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A space-time Legendre spectral tau method for the two-sided space-time Caputo fractional diffusion-wave equation

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Abstract The space-time fractional diffusion-wave equation (FDWE) is a generalization of classical diffusion and wave equations which is used in modeling practical phenomena of diffusion and wave in fluid flow, oil strata and others. This paper reports an accurate spectral tau method for solving the two-sided space and time Caputo FDWE with various types of nonhomogeneous boundary conditions. The proposed method is based on shifted Legendre tau (SLT) procedure in conjunction with the shifted Legendre operational matrices of Riemann-Liouville fractional integral, left-sided and right-sided fractional derivatives. We focus primarily on implementing this algorithm in both temporal and spatial discretizations. In addition, convergence analysis is provided theoretically for the Dirichlet boundary conditions, along with graphical analysis for several special cases using other conditions. These suggest that

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the Legendre Tau method converges exponentially provided that the data in the given FDWE are smooth. Finally, several numerical examples are given to demonstrate the high accuracy of the proposed method.

Keywords Fractional diffusion-wave equation · Tau method · Shifted legendre polynomials · Operational matrix · Convergence analysis · Riesz fractional derivative

1 Introduction

In recent years there has been a high level of interest in the field of fractional differential equations due to their important applications in science and engineering, such as modeling of anomalous diffusive and super-diffusive systems, description of fractional random walk and unification of diffusion and wave propagation phenomena [13, 27, 33, 35, 37–39]. In particular, the space-time fractional diffusion-wave equation (FDWE) is significant to applications of fractional differential equations. Gorenflo [20] and Mainardi [31] generalized the classical diffusion and wave equations by replacing the first-order or second-order time derivative term by a fractional derivative of order α with $0 < \alpha \leq 2$. They showed that as α increases from 0 to 2, the physical process changes from slow diffusion ($0 < \alpha < 1$) to a classical diffusion equation ($\alpha = 1$) to a diffusion-wave hybrid ($1 < \alpha < 2$) to a classical wave equation ($\alpha = 2$) processes.

In the last few years, many published papers (e.g., [9, 10, 14, 46, 53, 57]) have been devoted to numerical methods for solving the space and time fractional diffusion equation. Three kinds of finite difference schemes, (i) the explicit Euler scheme, (ii) the implicit Euler scheme, and (iii) the fractional Cranck-Nicholson scheme based on a shifted Grunwald formula were proposed in [32–34, 50, 51]. Liu et al. [28] implemented an efficient implicit numerical scheme for the fractional-order advection-dispersion models in which the authors studied five fractional models. Two finite difference approaches valid on spatially bounded domains were proposed and developed in [24, 58] for solving the fractional sub-diffusion equations subject to Neumann boundary conditions. The spectral method was implemented in both temporal and spatial discretizations for the diffusion problem in [26]. Saadatmandi et al. [44] proposed and developed an efficient numerical algorithm based on the Sinc-Legendre collocation method for the time-fractional convection-diffusion equation with variable coefficients on a finite domain. Recently, a Chebyshev spectral-tau method has been developed by Doha et al. [16] for fractional diffusion equations. The space-fractional advection diffusion equations was considered by Bhrawy and Baleanu in [6] using spectral Legendre collocation approximation for the spatial discretization.

Compared with the considerable numerical methods for the fractional diffusion equation, relatively little work [17, 45, 48, 52] has been done on numerical methods for the FDWEs. The fully discrete difference approximation has been proposed by Sun and Wu [47] for the numerical solution of FDWEs and sub-diffusion equation. Li et al. [25] proposed a finite difference scheme for time discretization and finite element scheme for space discretization for solving the time-space fractional sub-diffusion and super-diffusion problems. More recently, Zeng [56] proposed a second-order stable finite difference technique for solving the time FDWEs.

Most existing numerical schemes are implemented for fractional Dirichlet problems. However, fewer numerical schemes are proposed for solving problems with Neumann or Robin boundary conditions. Ren and Sun [40] investigated the fourth-order compact approach with high spatial accuracy for the FDWEs with Neumann Boundary Conditions. Huang et al. [21] introduced two finite difference schemes for the FDWEs. Langlands and Henry [24] considered an implicit numerical scheme for a fractional diffusion equation with Neumann boundary conditions, in which the backward Euler approximation is used to discretize the first order time derivative and the L^1 approximation for the Riemann-Liouville fractional derivative. Recently, Zhao and Sun [58] proposed a box-type scheme for solving a class of fractional sub-diffusion equation with Neumann boundary conditions. In the area of numerical methods of FDWEs, little work has been done using spectral methods compared to finite difference and finite element methods. Moreover, no spectral method has been investigated for solving two-sided space-time FDWEs. This partially motivates our interest in applying such methods.

The main goal of this paper is to present an efficient numerical algorithm for the solution of the two-sided space and time FDWEs with various kinds of nonhomogeneous boundary conditions. The shifted Legendre tau method, in combination with the operational matrices of left- and right-sided Caputo fractional derivatives and the operational matrix of Riemann-Liouville fractional integral of the shifted Legendre polynomials, is investigated for treating the temporal and spatial discretizations of the initial-boundary value problem of the two-sided space-time FDWE. Moreover, the convergence of the proposed method is analyzed, theoretically for the Dirichlet boundary conditions and graphically through examples for other types of conditions. Finally, several numerical simulations are given to confirm the high accuracy of the proposed algorithm.

The rest of this article is organized as follows. In Section 2, we present some fractional calculus preliminaries and properties of Legendre polynomials and then we construct the operational matrices of Legendre polynomials with a particular focus on the Caputo definition. In Section 3, by using tau spectral method, we construct and develop an algorithm for the solution of the two-sided space and time FDWE with Dirichlet conditions. Section 4 is devoted to extensions of this approach to more general boundary conditions. The convergence analysis is provided in Section 5 for the Dirichlet boundary conditions. In Section 6, some illustrative numerical experiments are given and some comparisons are made between our method and other methods in the open literature. Convergence of the method for other types of boundary conditions is explored through these examples. The paper ends with some conclusions and observations in Section 7.

2 Preliminaries and fundamentals

In this section, we recall some basic properties of fractional calculus theory. Then we present some properties of the Legendre polynomials [15, 43, 49].

2.1 The fractional integral and derivative

A complication associated with fractional derivatives is that there are several definitions of exactly what a fractional derivatives means, for example, Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, Weyl, Grunwald- Letnikov, Hadamard, and Chen. Here we recall some basic concepts of Caputo's fractional derivative. For proofs and details on the subject, we refer the readers to [39].

Let $f : [0, \ell] \to \mathbb{R}$ be an integrable function and Γ be the Gamma function. The left- and right-sided Riemann-Liouville fractional integral operators are respectively defined by [39]

$$\begin{aligned} {}_0I_x^{\nu}[f] &:= x \mapsto \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1} f(t)dt, \\ {}_xI_\ell^{\nu}[f] &:= x \mapsto \frac{1}{\Gamma(\nu)}\int_x^\ell (t-x)^{\nu-1} f(t)dt. \end{aligned} \tag{2.1}$$

where $\nu > 0$ is the order of the derivative. The operator ${}_0I_x^{\nu}$ satisfies the following properties

$$\begin{aligned} {}_0I_x^{\nu} \circ {}_0I_x^{\mu} &= {}_0I_x^{\nu+\mu}, \\ {}_0I_x^{\nu} \circ {}_0I_x^{\mu} &= {}_0I_x^{\mu} \circ {}_0I_x^{\nu}, \\ {}_0I_x^{\nu} x^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\nu+1)} x^{\beta+\nu}. \end{aligned} \tag{2.2}$$

The left- and right-sided Riemann-Liouville fractional derivative operators are defined, respectively, by [39]

$$\begin{aligned} {}_0D_x^{\nu} &:= \frac{d^n}{dx^n} \circ {}_0I_x^{n-\nu}, \\ {}_xD_\ell^{\nu} &:= (-1)^n \frac{d^n}{dx^n} \circ {}_xI_\ell^{n-\nu}. \end{aligned} \tag{2.3}$$

where $n = \lceil \nu \rceil$. Interchanging the composition of operators in the definition of Riemann-Liouville fractional derivatives, we obtain the left- and right-sided Caputo fractional derivatives:

$$\begin{aligned} {}_0^CD_x^{\nu} &:= {}_0I_x^{n-\nu} \circ \frac{d^n}{dx^n}, \\ {}_x^CD_\ell^{\nu} &:= {}_xI_\ell^{n-\nu} \circ (-1)^n \frac{d^n}{dx^n}. \end{aligned} \tag{2.4}$$

The Caputo and the Riemann-Liouville derivatives are equivalent under some conditions, which given as follows,

$${}_0^CD_x^{\nu} f(x) = {}_0D_x^{\nu} f(x) - \sum_{i=0}^{\lceil \nu \rceil - 1} \frac{f^{(i)}(0)}{\Gamma(i+1-\nu)} x^{i-\nu}, \tag{2.5}$$

$${}_x^CD_\ell^{\nu} f(x) = {}_xD_\ell^{\nu} f(x) - \sum_{i=0}^{\lceil \nu \rceil - 1} \frac{(-1)^i f^{(i)}(\ell)}{\Gamma(i+1-\nu)} (\ell - x)^{i-\nu}. \tag{2.6}$$

Therefore, if $f(0) = f'(0) = \ldots = f^{(\lceil\nu\rceil-1)}(0) = 0$, then ${}_0^C D_x^\nu f(x) = {}_0D_x^\nu f(x)$ and if $f(\ell) = f'(\ell) = \ldots = f^{(\lceil\nu\rceil-1)}(\ell) = 0$, then ${}_x^C D_\ell^\nu f(x) = {}_xD_\ell^\nu f(x)$.

These fractional operators are linear, i.e.,

$$\mathcal{P}(\mu f(t) + \lambda g(t)) = \mu\mathcal{P} f(t) + \lambda\mathcal{P} g(t), \tag{2.7}$$

where $\mathcal{P}$ is ${}_0D_x^\nu$, ${}_xD_\ell^\nu$, ${}_0^C D_x^\nu$, ${}_x^C D_\ell^\nu$, ${}_0I_x^\nu$ or ${}_xI_\ell^\nu$, and μ and λ are real numbers. The Caputo fractional derivative, nowadays the most popular fractional operator among engineers and applied scientists, was obtained by reformulating the classical definition of Riemann-Liouville fractional derivative in order to make possible the solution of fractional initial value problems with standard initial conditions. For the Riemann-Liouville definition, such conditions must be imposed on fractional derivatives which is often not available. For this reason we shall focus on the Caputo definition in this work. For the Caputo derivative, we have the following some basic properties which are needed in this paper [22].

$${}_0^C D_x^\nu C = {}_x^C D_\ell^\nu C = 0, \qquad \text{(C is constant)}, \tag{2.8}$$

$${}_0I_x^\nu \; {}_0^C D_x^\nu f(x) = f(x) - \sum_{i=0}^{\lceil\nu\rceil-1} f^{(i)}(0^+)\frac{x^i}{i!}, \tag{2.9}$$

$${}_0^C D_x^\nu x^i = \begin{cases} 0, & \text{for } i \in N_0 \text{ and } i < \lceil\nu\rceil, \\ \dfrac{\Gamma(i+1)}{\Gamma(i+1-\nu)}\, x^{i-\nu}, & \text{for } i \in N_0 \text{ and } i \geq \lceil\nu\rceil, \end{cases} \tag{2.10}$$

$${}_x^C D_\ell^\nu (x-\ell)^i = \begin{cases} 0, & \text{for } i \in N_0 \text{ and } i < \lceil\nu\rceil, \\ \dfrac{(-1)^i\Gamma(i+1)}{\Gamma(i+1-\nu)}\, (\ell-x)^{i-\nu}, & \text{for } i \in N_0 \text{ and } i \geq \lceil\nu\rceil, \end{cases} \tag{2.11}$$

where $\lceil.\rceil$ is the ceiling function and $N_0 = \{0, 1, 2, ...\}$.

The Riesz-Caputo fractional derivative of order ν of $f(x)$ is defined as,

$$\frac{\partial^\nu}{\partial|x|} f(x) = C_\nu({}_0D_x^\nu f(x) + {}_xD_\ell^\nu f(x)). \tag{2.12}$$

where

$$C_\nu = -\frac{1}{2\cos(\frac{\pi\nu}{2})}, \qquad \nu \neq 1.$$

2.2 Shifted Legendre polynomials

It is well-known that the classical Legendre polynomials are defined on $[-1, 1]$ by the three-term recurrence relation

$$L_0(x) = 1, \qquad L_1(x) = x,$$
$$L_{i+1}(x) = \frac{2i+1}{i+1} x L_i(x) - \frac{i}{i+1} L_{i-1}(x), \qquad i = 1, 2, \ldots.$$

Assume $x \in [x_a, x_b]$ and let $\tilde{x} = (2x - x_a - x_b)/(x_b - x_a)$. Then $\{L_i(\tilde{x})\}$ are called the shifted Legendre polynomials on $[x_a, x_b]$. In this paper, we mainly consider the shifted Legendre polynomials defined on $[0, \ell]$. For $x \in [0, \ell]$, let $L_{\ell,i}(x) =$

$L_i\left(\frac{2x}{\ell}-1\right),\ i=0,1,\ldots$. Then the shifted Legendre polynomials $\{L_{\ell,i}(x)\}$ are defined by

$$L_{\ell,0}(x)=1, \qquad L_{\ell,1}(x)=\frac{2x}{\ell}-1,$$
$$L_{\ell,i+1}(x)=\frac{(2i+1)(2x-\ell)}{(i+1)\ell}L_{\ell,i}(x)-\frac{i}{i+1}L_{\ell,i-1}(x), \qquad i=1,2,\ldots. \tag{2.13}$$

For $i=0,1,\ldots$, the analytic form of $L_{\ell,i}(x)$ is given by

$$L_{\ell,i}(x)=\sum_{k=0}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!\,(k!)^2\ell^k}x^k,$$
$$=\sum_{k=0}^{i}\frac{(i+k)!}{(i-k)!(k!)^2\,\ell^k}(x-\ell)^k. \tag{2.14}$$

The set of $L_{\ell,i}(x)$ is a complete $L^2(0,\ell)$-orthogonal system, namely

$$\int_0^\ell L_{\ell,i}(x)L_{\ell,j}(x)dx=\frac{\ell}{2i+1}\delta_{i,j}, \tag{2.15}$$

where $\delta_{i,j}$ the Kronecker symbol. So we define

$$\mathbb{P}_M=\text{span}\{L_{\ell,0},L_{\ell,1}\ldots,L_{\ell,M}\}. \tag{2.16}$$

Thus, for any $u(x)\in L^2(0,\ell)$, we write

$$u(x)=\sum_{i=0}^{\infty}c_iL_{\ell,i}(x),$$

from which the coefficients c_i are given by

$$c_i=\frac{2i+1}{\ell}\int_0^\ell u(x)L_{\ell,i}(x)dx, \qquad i=0,1,\cdots. \tag{2.17}$$

In practice, only the first $(M+1)$ terms of shifted Legendre polynomials are considered.

Hence we can write

$$u_M(x)\simeq\sum_{j=0}^{M}c_jL_{\ell,j}(x), \tag{2.18}$$

which alternatively may be written in the matrix form:

$$u_M(x)=\mathbf{C}^T\Phi_{\ell,M}(x), \qquad \mathbf{C}^T=[c_0,c_1,\cdots,c_M], \tag{2.19}$$

with

$$\Phi_{\ell,M}(x)=[L_{\ell,0}(x),L_{\ell,1}(x),\cdots,L_{\ell,M}(x)]^T, \tag{2.20}$$

where $(.)^T$ stands for the transpose.

Consequently, a function of two independent variables $u(x, t)$ which is infinitely differentiable in $[0, \ell] \times [0, \tau]$ may be expressed in terms of the double shifted Legendre polynomials as

$$u_{N,M}(x, t) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{ij} L_{\tau,i}(t) L_{\ell,j}(x) = \Phi_{\tau,N}^{T}(t) \mathbf{A} \Phi_{\ell,M}(x), \tag{2.21}$$

where the shifted Legendre vectors $\Phi_{\tau,N}(t)$ and $\Phi_{\ell,M}(x)$ are defined similarly to (2.20); also the shifted Legendre coefficient matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0M} \\ a_{10} & a_{11} & \cdots & a_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NM} \end{pmatrix}, \tag{2.22}$$

where

$$a_{ij} = \left(\frac{2i+1}{\tau}\right) \left(\frac{2j+1}{\ell}\right) \int_0^{\tau} \int_0^{\ell} u(x, t) L_{\tau,i}(t) L_{\ell,j}(x) dx dt,$$
$$i = 0, 1, \cdots, N, \quad j = 0, 1, \cdots, M. \tag{2.23}$$

2.3 Operational matrices of shifted Legendre polynomials

Operational matrices are used in several areas of numerical analysis and they hold particular importance for solving different kinds of problems in various subjects such as differential equations [23, 54], integral equations [55], integro-differential equations [3, 12, 36], ordinary and partial fractional differential equations [2, 4, 5, 41], optimal control problems [29] and etc. In what follows, we construct the operational matrix of Riemann-Liouville fractional integral of the shifted Legendre vector.

Theorem 2.1 *Let $\Phi_{\ell,M}(x)$ be the shifted Legendre vector and $\nu > 0$ then*

$${}_0I_x^{\nu} \Phi_{\ell,M}(x) \simeq \mathbf{P}_{\nu} \Phi_{\ell,M}(x), \tag{2.24}$$

where $\mathbf{P}_{\nu}$ is the $(M + 1) \times (M + 1)$ operational matrix of fractional integration of order ν and is defined as follows:

$$\mathbf{P}_{\nu} = \begin{pmatrix} \Omega_{\nu}(0,0) & \Omega_{\nu}(0,1) & \cdots & \Omega_{\nu}(0,M) \\ \Omega_{\nu}(1,0) & \Omega_{\nu}(1,1) & \cdots & \Omega_{\nu}(1,M) \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_{\nu}(i,0) & \Omega_{\nu}(i,1) & \cdots & \Omega_{\nu}(i,M) \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_{\nu}(M,0) & \Omega_{\nu}(M,1) & \cdots & \Omega_{\nu}(M,M) \end{pmatrix}, \tag{2.25}$$

where

$$\Omega_{\nu}(i, j) = \sum_{k=0}^{i} \frac{(-1)^{i+k}\, \ell^{\nu} (2j+1)\, (i+k)!\, (k-j+\nu+1)_j}{(i-k)!\, k!\, \Gamma(k+\nu+1)\, (k+\nu+1)_{j+1}}. \tag{2.26}$$

Proof The analytic form of the shifted Legendre polynomials $L_{\ell,i}(x)$ of degree i is given by (2.14). Using (2.1) and (2.2), and since the Riemann-Liouville's fractional integration is a linear operation, we get

$$\begin{aligned} {}_0I_x^{\nu}\, L_{\ell,i}(x) &= \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!\,(k!)^2\,\ell^k}\ {}_0I_x^{\nu} x^k \\ &= \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!\,k!\,\ell^k\,\Gamma(k+\nu+1)}\, x^{k+\nu}, \quad i = 0, 1, \cdots. \end{aligned} \tag{2.27}$$

Now, approximate $x^{k+\nu}$ by $M+1$ terms of shifted Legendre series, yields

$$x^{k+\nu} = \sum_{j=0}^{M} c_{kj} L_{\ell,j}(x), \tag{2.28}$$

where c_{kj} is given from (2.17) with $u(x) = x^{k+\nu}$, and

$$c_{kj} = (2j+1)\,\ell^{k+\nu} \sum_{r=0}^{j} \frac{(-1)^{r+j}\,(j+r)!}{(j-r)!\,(r!)^2\,(k+r+\nu+1)}. \tag{2.29}$$

In virtue of (2.28) and (2.27) we get

$${}_0I_x^{\nu}\, L_{\ell,i}(x) = \sum_{j=0}^{M} \Omega_\nu(i,j) L_{\ell,j}(x), \quad i = 0, 1, \cdots, \tag{2.30}$$

After some lengthly manipulation, $\Omega_\nu(i,j)$ may be put in the form as in (2.26). □

Theorem 2.2 *Let $\Phi_{\ell,M}(x)$ be the shifted Legendre vector and $\nu > 0$ then the left-sided Caputo fractional derivative of order $\nu > 0$ of $\Phi_{\ell,M}(x)$ can be expressed as*

$${}_0^C D_x^{\nu} \Phi_{\ell,M}(x) \simeq \mathbf{D}_\nu^{+} \Phi_{\ell,M}(x), \tag{2.31}$$

where $\mathbf{D}_\nu^{+}$ is the $(M+1)\times(M+1)$ Legendre operational matrix of the left- sided fractional derivatives of order ν in the Caputo sense and is defined as follows:

$$\mathbf{D}_\nu^{+} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ S_\nu^{+}(\lceil\nu\rceil,0) & S_\nu^{+}(\lceil\nu\rceil,1) & S_\nu^{+}(\lceil\nu\rceil,2) & \cdots & S_\nu^{+}(\lceil\nu\rceil,M) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\nu^{+}(i,0) & S_\nu^{+}(i,1) & S_\nu^{+}(i,2) & \cdots & S_\nu^{+}(i,M) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\nu^{+}(M,0) & S_\nu^{+}(M,1) & S_\nu^{+}(M,2) & \cdots & S_\nu^{+}(M,M) \end{pmatrix}, \tag{2.32}$$

where

$$S^{+}_{\nu}(i,j)=\sum_{k=\lceil\nu\rceil}^{i}\frac{(-1)^{i+k}\,(2j+1)\,(i+k)!\,(k-j-\nu+1)_{j}}{\ell^{\nu}\,(i-k)!\,k!\,\Gamma(k-\nu+1)\,(k-\nu+1)_{j+1}},\tag{2.33}$$

Proof For the proof (see [43]). □

Now, we state and prove a new theorem for the operational matrix of right-sided Caputo fractional derivative of any order for the shifted Legendre polynomials.

Lemma 2.3 *Let $L_{\ell,i}(x)$ be a shifted Legendre polynomial; then*

$${}^{C}_{x}D^{\nu}_{\ell}L_{\ell,i}(x)=0,\qquad i=0,1,...,\lceil\nu\rceil-1,\quad \nu>0.\tag{2.34}$$

Proof This lemma can be easily proved by making use of relations (2.11) and (2.14). □

Theorem 2.4 *Let $\Phi_{\ell,M}(x)$ be the shifted Legendre vector and $\nu>0$, then the right-sided Caputo fractional derivative of order ν of $\Phi_{\ell,M}(x)$ can be expressed as*

$${}^{C}_{x}D^{\nu}_{\ell}\Phi_{\ell,M}(x)\simeq\mathbf{D}^{-}_{\nu}\Phi_{\ell,M}(x),\tag{2.35}$$

where $\mathbf{D}^{-}_{\nu}$ is the $(M+1)\times(M+1)$ Legendre operational matrix of the right-sided fractional derivatives of order ν in the Caputo sense and is defined as follows:

$$\mathbf{D}^{-}_{\nu}=\begin{pmatrix}0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\cdots&\vdots\\ 0&0&0&\cdots&0\\ S^{-}_{\nu}(\lceil\nu\rceil,0)&S^{-}_{\nu}(\lceil\nu\rceil,1)&S^{-}_{\nu}(\lceil\nu\rceil,2)&\cdots&S^{-}_{\nu}(\lceil\nu\rceil,M)\\ \vdots&\vdots&\vdots&\cdots&\vdots\\ S^{-}_{\nu}(i,0)&S^{-}_{\nu}(i,1)&S^{-}_{\nu}(i,2)&\cdots&S^{-}_{\nu}(i,M)\\ \vdots&\vdots&\vdots&\cdots&\vdots\\ S^{-}_{\nu}(M,0)&S^{-}_{\nu}(M,1)&S^{-}_{\nu}(M,2)&\cdots&S^{-}_{\nu}(M,M)\end{pmatrix},\tag{2.36}$$

where

$$S^{-}_{\nu}(i,j)=\sum_{k=\lceil\nu\rceil}^{i}\frac{(-1)^{k+j}\,(2j+1)\,(i+k)!\,(-k+\nu)_{j}}{\ell^{\nu}\,(i-k)!\,k!\,\Gamma(k-\nu+1)\,(k-\nu+1)_{j+1}}.\tag{2.37}$$

Proof The analytic form of the shifted Legendre polynomials $L_{\ell,i}(x)$ of degree i is given by (2.14), using (2.11) we have

$$\begin{aligned}{}^{C}_{x}D^{\nu}_{\ell}L_{\ell,i}(x)&=\sum_{k=0}^{i}\frac{(i+k)!}{(i-k)!\,(k!)^{2}\,\ell^{k}}\;{}^{C}_{x}D^{\nu}_{\ell}(x-\ell)^{k}\\&=\sum_{k=\lceil\nu\rceil}^{i}\frac{(-1)^{k}\,(i+k)!}{(i-k)!\,k!\,\ell^{k}\,\Gamma(k-\nu+1)}\,(\ell-x)^{k-\nu},\quad i=\lceil\nu\rceil,\lceil\nu\rceil+1,\cdots.\end{aligned}\tag{2.38}$$

Now, $(\ell-x)^{k-\nu}$ may be expressed in terms of shifted Legendre series, so we have

$$(\ell - x)^{k-\nu} = \sum_{j=0}^{M} b_{kj} L_{\ell,j}(x), \tag{2.39}$$

where b_{kj} is given from (2.17) with $u(x) = (\ell - x)^{k-\nu}$, and

$$b_{kj} = (2j+1)\ \ell^{k-\nu} \sum_{r=0}^{j} \frac{(-1)^r\ (j+r)!}{(j-r)!\ (r!)^2\ (k+r-\nu+1)}. \tag{2.40}$$

Using (2.39), (2.40) in (2.38) it follows that

$${}_{x}^{C}D_{\ell}^{\nu} L_{\ell,i}(x) = \sum_{j=0}^{M} S_{\nu}^{-}(i,j) L_{\ell,j}(x), \quad i = \lceil\nu\rceil, \lceil\nu\rceil+1, \cdots. \tag{2.41}$$

After some lengthly manipulation, $S_{\nu}^{-}(i,j)$ may be put in the form as in (2.37). □

3 The shifted Legendre spectral tau method

As we know, the FWE is a generalization of the classical wave equation and it models the practical phenomena of a wave in fluid flow, oil strata and others that cannot be modeled accurately by the second-order wave equation. The spectral method has been an efficient tool for computing approximate solutions of differential equations because of its high-order accuracy. The use of the spectral method both temporal and spatial discretizations of FPDEs may significantly reduce the storage requirement because, as compared to low order methods, much fewer time and space levels are needed to compute a smooth solution.

In this section, a new algorithm for solving two-sided one dimensional space-time FDWE is proposed based on Legendre-tau spectral method in conjunction with the operational matrices of left- and right-sided Caputo fractional derivatives and the operational matrix of Riemann-Liouville fractional integral of the Legendre polynomials. A FDWE with damping is given by [11, 30]:

$${}_{0}^{C}D_{t}^{\alpha} u(x,t) + \gamma \frac{\partial u(x,t)}{\partial t} = c_{+}(x)\ {}_{0}^{C}D_{x}^{\nu} u(x,t) + c_{-}(x)\ {}_{x}^{C}D_{\ell}^{\nu} u(x,t) + q(x,t). \tag{3.1}$$

with initial conditions:

$$u(x,0) = f_0(x), \qquad \frac{\partial u(x,0)}{\partial t} = f_1(x), \qquad 0 < x < \ell, \tag{3.2}$$

and the Dirichlet boundary conditions:

$$u(0,t) = g_0(t), \qquad u(\ell,t) = g_1(t), \qquad 0 < t \le \tau, \tag{3.3}$$

where $1 < \alpha, \nu \le 2$, $c_{+}(x) \ge 0$, $c_{-}(x) \ge 0$, and $q(x,t)$ is a source term.

Particular cases of (2.13) are summarized as:

When $\alpha = 2,\ \nu = 2, c(x) = c_+(x) + c_-(x)$, (2.13) is the telegraph equation which governs electrical transmission in a telegraph cable [11],

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \gamma \frac{\partial u(x,t)}{\partial t} = c(x)\frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t).$$

Let $\gamma = 0,\ \alpha = 2,\ \nu = 2,\ c(x) = c_+(x) + c_-(x)$, (2.13) becomes the classical wave equation of the following form,

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c(x)\frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t).$$

If $\gamma = 0,\ c_-(x) = 0$, then (2.13) has no right-sided fractional derivative, (2.13) becomes the space-time fractional diffusion-wave equations [17, 21, 25, 47]

$${}_0^C D_t^{\alpha} u(x,t) = c_+(x)\ {}_0^C D_x^{\nu} u(x,t) + q(x,t).$$

If $\gamma = 0,\ \alpha = 2$, (2.13) reduces to two-sided space fractional wave equation [45, 48]

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c_+(x)\ {}_0^C D_x^{\nu} u(x,t) + c_-(x)\ {}_x^C D_\ell^{\nu} u(x,t) + q(x,t).$$

Let us start our algorithm to solve (3.1)-(3.3) by applying the left- sided Riemann-Liouville integral of order α on (3.1), and making use of (2.9), then we get the integrated form of (3.1)

$$\begin{aligned} u(x,t) - f(x,t) + \gamma\ {}_0I_t^{\alpha-1} u(x,t) = &\ c_+(x)\ {}_0I_t^{\alpha}\left[{}_0^C D_x^{\nu} u(x,t)\right] \\ &+ c_-(x)\ {}_0I_t^{\alpha}\left[{}_x^C D_\ell^{\nu} u(x,t)\right] + {}_0I_t^{\alpha}\left[q(x,t)\right], \\ &u(0,t) = g_0(t),\ u(\ell,t) = g_1(t), 0 < t \le \tau, \end{aligned} \tag{3.4}$$

where $f(x,t) = f_0(x) + t f_1(x) + \gamma\ {}_0I_t^{\alpha-1} f_0(x)$.

It is clear that the initial condition (3.2) is satisfied exactly in (3.4). Now we approximate $u(x,t)$, $c_+(x)$, $c_-(x)$, $q(x,t)$ and $f(x,t)$ by the shifted Legendre polynomials as

$$\begin{aligned} u_{N,M}(x,t) &= \Phi_{\tau,N}^T(t)\mathbf{A}\Phi_{\ell,M}(x), \\ c_{+M}(x) &= \mathbf{C}_+^T\Phi_{\ell,M}(x), \\ c_{-M}(x) &= \mathbf{C}_-^T\Phi_{\ell,M}(x), \\ q_{N,M}(x,t) &= \Phi_{\tau,N}^T(t)\mathbf{Q}\Phi_{\ell,M}(x), \\ f_{N,M}(x,t) &= \Phi_{\tau,N}^T(t)\mathbf{F}\Phi_{\ell,M}(x), \end{aligned} \tag{3.5}$$

where $\mathbf{A}$ is an unknown $(N+1)\times(M+1)$ matrix, but $\mathbf{C}_+^T$, $\mathbf{C}_-^T$, $\mathbf{Q}$ and $\mathbf{F}$ are known matrices which can be written as

$$\mathbf{C}_+^T = [c_{+0}, c_{+1}, \cdots, c_{+M}],$$
$$\mathbf{C}_-^T = [c_{-0}, c_{-1}, \cdots, c_{-M}],$$
$$\mathbf{Q} = \begin{pmatrix} q_{00} & q_{01} & \cdots & q_{0M} \\ q_{10} & q_{11} & \cdots & q_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ q_{N0} & q_{N1} & \cdots & q_{NM} \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} f_{00} & f_{01} & \cdots & f_{0M} \\ f_{10} & f_{11} & \cdots & f_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ f_{N0} & f_{N1} & \cdots & f_{NM} \end{pmatrix}, \tag{3.6}$$

where $c_{\pm j}$ are given as in (2.17) but q_{ij} and f_{ij} are given as in (2.23). Using (2.24), (2.31), (2.35) and (3.5) it easy to obtain that

$$\begin{aligned} {}_0I_t^\alpha\left[c_+(x)\,{}_0^CD_x^\nu u(x,t)\right] &\simeq \left(\mathbf{C}_+^T\Phi_{\ell,M}(x)\right)\left({}_0I_t^\alpha\left[\Phi_{\tau,N}^T(t)\right]\right)\mathbf{A}\left({}_0^CD_x^\nu\Phi_{\ell,M}(x)\right) \\ &= \left(\mathbf{C}_+^T\Phi_{\ell,M}(x)\right)\left(\Phi_{\tau,N}^T(t)\mathbf{P}_\alpha^T\mathbf{A}\mathbf{D}_\nu^+\Phi_{\ell,M}(x)\right) \\ &= \Phi_{\tau,N}^T(t)\mathbf{P}_\alpha^T\mathbf{A}\mathbf{D}_\nu^+\Phi_{\ell,M}(x)\Phi_{\ell,M}^T(x)\mathbf{C}_+ \\ &= \Phi_{\tau,N}^T(t)\mathbf{P}_\alpha^T\mathbf{A}\mathbf{D}_\nu^+\mathbf{H}^+\Phi_{\ell,M}(x), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} {}_0I_t^\alpha\left[c_-(x)\,{}_x^CD_\ell^\nu u(x,t)\right] &\simeq \left(\mathbf{C}_-^T\Phi_{\ell,M}(x)\right)\left({}_0I_t^\alpha\left[\Phi_{\tau,N}^T(t)\right]\right)\mathbf{A}\left({}_x^CD_\ell^\nu\Phi_{\ell,M}(x)\right) \\ &= \left(\mathbf{C}_-^T\Phi_{\ell,M}(x)\right)\left(\Phi_{\tau,N}^T(t)\mathbf{P}_\alpha^T\mathbf{A}\mathbf{D}_\nu^-\Phi_{\ell,M}(x)\right) \\ &= \Phi_{\tau,N}^T(t)\mathbf{P}_\alpha^T\mathbf{A}\mathbf{D}_\nu^-\Phi_{\ell,M}(x)\Phi_{\ell,M}^T(x)\mathbf{C}_- \\ &= \Phi_{\tau,N}^T(t)\mathbf{P}_\alpha^T\mathbf{A}\mathbf{D}_\nu^-\mathbf{H}^-\Phi_{\ell,M}(x), \end{aligned} \tag{3.8}$$

where

$$\Phi_{\ell,M}(x)\Phi_{\ell,M}^T(x)\mathbf{C}_\pm \simeq \mathbf{H}^\pm\Phi_{\ell,M}(x), \tag{3.9}$$

and $\mathbf{H}^\pm$ is a $(M+1)\times(M+1)$ matrix. To explain the construction of $\mathbf{H}^\pm$, making use of (3.9) and the orthogonal property (2.15) the elements $\left\{\mathbf{H}_{ij}^\pm\right\}_{i,j=0}^M$ can be calculated from

$$\mathbf{H}_{ij}^\pm = \frac{2j+1}{\ell}\sum_{k=0}^M c_{\pm k}\mathrm{h}_{ijk}^\pm,$$

where $\mathrm{h}_{\pm ijk}$ is given by

$$\mathrm{h}_{ijk}^\pm = \int_0^\ell L_{\ell,i}(x)L_{\ell,j}(x)L_{\ell,k}(x)dx.$$

By using the well-known Neumann-Adams formula, we can expresses the product of two shifted Legendre polynomials as a sum of such polynomials as follows

$$L_{\ell,i}(x)L_{\ell,j}(x) = \sum_{s=0}^{\min(i,j)} \frac{d_{j-s}d_sd_{i-s}}{d_{i+j-s}}\frac{2i+2j-4s+1}{2i+2j-2s+1}L_{\ell,i+j-2s}(x),$$

where $d_s = \frac{(2s)!}{2^s (s!)^2}$. Multiplying both sides of the above equation by $L_{\ell,k}(x),\ k = 0, 1, \cdots, M$, integrating the result from 0 to ℓ, and using the orthogonal property, we obtain

$$\mathrm{h}^{\pm}_{ijk} = \sum_{s=0}^{\min(i,j)} \frac{\ell\, d_{j-s}\, d_s\, d_{i-s}}{(2i + 2j - 2s + 1)\, d_{i+j-s}} \delta_{i+j-2s,k}, \tag{3.10}$$

Making use of (2.24) and (3.5) we get

$${}_0I_t^{(\alpha-1)}\left[u(x,t)\right] = \Phi^T_{\tau,N}(t)\mathbf{P}^T_{(\alpha-1)}\mathbf{A}\Phi_{\ell,M}(x), \tag{3.11}$$

and

$${}_0I_t^{\alpha}\left[q_{N,M}(x,t)\right] = \Phi^T_{\tau,N}(t)\mathbf{P}^T_{\alpha}\mathbf{Q}\Phi_{\ell,M}(x). \tag{3.12}$$

Equations (3.7), (3.8), (3.11), (3.12) with (3.5), enable us to write the residual $R_{N,M}(x,t)$ for (3.4) in the form

$$\begin{aligned} R_{N,M}(x,t) &= \Phi^T_{\tau,N}(t)\big[\mathbf{A} - \mathbf{F} + \gamma\ \mathbf{P}^T_{(\alpha-1)}\mathbf{A} - \mathbf{P}^T_{\alpha}\mathbf{A}\mathbf{D}^{+}_{\nu}\mathbf{H}^{+} - \mathbf{P}^T_{\alpha}\mathbf{A}\mathbf{D}^{-}_{\nu}\mathbf{H}^{-} \\ &\quad -\mathbf{P}^T_{\alpha}\mathbf{Q}\big]\Phi_{\ell,M}(x) \\ &= \Phi^T_{\tau,N}(t)\mathbf{E}\Phi_{\ell,M}(x), \end{aligned} \tag{3.13}$$

where

$$\mathbf{E} = \mathbf{A} - \mathbf{F} + \gamma\ \mathbf{P}^T_{(\alpha-1)}\mathbf{A} - \mathbf{P}^T_{\alpha}\mathbf{A}\mathbf{D}^{+}_{\nu}\mathbf{H}^{+} - \mathbf{P}^T_{\alpha}\mathbf{A}\mathbf{D}^{-}_{\nu}\mathbf{H}^{-} - \mathbf{P}^T_{\alpha}\mathbf{Q}.$$

According to the typical tau method see [16], we generate $(N + 1) \times (M - 1)$ linear algebraic equations in the unknown expansion coefficients, $a_{ij},\ i = 0, 1, \cdots, N;\ j = 0, 1, \cdots, M - 2$, namely

$$\int_0^{\ell}\int_0^{\tau} R_{N,M}(x,t)\ L_{\tau,i}(t)L_{\ell,j}(x)\,dt\,dx = 0, \quad \begin{aligned} i &= 0, 1, \cdots, N, \\ j &= 0, 1, \cdots, M - 2. \end{aligned} \tag{3.14}$$

and the rest of linear algebraic equations are obtained from the boundary conditions (3.3), as

$$\begin{aligned} \Phi^T_{\tau,N}(t)\mathbf{A}\Phi_{\ell,M}(0) &= g_0(t), \\ \Phi^T_{\tau,N}(t)\mathbf{A}\Phi_{\ell,M}(\ell) &= g_1(t), \end{aligned} \tag{3.15}$$

respectively. Equation (3.15) are collocated at $(N+1)$ points. For suitable collocation points we use the shifted Legendre roots $t_i,\ i = 1, 2, \cdots, N + 1$ of $L_{\tau,N+1}(t)$. The number of the unknown coefficients a_{ij} is equal to $(N + 1) \times (M + 1)$ and can be obtained from (3.14)-(3.15). Consequently $u_{N,M}(x,t)$ given in (3.5) can be calculated.

In our implementation, we have solved this system using the Mathematica function FindRoot, which uses Newton's method as the default method. In all the considered examples, this function has succeeded to obtain an accurate approximate solution of the system, even starting with a zero initial approximation.

4 Extensions to more general boundary conditions

Since many application problems in science and engineering involve Neumann and Robin boundary conditions [15, 20, 24, 40, 42, 58], such as zero flow or specified flow flux condition, it is important to extend the result of the present section to account for more general boundary conditions so that the shifted Legendre tau technique can be used efficiently to simulate these models.

Let us consider the FDWEs (3.1) subject to (3.2) and the Robin boundary conditions

$$\begin{aligned} \lambda_1 u(0,t) + \lambda_2 \frac{\partial u(0,t)}{\partial x} &= g_0(t), \\ \mu_1 u(\ell,t) + \mu_2 \frac{\partial u(\ell,t)}{\partial x} &= g_1(t), \end{aligned} \qquad 0 < t \leq \tau, \tag{4.1}$$

In general, the Dirichlet or Neumann boundary conditions may be obtained as a special case from the general boundary conditions.

If we apply the shifted Legendre tau technique based on operational matrices to the modified (3.1), we get $(N+1)\times(M-1)$ linear algebraic equations system from

$$E_{ij} = 0, \qquad i = 0,1,\cdots,N, \quad j = 0,1,\cdots,M-2. \tag{4.2}$$

The operational matrix formulation of the Robin conditions (4.1) is:

$$\begin{aligned} \lambda_1 \Phi^T_{\tau,N}(t)\mathbf{A}\Phi_{\ell,M}(0) + \lambda_2\, \Phi^T_{\tau,N}(t)\mathbf{A}\mathbf{D}^{(1)}\Phi_{\ell,M}(0) &= g_0(t) \\ \mu_1 \Phi^T_{\tau,N}(t)\mathbf{A}\Phi_{\ell,M}(\ell) + \mu_2\, \Phi^T_{\tau,N}(t)\mathbf{A}\mathbf{D}^{(1)}\Phi_{\ell,M}(\ell) &= g_1(t) \end{aligned} \tag{4.3}$$

Which generates a $(N+1)\times(2)$ linear algebraic equations by collocating these two equations at the zeros $t_i,\ i = 1,2,\cdots,N+1$ of $L_{\tau,N+1}(t)$. Consequently the approximate solution can be obtained from solving the generated algebraic system.

5 Convergence analysis

In this section we present a general approach to the convergence analysis of the Legendre Tau method for the FDWE. The convergence analysis of this method is based on the Legendre orthogonal polynomials using the operator theory. The convergence of the proposed method is established in the $L^2(I_t; L^2(I_x))$-norm. Here, we will confine ourselves to the FDWE of the form

$$\begin{cases} {}_0^C D_t^{\alpha} u(x,t) = c_+(x)\ {}_0^C D_x^{\nu} u(x,t) + c_-(x)\ {}_x^C D_\ell^{\nu} u(x,t) + q(x,t), \\ u(x,0) = \dfrac{\partial u(x,0)}{\partial t} = 0, \qquad 0 < x < 1, \\ u(0,t) = u(1,t) = 0, \qquad 0 < t \leq 1, \end{cases} \tag{5.1}$$

noting that more complicated boundary conditions will be shown to converge on a case-by-case basic in the examples provided in the next section. Let us define $e_{N,M}(x,t) = u_{N,M}(x,t) - u(x,t)$ as the error function of the Tau approximation, where $u(x,t)$ is the exact solution of (5.1) and $u_{N,M}(x,t)$ is the Legendre Tau approximation for $u(x,t)$.

First, we provide some definitions and lemmas which are important for the derivation of the main results in this section. Through this section denote by $C_i, i = 1, 2, \ldots,$ generic positive constant independent of N, M and any function.

Definition 5.1 For $p < +\infty$ and $I = (a, b)$, we denote by $L^p(I)$ the Banach space of the measurable functions $u : (a, b) \to \mathbb{R}$ such that $\int_a^b |u(x)|^p \, dx < +\infty$. It is endowed with the norm

$$\|u\|_{L^p(I)} = \left(\int_a^b |u(x)|^p dx\right)^{1/p}.$$

The space $L^2(I)$ is a Hilbert space with the inner product

$$(u, v) = \int_a^b u(x)v(x)dx,$$

and the norm

$$\|u\|_{L^2(I)} = \left(\int_a^b |u(x)|^2 dx\right)^{1/2}.$$

Definition 5.2 (see [8]) For any integer $k \geq 0$. We define

$$H^m(I) = \left\{u \in L^2(I);\ \frac{\partial^k u}{\partial x^k} \in L^2(I),\ 0 \leq k \leq m\right\}.$$

$H^m(I)$ is endowed with the inner product

$$(u, v)_m = \sum_{j=0}^{m} \left(\frac{d^j u(x)}{dx^j}, \frac{d^j v(x)}{dx^j}\right),$$

for which $H^m(I)$ is a Hilbert space. The associated norm is

$$\|u\|_{H^m(I)} = \left(\sum_{j=0}^{m} \left\|\frac{d^j u}{dx^j}\right\|^2_{L^2(I)}\right)^{1/2}.$$

Definition 5.3 $\Pi_M : L^2(I) \to \mathbb{P}_M$ is an orthogonal projection if and only if for any $u \in L^2(I)$, we have

$$\left(\Pi_M u - u, v\right) = 0, \qquad \forall v \in \mathbb{P}_M.$$

Lemma 5.1 *(see [8]) For all $u \in H^k(I)$, $k \geqslant 0$, there exists a constant C independent of M, such that*

$$\|u - \Pi_M u\|_{H^k(I)} \leq C M^{\sigma(k)-m} |u|_{H^{m;M}(I)} \quad 0 \leq k \leq m, \tag{5.2}$$

where $\sigma(k) = 0$ *if* $k = 0$ *and* $\sigma(k) = 2k - \frac{1}{2}$ *for* $k > 0$*. The seminorm on the right-hand side is defined as*

$$|u|_{H^{m;M}(I)} = \left(\sum_{j=\min(m,M+1)}^{m} \left\| \frac{d^j u}{dx^j} \right\|^2_{L^2(I)} \right)^{1/2};$$

note that whenever $M \geq m - 1$*, one has*

$$|u|_{H^{m;M}(I)} = \left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^2(I)} = \|u\|_{H^m(I)}.$$

We consider here a two-dimensional domain, say $\Omega = I_t \times I_x = (0, 1)^2$ and consider

$$\mathbb{P}_{N,M} = \operatorname{span}\{L_{\ell,i}(x)L_{\ell,j}(t), \quad i = 0, 1, \cdots, M, \quad j = 0, 1, \cdots, N\}.$$

Let us denote by $H^{r,0} = L^2(I_t; H^r(I_x))$ the space of the measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{H^{r,0}} = \left(\int_0^1 \|u(.,t)\|^2_{H^r(I_x)} dt \right)^{1/2} < +\infty.$$

For $r = 0$, this norm will be denoted briefly by

$$\|u\|_{L^2(\Omega)} = \left(\int_0^1 \int_0^1 |u(x,y)|^2 dx dt \right)^{12},$$

where $L^2(\Omega) = H^{0,0} = L^2(I_t; L^2(I_x))$.

Moreover, for any positive integer s we define

$$H^{0,s} = H^s(I_t; L^2(I_x)) = \left\{ u \in L^2(\Omega) \,\middle|\, \frac{\partial^j u}{\partial t^j} \in L^2(\Omega),\ 0 \leq j \leq s \right\};$$

the norm is given by

$$\|u\|_{H^{0,s}} = \left(\sum_{j=0}^{s} \left\| \frac{\partial^j u}{\partial t^j} \right\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Definition 5.4 We define

$$H^{r,s}(\Omega) = H^s(I_t; H^r(I_x)) = \left\{ u \in L^2(\Omega) \,\middle|\, \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \in L^2(\Omega), 0 \leq i \leq r, 0 \leq j \leq s \right\}.$$

$H^{r,s}(\Omega)$ is a Hilbert space equipped with the inner product and the norm as

$$(u, v)_{r,s} = \sum_{i=0}^{r}\sum_{j=0}^{s}\int_0^1\int_0^1 \frac{\partial^{i+j}u(x,t)}{\partial x^i \partial t^j}\frac{\partial^{i+j}v(x,t)}{\partial x^i \partial t^j}dxdt,$$

$$\|u\|_{H^{r,s}} = \left(\sum_{i=0}^{r}\sum_{j=0}^{s}\left\|\frac{\partial^{i+j}u(x,t)}{\partial x^i \partial t^j}\right\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Theorem 5.2 *For any function $u \in L^2(\Omega)$, let $\Pi_{N,M}u$ denote the projection of u upon $\mathbb{P}_{N,M}$, i.e.,*

$$(\Pi_{N,M}u)(x,t) = u_{N,M}(x,t).$$

Then, for all $r, s \geq 0$, we have

$$\left\|u - \Pi_{N,M}u\right\|_{L^2(\Omega)} \leq C_1 M^{-r}\|u\|_{H^{r,0}} + C_2 N^{-s}\|u\|_{H^{0,s}},$$

for all u for which the norms on the right-hand side are finite.

Proof Let Π_N and Π_M be the one-dimensional orthogonal projections defined in Definition 5.3. Then,

$$\Pi_{N,M}u = \Pi_N \circ \left(\Pi_M u\right).$$

Hence, using Lemma 5.1 leads to

$$\begin{aligned}
\left\|u - \Pi_{N,M}u\right\|_{L^2(\Omega)} &\leq \|u - \Pi_N u\|_{L^2(\Omega)} + \left\|\Pi_N \circ \left(u - \Pi_M u\right)\right\|_{L^2(\Omega)} \\
&\leq \|u - \Pi_N u\|_{L^2(\Omega)} + C_2\|u - \Pi_M u\|_{L^2(\Omega)} \\
&\leq M^{-r}\left\|\frac{\partial^k u}{\partial x^r}\right\|_{L^2(\Omega)} + C_2 N^{-s}\left\|\frac{\partial^\ell u}{\partial t^\ell}\right\|_{L^2(\Omega)} \\
&\leq C_1 M^{-r}\|u\|_{H^{r,0}} + C_2 N^{-s}\|u\|_{H^{0,s}}. \qquad (5.3)
\end{aligned}$$

This ends the proof. □

Theorem 5.3 *(The convergence theorem) Suppose u is the exact solution of (5.1) and $\Pi_{N,M} = u_{N,M}$ is the solution of (3.14) which is obtained by the method presented in Section 3. Assume that $u \in L^2(\Omega)$. Then, for sufficiently smooth functions $c_+(x)$, $c_-(x)$ and $q(x,t)$ in (5.1) and for all sufficiently Large N and M we have*

$$\left\|u_{N,M}(x,t) - u(x,t)\right\|_{L^2(\Omega)} \to 0.$$

Proof Let us define the error functions $e_{N,M}(u(x,t)) = \Pi_{N,M}(u(x,t)) - u(x,t)$ where $u(x,t)$ is a continuous function on Ω. According to the proposed method, we obtain

$$\begin{aligned} u_{N,M}(x,t) = & \frac{c_+(x)}{\Gamma(\alpha)\Gamma(m-\nu)}\Pi_{N,M} \\ & \times\left(\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\nu-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right) \\ & +\frac{c_-(x)}{\Gamma(\alpha)\Gamma(m-\nu)}\Pi_{N,M} \\ & \times\left(\int_0^t\int_x^1 (t-s)^{\alpha-1}(r-x)^{m-\nu-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right). \end{aligned} \tag{5.4}$$

Again, we rewrite (5.1) as

$$\begin{aligned} u(x,t) = & \frac{c_+(x)}{\Gamma(\alpha)\Gamma(m-\nu)}\left(\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\nu-1}\frac{\partial^m u(r,s)}{\partial r^m}drds\right) \\ & +\frac{c_-(x)}{\Gamma(\alpha)\Gamma(m-\nu)}\left(\int_0^t\int_x^1 (t-s)^{\alpha-1}(r-x)^{m-\nu-1}\frac{\partial^m u(r,s)}{\partial r^m}drds\right). \end{aligned} \tag{5.5}$$

Subtracting (5.4) from (5.5), we obtain

$$e_{N,M}(x,t) = G_1 + G_2 + G_3 + G_4 \tag{5.6}$$

where

$$\begin{aligned} G_1 = & \frac{c_+(x)}{\Gamma(\alpha)\Gamma(m-\nu)}\, e_{N,M}\left(\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\nu-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right), \\ G_2 = & \frac{c_+(x)}{\Gamma(\alpha)\Gamma(m-\nu)}\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\nu-1}\frac{\partial^m e_{N,M}(r,s)}{\partial r^m}drds, \\ G_3 = & \frac{c_-(x)}{\Gamma(\alpha)\Gamma(m-\nu)}\, e_{N,M}\left(\int_0^t\int_x^1 (t-s)^{\alpha-1}(r-x)^{m-\nu-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right), \\ G_4 = & \frac{c_-(x)}{\Gamma(\alpha)\Gamma(m-\nu)}\int_0^t\int_x^1 (t-s)^{\alpha-1}(r-x)^{m-\nu-1}\frac{\partial^m e_{N,M}(r,s)}{\partial r^m}drds. \end{aligned} \tag{5.7}$$

Then we can write

$$\left\|e_{N,M}(x,t)\right\|_{L^2(\Omega)} \le \sum_{i=1}^{4}\|G_i\|_{L^2(\Omega)}. \tag{5.8}$$

Now, it is sufficient to show that the right hand of (5.8) tends to zero for sufficiently large N and M. To this end, in virtue of (5.3), we obtain

$$\begin{aligned} \|G_1\|_{L^2(\Omega)} \le & C_1 M^{-1}\left\|\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\nu-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right\|_{H^{1,0}} \\ & + C_2 N^{-1}\left\|\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\nu-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right\|_{H^{0,1}}. \end{aligned} \tag{5.9}$$

The first term on the right-hand side of the above equation can be written as follows

$$\begin{aligned}
&\left\|\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\upsilon-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right\|^2_{H^{1,0}}\\
&\quad=\left\|\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\upsilon-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right\|^2_{L^2(\Omega)}\\
&\qquad+\left\|\frac{\partial}{\partial x}\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\upsilon-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right\|^2_{L^2(\Omega)}. \qquad (5.10)
\end{aligned}$$

Due to (2.5), we obtain following relation

$$\frac{\partial}{\partial x}\int_0^x (x-r)^{m-\nu-1}u(r)dr=\int_0^x (x-r)^{m-\nu-1}\frac{\partial u(r)}{\partial r}dr,$$

and using Young inequality [1, 18], we obtain

$$\begin{aligned}
&\left\|\int_0^t\int_0^x (t-s)^{\alpha-1}(x-r)^{m-\upsilon-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds\right\|^2_{H^{1,0}}\\
&\quad=\left\|\int_0^x (x-r)^{m-\upsilon-1}\left(\int_0^t (t-s)^{\alpha-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}ds\right)dr\right\|^2_{L^2(\Omega)}\\
&\qquad+\left\|\int_0^x (x-r)^{m-\upsilon-1}\left(\int_0^t (t-s)^{\alpha-1}\frac{\partial^{m+1} u_{N,M}(r,s)}{\partial r^{m+1}}ds\right)dr\right\|^2_{L^2(\Omega)}\\
&\quad\le\left\|(x-r)^{m-\upsilon-1}\right\|^2_{L^1(I_x)}\left\|\int_0^t (t-s)^{\alpha-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}ds\right\|^2_{L^2(\Omega)}\\
&\qquad+\left\|(x-r)^{m-\upsilon-1}\right\|^2_{L^1(I_x)}\left\|\left(\int_0^t (t-s)^{\alpha-1}\frac{\partial^{m+1} u_{N,M}(r,s)}{\partial r^{m+1}}ds\right)\right\|^2_{L^2(\Omega)}\\
&\quad\le\left\|(x-r)^{m-\upsilon-1}\right\|^2_{L^1(I_x)}\left\|(t-s)^{\alpha-1}\right\|^2_{L^1(I_t)}\left\|\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}\right\|^2_{L^2(\Omega)}\\
&\qquad+\left\|(x-r)^{m-\upsilon-1}\right\|^2_{L^1(I_x)}\left\|(t-s)^{\alpha-1}\right\|^2_{L^1(I_t)}\left\|\frac{\partial^{m+1} u_{N,M}(r,s)}{\partial r^{m+1}}\right\|^2_{L^2(\Omega)}\\
&\quad\le C_3\left\|\frac{\partial^m u_{N,M}(x,t)}{\partial x^m}\right\|^2_{L^2(\Omega)}+C_3\left\|\frac{\partial^{m+1} u_{N,M}(x,t)}{\partial x^{m+1}}\right\|^2_{L^2(\Omega)}\\
&\quad=C_3\left\|\frac{\partial^m u_{N,M}(x,t)}{\partial x^m}\right\|^2_{H^{1,0}}\\
&\quad\le C_3\left\|u_{N,M}(x,t)\right\|^2_{H^{m+1,0}}=C_3\left\|e_{N,M}(x,t)-u(x,t)\right\|^2_{H^{m+1,0}}\\
&\quad\le C_3\left(\left\|e_{N,M}(x,t)\right\|_{H^{m+1,0}}+\left\|u(x,t)\right\|_{H^{m+1,0}}\right)^2. \qquad (5.11)
\end{aligned}$$

Also we can obtain

$$
\begin{aligned}
&\left\| \int_0^t \int_0^x (t-s)^{\alpha-1}(x-r)^{m-\nu-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds \right\|^2_{H^{0,1}} \\
&\qquad\qquad \le C_4 \left\| \frac{\partial^m u_{N,M}(r,s)}{\partial x^m} \right\|^2_{L^2(\Omega)} + C_4 \left\| \frac{\partial^{m+1} u_{N,M}(r,s)}{\partial x^m \partial t} \right\|^2_{L^2(\Omega)} \\
&\qquad\qquad = C_4 \left\| u_{N,M}(x,t) \right\|^2_{H^{m,1}} \\
&\qquad\qquad \le C_4 \left\| e_{N,M}(x,t) - u(x,t) \right\|^2_{H^{m,1}} \\
&\qquad\qquad \le C_4 \left(\left\| e_{N,M}(x,t) \right\|_{H^{m,1}} + \| u(x,t) \|_{H^{m,1}} \right)^2 .
\end{aligned} \tag{5.12}
$$

According to (5.9), (5.11) and (5.12), we obtain

$$
\begin{aligned}
\|G_1\|_{L^2(\Omega)} \le\ & C_3 M^{-1} \left(\left\| e_{N,M}(x,t) \right\|_{H^{m+1,0}} + \| u(x,t) \|_{H^{m+1,0}} \right) \\
&+ C_4 N^{-1} \left(\left\| e_{N,M}(x,t) \right\|_{H^{m,1}} + \| u(x,t) \|_{H^{m,1}} \right).
\end{aligned} \tag{5.13}
$$

From (5.2), we can conclude $\|G_1\|_{L^2(\Omega)} \to 0$ for sufficiently large N and M.

Further, consider the second term of (5.8). In a similar manner with G_1, employing Young inequality and properties of Sobolev norm we have

$$
\begin{aligned}
\|G_2\|_{L^2(\Omega)} &\le \left\| \int_0^t \int_0^x (t-s)^{\alpha-1}(x-r)^{m-\nu-1}\frac{\partial^m e_{N,M}(r,s)}{\partial r^m}drds \right\|_{L^2(\Omega)} \\
&\le \left\| (x-r)^{m-\nu-1} \right\|_{L^1(I_x)} \left\| (t-s)^{\alpha-1} \right\|_{L^1(I_t)} \left\| \frac{\partial^m e_{N,M}(r,s)}{\partial r^m} \right\|_{L^2(\Omega)} \\
&\le C_5 \left\| \frac{\partial^m e_{N,M}(x,t)}{\partial x^m} \right\|^2_{L^2(\Omega)} \le C_5 \left\| e_{N,M}(x,t) \right\|_{H^{m,0}} .
\end{aligned} \tag{5.14}
$$

Making use of (5.2), we conclude that $\|G_2\|_{L^2(\Omega)} \to 0$. For the third term, we may write

$$
\begin{aligned}
\|G_3\|_{L^2(\Omega)} \le\ & C_1 M^{-1} \left\| \int_0^t \int_x^1 (t-s)^{\alpha-1}(r-x)^{m-\nu-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds \right\|_{H^{1,0}} \\
&+ C_2 N^{-1} \left\| \int_0^t \int_x^1 (t-s)^{\alpha-1}(r-x)^{m-\nu-1}\frac{\partial^m u_{N,M}(r,s)}{\partial r^m}drds \right\|_{H^{0,1}} .
\end{aligned} \tag{5.15}
$$

Due to (2.6), we obtain following relation

$$
\frac{\partial}{\partial x}\int_x^1 (r-x)^{m-\nu-1}u(r)dr = \int_x^1 (x-r)^{m-\nu-1}\frac{\partial u(r)}{\partial r}dr.
$$

The first term on the right-hand side of (5.15) can be written as

$$\begin{aligned}
&\left\| \int_0^t \int_x^1 (t-s)^{\alpha-1}(r-x)^{m-\nu-1} \frac{\partial^m u_{N,M}(r,s)}{\partial r^m} dr ds \right\|_{H^{1,0}}^2 \\
&= \left\| \int_x^1 (r-x)^{m-\nu-1} \left(\int_0^t (t-s)^{\alpha-1} \frac{\partial^m u_{N,M}(r,s)}{\partial r^m} ds \right) dr \right\|_{L^2(\Omega)}^2 \\
&\quad + \left\| \int_x^1 (r-x)^{m-\nu-1} \left(\int_0^t (t-s)^{\alpha-1} \frac{\partial^{m+1} u_{N,M}(r,s)}{\partial r^{m+1}} ds \right) dr \right\|_{L^2(\Omega)}^2 \\
&\leq C_6 \left\| \frac{\partial^m u_{N,M}(x,t)}{\partial x^m} \right\|_{L^2(\Omega)}^2 + C_6 \left\| \frac{\partial^{m+1} u_{N,M}(x,t)}{\partial x^{m+1}} \right\|_{L^2(\Omega)}^2 \\
&= \left\| u_{N,M}(x,t) \right\|_{H^{m+1,0}}^2 \leq \left\| e_{N,M}(x,t) - u(x,t) \right\|_{H^{m+1,0}}^2 \\
&\leq C_6 \left(\left\| e_{N,M}(x,t) \right\|_{H^{m+1,0}} + \| u(x,t) \|_{H^{m+1,0}} \right)^2 . \qquad (5.16)
\end{aligned}$$

Moreover, the second term can be estimated as follows

$$\begin{aligned}
&\left\| \int_0^t \int_x^1 (t-s)^{\alpha-1}(r-x)^{m-\nu-1} \frac{\partial^m u_{N,M}(r,s)}{\partial r^m} dr ds \right\|_{H^{0,1}}^2 \\
&\leq C_7 \left\| \frac{\partial^m u_{N,M}(r,s)}{\partial x^m} \right\|_{L^2(\Omega)}^2 \\
&\quad + C_7 \left\| \frac{\partial^{m+1} u_{N,M}(r,s)}{\partial x^m \partial t} \right\|_{L^2(\Omega)}^2 = C_7 \left\| \frac{\partial^m u_{N,M}(r,s)}{\partial x^m} \right\|_{H^{0,1}}^2 \\
&\leq C_7 \left\| u_{N,M}(x,t) \right\|_{H^{m,1}}^2 \leq C_7 \left\| e_{N,M}(x,t) - u(x,t) \right\|_{H^{m,1}}^2 \\
&\leq C_7 \left(\left\| e_{N,M}(x,t) \right\|_{H^{m,1}} + \| u(x,t) \|_{H^{m,1}} \right)^2 . \qquad (5.17)
\end{aligned}$$

Making use of (5.16), (5.17) and (5.15), we obtain

$$\begin{aligned}
\| G_3 \|_{L^2(\Omega)} &\leq C_6 M^{-1} \left(\left\| e_{N,M}(x,t) \right\|_{H^{m+1,0}} + \| u(x,t) \|_{H^{m+1,0}} \right) \\
&\quad + C_7 N^{-1} \left(\left\| e_{N,M}(x,t) \right\|_{H^{m,1}} + \| u(x,t) \|_{H^{m,1}} \right) . \qquad (5.18)
\end{aligned}$$

Finally, we can obtain

$$\begin{aligned}
\| G_4 \|_{L^2(\Omega)} &\leq \left\| \int_0^t \int_x^1 (t-s)^{\alpha-1}(r-x)^{m-\nu-1} \frac{\partial^m e_{N,M}(r,s)}{\partial r^m} dr ds \right\|_{L^2(\Omega)} \\
&\leq C_8 \left\| \frac{\partial^m e_{N,M}(x,t)}{\partial x^m} \right\|_{L^2(\Omega)} \leq C_8 \left\| e_{N,M}(x,t) \right\|_{H^{m,0}} . \qquad (5.19)
\end{aligned}$$

From (5.2), we can conclude $\| G_3 \|_{L^2(\Omega)}$ and $\| G_4 \|_{L^2(\Omega)} \to 0$ for sufficiently large N, M.

The desired convergence result for the proposed space-time Legendre tau scheme is obtained by combining (5.13), (5.14), (5.18), (5.19) and (5.8). The presented results show that the convergence of the approximate solution $u_{N,M}(x,t)$ to $u(x,t)$

as $N, M \to \infty$ depends on how many times $u(x,t)$ is differentiable with respect to x and t. □

6 Numerical results and comparisons

In this section, we present four numerical examples to demonstrate the accuracy and applicability of the proposed method. The obtained results of these examples show that SLT method, by selecting a few number of shifted Legendre polynomials, is more accurate than explicit difference approximation [48], implicit difference approximation [11, 48], compact finite difference method [40] and the Crank-Nicolson method [40].

Example 1 As a first application, we offer the following one-dimensional space-time fractional wave equation [19]:

$$ {}_0^C D_t^{\alpha} u(x,t) = \frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + q(x,t), \qquad 0 < x, t < 1, \ 1 < \alpha, \beta < 2, \tag{6.1} $$

with initial conditions

$$ u(x,0) = 1 - x^2, \qquad \frac{\partial u(x,0)}{\partial t} = 0, \tag{6.2} $$

and Robin boundary conditions

$$ \begin{aligned} u(0,t) + \frac{\partial u(0,t)}{\partial x} &= 1 + t^{2\alpha}, \\ u(1,t) - \frac{\partial u(1,t)}{\partial x} &= 2(1 + t^{2\alpha}), \end{aligned} \tag{6.3} $$

where

$$ q(x,t) = \frac{\Gamma(3+\alpha)(1-x^2)}{\Gamma(3)} t^2 + \frac{2C_{\beta}(1+t^{2+\alpha})}{\Gamma(3-\beta)} (x^{2-\beta} + (1-x)^{2-\beta}). $$

The exact solution of this problem is $u(x,t) = (1 + t^{2+\alpha})(1 - x^2)$.

The maximum absolute errors (MAEs) between the exact solution $u(x,t)$ and the approximate solution $u_{N,M}(x,t)$ with various choices of N $(N = M)$ and three choices of the fractional derivatives α and β are given in Table 1. We see in this table that the results are accurate for even small choices of N and M. Fig. 1 shows the error functions $u(x, 0.9) - u_{10,10}(x, 0.9)$ and $u(0.6, t) - u_{10,10}(0.6, t)$ with $\alpha = \beta = 1.9$. In addition, to demonstrate the convergence of the proposed method, in Fig. 2, we plot the logarithmic graphs of MAEs ($log_{10} Error$) at various values of the fractional derivatives α and β. From these figures, we conclude that the numerical errors for all chosen fractional derivatives (α, β) decay rapidly as N and M increase. We observe also that the suggested algorithm provides accurate and stable numerical results.

Table 1 Maximum absolute errors at various choices of N, M, α and β for Example 1

$N = M$	$\alpha = \beta = 1.1$	$\alpha = \beta = 1.3$	$\alpha = \beta = 1.5$	$\alpha = \beta = 1.7$
3	6.299×10^{-3}	3.348×10^{-3}	6.133×10^{-3}	9.227×10^{-3}
5	3.366×10^{-4}	7.827×10^{-5}	8.609×10^{-5}	6419×10^{-5}
7	1.920×10^{-5}	9.016×10^{-6}	8.361×10^{-6}	5.214×10^{-6}
9	1.859×10^{-6}	1.850×10^{-6}	1.535×10^{-6}	8.550×10^{-7}
11	3.461×10^{-7}	5.249×10^{-7}	4.025×10^{-7}	2.052×10^{-7}
13	1.627×10^{-7}	1.187×10^{-7}	1.860×10^{-7}	4.829×10^{-8}

Example 2 Consider the following two-sided space-fractional wave equation [48]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c_+(x)\ {}_0^C D_x^{1.8} u(x,t) + c_-(x)\ {}_x^C D_\ell^{1.8} u(x,t) + q(x,t), \tag{6.4}$$

with the initial and boundary conditions:

$$\begin{aligned} u(x,0) &= 4x^2(2-x)^2, & \frac{\partial u(x,0)}{\partial t} &= -4x^2(2-x)^2, & x &\in (0,2), \\ u(0,t) &= 0, & u(2,t) &= 0, & t &\in (0,\tau], \end{aligned} \tag{6.5}$$

where $c_+(x) = \Gamma(1.2)\ x^{1.8}$, and $c_-(x) = \Gamma(1.2)\ (2-x)^{1.8}$, while

$$\begin{aligned} q(x,t) = {} & 4e^{-t}x^2(2-x)^2 - 32e^{-t}\Big(x^2 + (2-x)^2 - 2.5(x^3 + (2-x)^3) \\ & + \frac{25}{22}(x^4 + (2-x)^4)\Big). \end{aligned}$$

The exact solution of this problem is $u(x,t) = 4e^{-t}x^2(2-x)^2$.

In [48], Sweilam et al. applied the implicit and explicit methods to introduce an approximate solution of this problem at $\tau = 2$ with several choices of Δx and Δt, where Δx and Δt are space and time step sizes, respectively. Regarding problem (6.4), in [48], the best result is achieved with the implicit method at $\Delta t = 4 \times 10^{-3}$ and $\Delta x = 2 \times 10^{-6}$ and the maximum absolute error is 3×10^{-3}. In Table 2, we

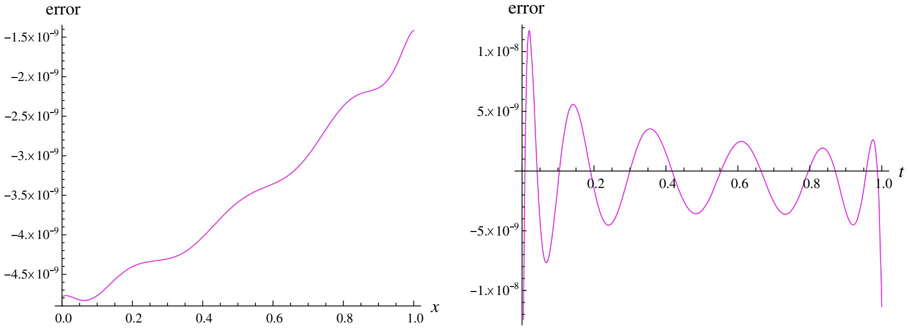

Fig. 1 Errors at $t = 0.9$ (*left*) and $x = 0.6$ (*right*) with $N = M = 11$, $\alpha = \beta = 1.9$ for Example 1

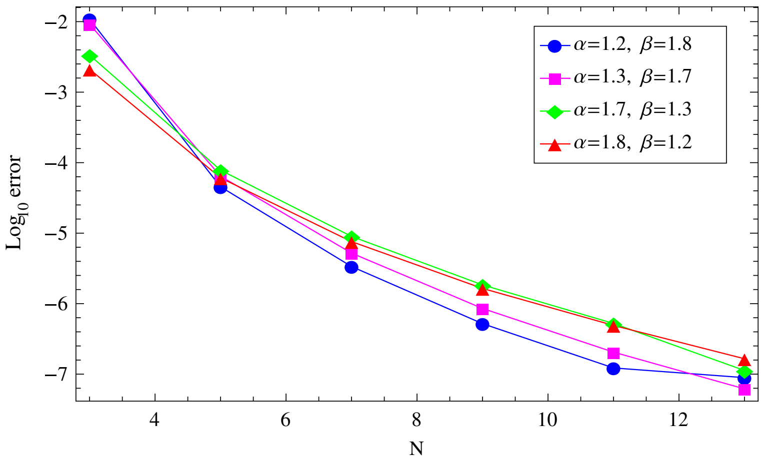

Fig. 2 Convergence rates of the numerical method for problem (6.1) at various choices of α and β

list MAEs at $\tau = 2$ with various choices of N and M. Our method is more accurate than the implicit and explicit methods [48]. We see in this table that the results are accurate for even small choices of N and M.

Example 3 Consider the following fractional wave equation with damping ($\gamma = 1$)[11]:

$$ {}_0^C D_t^{\alpha} u(x,t) + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t), \tag{6.6} $$

with the initial and boundary conditions:

$$ \begin{aligned} u(x,0) &= 0, & \frac{\partial u(x,0)}{\partial t} &= 0, & 0 < x < \ell, \\ u(0,t) &= 0, & u(\ell,t) &= 0, & 0 < t \le \tau, \end{aligned} \tag{6.7} $$

where

$$ q(x,t) = \frac{2x(\ell - x)}{\Gamma(3-\alpha)} t^{2-\alpha} + 2tx(\ell - x) + 2t^2. $$

The exact solution is $u(x,t) = t^2 x(\ell - x)$.

Recently, Chen et al. [11] applied the method of separation of variables with constructing the implicit difference approximation to introduce an approximate solution of this problem at $\ell = 2$, $\tau = 1$ and $\alpha = 1.7$ using the Caputo form of the fractional derivative. Regarding problem (6.6), in [11], the best result is achieved at

Table 2 Maximum absolute errors with $\tau = 2$ and various choices of N and M for Example 2

$N = M$	4	5	7	9	11
MAE	2.315×10^{-3}	2.291×10^{-4}	3.407×10^{-5}	1.015×10^{-5}	3.973×10^{-6}

$\ell = 2$, $\tau = 1$ and $\alpha = 1.7$ with 640 steps and the maximum absolute error is 5.316×10^{-5}. In Table 3, we list MAEs at $\tau = 1$, $\ell = 2$ and various choices of N, M and α. Our method is more accurate than the method of separation of variables combined with the implicit difference approximation [11]. We see in this table that the results are accurate for even small choices of N and M.

In addition, to demonstrate the convergence of the proposed method, in Fig. 3, we plot the logarithmic graphs of MAEs ($log_{10}Error$) at $\tau = 1$, $\ell = 2$ and various values of the fractional derivative α with various values of N ($N = M$); by using the presented algorithm. Clearly, the numerical errors decay rapidly as N and M increase.

Example 4 Consider the time-fractional diffusion-wave problem [40]:

$$ {}_0^C D_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t), \quad 1 < \alpha < 2, \quad 0 < x < 1, \quad 0 < t < 1, \tag{6.8} $$

with initial conditions

$$ u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \tag{6.9} $$

and Neumann boundary conditions

$$ \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x} = 0. \tag{6.10} $$

The forcing term is

$$ q(x,t) = \frac{\Gamma(\alpha+3)}{2} t^2 e^x x^2 (1-x)^2 - e^x t^{\alpha+2} (2 - 8x + x^2 + 6x^3 + x^4). $$

The exact solution of this problem has the form:

$$ u(x,t) = e^x x^2 (1-x)^2 t^{\alpha+2}. $$

Table 3 Maximum absolute errors at $\tau = 1$, $\ell = 2$ and various choices of N, M and α for Example 3

$N = M$	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
3	3.685×10^{-4}	7.757×10^{-4}	1.039×10^{-3}	8.196×10^{-4}
5	8.735×10^{-5}	2.150×10^{-4}	3.053×10^{-4}	2.525×10^{-4}
7	3.166×10^{-5}	8.303×10^{-5}	1.243×10^{-4}	1.081×10^{-4}
9	1.439×10^{-5}	3.960×10^{-5}	6.169×10^{-5}	5.554×10^{-5}
11	7.506×10^{-6}	2.145×10^{-5}	3.447×10^{-5}	3.198×10^{-5}
13	4.315×10^{-6}	1.273×10^{-5}	2.101×10^{-5}	2.002×10^{-5}
15	2.665×10^{-6}	8.082×10^{-6}	1.365×10^{-5}	1.329×10^{-5}
17	1.739×10^{-6}	5.402×10^{-6}	9.314×10^{-6}	9.260×10^{-6}
19	1.186×10^{-6}	3.762×10^{-6}	6.606×10^{-6}	6.689×10^{-6}
21	8.380×10^{-7}	2.709×10^{-6}	4.837×10^{-6}	4.980×10^{-6}
23	7.957×10^{-7}	2.219×10^{-6}	2.325×10^{-6}	2.021×10^{-6}
25	7.950×10^{-7}	2.212×10^{-6}	2.325×10^{-6}	2.021×10^{-6}

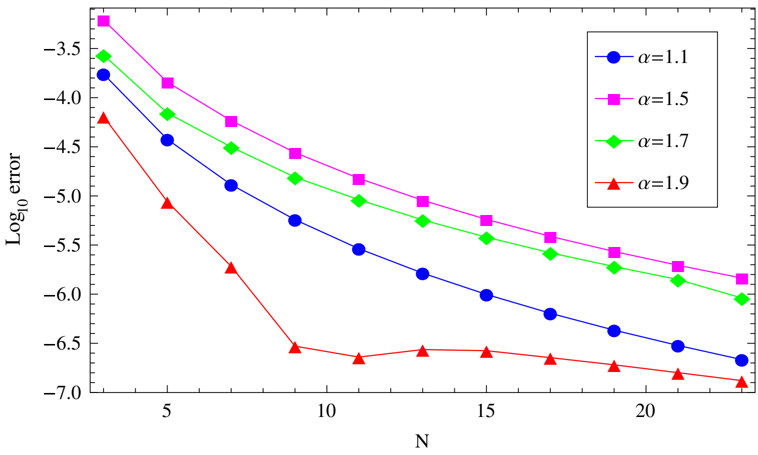

Fig. 3 Convergence rates of the numerical method for problem (6.6) at $\tau = 1,\ \alpha = 1.1,\ 1.5,\ 1.7,\ 1.9$

Table 4 displays maximum absolute errors using SLT method with $N = 8$ together with the results obtained by using the compact finite difference method (CFDM [40]) and the Crank-Nicolson method (CNM [40]), for different choices of α. From the results of this example, we observe that the approximate solution by SLT method is more better than those obtained by CFDM [40] and CNM [40].

Example 5 Consider the exact solution of the problem (6.8) as follows [40],

$$u(x, t) = t^{2+\alpha} \sin x, \tag{6.11}$$

defined on $0 < x < \pi$, and $0 < t < \tau$, then the corresponding forcing term is

$$q(x, t) = \left(\frac{\Gamma(3+\alpha)}{2} t^2 + t^{2+\alpha} \right) \sin x,$$

Table 4 MAEs of the SLT method and methods of [40] at different values of α for Example 4

	SLT method			CFD method [40]		CN method [40]	
α	$M = 4$	$M = 8$	$M = 12$	$M = 10$	$M = 160$	$M = 10$	$M = 160$
1.2	2.80×10^{-5}	4.58×10^{-7}	3.82×10^{-7}	–	–	–	–
1.3	3.83×10^{-5}	4.65×10^{-7}	4.07×10^{-7}	$1.15.10^{-3}$	$1.04.10^{-5}$	1.15×10^{-3}	1.06×10^{-5}
1.4	4.53×10^{-5}	5.79×10^{-7}	4.53×10^{-7}	–	–	–	–
1.5	4.85×10^{-5}	5.51×10^{-7}	4.37×10^{-7}	$2.54.10^{-3}$	$3.99.10^{-5}$	2.54×10^{-3}	4.00×10^{-5}
1.6	4.77×10^{-5}	4.47×10^{-7}	4.47×10^{-7}	–	–	–	–
1.7	4.26×10^{-5}	4.42×10^{-7}	3.75×10^{-7}	5.20×10^{-3}	1.40×10^{-4}	5×20.10^{-3}	1×40.10^{-4}
1.8	3.29×10^{-5}	6.23×10^{-7}	5.74×10^{-7}	–	–	–	–
2	4.09×10^{-7}	4.05×10^{-7}	4.04×10^{-7}	–	–	–	–

Table 5 MAEs of the presented method and CFD method [40] at different values of α for Example 5

	SLT method			CFD method [40]	
α	$N = 5$	$N = 9$	$N = 13$	$N = 10$	$M = 160$
1.3	1.99×10^{-3}	1.84×10^{-6}	$1.07.10^{-7}$	1.55×10^{-2}	1.41×10^{-4}
1.5	1.88×10^{-3}	1.52×10^{-6}	$2.10.10^{-7}$	3.61×10^{-2}	5.70×10^{-3}
1.7	1.75×10^{-3}	8.49×10^{-7}	$5.50.10^{-8}$	7.62×10^{-2}	2.08×10^{-3}

with initial conditions

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0, \tag{6.12}$$

and the nonhomogeneous Robin boundary conditions

$$\begin{aligned} u(0,t) - \frac{\partial u(0,t)}{\partial x} &= -t^{2+\alpha}, \\ u(\pi,t) + \frac{\partial u(\pi,t)}{\partial x} &= t^{2+\alpha}. \end{aligned} \tag{6.13}$$

In this example, we implement the SLT method to solve the time-fractional diffusion-wave Robin problem [40]. In Table 5, we make a comparison of the presented algorithm with the CFDM proposed in [40]. Obviously, our method is more accurate than CFDM [40]. From the results of this table, the best result we have achieved is at $N = M = 13$, we may conclude also that the obtained results are excellent in terms of accuracy for problems with nonhomogeneous boundary conditions.

7 Conclusions

We have presented a new space-time spectral algorithm based on shifted Legendre tau approximation in conjunction with the operational matrices of left-sided Caputo, right-sided Caputo fractional derivatives and Riemann-Liouville fractional integrals. This method is implemented for solving the two-sided space-time Caputo FDWE with damping subject to various boundary conditions. The fractional derivatives and integrals were given in the Caputo description. In particular, the fractional derivative is described in the Caputo sense to avoid hyper-singular improper integrals, mass balance error, non-zero derivative of constant, and fractional derivative involved in the initial data which is often ill-defined. The fractional derivative includes the left- and right-sided Caputo derivatives that allow for the modeling of flow regime impacts from either side of the domain. The method provided a very accurate approximate solution using few terms of the shifted Legendre polynomial expansion. From the numerical results given in Section 6, we may conclude that the obtained results are excellent in terms of accuracy for all tested problems.

With this paper we have outlined the implementation of a shifted Legendre tau approximation based on the fractional derivatives and fractional integrals operational matrices for solving the two-sided space-time Caputo FDWEs. In principle, this method may be extended to related problems, such as coupled two-sided space-time Caputo FDWEs. One might also consider other two-sided space-time Caputo partial differential equations subject to Dirichlet, Robin and/or non-local conditions. We should note that, as a numerical method, we are restricted to solving problems over a finite domain. Hence, this method is particularly well suited for boundary value problems with finite spatial intervals.

It is possible to use other orthogonal polynomials, say Chebyshev polynomials, or Jacobi polynomials to solve the two-sided space-time Caputo FDWEs. Furthermore, the proposed spectral method might be developed by considering the generalized Laguerre [7] or modified generalized Laguerre polynomials to solve similar problems in a semi-infinite spatial intervals. This is one possible area of future work.

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