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ORIGINAL PAPER

A study on the local convergence and the dynamics of Chebyshev–Halley–type methods free from second derivative

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Abstract We study the local convergence of Chebyshev-Halley-type methods of convergence order at least five to approximate a locally unique solution of a nonlinear equation. Earlier studies such as Behl (2013), Bruns and Bailey (Chem. Eng. Sci **32**, 257–264, 1977), Candela and Marquina (Computing **44**, 169–184, 1990), (Computing **45**(4):355–367, 1990), Chicharro et al. (2013), Chun (Appl. Math. Comput, **190**(2):1432–1437, 1990), Cordero et al. (Appl.Math. Lett. **26**, 842–848, 2013), Cordero et al. (Appl. Math. Comput. **219**, 8568–8583, 2013), Cordero and Torregrosa (Appl. Math. Comput. **190**, 686–698, 2007), Ezquerro and Hernández (Appl. Math. Optim. **41**(2):227–236, 2000), (BIT Numer. Math. **49**, 325–342, 2009), (J. Math. Anal. Appl. **303**, 591–601, 2005), Gutiérrez and Hernández (Comput. Math. Applic. **36**(7):1–8, 1998), Ganesh and Joshi (IMA J. Numer. Anal. **11**, 21–31, 1991), Hernández (Comput. Math. Applic. **41**(3–4):433–455, 2001), Hernández and Salanova (Southwest J. Pure Appl. Math. **1**, 29–40, 1999), Jarratt (Math. Comput. **20**(95):434–437, 1966), Kou and Li (Appl. Math. Comput. **189**, 1816–1821, 2007), Li (Appl. Math. Comput. **235**, 221–225, 2014), Ren et al. (Numer. Algorithm. **52**(4):585–603, 2009), Wang et al. (Numer. Algorithm. **57**, 441–456, 2011), Kou et al. (Numer. Algorithm. **60**, 369–390, 2012) show convergence under hypotheses on the third derivative or even higher. The convergence in this study is shown under hypotheses on the first derivative. Hence, the applicability of the method is expanded. The dynamical analyses of these methods are also studied. Finally, numerical examples are also provided to show that our results apply to solve equations in cases where earlier studies cannot apply.

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1 Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where F is a differentiable function defined on a convex subset D of S with values in S, where S is $\mathbb{R}$ or $\mathbb{C}$.

Many problems from Applied Sciences including engineering can be solved by means of finding the solutions of equations in a form like (1.1) using Mathematical Modelling [4, 5, 28, 31]. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. Except in special cases, the solutions of these equations can be found in closed form. This is the main reason why the most commonly used solution methods are usually iterative. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures. A very important problem in the study of iterative procedures is the convergence domain. In general the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypothesis. Another important problem is to find more precise error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$. These are with the study of the dynamical behavior our objectives in this paper.

The dynamical properties related to an iterative method applied to polynomials give important information about its stability and reliability. In recently studies, authors such as Cordero et al. [10–14], Amat et al [1, 2, 5], Gutiérrez et al. [18], Chun et al. [11], Magreñán [25, 26], and many others [6–9, 15–24, 29–33] have found interesting dynamical planes, including periodical behavior and others anomalies. One of our main interests in this paper is the study of the parameter spaces associated to a family of iterative methods, which allow us to distinguish between the good and bad methods in terms of its numerical properties. Recently, D. Li, P. Liu and J. Kou studied the local convergence of the method in [24] defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - (1 + (F(x_n) - 2\alpha F(y_n))^{-1}F(y_n))F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - (F'(x_n) + \bar{F}''(x_n)(z_n - x_n))^{-1}F(z_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point, $\alpha \in S$ a given parameter and $\bar{F}''(x_n) = 2F(y_n)F'(x_n)^2F(x_n)^{-2}$.

The order of convergence was shown to be at least five and if $\alpha = 1$ the order of convergence is six.

This method includes the modifications of Chebyshev's method ($\alpha = 0$), Halley's method ($\alpha = 1/2$) and super-Halley method ($\alpha = 1$). Method (1.2) is a usefull alternative to the third order Chebyshev-Halley-methods [15–21] defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}(1 - \alpha K_F(x_n)\right)^{-1} K_F(x_n))F'(x_n)^{-1}F(x_n), \tag{1.3}$$

where

$$K_F(x_n) = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n),$$

since the computation of $F''(x_n)$ is being avoided.

However, the convergence of the method (1.2) has been shown under hypotheses on at least the third derivative although only the first derivative appears in this method. These hypotheses limit the applicability of method (1.2). For a motivational example, define function F on $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $D = \bar{U}(0, 1)$ by

$$F(x) = \begin{cases} c_1x^3 \ln x^2 + c_2x^5 + c_3x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

where $c_1 \neq 0$, c_2 and c_3 are real parameters. Then, we have that

$$F'(x) = 3c_1x^2 \ln x^2 + 5c_2x^4 + 4c_3x^3 + 2c_1x^2,$$

$$F''(x) = 6c_1x \ln x^2 + 20c_2x^3 + 12c_3x^2 + 10c_1x$$

and

$$F'''(x) = 6c_1 \ln x^2 + 60c_2x^2 + 24c_3x + 22c_1.$$

Then, obviously, function $F'''(x)$ is unbounded on D. Hence, the results in [24], cannot apply to show the convergence of method (1.2) or its special cases requiring hypotheses on the third derivative of function F or higher. In particular, there is a plethora of iterative methods for approximating solutions of nonlinear equations defined in $\mathbb{R}$ or $\mathbb{C}$ [1–33].

These results show that if the initial point x_0 is sufficient close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* the initial guess x_0 should be? The local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) in Section 2. The same technique can be used to other methods.

In this paper we present the local convergence analysis of method (1.2) using hypotheses only on the first derivative of function F. Hence, the applicability of these methods is expanded under less restrictive conditions.

The dynamics of this family applied to an arbitrary quadratic polynomial $p(z) = (z - A)(z - B)$ will also be analyzed. The study of the dynamics of families of iterative methods has grown in the last years due to the fact that this study allows to know the best choices of the parameter in terms of stability and to find the values of the parameter for which appear anomalies, such as convergence to cycles, divergence to infinity, etc. The graphic tool used to obtain the parameter space and the different dynamical planes have been introduced by Magreñán in [25, 26], but there exist other techniques such as the one given by Chicharro et al in [10].

The rest of the paper is organized as follows: in Section 2 we present the local convergence analysis of method (1.2). The dynamics of method (1.2) are given in Section 3. Finally, the numerical examples are presented in the concluding Section 4.

2 Local convergence

In this Section we present the local convergence analysis of method (1.2). Let $U(v, \rho)$, $\bar{U}(v, \rho)$ stand for the open and closed balls in S, respectively with center $v \in S$ and of radius $\rho > 0$. Let $L_0 > 0$, $L > 0$, $M > 0$ and $\alpha \in S$ be given parameters with $L_0 \leq L$. It is convenient for the local convergence analysis that follows to introduce some functions and parameters. Define function on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)}$$

and parameter

$$r_1 = \frac{2}{2L_0 + L} < \frac{1}{L_0}. \tag{2.1}$$

Notice that $g_1(r_1) = 1$. Define functions g_2 and h_2 on the interval $[0, \frac{1}{L_0})$ by

$$g_2(t) = \frac{L_0 t}{2} + \frac{|\alpha| ML}{1 - L_0 t},$$

and

$$h_2(t) = g_2(t) - 1.$$

Suppose that

$$0 < |\alpha| ML < 1. \tag{2.2}$$

Then, we have by (2.2) that $h_2(0) = |\alpha| ML - 1 < 0$ and $h_2(t) \to \infty$ as $t \to (\frac{1}{L_0})^-$. It follows from the Intermediate Value Theorem that function h_2 has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_2 the smallest such zero. The value of r_2 can be obtained explicitly as follows:

$$h_2(t) = 0 \Leftrightarrow g_2(t) = 1 \Leftrightarrow \frac{L_0 t}{2} + \frac{|\alpha| ML}{1 - L_0 t} = 0.$$

The solutions are

$$t = \frac{3 \pm \sqrt{1 + 8|\alpha| ML}}{2L_0}.$$

Since $|\alpha| ML < 1$, $1 < \sqrt{1 + 8|\alpha| ML} < 3$. Thus

$$0 < \frac{3 - \sqrt{1 + 8|\alpha| ML}}{2L_0} < \frac{1}{L_0} < \frac{2}{L_0} < \frac{3 + \sqrt{1 + 8|\alpha| ML}}{2L_0} < \frac{3}{L_0}.$$

Therefore

$$r_2 = \frac{3 - \sqrt{1 + 8|\alpha| ML}}{2L_0}.$$

Moreover, define functions g_3 and h_3 on the interval $(0, r_2)$ by

$$g_3(t) = g_1(t)\left[1 + \frac{2M^2}{2(1 - L_0 t) - L_0 t(1 - L_0 t) - 2|\alpha|ML}\right]$$

and

$$h_3(t) = g_3(t) - 1.$$

Then, we have that $h_3(0) = -1$ and $h_3(t) \to \infty$ as $t \to r_2^-$. Hence, function h_3 has zeros in the interval $(0, r_2)$. Denote by r_3 the smallest such zero.

Furthermore, define functions g_4 and h_4 on the interval $[0, r_2)$ by

$$g_4(t) = \left[L_0 + \frac{M^3 L(1 + g_3(t))}{(1 - L_0 t)(1 - \frac{L_0}{2}t)^2}\right] t$$

and

$$h_4(t) = g_4(t) - 1.$$

We have that $h_4(0) = -1$ and $h_4(t) \to \infty$ as $t \to r_2^-$. Hence, function h_4 has zeros in the interval $(0, r_2)$. Denote by r_4 the smallest such zero.

Finally, define functions g_5 and h_5 on the interval $(0, r_3)$ if $r_3 < r_4$ and on the interval $(0, r_4)$ if $r_4 \leq r_3$ by

$$g_5(t) = \left(1 + \frac{M}{1 - g_4(t)}\right) g_3(t)$$

and

$$h_5(t) = g_5(t) - 1.$$

If $r_3 < r_4$, we have that $h_5(0) = -1 < 0$ and $h_5(r_3) = \frac{M_3}{1-g_4(r_3)} > 0$, since $g_3(r_3) = 1$ and $g_4(r_3) < 1$. Moreover, if $r_4 \leq r_3$, then $r_3 < r_2$ and we also have that $h_5(0) = -1 < 0$ and $h_5(t) \to \infty$ as $t \to r_4^-$ (since $g_3(r_4) > 0$). Hence, function h_5 has zeros in these intervals. Denote by r_5 the smallest such zero in either case.

Set

$$r = \min\{r_1, r_3, r_4, r_5\} < \frac{1}{L_0}. \tag{2.3}$$

Then, we have that for each $t \in [0, r)$

$$0 \leq g_1(t) < 1, \tag{2.4}$$

$$0 \leq g_2(t) < 1, \tag{2.5}$$

$$0 \leq g_3(t) < 1, \tag{2.6}$$

$$0 \leq g_4(t) < 1, \tag{2.7}$$

and

$$0 \leq g_5(t) < 1. \tag{2.8}$$

Using the preceding notation we can show the main local convergence result for method (1.2).

Theorem 1 *Let $F : \mathbb{D} \subset S \to S$ be a differentiable function. Let $L_0 > 0$, $L > 0$, $M > 0$, $\alpha \in S$ be given parameters. Suppose that there exists $x^* \in \mathbb{D}$ such that for all $x, y \in D$ the following hold:*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(S, S),$$

$$|\alpha| M L < 1,$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*)\| \leq L_0 \|x - x^*\|, \tag{2.9}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y)\| \leq L \|x - y\|, \tag{2.10}$$

$$\|F'(x^*)^{-1} F'(x)\| \leq M, \tag{2.11}$$

and

$$\bar{U}(x^*, r) \subseteq \mathbb{D}, \tag{2.12}$$

where r is given in (2.3). Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r) \setminus \{x^*\}$ is well defined, remains in $\bar{U}(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.13}$$

$$\|z_n - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.14}$$

and

$$\|x_{n+1} - x^*\| \leq g_5(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.15}$$

where the "g" functions are defined above Theorem 2.1. Furthermore, suppose that there exists $R \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, R) \subseteq \mathbb{D}$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, R)$.

Proof Using (2.9), the definition of r and the hypothesis $x_0 \in U(x^*, r)$, we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0 \|x_0 - x^*\| < L_0 r < 1.$$

It follows from the preceding inequality and the Banach lemma on invertible functions [4, 5, 28] that $F'(x_0)^{-1} \in L(S, S)$ and

$$\|F'(x_0)^{-1} F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} \leq \frac{1}{1 - L_0 r}. \tag{2.16}$$

Then, y_0 is well defined by the first substep of the method (1.2) for $n = 0$. By the first substep of method (1.2) for $n = 0$, we get the approximation

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1} F(x_0) \\ &= -F'(x_0)^{-1} F'(x^*) \int_0^1 \left[F'(x^* + \theta(x_0 - x^*)) - F'(x_0) \right] (x_0 - x^*) d\theta \end{aligned} \tag{2.17}$$

Using (2.3), (2.4), (2.10),(2.16) and (2.17), we obtain in turn that

$$\begin{aligned}\|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\|\int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]\|d\theta\|x_0 - x^*\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} = g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,\end{aligned}$$

which shows (2.13) for $n = 0$ and $y_0 \in U(x^*, r)$.

In view of (2.3), (2.5), (2.10), (2.11), (2.13), (2.16) and (2.17), we get that

$$\begin{aligned}&\|(F'(x^*)(x_0 - x^*))^{-1}\left[F(x_0) - F(x^*) - 2\alpha F(y_0) - F'(x^*)(x_0 - x^*)\right]\| \\ &\leq \frac{1}{\|x_0 - x^*\|}\|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x^*))(x_0 - x^*)d\theta\| \\ &\quad +2|\alpha|\|\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))d\theta \\ &F'(x_0)^{-1}F'(x^*)\int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta\| \\ &\leq \frac{L_0\|x_0 - x^*\|}{2} + \frac{|\alpha|ML}{1 - L_0\|x_0 - x^*\|} = g_2(\|x_0 - x^*\|) \\ &< g_2(r) < 1,\end{aligned} \tag{2.18}$$

where we used that

$$F'(x^*)^{-1}F(y_0)) = F'(x^*)^{-1}(F(y_0) - F(x^*)) = \int_0^1 F'(x^*)^{-1}F'(x^* + \theta(y_0 - x^*))(y_0 - x^*)d\theta, \tag{2.19}$$

so

$$\|F'(x^*)^{-1}F(y_0)\| \leq M\|y_0 - x^*\| \leq Mg_1(\|x_0 - x^*\|)\|x_0 - x^*\|$$

and

$$\|x^* + \theta(y_0 - x^*) - x^*\| = \theta\|y_0 - x^*\| \leq \|y_0 - x^*\| < r.$$

Hence, $(F(x_0) - 2\alpha F(y_0))^{-1} \in L(S, S)$ and

$$\|(F(x_0) - 2\alpha F(y_0))^{-1}F'(x^*)\| \leq \frac{1}{\|x_0 - x^*\|(1 - g_2(\|x_0 - x^*\|))}. \tag{2.20}$$

Hence, z_0 is well defined. Using the second substep of method (1.2) for $n = 0$, (2.3), (2.5), (2.13), (2.16), (2.19) (for $y_0 = x_0$) and (2.20) we obtain that

$$\begin{aligned}&\|z_0 - x^*\| \leq \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\ &\quad +\|F'(x^*)^{-1}F(y_0)\|\|F'(x^*)^{-1}F(x_0)\|\|F'(x_0)^{-1}F'(x^*)\|\|(F(x_0) - 2\alpha F(y_0))^{-1}F'(x^*)\| \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| + \frac{M^2\|x_0 - x^*\|\|y_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)\|x_0 - x^*\|[1 - \frac{1}{2}(L_0\|x_0 - x^*\| + \frac{2|\alpha|ML}{1 - L_0\|x_0 - x^*\|})]} \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\|\left[1 + \frac{2M^2}{2(1 - L_0\|x_0 - x^*\|) - L_0\|x_0 - x^*\|(1 - L_0\|x_0 - x^*\|) - 2\alpha ML}\right] \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,\end{aligned}$$

which shows (2.14) for $n = 0$ and $z_0 \in U(x^*, r)$.

It follows from the definition of r and (2.9) that

$$\|(F'(x^*)(x_0-x^*))^{-1}(F(x_0)-F(x^*)-F'(x^*)(x_0-x^*))\| \leq \frac{L_0}{2}\|x_0-x^*\| < \frac{L_0}{2}r < 1. \quad (2.21)$$

It follows from (2.21) that $F(x_0)^{-1} \in L(S, S)$ and

$$\|F(x_0)^{-1}F'(x^*)\| \leq \frac{1}{\|x_0 - x^*\|[1 - \frac{L_0}{2}\|x_0 - x^*\|]}. \quad (2.22)$$

Next, we shall show that

$$(F'(x_0) + 2F(y_0)F'(x_0)^2 F(x_0)^{-2}(z_0 - x_0))^{-1} \in L(S, S). \quad (2.23)$$

Using (2.3), (2.8), (2.9), (2.11), (2.13), (2.14), (2.19) and (2.22), we have in turn that

$$\begin{aligned}
&\|F'(x^*)^{-1}[F'(x_0) - F'(x^*) + 2F(y_0)F'(x_0)^2F(x_0)^{-2}(z_0 - x_0)]\| \\
&\leq \|F'(x^*)^{-1}[F'(x_0) - F'(x^*)]\| + 2\|F'(x^*)^{-1}F(y_0)\|\|F'(x^*)^{-1}F'(x_0)\|^2\|F'(x_0)^{-1}F'(x^*)\|^2\|z_0 - x_0\| \\
&\leq L_0\|x^* - x_0\| + 2\|\textstyle\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(y_0 - x^*))d\theta(y_0 - x^*)\| \\
&\|F'(x^*)^{-1}F'(x_0)\|^2\|F(x_0)^{-1}F'(x^*)\|^2(\|z_0 - x^*\| + \|x_0 - x^*\|) \\
&\leq L_0\|x^* - x_0\| + \frac{2M^3\|y_0 - x^*\|(\|z_0 - x^*\| + \|x_0 - x^*\|)}{\|x_0 - x^*\|^2(1 - \frac{L_0}{2}\|x_0 - x^*\|)^2} \\
&\leq L_0\|x^* - x_0\| + \frac{2M^3 g_1(\|x_0 - x^*\|)(1 + g_3(\|x_0 - x^*\|)\|x_0 - x^*\|^2}{\|x_0 - x^*\|^2(1 - \frac{L_0}{2}\|x_0 - x^*\|)^2} \\
&\leq L_0\|x^* - x_0\| + \frac{2M^3 L\|x_0 - x^*\|(1 + g_3(\|x_0 - x^*\|))}{2(1 - L_0\|x_0 - x^*\|)(1 - \frac{L_0}{2}\|x_0 - x^*\|)^2} \\
&= g_4(\|x_0 - x^*\|) < 1.
\end{aligned} \quad (2.24)$$

It follows from (2.24) that (2.23) holds and

$$\|(F'(x_0) + 2F(y_0)F'(x_0)^2F(x_0)^{-2}(z_0 - x_0))^{-1}F'(x^*)\| \leq \frac{1}{1 - g_4(\|x_0 - x^*\|)}. \quad (2.25)$$

Hence, x_1 is well defined. Then, using the last substep of method (1.2) for $n = 0$ and $x_1 \in U(x^*, r)$, (2.3), (2.8), (2.11), (2.14), (2.19) (for $y_0 = z_0$), and (2.25), we get that

$$\begin{aligned}
\|x_1 - x^*\| &\leq \|z_0 - x^*\| + \frac{M\|z_0 - x^*\|}{1 - g_4(\|x_0 - x^*\|)} \\
&= \left[1 + \frac{M}{1 - g_4(\|x_0 - x^*\|)}\right]\|z_0 - x^*\| \\
&\leq \left[1 + \frac{M}{1 - g_4(\|x_0 - x^*\|)}\right]g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= g_5(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned}$$

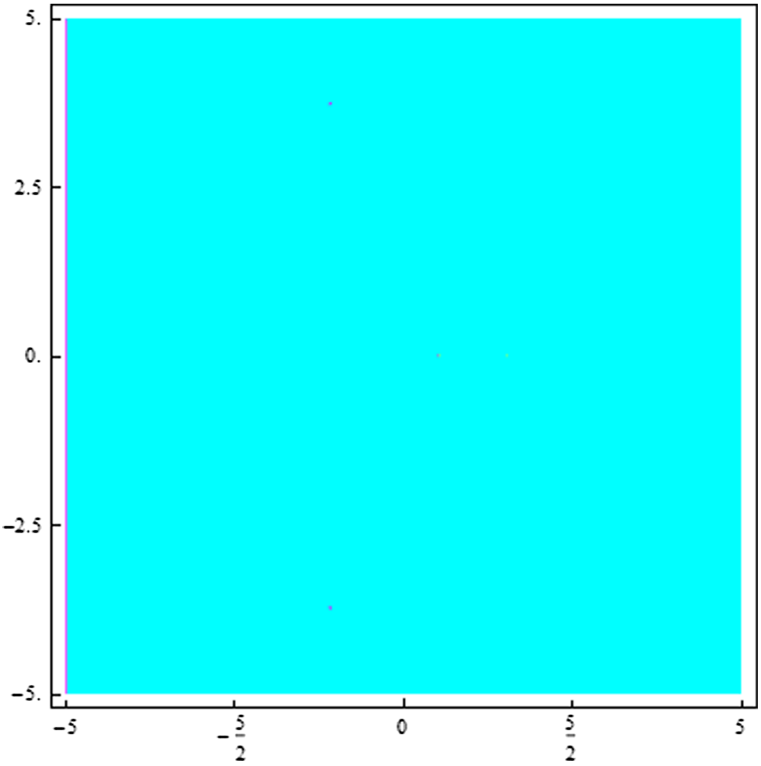

Fig. 1 Parameter space associated to the free critical point $cr_1(\alpha)$

which shows (2.15) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at (2.13)–(2.15). Using the estimates

$$\|x_{k+1} - x^*\| < \|x_k - x^*\| < r,$$

we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$.

Finally to show the uniqueness part, let $T = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$ for some $y^* \in \bar{U}(x^*, R)$ with $F(y^*) = 0$. In view of (2.9), we get in turn that

$$\begin{aligned} &\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \|\int_0^1 L_0\|y^* + \theta(x^* - y^*)\|d\theta \\ &= L_0\int_0^1 (1-\theta)\|y^* - x^*\| = \frac{L_0}{2}R < 1. \end{aligned} \tag{2.26}$$

It follows from (2.26) that $T^{-1} \in L(S, S)$. Then, from the identity $0 = F(y^*) - F(x^*) = T(y^* - x^*)$, we conclude that $x^* = y^*$. □

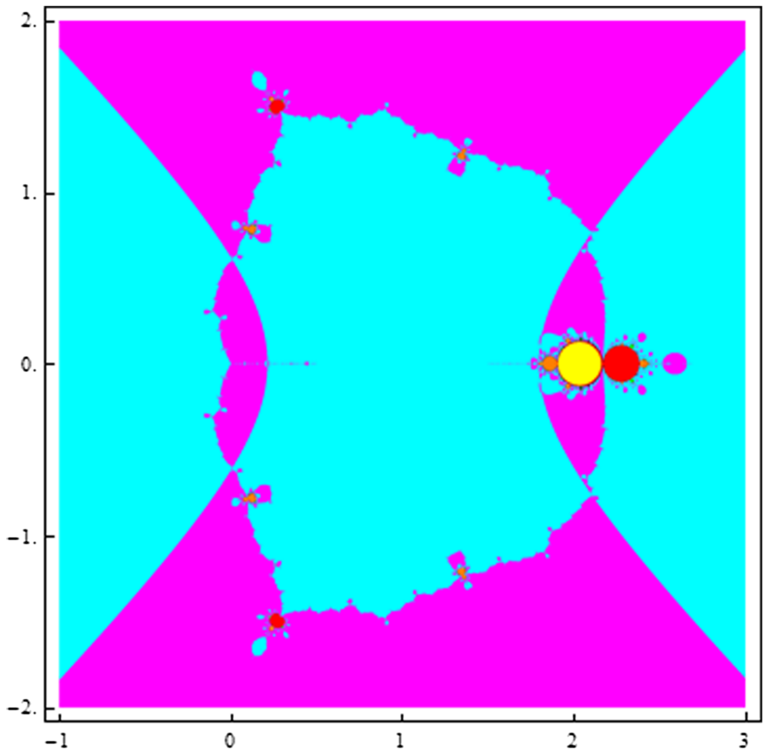

Fig. 2 Parameter space associated to the free critical point $cr_2(\alpha)$

Remark 1 1. In view of (2.10) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x)-F'(x^*))+I\| \\ &\leq 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\| \\ &\leq 1+L_0\|x_0-x^*\| \end{aligned}$$

condition (2.12) can be dropped and M can be replaced by

$$M(t)=1+L_0t.$$

Moreover, condition (2.12) can be replaced by the popular but stronger conditions

$$\|F'(x^*)^{-1}(F'(x)-F'(y))\| \leq L\|x-y\| \quad \text{for each } x, y \in D \tag{2.27}$$

or

$$\|F'(x^*)^{-1}(F'(x^*+\theta(x-x^*))-F'(x))\| \leq L(1-\theta)\|x-x^*\| \quad \text{for each } x \in D \text{ and } \theta \in [0,1]. \tag{2.28}$$

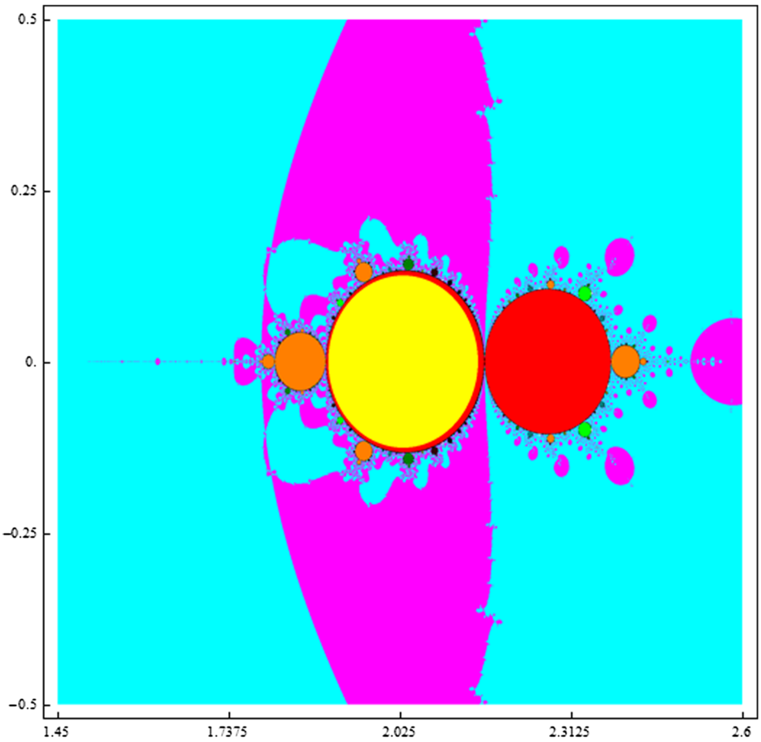

Fig. 3 Detail of the parameter space associated to the free critical point $cr_2(\alpha)$

2. The results obtained here can be used for operators F satisfying the autonomous differential equation [4, 5] of the form

$$F'(x) = P(F(x)),$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$.

3. The radius r_1 was shown in [4], [5] to be the convergence radius for Newton's method under conditions (2.11) and (2.30)

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \qquad \text{for each } n = 0, 1, 2 \ldots. \tag{2.29}$$

It follows from (2.6) and the definition of r_1 that the convergence radius r of the method (2.1) cannot be larger than the convergence radius r_1 of the second order Newton's method (2.29). As already noted in r_1 is at least as the convergence ball give by Rheinboldt [28]

$$r_R = \frac{2}{3L}. \tag{2.30}$$

In particular, for $L_0 < L$ we have that

$$r_R < r_1$$

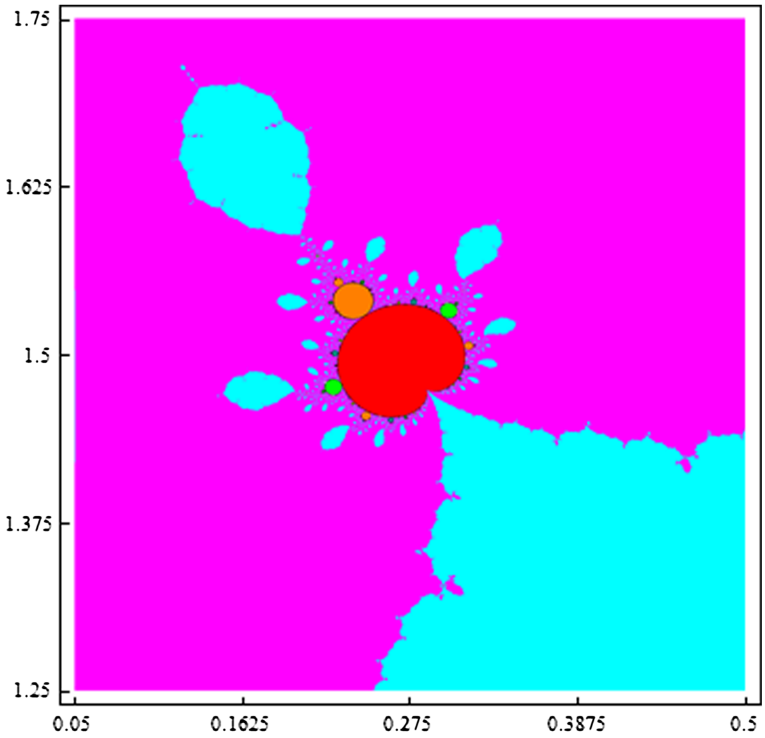

Fig. 4 Detail of the parameter space associated to the free critical point $cr_2(\alpha)$

and

$$\frac{r_R}{r_1} \to \frac{1}{3} \quad \text{as} \quad \frac{L_0}{L} \to 0.$$

That is our convergence ball r_1 is at most three times larger than Rheinboldt's. The same value for r_R given by Traub [29].

4. It is worth noticing that method (2.1) is not changing if we use the conditions of Theorem 2.1 instead of the stronger conditions given in [24]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [4, 5]

$$\xi = \sup \frac{ln \frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \qquad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC) [14]

$$\xi^* = \sup \frac{ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \qquad \text{for each } n = 0, 1, 2, \ldots$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates higher than the first Fréchet derivative.

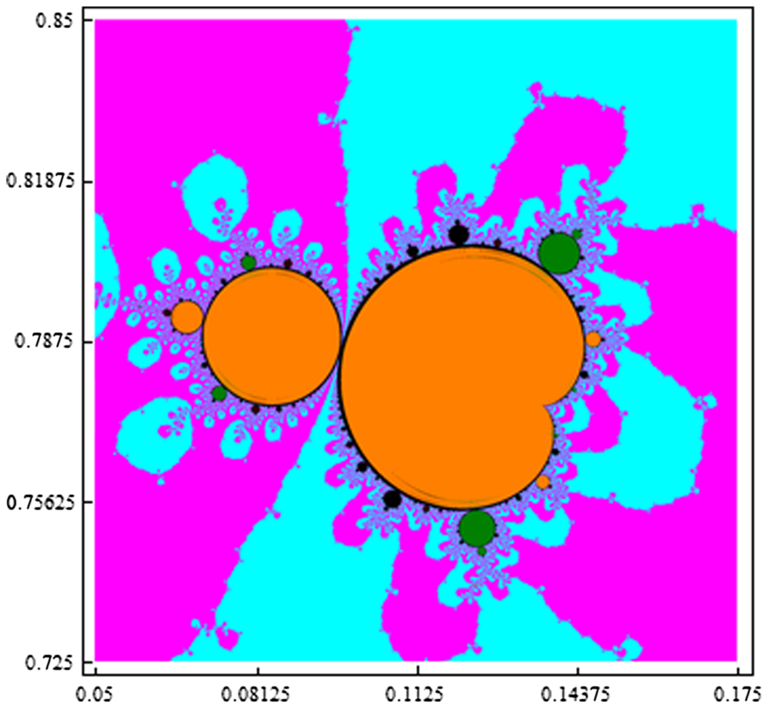

Fig. 5 Detail of the parameter space associated to the free critical point $cr_2(\alpha)$

3 Dynamical study of the method (1.2)

Firstly, some dynamical concepts of complex dynamics that are used in this work are shown. Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as

$$\left\{z_0,\ R(z_0),\ R^2(z_0),\ \ldots,\ R^n(z_0),\ \ldots\right\}.$$

A point $z_0 \in \hat{\mathbb{C}}$, is called a *fixed point* of $R(z)$ if it verifies that $R(z) = z$. Moreover, z_0 is called a *periodic point* of period $p > 1$ if it is a point such that $R^p(z_0) = z_0$ but $R^k(z_0) \neq z_0$, for each $k < p$. Moreover, a point z_0 is called *pre-periodic* if it is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic.

There exist different types of fixed points depending on its associated multiplier $|R'(z_0)|$. Taking the associated multiplier into account, a fixed point z_0 is called:

- *superattractor* if $|R'(z_0)| = 0$,
- *attractor* if $|R'(z_0)| < 1$,
- *repulsor* if $|R'(z_0)| > 1$,
- and *parabolic* if $|R'(z_0)| = 1$.

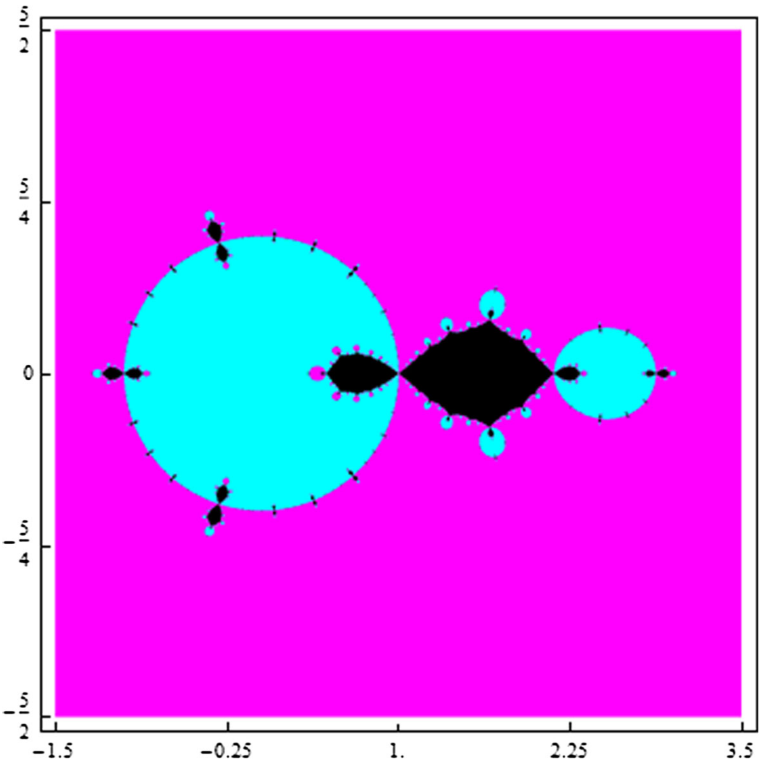

Fig. 6 Basins of attraction associated to the method with $\alpha = 2.3$

The fixed points that do not correspond to the roots of the polynomial $p(z)$ are called *strange fixed points*. On the other hand, a *critical point* z_0 is a point which satisfies that $R'(z_0) = 0$.

The basin of attraction of an attractor α is defined as

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} \ : \ R^n(z_0) \to \alpha, \ n \to \infty\}.$$

The *Fatou set* of the rational function R, $\mathcal{F}(R)$, is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{\mathbb{C}}$ is the *Julia set,* $\mathcal{J}(R)$. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

In this section we are going to study the complex dynamics of the method (1.2). By applying this operator on a generic polynomial $p(z) = (z - A)(z - B)$, and by using the Möebius map $h(z) = \frac{z-A}{z-B}$, whose properties are

$$\text{i)} \ \ h(\infty) = 1, \quad \text{ii)} \ \ h(A) = 0, \quad \text{iii)} \ \ h(B) = \infty,$$

the rational operator associated to the family of iterative schemes is finally

$$G(z, \alpha) = \frac{z^6(2 - 2\alpha + z)^2}{(1 - 2(-1 + \alpha)z)^2}. \tag{3.1}$$

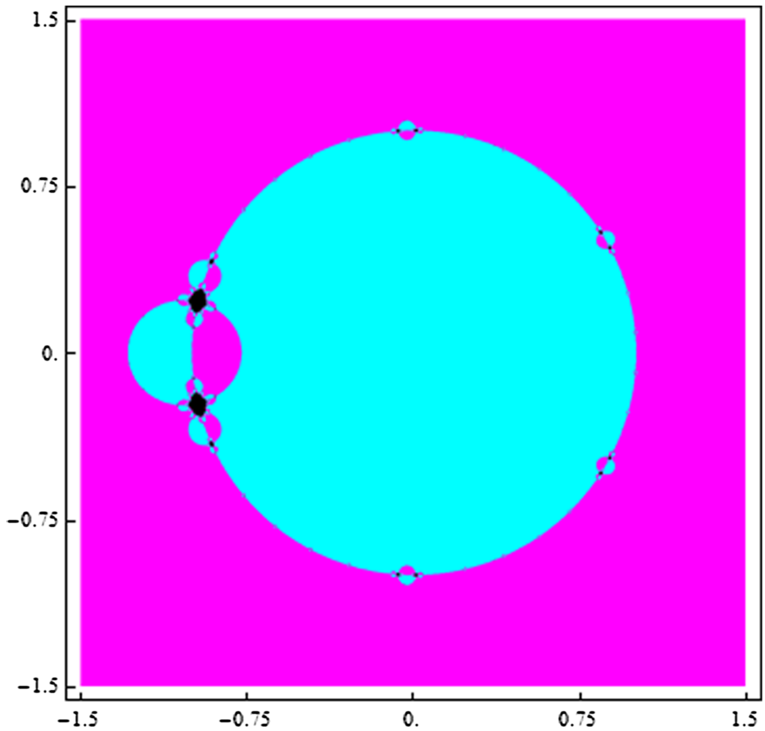

Fig. 7 Basins of attraction associated to the method with $\alpha = 0.452$

3.1 Study of the fixed points and their stability

It is clear that $z = 0$ and $z = \infty$ are fixed points of $G(z, \alpha)$ which are related to the root A and B respectively. Now, focussing the attention on the extraneous fixed points (those points which are fixed points but are not solution of the equation $f(z) = 0$. First of all, we notice that $z = 1$ is a strange fixed point, which is associated with the original convergence to infinity. Moreover, there are also other strange fixed which are the solutions of the polynomial

$$p(z) = 1+5z-4\alpha z+9z^2-12\alpha z^2+4\alpha^2 z^2+9z^3-12\alpha z^3+4\alpha^2 z^3+9z^4-12\alpha z^4+4\alpha^2 z^4+5z^5-4\alpha z^5+z^6$$

It is easy to see that the solutions of this polynomial depend on the value of the parameter α.

3.2 Study of the critical points and parameter spaces

In this section, the critical points will be calculated and the parameter spaces associated to the free critical points will be shown. It is well known that there is at least one critical point associated with each invariant Fatou component. The critical points of

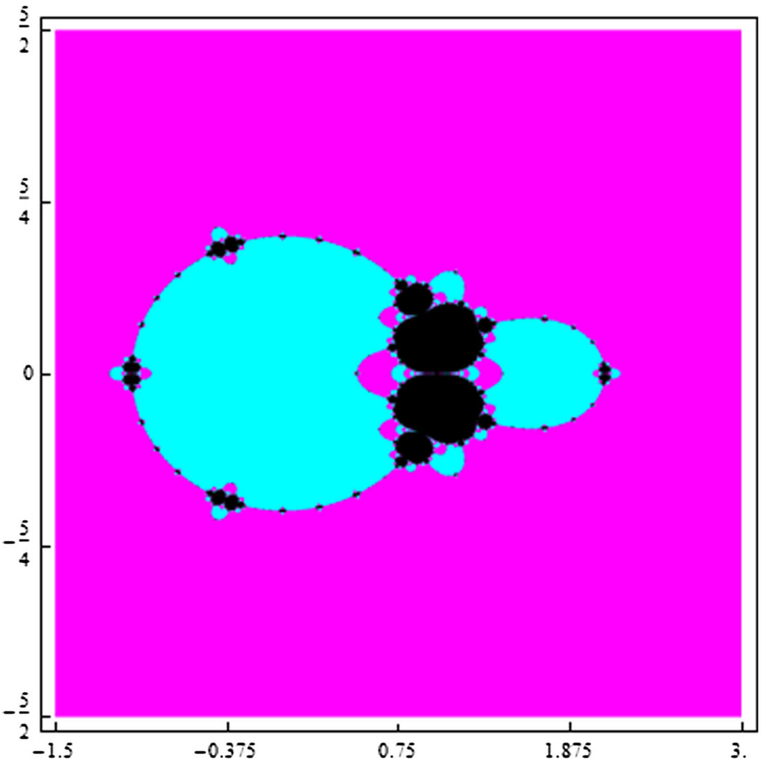

Fig. 8 Basins of attraction associated to the method with $\alpha = 1.9$

the family are the solutions of is $G'(z, \alpha) = 0$, where

$$G'(z,\alpha) = \frac{4(-2+2\alpha-z)z^5\left(3-3\alpha+6z-8\alpha z+4\alpha^2 z+3z^2-3\alpha z^2\right)}{(-1-2z+2\alpha z)^3}.$$

By solving this equation, it is clear that $z = 0$ and $z = \infty$ are critical points, which are related to the roots of the polynomial $p(z)$ and they have associated their own Fatou component. Moreover, there exist critical points no related to the roots, these points are called free critical points. Their expressions are:

$$\begin{aligned} cr_1(\alpha) &= 2(-1+\alpha) \\ cr_2(\alpha) &= \frac{3-4\alpha+2\alpha^2-\sqrt{-6\alpha+19\alpha^2-16\alpha^3+4\alpha^4}}{3(-1+\alpha)} \\ cr_3(\alpha) &= \frac{3-4\alpha+2\alpha^2+\sqrt{-6\alpha+19\alpha^2-16\alpha^3+4\alpha^4}}{3(-1+\alpha)} \end{aligned}$$

The relations between the free critical points are described in the following result.

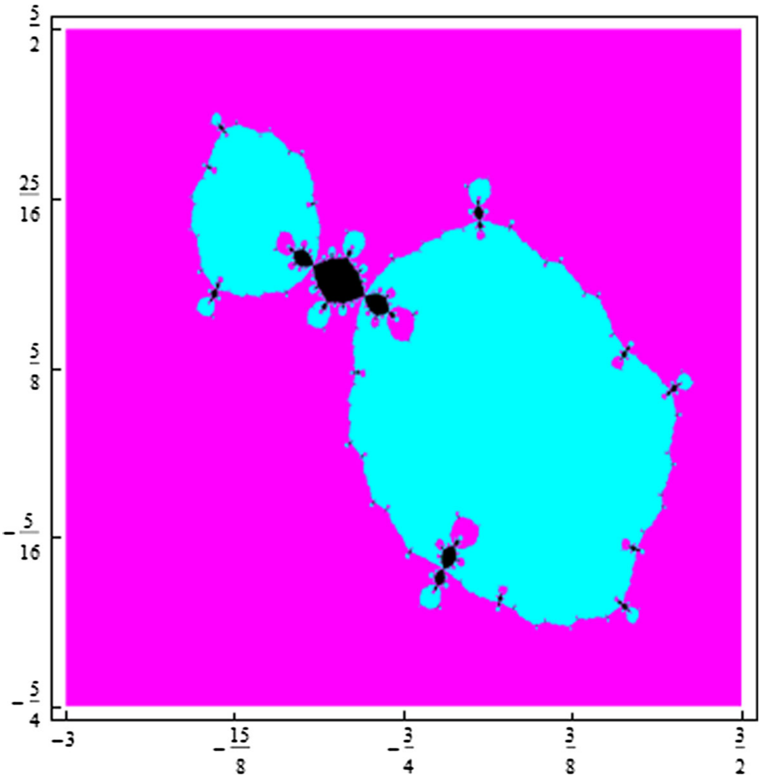

Fig. 9 Basins of attraction associated to the method with $\alpha = 0.08125 + 0.7875i$

Lemma 1 *a)* *If* $\alpha = \frac{1}{2}$

(i) $cr_1(\alpha) = cr_2(\alpha) = cr_3(\alpha) = -1.$

b) *If* $\alpha = \frac{3}{2}$

(i) $cr_1(\alpha) = cr_2(\alpha) = cr_3(\alpha) = 1.$

c) *If* $\alpha = 0$

(i) $cr_1(\alpha) = -2$ *and* $cr_2(\alpha) = cr_3(\alpha) = -1.$

d) *If* $\alpha = 2$

(i) $cr_1(\alpha) = 2$ *and* $cr_2(\alpha) = cr_3(\alpha) = 1.$

Moreover, it is clear that for every value of α $cr2(\alpha) = \frac{1}{cr_3(\alpha)}$

So, there are only three independent free critical points, without loss of generality, we consider in this paper the free critical point $cr_2(\alpha)$. In order to find the best

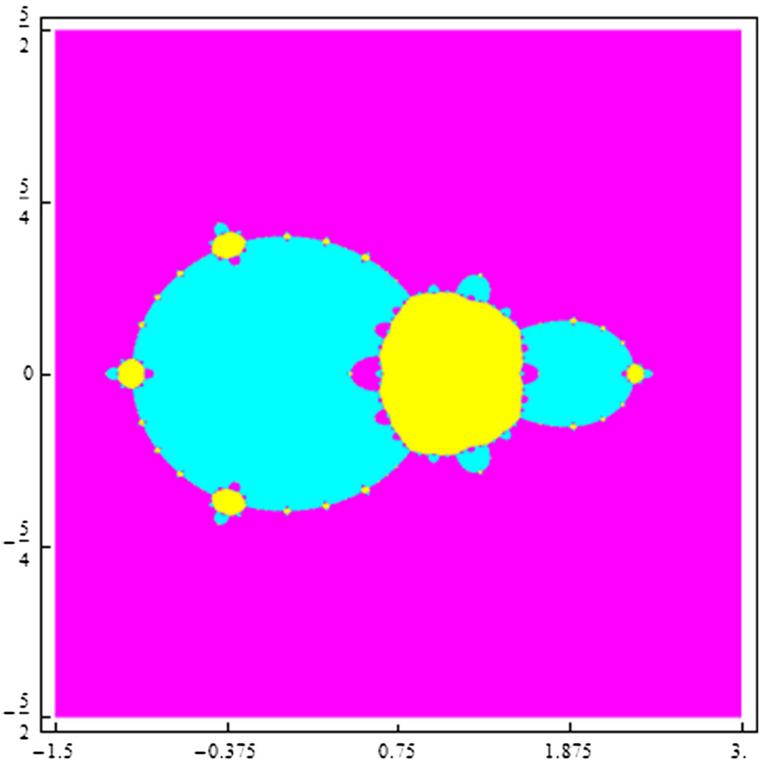

Fig. 10 Basins of attraction associated to the method with $\alpha = 2$

members of the family in terms of stability, the parameter space corresponding to this independent free critical point will be shown.

The study of the orbits of the critical points gives rise about the dynamical behavior of an iterative method. In concrete, to determinate if there exists any attracting strange fixed point or periodic orbit, the following question must be answered: For which values of the parameters, the orbits of the free critical points are attracting periodic orbits? In order to answer this question we are going to draw the parameter spaces. When the critical point is used as an initial estimation, for each value of the parameter, the color of the point tell us about the place it has converged to: to a fixed point, to an attracting periodic orbit or even the infinity.

In Fig. 1, the parameter space associated to $cr_1(\alpha)$ is shown and in Figs. 2, 3, 4 5 the parameter spaces associated to $cr_2(\alpha)$ are shown. A point is painted in cyan if the iteration of the method starting in $z_0 = cr_1(\alpha)$ converges to the fixed point 0 (related to root A), in magenta if it converges to ∞ (related to root B) and in yellow if the iteration converges to 1 (related to ∞). Moreover, it appears in red the convergence, after a maximum of 2000 iterations and with a tolerance of 10^{-6}, to any of the strange fixed points, in orange the convergence to 2-cycles, in light green the convergence to

3-cycles, in dark red to 4-cycles, in dark blue to 5-cycles, in dark green to 6-cycles, dark yellow to 7-cycles, and in white the convergence to 8-cycles. The regions in black correspond to zones of convergence to other cycles. As a consequence, every point of the plane which is neither cyan nor magenta is not a good choice of α in terms of numerical behavior.

Once the values of the parameters where anomalies appear have been detected, the next step consist on finding them in the dynamical planes. In these dynamical planes the convergence to 0 appear in magenta, in cyan it appears the convergence to ∞ and in black the zones with no convergence to the roots.

Then, focussing the attention in the region shown in Fig. 2 it is evident that there exist members of the family with complicated behavior. In Fig. 6, the dynamical planes of a member of the family with regions of convergence to any of the strange fixed points is shown

In Figs. 7, 8 and 9 dynamical planes of members of the family with regions of convergence to an attracting 2-cycle is shown.

On the other hand, in Fig. 10, a dynamical planes of a member of the family with regions of convergence to $z = 1$, related to ∞ is shown.

Other special cases are shown in Figs. 11, 12 and 13.

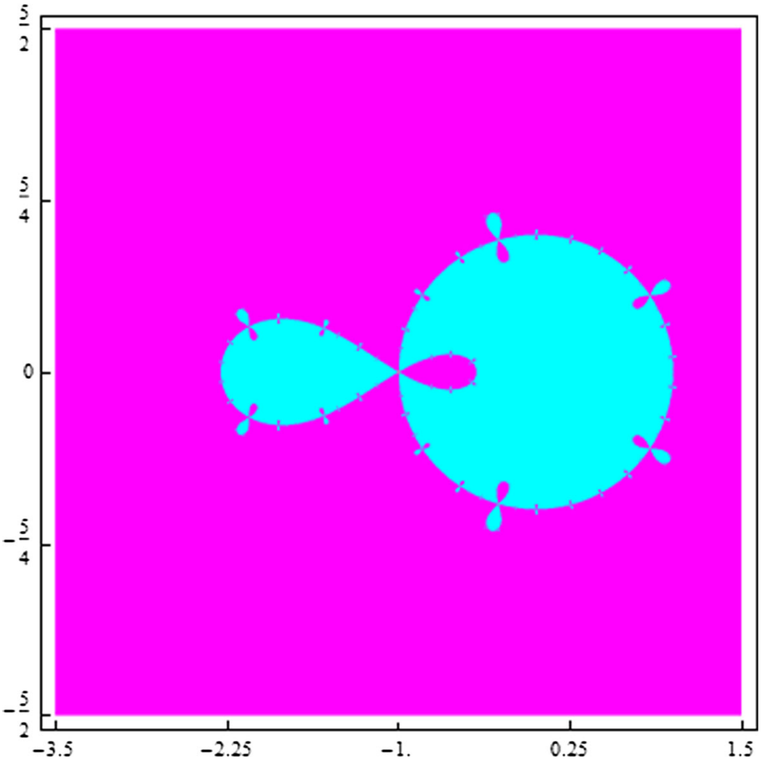

Fig. 11 Basins of attraction associated to the method with $\alpha = 0$

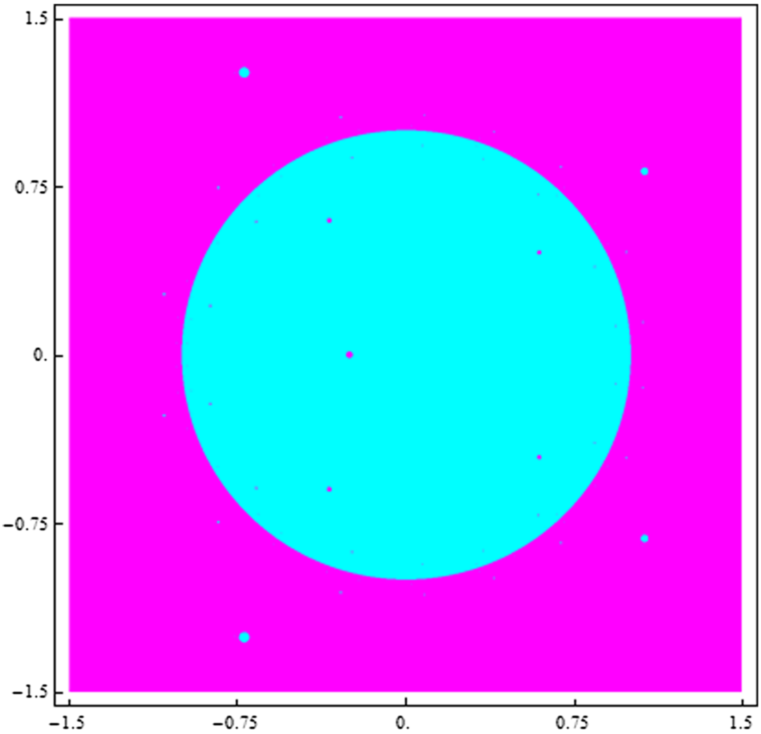

Fig. 12 Basins of attraction associated to the method with $\alpha = -1$

4 Numerical example and applications

We present numerical examples in this section.

Example 4.1 Let $S = \mathbb{R}$, $D = [-1, 1]$, $x^* = 0$ and define function F on D by

$$F(x) = sin(x). \tag{4.1}$$

Then, choosing

$$\alpha = 0.75$$

we get

$$L_0 = 1,$$
$$L = 1$$

and

$$M = 1.$$

Then, by the definition of the "g" functions we obtain

$r_1 = 0.666666\ldots \quad , r_3 = 0.121895\ldots \quad , r_4 = 0.134343\ldots \quad , r_5 = 0.107424\ldots$

and as a consequence

$$r = r_5 = 0.107424\ldots$$

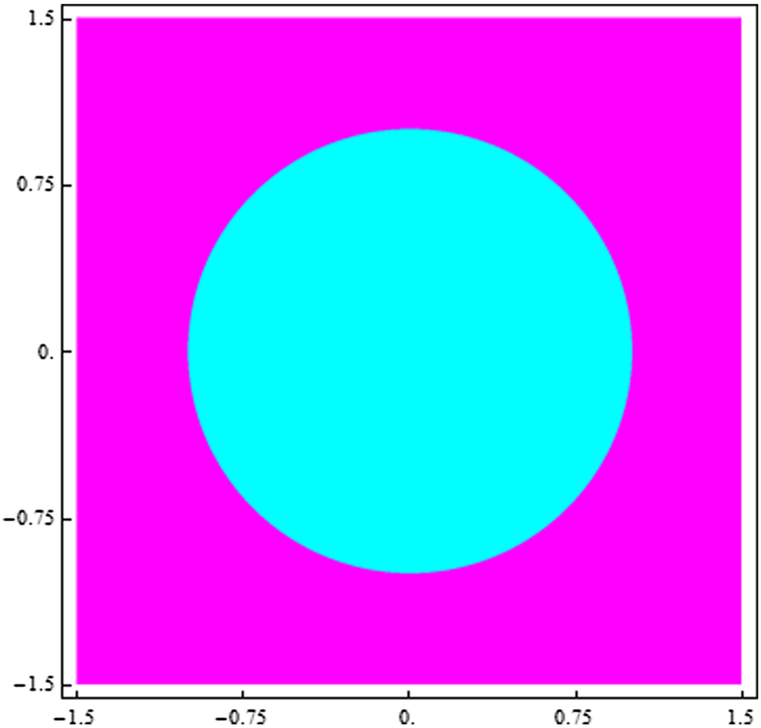

Fig. 13 Basins of attraction associated to the method with $\alpha = 1$

So we can ensure the convergence of the method (1.2) with $\alpha = 0.75$ by Theorem 1.

Example 4.2 Let $S = \mathbb{R}$, $D = [-1, 1]$, $x^* = 0$ and define function F on D by

$$F(x) = e^x - 1. \tag{4.2}$$

Then, choosing

$$\alpha = 0.125$$

we get

$$L_0 = e - 1,$$
$$L = e$$

and

$$M = e.$$

Then, by the definition of the "g" functions we obtain

$r_1 = 0.324947\ldots \quad, r_3 = 0.005966\ldots \quad, r_4 = 0.007504\ldots \quad, r_5 = 0.002110\ldots$

and as a consequence

$$r = r_5 = 0.002110\ldots$$

So we can ensure the convergence of the method (1.2) with $\alpha = 0.125$ by Theorem 1.

Example 4.3 Returning back to the motivation example at the introduction on this paper, we have $L = L_0 = 146.6629073\ldots$ and $M = 101.5578008$. Choosing $\alpha = 0.00006$ and by the definition of the "g" functions we obtain

$$r_1 = 0.004545\ldots \quad , r_3 = 1.40519 \times 10^{-7} \quad , r_4 = 6.23297 \times 10^{-9} \quad , r_5 = 1.96801 \times 10^{-9}$$

and as a consequence

$$r = r_5 = 1.96801 \times 10^{-9}.$$

So we can ensure the convergence of the method (1.2) with $\alpha = 0.00006$ by Theorem 1, while with the conditions in [24] as already noted in the Introduction of this study the convergence is not guaranteed.

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Conflict of interests The authors declare that they have no conflict of interest.

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