multiplying both sides through by h^2. In fact, this complex symmetric system of linear equations arises in direct frequency domain analysis of an n-degree-of-freedom (n-DOF) linear system. For more details, we refer to [2, 23].

- 4.2 Experimental results

The APMHSS iteration method is compared with the MHSS and PMHSS methods. For the tests reported in this section we used the optimal values of the parameter α, β (denoted by $\alpha_{opt}, \beta_{opt}$, respectively) for the APMHSS, PMHSS and MHSS iteration methods. The experimentally found optimal parameters $\alpha_{opt}, \beta_{opt}$ are the ones resulting in the least numbers of iterations for the three methods for each of the numerical examples and for each choice of the spatial mesh-sizes. For the APMHSS method, the matrix P was taken as W and I in Example 1 and Example 2, respectively. The matrix V in the PMHSS method is taken as W.

With respect to different sizes of the coefficient matrix, we list IT, CPU and RES about the APMHSS, PMHSS and MHSS methods in Tables 1 and 2. And the results in Tables 1 and 2 are the IT, CPU and RES for the for Example 1 and Example 2, respectively. By comparing the results in Table 1, we observe that the APMHSS iteration method outperforms the PMHSS and MHSS methods, as it requires much less time and iteration steps to achieve the stopping criterion. These results can also be seen in Table 2.

In Fig. 1, We show the spectral radius $\rho(H(\alpha, \beta))$ (or $\rho(H(\alpha))$) of the iteration matrices $H(\alpha, \beta)$ (or $H(\alpha)$) for APMHSS, PMHSS and MHSS methods of Example 1 and Example 2. Obviously, we can see from the figures that the spectral radius of the APMHSS method $\rho(H(\alpha, \beta))$ is always less than that of the PMHSS and MHSS methods. These results show that the APMHSS iteration always converges faster than the PMHSS and MHSS methods. It is worth noting that we let $\beta = \alpha \pm c$ in the experiment, where c is a positive constant, so the spectral radius $\rho(H(\alpha, \beta))$ reduces to $\rho(H(\alpha))$.

In Fig. 2, we depict the upper bound $\sigma(\alpha, \beta)$ (or $\sigma(\alpha)$) of the spectral radius for the three different iteration methods. We can see that the the upper bound $\sigma(\alpha, \beta)$ of the spectral radius of the APMHSS method is less than that the upper bound $\sigma(\alpha)$ of the PMHSS and MHSS methods. It is also worth noting that analogously to the spectral radius $\rho(H(\alpha, \beta))$, we let $\beta = \alpha \pm c$ in the experiment, where c is a positive constant, so $\sigma(H(\alpha, \beta))$ reduces to $\sigma(H(\alpha))$ in the experiment. In Fig. 3, we depict the eigenvalue distributions of the iteration matrices for the APMHSS, PMHSS and MHSS iteration methods.

5 Conclusions

In this paper, we have studied an accelerated PMHSS iteration method for a class of complex symmetric linear systems. The APMHSS iteration method not only presents a more general framework, but also yields much better theoretical and numerical properties than the PMHSS iteration method. Moreover, the analysis for the APMHSS iteration method is proposed, which includes the convergence theory of this new method. Numerical experiments show that the APMHSS method outperforms the MHSS method and PMHSS methods. This illustrates that the APMHSS iteration method is a very efficient method for solving complex symmetric linear system (1.1) which arises in many important problems in scientific computing and engineering applications. However, the theoretical analysis for the optimal value or quasi-optimal value of the two parameters α and β should be studied in the further.

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