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A compact finite difference method for a class of time fractional convection-diffusion-wave equations with variable coefficients

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Abstract This paper is concerned with numerical methods for a class of time fractional convection-diffusion-wave equations. The convection coefficient in the equation may be spatially variable and the time fractional derivative is in the Caputo sense with the order α ($1 < \alpha < 2$). The class of the equations includes time fractional convection-diffusion-wave/diffusion-wave equations with or without damping as its special cases. In order to overcome the difficulty caused by variable coefficient problems, we first transform the original equation into a special and equivalent form, which is then discretized by a fourth-order compact finite difference method for the spatial derivative and by the L_1 approximation coupled with the Crank-Nicolson technique for the time derivative. The local truncation error and the solvability of the method are discussed in detail. A rigorous theoretical analysis of the stability and convergence is carried out using a discrete energy analysis method. The optimal error estimates in the discrete H^1, L^2 and L^∞ norms are obtained under the mild condition that the time step is smaller than a positive constant, which depends solely upon physical parameters involved (this condition is no longer required for the special case of constant coefficients). Applications using three model problems give numerical results that demonstrate the effectiveness and the accuracy of the proposed method.

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1 Introduction

Fractional differential equations have been successfully used in the modeling of many different processes and systems. The monograph [43] presents a detailed description for different applications of derivatives and integrals of fractional order in physics, chemistry, engineering, astrophysics, and so on. Some applications of fractional differential equations in classical mechanics, quantum mechanics, nuclear physics, hadron spectroscopy, and quantum field theory can be found in [17, 18, 36]. For other interesting models related to fractional differential equations we refer the reader to [1, 3, 15, 29, 31, 40, 47]. Among different applications, models for anomalous transport processes in the form of time and/or space fractional convection-diffusion-wave equations enjoyed a particular attention and have been considered by a number of researchers (see [13, 14, 21, 30, 32]).

In general cases, numerical methods have become important in obtaining the approximate solutions of fractional differential equations [2, 4–6, 16, 22, 24, 25, 48, 51, 52]. Various numerical methods have been developed for fractional convection-diffusion equations such as explicit and implicit finite difference methods [7, 9, 26, 27, 41, 49], compact finite difference methods [11, 34], finite element methods [53], Sinc-Legendre collocation methods [38], and radial basis function approximation methods [28, 44]. Significant progress has also already been made in numerical methods for time fractional diffusion-wave equations. Sun and Wu [42] proposed and analyzed a finite difference method for a one-dimensional time fractional diffusion-wave equation. Du et al. [12] improved the spatial accuracy of the method in [42] by introducing a fourth-order compact finite difference discretization in the spatial direction. Based on an equivalent partial integro-differential equation, Huang et al. [20] constructed two finite difference methods for solving a similar time fractional diffusion-wave equation to that in [42]. Li et al. [23] developed a numerical scheme combining a finite difference method in the temporal direction and a finite element method in the spatial direction for a one-dimensional time-space fractional diffusion-wave equation. Hu and Zhang [19] presented a finite difference method for a fourth-order time fractional diffusion-wave equation.

There is relatively little discussion on numerical methods for time fractional convection-diffusion-wave equations. The most recent work on this subject was given in [27], where an implicit finite difference method with the first-order spatial accuracy was established, and the discussions were limited to the case of constant coefficients. In practical computations, a proper high-order numerical method is required for the more accurate numerical simulation. On the other hand, the coefficients in the equations are usually spatially and/or temporally variable. Thus, we were motivated in this paper to propose and analyze a high-order compact finite difference method for a class of time fractional convection-diffusion-wave equations with variable convection coefficients. The class of equations under consideration is given by

$$\beta_2 \frac{\partial^\alpha v}{\partial t^\alpha}(x,t) + \beta_1 \frac{\partial v}{\partial t}(x,t) = d\frac{\partial^2 v}{\partial x^2}(x,t) - p(x)\frac{\partial v}{\partial x}(x,t) + f(x,t),$$
$$(x,t) \in (0,L) \times (0,T] \tag{1.1}$$

with the boundary conditions

$$v(0,t) = \phi_0(t), \quad v(L,t) = \phi_L(t), \qquad t \in (0,T] \tag{1.2}$$

and the initial conditions

$$v(x,0) = \varphi(x), \quad \frac{\partial v}{\partial t}(x,0) = \psi(x), \qquad x \in [0,L], \tag{1.3}$$

where β_1, β_2 and d are known parameters with $\beta_1 \geq 0$, $\beta_2 \geq 0$, $\beta_1 + \beta_2 \neq 0$ and $d > 0$. The fractional derivative $\frac{\partial^\alpha v}{\partial t^\alpha}$ in (1.1) is given in the Caputo sense:

$$\frac{\partial^\alpha v}{\partial t^\alpha}(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\partial^2 v}{\partial s^2}(x,s)(t-s)^{1-\alpha}\mathrm{d}s, \qquad 1 < \alpha < 2. \tag{1.4}$$

In terms of convection-diffusion problems, the first two terms on the right-hand side of (1.1) describe "diffusion" and "convection", respectively. In particular, d is referred to as the diffusivity or diffusion coefficient and $p(x)$ is called the average convective velocity or convection coefficient. Compared to the commonly discussed time fractional convection-diffusion-wave equation, the equation (1.1) is more physically flexible due to the new term $\frac{\partial v}{\partial t}(x,t)$ and the additional parameter β_1 (see [50]). Since the term $\frac{\partial v}{\partial t}(x,t)$ describes a damping effect, the equation (1.1) with $p(x) \not\equiv 0$ and $\beta_2 \neq 0$ is called the time fractional convection-diffusion-wave equation with damping ($\beta_1 \neq 0$) or without damping ($\beta_1 = 0$) (see [27, 50] and the references therein). Similarly, if $p(x) \equiv 0$ and $\beta_2 \neq 0$, the equation (1.1) is referred to as the time fractional diffusion-wave equation with damping ($\beta_1 \neq 0$) or without damping ($\beta_1 = 0$) (see [8] and the references therein). In the equation (1.1), we allow $\beta_2 = 0$. This implies that it may be a classical convection-diffusion equation of integer order with a variable convection coefficient. In the following discussions, we also include this equation as special case of the equation (1.1). In this case, the second initial condition in (1.3) will be removed.

When the coefficient $p(x) \equiv p$ is independent of the variable x, some numerical treatments to the equation (1.1) with the boundary and initial conditions (1.2) and (1.3) were given in [8, 27, 46]. Specifically, the works in [8, 46] give two different implicit finite difference methods for the special case of $p = 0$. For any constant p, a stable implicit numerical method by the basic finite difference discretization was presented in [27]. But the accuracy of the method proposed there is only of order $\mathcal{O}(\tau + h)$ for $\beta_1 \neq 0$, where τ is the time step and h is the spatial step. In this paper, we propose a high-order compact finite difference method for the problem (1.1)–(1.3), where the coefficient $p(x)$ may be spatially variable. In our method, we use a fourth-order compact finite difference approximation for the spatial discretization and apply the L_1 approximation [10, 35, 42] coupled with the Crank-Nicolson technique for the temporal discretization. The resulting finite difference scheme from this new method has the local truncation error $\mathcal{O}(\beta_2\tau^{3-\alpha} + \beta_1\tau^2 + h^4)$. Moreover, it is stable and convergent with the same order as the truncation error under the mild condition that the time step τ is smaller than a positive constant, which depends solely upon physical parameters involved. This condition is no longer required if the coefficient $p(x)$ is reduced to a constant.

In general, a direct discretization of the equation (1.1) by a high-order compact difference is much more complicated due to the dependence of $p(x)$ on the spatial variable x. One inconvenience is that it is often not clear how to analyze theoretically the resulting scheme. In order to overcome this difficulty, we here use an indirect approach by transforming (1.1) into a special and equivalent form, which is then discretized by a high-order compact finite difference method. The main advantage behind this approach is that it yields a very simple and effective high-order scheme for (1.1), especially when the equation is not convection-dominated. More importantly, it is very convenient for us to use a discrete energy analysis method to carry out the stability and convergence analysis of the derived scheme for the present variable coefficient problem.

The outline of the paper is as follows. In Section 2, we transform the equation (1.1) into a special and equivalent form, and then discretize the equivalent form into a compact finite difference system. The local truncation error and the solvability of the resulting finite difference scheme are discussed in Section 3. In Section 4, we use a discrete energy analysis method to prove the stability and convergence of the method, and provide the optimal error estimates (i.e., the error estimate with the same order as the truncation error) of the numerical solution in the discrete H^1, L^2 and L^∞ norms. In Section 5, we give some applications to three model problems. We use numerical results to confirm the theoretical analysis and to illustrate the effectiveness of the proposed method. The final section contains some concluding remarks.

2 Compact finite difference method

Assume that the coefficient $p(x)$ is differentiable in $[0, L]$. Let

$$k(x) = \exp\left(-\frac{1}{2d}\int_0^x p(s)\mathrm{d}s\right), \qquad v(x,t) = u(x,t)/k(x).$$

We transform the problem (1.1)–(1.3) into

$$\begin{cases} \beta_2\dfrac{\partial^\alpha u}{\partial t^\alpha}(x,t) + \beta_1\dfrac{\partial u}{\partial t}(x,t) = d\dfrac{\partial^2 u}{\partial x^2}(x,t) + q(x)u(x,t) + g(x,t), & (x,t) \in (0,L)\times(0,T], \\ u(0,t) = \phi_0^*(t), \quad u(L,t) = \phi_L^*(t), & t \in (0,T], \\ u(x,0) = \varphi^*(x), \quad \dfrac{\partial u}{\partial t}(x,0) = \psi^*(x), & x \in [0,L], \end{cases} \tag{2.1}$$

where

$$
\begin{aligned}
&q(x)=\frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}(x)-\frac{p^2(x)}{2d}\right),\\
&g(x,t)=k(x)f(x,t), \qquad \phi_0^*(t)=\phi_0(t),\\
&\phi_L^*(t)=k(L)\phi_L(t), \qquad \varphi^*(x)=k(x)\varphi(x), \qquad \psi^*(x)=k(x)\psi(x).
\end{aligned} \tag{2.2}
$$

It is clear that $v(x,t)$ is a solution of (1.1)–(1.3) if and only if $u(x,t)=k(x)v(x,t)$ is a solution of (2.1).

Our compact finite difference method for the problem (1.1)–(1.3) is based on the above equivalent form (2.1). For a positive integer N, we let $\tau=T/N$ be the time step. Denote $t_n=n\tau(0\le n\le N)$ and $t_{n-\frac{1}{2}}=(n-\frac{1}{2})\tau(1\le n\le N)$. Given a grid function $w=\{w^n\mid 0\le n\le N\}$, we define

$$
w^{n-\frac{1}{2}}=\frac{1}{2}\left(w^n+w^{n-1}\right), \qquad \delta_t w^{n-\frac{1}{2}}=\frac{1}{\tau}\left(w^n-w^{n-1}\right).
$$

Let $h=L/M$ be the spatial step, where M is a positive integer. We partition $[0,L]$ into a mesh by the mesh points $x_i=ih(0\le i\le M)$. Denote $x_{i-\frac{1}{2}}=(i-\frac{1}{2})h(1\le i\le M)$. For any grid function $w=\{w_i\mid 0\le i\le M\}$, we define spatial difference operators

$$
\delta_x w_{i-\frac{1}{2}}=\frac{1}{h}\left(w_i-w_{i-1}\right), \quad \delta_x^2 w_i=\frac{1}{h^2}\left(w_{i+1}-2w_i+w_{i-1}\right), \quad \mathcal{H}_x w_i=\left(I+\frac{h^2}{12}\delta_x^2\right)w_i,
$$

where I denotes the identical operator. Let $u(x,t)$ be the solution of (2.1), and define the grid functions

$$
\begin{aligned}
&U_i^n=u(x_i,t_n),\ W_i^n=\frac{\partial u}{\partial t}(x_i,t_n),\ Z_i^n=\frac{\partial^2 u}{\partial x^2}(x_i,t_n),\ q_i=q(x_i), \quad g_i^n=g(x_i,t_n),\\
&\phi_0^{*,n}=\phi_0^*(t_n),\ \ \phi_L^{*,n}=\phi_L^*(t_n), \quad \varphi_i^*=\varphi^*(x_i), \quad \psi_i^*=\psi^*(x_i).
\end{aligned}
$$

We now discretize (2.1) into a compact finite difference system. Let $\mu=\tau^{\alpha-1}\Gamma(3-\alpha)$ and let

$$
a_k=(2-\alpha)\int_k^{k+1}t^{1-\alpha}\mathrm{d}t=(k+1)^{2-\alpha}-k^{2-\alpha}, \qquad k=0,1,\ldots.
$$

Using the L_1 approximation of $\frac{\partial^\alpha u}{\partial t^\alpha}(x,t)$ at (x_i,t_n) (see [10, 35, 42]), we have

$$
\frac{\partial^\alpha u}{\partial t^\alpha}(x_i,t_n)=\frac{1}{\mu}\left(W_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})W_i^k-a_{n-1}W_i^0\right)-(R_t^\alpha)_i^n, \tag{2.3}
$$

where the truncation error $(R_t^\alpha)_i^n$ satisfies

$$
\left|(R_t^\alpha)_i^n\right|\le\frac{\tau^{3-\alpha}}{\Gamma(3-\alpha)}\left(\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-1-2^{1-\alpha}\right)\max_{t\in[0,t_n]}\left|\frac{\partial^3 u}{\partial t^3}(x_i,t)\right|. \tag{2.4}
$$

Substituting (2.3) into the first equation of (2.1), we obtain

$$\begin{aligned}\frac{\beta_2}{\mu}\left(W_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})W_i^k-a_{n-1}W_i^0\right)+\beta_1 W_i^n\\ =dZ_i^n+q_iU_i^n+g_i^n+\beta_2(R_t^\alpha)_i^n,\qquad 1\le i\le M-1,\ \ 1\le n\le N.\end{aligned}\tag{2.5}$$

Similarly, on the time level $n-1$, we have

$$\begin{aligned}\frac{\beta_2}{\mu}\left(W_i^{n-1}-\sum_{k=1}^{n-2}(a_{n-k-2}-a_{n-k-1})W_i^k-a_{n-2}W_i^0\right)+\beta_1 W_i^{n-1}\\ =dZ_i^{n-1}+q_iU_i^{n-1}+g_i^{n-1}+\beta_2(R_t^\alpha)_i^{n-1},\qquad 1\le i\le M-1,\ \ 2\le n\le N.\end{aligned}\tag{2.6}$$

Since

$$-\sum_{k=1}^{n-2}(a_{n-k-2}-a_{n-k-1})W_i^k-a_{n-2}W_i^0=-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})W_i^{k-1}-a_{n-1}W_i^0,$$

the equation (2.6) can be reformulated as

$$\begin{aligned}\frac{\beta_2}{\mu}\left(W_i^{n-1}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})W_i^{k-1}-a_{n-1}W_i^0\right)+\beta_1 W_i^{n-1}\\ =dZ_i^{n-1}+q_iU_i^{n-1}+g_i^{n-1}+\beta_2(R_t^\alpha)_i^{n-1},\qquad 1\le i\le M-1,\ \ 2\le n\le N.\end{aligned}\tag{2.7}$$

Letting $t=0$ in the first equation of (2.1), it holds that $\beta_1 W_i^0=dZ_i^0+q_iU_i^0+g_i^0$ which implies that (2.7) is true also for $n=1$ with $(R_t^\alpha)_i^0=0$. Taking the arithmetic mean of (2.5) and (2.7), we conclude that

$$\begin{aligned}\frac{\beta_2}{\mu}\left(W_i^{n-\frac12}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})W_i^{k-\frac12}-a_{n-1}W_i^0\right)+\beta_1 W_i^{n-\frac12}\\ =dZ_i^{n-\frac12}+q_iU_i^{n-\frac12}+g_i^{n-\frac12}+\beta_2(R_t^\alpha)_i^{n-\frac12},\qquad 1\le i\le M-1,\ \ 1\le n\le N.\end{aligned}\tag{2.8}$$

An application of the Crank-Nicolson technique (see, e.g., [51]) gives

$$W_i^{n-\frac12}=\delta_t U_i^{n-\frac12}+(R_t^c)_i^{n-\frac12},\qquad 1\le i\le M-1,\ \ 1\le n\le N,\tag{2.9}$$

where

$$(R_t^c)_i^{n-\frac12}=\frac{\tau^2}{16}\int_0^1\left(\frac{\partial^3 u}{\partial t^3}\left(x_i,t_{n-\frac12}+\frac{s\tau}{2}\right)+\frac{\partial^3 u}{\partial t^3}\left(x_i,t_{n-\frac12}-\frac{s\tau}{2}\right)\right)(1-s^2)\mathrm{d}s.\tag{2.10}$$

This implies that

$$\begin{aligned}\frac{\beta_2}{\mu}\left(\delta_t U_i^{n-\frac12}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\delta_t U_i^{k-\frac12}-a_{n-1}W_i^0\right)+\beta_1\delta_t U_i^{n-\frac12}\\ =dZ_i^{n-\frac12}+q_iU_i^{n-\frac12}+g_i^{n-\frac12}+(R_t)_i^{n-\frac12},\qquad 1\le i\le M-1,\ \ 1\le n\le N,\end{aligned}\tag{2.11}$$

where

$$(R_t)_i^{n-\frac12}=\beta_2(R_t^\alpha)_i^{n-\frac12}-\frac{\beta_2}{\mu}\left((R_t^c)_i^{n-\frac12}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(R_t^c)_i^{k-\frac12}\right)-\beta_1(R_t^c)_i^{n-\frac12}.\tag{2.12}$$

For the second-order spatial derivative Z_i^n, we adopt the following fourth-order compact approximation (see, e.g., [51]):

$$\mathcal{H}_x Z_i^n = \delta_x^2 U_i^n + (R_x)_i^n, \tag{2.13}$$

where

$$(R_x)_i^n = \frac{h^4}{360}\int_0^1 \left(\frac{\partial^6 u}{\partial x^6}(x_i - sh, t_n) + \frac{\partial^6 u}{\partial x^6}(x_i + sh, t_n)\right)\zeta(s)\mathrm{d}s \tag{2.14}$$

with $\zeta(s) = 5(1-s)^3 - 3(1-s)^5$. Multiplying (2.11) by μ and then applying $\mathcal{H}_x$ to both sides yields

$$\begin{aligned} &\beta_2\mathcal{H}_x\left(\delta_t U_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\delta_t U_i^{k-\frac{1}{2}} - a_{n-1}W_i^0\right) + \mu\beta_1\mathcal{H}_x\delta_t U_i^{n-\frac{1}{2}} \\ &= \mu\left(d\delta_x^2 U_i^{n-\frac{1}{2}} + \mathcal{H}_x\left(q_i U_i^{n-\frac{1}{2}}\right) + \mathcal{H}_x g_i^{n-\frac{1}{2}} + (R_{xt})_i^{n-\frac{1}{2}}\right), \quad 1 \le i \le M-1, \quad 1 \le n \le N, \end{aligned} \tag{2.15}$$

where

$$(R_{xt})_i^{n-\frac{1}{2}} = \mathcal{H}_x(R_t)_i^{n-\frac{1}{2}} + d(R_x)_i^{n-\frac{1}{2}}. \tag{2.16}$$

Omitting the small term $\mu(R_{xt})_i^{n-\frac{1}{2}}$ and replacing W_i^0 by ψ_i^* in (2.15), we obtain the following compact finite difference scheme:

$$\begin{cases} \beta_2\mathcal{H}_x\left(\delta_t u_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\delta_t u_i^{k-\frac{1}{2}} - a_{n-1}\psi_i^*\right) + \mu\beta_1\mathcal{H}_x\delta_t u_i^{n-\frac{1}{2}} \\ \quad = \mu\left(d\delta_x^2 u_i^{n-\frac{1}{2}} + \mathcal{H}_x\left(q_i u_i^{n-\frac{1}{2}}\right) + \mathcal{H}_x g_i^{n-\frac{1}{2}}\right), \qquad 1 \le i \le M-1, \quad 1 \le n \le N, \\ u_0^n = \phi_0^{*,n}, \quad u_M^n = \phi_L^{*,n}, \qquad 1 \le n \le N, \\ u_i^0 = \varphi_i^*, \qquad 0 \le i \le M, \end{cases} \tag{2.17}$$

where u_i^n denotes the finite difference approximation to U_i^n.

3 Truncation error and solvability

We now estimate the truncation error $(R_{xt})_i^{n-\frac{1}{2}}$. Assume that the solution $u(x,t)$ of the problem (2.1) is in $\mathcal{C}^{(6,3)}((0,L)\times[0,T])$. It follows from (2.10) that

$$\left|(R_t^c)_i^{n-\frac{1}{2}}\right| \le \frac{\tau^2}{12}\max_{t\in[0,T]}\left|\frac{\partial^3 u}{\partial t^3}(x_i,t)\right|, \qquad 1 \le i \le M-1, \quad 1 \le n \le N. \tag{3.1}$$

We thus have from (2.4) and (2.12) that

$$\left|(R_t)_i^{n-\frac{1}{2}}\right| \le \left(\frac{\beta_2 C_\alpha}{\Gamma(3-\alpha)}\tau^{3-\alpha} + \frac{\beta_1}{12}\tau^2\right)\max_{t\in[0,T]}\left|\frac{\partial^3 u}{\partial t^3}(x_i,t)\right|, \quad 1 \le i \le M-1, \quad 1 \le n \le N, \tag{3.2}$$

where $C_\alpha = \frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - 2^{1-\alpha} - \frac{5}{6}$. Since $\mathcal{H}_x w_i = \frac{1}{12}(w_{i-1} + 10w_i + w_{i+1})$, we apply the estimates (2.14) and (3.2) in (2.16) to get the following result immediately.

Theorem 3.1 *Assume that the solution $u(x,t)$ of problem (2.1) is in $\mathcal{C}^{(6,3)}((0,L)\times[0,T])$. Then the truncation error $(R_{xt})_i^{n-\frac{1}{2}}$ of the scheme (2.17) satisfies*

$$\left|(R_{xt})_i^{n-\frac{1}{2}}\right| \le C^* \left(\beta_2 \tau^{3-\alpha} + \beta_1 \tau^2 + h^4\right), \qquad 1 \le i \le M-1, \quad 1 \le n \le N, \tag{3.3}$$

where C^* is a positive constant independent of the step sizes τ and h and the time level n.

For implementing the scheme (2.17), it is more convenient to consider its matrix form. To do this, we define the following column vectors:

$$\mathbf{u}^n = \left(u_1^n, u_2^n, \ldots, u_{M-1}^n\right)^T, \qquad \mathbf{g}^{n-\frac{1}{2}} = \left(g_1^{n-\frac{1}{2}}, g_2^{n-\frac{1}{2}}, \ldots, g_{M-1}^{n-\frac{1}{2}}\right)^T,$$
$$\mathbf{u}^{n-1,*} = \left(u_1^{n-1,*}, u_2^{n-1,*}, \ldots, u_{M-1}^{n-1,*}\right)^T,$$

where

$$u_i^{n-1,*} = \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\delta_t u_i^{k-\frac{1}{2}} + a_{n-1}\psi_i^*, \qquad 1 \le i \le M-1.$$

We also define the following $(M-1)$-order tridiagonal or diagonal matrices:

$$A = \text{tridiag}(-1, 2, -1),\ B = \text{tridiag}\left(\frac{1}{12}, \frac{5}{6}, \frac{1}{12}\right),\ Q = \text{diag}\left(q_1, q_2, \ldots, q_{M-1}\right).$$

A simple process shows that the scheme (2.17) can be expressed in the matrix form as

$$\begin{aligned}&\left((\beta_2 + \mu\beta_1)B + \frac{d}{2}\frac{\mu\tau}{h^2}A - \frac{\mu\tau}{2}BQ\right)\mathbf{u}^n \\ &\quad = \left((\beta_2 + \mu\beta_1)B - \frac{d}{2}\frac{\mu\tau}{h^2}A + \frac{\mu\tau}{2}BQ\right)\mathbf{u}^{n-1} + \tau B\left(\beta_2\mathbf{u}^{n-1,*} + \mu\mathbf{g}^{n-\frac{1}{2}}\right) + \mathbf{r}^n,\end{aligned} \tag{3.4}$$

where $\mathbf{r}^n$ absorbs the boundary values of the solution vector and the source terms.

Theorem 3.2 *The compact finite difference scheme (2.17) is uniquely solvable if and only if the matrix*

$$Q^* \equiv (\beta_2 + \mu\beta_1)B + \frac{d}{2}\frac{\mu\tau}{h^2}A - \frac{\mu\tau}{2}BQ \tag{3.5}$$

is nonsingular.

Define

$$\overline{q} = \max_{x\in[0,L]} q(x), \qquad \underline{q} = \min_{x\in[0,L]} q(x). \tag{3.6}$$

A sufficient condition for the matrix Q^* to be nonsingular is given by

$$\mu\tau \max\left\{\frac{\overline{q}}{2}, \frac{5\overline{q}-\underline{q}}{8}\right\} \leq \beta_2 + \mu\beta_1. \tag{3.7}$$

Corollary 3.1 *The compact finite difference scheme (2.17) is uniquely solvable if the condition (3.7) is true.*

Proof In fact, $Q^* = \text{tridiag}(p_{i-1}^*, q_i^*, p_{i+1}^*)$ with $p_0^* = p_M^* = 0$ and

$$\begin{aligned} p_i^* &= \frac{1}{12}(\beta_2+\mu\beta_1) - \frac{d}{2}\frac{\mu\tau}{h^2} - \frac{q_i}{24}\mu\tau, \\ q_i^* &= \frac{5}{6}(\beta_2+\mu\beta_1) + d\frac{\mu\tau}{h^2} - \frac{5q_i}{12}\mu\tau \quad (1 \leq i \leq M-1). \end{aligned}$$

The condition (3.7) implies that $q_i^* > 0$ for each $1 \leq i \leq M-1$.

Case 1 Assume that $p_i^* \neq 0$ for all $1 \leq i \leq M-1$. In this case, the matrix Q^* is irreducible. By the condition (3.7), we have that for $2 \leq i \leq M-2$,

$$\begin{aligned} |p_{i-1}^*| + |p_{i+1}^*| &\leq \frac{1}{6}(\beta_2+\mu\beta_1) + d\frac{\mu\tau}{h^2} - \frac{q_{i-1}+q_{i+1}}{24}\mu\tau \\ &\leq \frac{1}{6}(\beta_2+\mu\beta_1) + d\frac{\mu\tau}{h^2} - \frac{\mu\tau}{12}\underline{q} \\ &\leq \frac{5}{6}(\beta_2+\mu\beta_1) + d\frac{\mu\tau}{h^2} - \frac{5q_i}{12}\mu\tau = |q_i^*|. \end{aligned}$$

Similarly,

$$\begin{aligned} |p_2^*| &\leq \frac{1}{12}(\beta_2+\mu\beta_1) + \frac{d}{2}\frac{\mu\tau}{h^2} - \frac{q_2}{24}\mu\tau \leq \frac{1}{12}(\beta_2+\mu\beta_1) + \frac{d}{2}\frac{\mu\tau}{h^2} - \frac{\mu\tau}{24}\underline{q} \leq \frac{q_1^*}{2} < |q_1^*|, \\ |p_{M-2}^*| &\leq \frac{1}{12}(\beta_2+\mu\beta_1) + \frac{d}{2}\frac{\mu\tau}{h^2} - \frac{q_{M-2}}{24}\mu\tau \leq \frac{1}{12}(\beta_2+\mu\beta_1) + \frac{d}{2}\frac{\mu\tau}{h^2} - \frac{\mu\tau}{24}\underline{q} \leq \frac{q_{M-1}^*}{2} < |q_{M-1}^*|. \end{aligned}$$

This proves that Q^* is irreducibly diagonally dominant and thus nonsingular (see [45]). □

Case 2 Assume that $p_{i_0}^* = 0$ for some $1 \leq i_0 \leq M-1$. In this case, we complete the proof by partitioning Q^* and considering the submatrices of Q^*.

Corollary 3.2 *The compact finite difference scheme (2.17) is uniquely solvable if the function $q(x)$ is nonpositive and convex in* $[0, L]$.

Proof We write $Q^* = \text{tridiag}(p_{i-1}^*, q_i^*, p_{i+1}^*)$ as in the proof of Corollary 3.1. Since the function $q(x)$ is nonpositive and convex, we have that for $2 \leq i \leq M-2$,

$$\begin{aligned} |p_{i-1}^*| + |p_{i+1}^*| &\leq \frac{1}{6}(\beta_2+\mu\beta_1) + d\frac{\mu\tau}{h^2} - \frac{q_{i-1}+q_{i+1}}{24}\mu\tau \\ &< \frac{5}{6}(\beta_2+\mu\beta_1) + d\frac{\mu\tau}{h^2} - \frac{5q_i}{12}\mu\tau = |q_i^*|, \end{aligned}$$

and

$$
\begin{aligned}
|p_2^*| &\le \frac{1}{12}(\beta_2+\mu\beta_1)+\frac{d}{2}\frac{\mu\tau}{h^2}-\frac{q_2}{24}\mu\tau \\
&< \frac{5}{6}(\beta_2+\mu\beta_1)+d\frac{\mu\tau}{h^2}-\frac{5q_1}{12}\mu\tau=|q_1^*|, \\
|p_{M-2}^*| &\le \frac{1}{12}(\beta_2+\mu\beta_1)+\frac{d}{2}\frac{\mu\tau}{h^2}-\frac{q_{M-2}}{24}\mu\tau \\
&< \frac{5}{6}(\beta_2+\mu\beta_1)+d\frac{\mu\tau}{h^2}-\frac{5q_{M-1}}{12}\mu\tau=|q_{M-1}^*|.
\end{aligned}
$$

This shows that the matrix Q^* is strictly diagonally dominant and thus nonsingular (see [45]). □

Remark 3.1 When $q(x) \equiv q$ is independent of x and $q \le 0$, the conditions in Corollaries 3.1 and 3.2 are trivially satisfied. We notice that if the convection coefficient $p(x)$ in the original equation (1.1) is independent of x, i.e., $p(x) \equiv p$, we must have $q(x) \equiv -\frac{p^2}{4d} \le 0$. Therefore, for the fractional convection-diffusion-wave equation (1.1) with constant coefficients, the corresponding compact finite difference scheme (2.17) is always uniquely solvable without any additional constraints.

4 Stability and convergence

We now carry out the stability and convergence analysis of the compact finite difference scheme (2.17) using a technique of discrete energy analysis. Let $\mathcal{S}_h = \{u \mid u = (u_0, u_1, \ldots, u_M), u_0 = u_M = 0\}$ be the space of the grid functions defined in the spatial mesh and vanishing on two boundary points. For any grid functions $v, w \in \mathcal{S}_h$, we define the inner product (v, w), L^2 norm $\|v\|$ and L^∞ norm $\|v\|_\infty$ by

$$
(v,w)=h\sum_{i=1}^{M-1} v_i w_i, \qquad \|v\|=(v,v)^{\frac{1}{2}}, \qquad \|v\|_\infty=\max_{0\le i\le M}|v_i|.
$$

We also define

$$
(\delta_x v,\delta_x w]=h\sum_{i=1}^{M}\delta_x v_{i-\frac{1}{2}}\delta_x w_{i-\frac{1}{2}}, \qquad |v|_1=(\delta_x v,\delta_x v]^{\frac{1}{2}}.
$$

For any $v \in \mathcal{S}_h$, its H^1 norm is defined by $\|v\|_1 = \left(\|v\|^2+|v|_1^2\right)^{\frac{1}{2}}$. Some simple calculations show that for any grid functions $v, w \in \mathcal{S}_h$,

$$
(\delta_x^2 v, w) = -(\delta_x v, \delta_x w], \qquad h\|\delta_x^2 v\| \le 2|v|_1, \qquad h|v|_1 \le 2\|v\|. \tag{4.1}
$$

The inverse estimate $h\|\delta_x^2 v\| \le 2|v|_1$ in (4.1) implies that $|v|_1^2 - \frac{h^2}{12}\|\delta_x^2 v\|^2 \ge \frac{2}{3}|v|_1^2$. For convenience, we introduce the following notation:

$$
\|v\|_* = \left(|v|_1^2 - \frac{h^2}{12}\|\delta_x^2 v\|^2\right)^{\frac{1}{2}}.
$$

Then we have

$$\frac{2}{3}|v|_1^2 \le \|v\|_*^2 \le |v|_1^2. \tag{4.2}$$

Lemma 4.1 *For any grid function $v \in \mathcal{S}_h$,*

$$\|v\|^2 \le \frac{3L^2}{16}\|v\|_*^2, \qquad \|v\|_\infty^2 \le \frac{3L}{8}\|v\|_*^2, \qquad \|v\|_1^2 \le \frac{3(8+L^2)}{16}\|v\|_*^2,$$
$$\|\mathcal{H}_x v\|^2 \le \frac{3L^2}{16}\|v\|_*^2.$$

Proof It is known that $\|v\|^2 \le \frac{L^2}{8}|v|_1^2$ and $\|v\|_\infty^2 \le \frac{L}{4}|v|_1^2$ (see [39], pp. 111 and 112). On the other hand, we have from (4.1) that

$$\|\mathcal{H}_x v\|^2 = \|v\|^2 - \frac{h^2}{6}|v|_1^2 + \frac{h^4}{144}\|\delta_x^2 v\|^2 \le \|v\|^2 - \frac{5h^2}{36}|v|_1^2 \le \|v\|^2. \tag{4.3}$$

Thus, the desired inequalities follow from $|v|_1^2 \le \frac{3}{2}\|v\|_*^2$ in (4.2) immediately. □

Lemma 4.2 *Let $\gamma(x)$ be a continuous function in $[0, L]$. For any grid function $v \in \mathcal{S}_h$, we have $\|\mathcal{H}_x(\gamma v)\| \le \|\gamma\|_\infty \|v\|$.*

Proof By (4.3), $\|\mathcal{H}_x(\gamma v)\|^2 \le \|\gamma v\|^2$. It is clear that $\|\gamma v\|^2 \le \|\gamma\|_\infty^2 \|v\|^2$. This completes the proof. □

Lemma 4.3 *(Discrete Gronwall lemma [37]). Assume that $\{k_n\}$ and $\{s_n\}$ are nonnegative sequences, and that the sequence $\{\phi_n\}$ satisfies*

$$\phi_0 \le g_0, \quad \phi_n \le g_0 + \sum_{l=0}^{n-1} s_l + \sum_{l=0}^{n-1} k_l \phi_l, \qquad n \ge 1,$$

where $g_0 \ge 0$. Then the sequence $\{\phi_n\}$ satisfies

$$\phi_n \le \left(g_0 + \sum_{l=0}^{n-1} s_l\right) \exp\left(\sum_{l=0}^{n-1} k_l\right), \qquad n \ge 1.$$

Based on the above lemmas, we now discuss the stability of the compact finite difference scheme (2.17) with respect to the initial values φ^*, ψ^* and the forcing term g.

Theorem 4.1 *Let $u^n = (u_0^n, u_1^n, \ldots, u_M^n)$ be the solution of the compact finite difference scheme (2.17) with $u_0^n = u_M^n = 0$. Then when $\tau\|q\|_\infty^2 \le \frac{4d}{3CL^2}$, it holds that*

$$\left\|u^n\right\|_*^2 \le \left(G^0 + \frac{8\tau C}{d}\sum_{k=1}^{n}\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2\right)\exp\left(\frac{3TC\|q\|_\infty^2 L^2}{2d}\right), 1 \le n \le N, \tag{4.4}$$

where

$$G^0 = 2\left\|\varphi^*\right\|_*^2 + \frac{4\tau C\|q\|_\infty^2}{d}\left\|\varphi^*\right\|^2 + \frac{4\beta_2 T^{2-\alpha}}{d\Gamma(3-\alpha)}\|\mathcal{H}_x\psi^*\|^2,$$
$$C = \frac{T^{\alpha-1}\Gamma(2-\alpha)}{\beta_2 + 4\beta_1 T^{\alpha-1}\Gamma(2-\alpha)}. \tag{4.5}$$

Proof Let $b_{n,k} = a_{n-k-1} - a_{n-k}$. It is clear that $b_{n,k} \geq 0$. Taking the inner product of (2.17) with $\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}$ gives

$$\begin{aligned}(\beta_2+\mu\beta_1)\left\|\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right\|^2 &- \beta_2\sum_{k=1}^{n-1} b_{n,k}\left(\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right)\\ &-\beta_2 a_{n-1}\left(\mathcal{H}_x\psi^*, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right)\\ &= \mu d\left(\delta_x^2 u^{n-\frac{1}{2}}, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right) + \mu\left(\mathcal{H}_x(q u^{n-\frac{1}{2}}), \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right)\\ &+\mu\left(\mathcal{H}_x g^{n-\frac{1}{2}}, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right). \end{aligned} \tag{4.6}$$

By the Cauchy-Schwarz inequality and the relation $(\delta_x^2 v, w) = -(\delta_x v, \delta_x w]$ in (4.1),

$$\begin{aligned}&\beta_2\sum_{k=1}^{n-1} b_{n,k}\left(\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right)\\ &\leq \frac{\beta_2}{2}\sum_{k=1}^{n-1} b_{n,k}\left(\left\|\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}\right\|^2 + \left\|\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right\|^2\right)\\ &= \frac{\beta_2}{2}\left(\sum_{k=1}^{n-1} a_{n-k-1}\left\|\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}\right\|^2 - \sum_{k=1}^{n-1} a_{n-k}\left\|\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}\right\|^2 + (1-a_{n-1})\left\|\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right\|^2\right),\\ &\mu d\left(\delta_x^2 u^{n-\frac{1}{2}}, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right) = \mu d\left(\delta_x^2 u^{n-\frac{1}{2}}, \delta_t u^{n-\frac{1}{2}}\right) + \mu d\left(\delta_x^2 u^{n-\frac{1}{2}}, \frac{h^2}{12}\delta_x^2\delta_t u^{n-\frac{1}{2}}\right)\\ &\qquad = -\mu d\left(\delta_x u^{n-\frac{1}{2}}, \delta_t\delta_x u^{n-\frac{1}{2}}\right] + \frac{\mu d h^2}{12}\left(\delta_x^2 u^{n-\frac{1}{2}}, \delta_t\delta_x^2 u^{n-\frac{1}{2}}\right)\\ &\qquad = -\frac{\mu d}{2\tau}\left(\left|u^n\right|_1^2 - \left|u^{n-1}\right|_1^2\right) + \frac{\mu d h^2}{24\tau}\left(\left\|\delta_x^2 u^n\right\|^2 - \left\|\delta_x^2 u^{n-1}\right\|^2\right)\\ &\qquad = -\frac{\mu d}{2\tau}\left(\left\|u^n\right\|_*^2 - \left\|u^{n-1}\right\|_*^2\right)\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\beta_2 a_{n-1}\left(\mathcal{H}_x\psi^*, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right) \leq \beta_2 a_{n-1}\left(\|\mathcal{H}_x\psi^*\|^2 + \frac{1}{4}\left\|\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right\|^2\right),\\ &\mu\left(\mathcal{H}_x g^{n-\frac{1}{2}}, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right)\\ &\qquad \leq \frac{2\mu^2}{\beta_2 a_{n-1} + 4\mu\beta_1}\left\|\mathcal{H}_x g^{n-\frac{1}{2}}\right\|^2 + \frac{1}{8}\left(\beta_2 a_{n-1} + 4\mu\beta_1\right)\left\|\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right\|^2.\end{aligned} \tag{4.8}$$

We have from Lemma 4.2 that

$$\begin{aligned}\mu\left(\mathcal{H}_x(qu^{n-\frac{1}{2}}),\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right)&\\ \le \frac{2\mu^2}{\beta_2 a_{n-1}+4\mu\beta_1}\left\|\mathcal{H}_x(qu^{n-\frac{1}{2}})\right\|^2+\frac{1}{8}\left(\beta_2 a_{n-1}+4\mu\beta_1\right)\left\|\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right\|^2&\\ \le \frac{2\mu^2\|q\|_\infty^2}{\beta_2 a_{n-1}+4\mu\beta_1}\left\|u^{n-\frac{1}{2}}\right\|^2+\frac{1}{8}\left(\beta_2 a_{n-1}+4\mu\beta_1\right)\left\|\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right\|^2.&\end{aligned}\tag{4.9}$$

Substituting the above inequalities (4.7)–(4.9) into (4.6), we obtain

$$\begin{aligned}\beta_2\tau\sum_{k=1}^{n}a_{n-k}\left\|\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}\right\|^2+\mu d\left\|u^n\right\|_*^2\le \beta_2\tau\sum_{k=1}^{n-1}a_{n-k-1}\left\|\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}\right\|^2+\mu d\left\|u^{n-1}\right\|_*^2\\ +\frac{4\tau\mu^2}{\beta_2 a_{n-1}+4\mu\beta_1}\left(\|q\|_\infty^2\left\|u^{n-\frac{1}{2}}\right\|^2+\left\|\mathcal{H}_x g^{n-\frac{1}{2}}\right\|^2\right)+2\tau\beta_2 a_{n-1}\|\mathcal{H}_x\psi^*\|^2.\end{aligned}$$

Let

$$F^0=\mu d\left\|u^0\right\|_*^2,\qquad F^n=\beta_2\tau\sum_{k=1}^{n}a_{n-k}\left\|\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}\right\|^2+\mu d\left\|u^n\right\|_*^2\ (1\le n\le N).$$

Then

$$F^n\le F^{n-1}+\frac{4\tau\mu^2}{\beta_2 a_{n-1}+4\mu\beta_1}\left(\|q\|_\infty^2\left\|u^{n-\frac{1}{2}}\right\|^2+\left\|\mathcal{H}_x g^{n-\frac{1}{2}}\right\|^2\right)+2\tau\beta_2 a_{n-1}\|\mathcal{H}_x\psi^*\|^2$$

or equivalently,

$$F^n\le F^0+\sum_{k=1}^{n}\frac{4\tau\mu^2}{\beta_2 a_{k-1}+4\mu\beta_1}\left(\|q\|_\infty^2\left\|u^{k-\frac{1}{2}}\right\|^2+\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2\right)+2\tau\beta_2 n^{2-\alpha}\|\mathcal{H}_x\psi^*\|^2.\tag{4.10}$$

Since $a_{k-1}\ge(2-\alpha)k^{1-\alpha}\ge(2-\alpha)n^{1-\alpha}$ for $k\le n$ and $\mu=\tau^{\alpha-1}\Gamma(3-\alpha)$, we obtain

$$\beta_2 a_{k-1}+4\mu\beta_1\ge\frac{\mu\left(\beta_2+4\beta_1T^{\alpha-1}\Gamma(2-\alpha)\right)}{T^{\alpha-1}\Gamma(2-\alpha)}=\frac{\mu}{C},\qquad \tau n^{2-\alpha}\le\frac{\mu T^{2-\alpha}}{\Gamma(3-\alpha)}.$$

Applying these two inequalities to (4.10) leads to

$$F^n\le F^0+4\tau\mu C\|q\|_\infty^2\sum_{k=1}^{n}\left\|u^{k-\frac{1}{2}}\right\|^2+4\tau\mu C\sum_{k=1}^{n}\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2+\frac{2\mu\beta_2T^{2-\alpha}}{\Gamma(3-\alpha)}\|\mathcal{H}_x\psi^*\|^2.\tag{4.11}$$

Furthermore, by the relation $\|u^{k-\frac{1}{2}}\|^2\le\frac{1}{2}(\|u^k\|^2+\|u^{k-1}\|^2)$, we have

$$\begin{aligned}F^n\le F^0+2\tau\mu C\|q\|_\infty^2\left(\left\|u^0\right\|^2+2\sum_{k=1}^{n-1}\left\|u^k\right\|^2+\left\|u^n\right\|^2\right)\\ +4\tau\mu C\sum_{k=1}^{n}\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2+\frac{2\mu\beta_2T^{2-\alpha}}{\Gamma(3-\alpha)}\|\mathcal{H}_x\psi^*\|^2.\end{aligned}\tag{4.12}$$

In view of the definitions of F^n and F^0, it follows that

$$d\left\|u^n\right\|_*^2 - 2\tau C\|q\|_\infty^2\left\|u^n\right\|^2 \le d\left\|u^0\right\|_*^2 + 2\tau C\|q\|_\infty^2\left\|u^0\right\|^2 + 4\tau C\|q\|_\infty^2\sum_{k=1}^{n-1}\left\|u^k\right\|^2 + 4\tau C\sum_{k=1}^{n}\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2 + \frac{2\beta_2 T^{2-\alpha}}{\Gamma(3-\alpha)}\|\mathcal{H}_x\psi^*\|^2. \tag{4.13}$$

An application of Lemma 4.1 gives

$$\left(d - \frac{3\tau C\|q\|_\infty^2 L^2}{8}\right)\left\|u^n\right\|_*^2 \le d\left\|u^0\right\|_*^2 + 2\tau C\|q\|_\infty^2\left\|u^0\right\|^2 + \frac{3\tau C\|q\|_\infty^2 L^2}{4}\sum_{k=1}^{n-1}\left\|u^k\right\|_*^2 + 4\tau C\sum_{k=1}^{n}\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2 + \frac{2\beta_2 T^{2-\alpha}}{\Gamma(3-\alpha)}\|\mathcal{H}_x\psi^*\|^2. \tag{4.14}$$

When $\tau\|q\|_\infty^2 \le \frac{4d}{3CL^2}$, we have

$$\left\|u^n\right\|_*^2 \le G^0 + \frac{3\tau C\|q\|_\infty^2 L^2}{2d}\sum_{k=1}^{n-1}\left\|u^k\right\|_*^2 + \frac{8\tau C}{d}\sum_{k=1}^{n}\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2. \tag{4.15}$$

The estimate (4.4) follows from Lemma 4.3 (Discrete Gronwall lemma) immediately. □

Theorem 4.1 shows that the compact finite difference scheme (2.17) is almost unconditionally stable to the initial values φ^*, ψ^* and the forcing term g, or more precisely, it is stable under the mild condition $\tau\|q\|_\infty^2 \le \frac{4d}{3CL^2}$ for the general $q(x)$. For the special case when $q(x) \equiv q$ is independent of x and $q < \frac{16d}{3L^2}$, this mild condition is no longer required to obtain the unconditional stability of the compact finite difference scheme (2.17). Specifically, we have the following result.

Theorem 4.2 *Let $u^n = (u_0^n, u_1^n, \ldots, u_M^n)$ be the solution of the compact finite difference scheme (2.17) with $u_0^n = u_M^n = 0$. Assume that $q(x) \equiv q$ is independent of x and that $q < \frac{16d}{3L^2}$. Then we have that for $1 \le n \le N$,*

$$\left\|u^n\right\|_*^2 \le \frac{1}{C_1}\left(d\left\|\varphi^*\right\|_*^2 - q\left\|\mathcal{H}_x\varphi^*\right\|^2 + 2\tau C\sum_{k=1}^{n}\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2 + \frac{2\beta_2 T^{2-\alpha}}{\Gamma(3-\alpha)}\|\mathcal{H}_x\psi^*\|^2\right), \tag{4.16}$$

where C is the constant in (4.5) and

$$C_1 = \frac{16d - 3qL^2}{16}. \tag{4.17}$$

Proof The proof follows from the similar argument as that in the proof of Theorem 4.1. When $q(x) \equiv q$ is independent of x, we have

$$\mu\left(\mathcal{H}_x(qu^{n-\frac{1}{2}}), \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right) = \frac{\mu q}{2\tau}\left(\left\|\mathcal{H}_x u^n\right\|^2 - \left\|\mathcal{H}_x u^{n-1}\right\|^2\right). \tag{4.18}$$

By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\beta_2 a_{n-1}\left(\mathcal{H}_x\psi^*, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right) \le \beta_2 a_{n-1}\left(\|\mathcal{H}_x\psi^*\|^2 + \frac{1}{4}\left\|\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right\|^2\right),\\
&\mu\left(\mathcal{H}_x g^{n-\frac{1}{2}}, \mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right)\\
&\quad\le \frac{\mu^2}{\beta_2 a_{n-1} + 4\mu\beta_1}\left\|\mathcal{H}_x g^{n-\frac{1}{2}}\right\|^2 + \frac{1}{4}\left(\beta_2 a_{n-1} + 4\mu\beta_1\right)\left\|\mathcal{H}_x\delta_t u^{n-\frac{1}{2}}\right\|^2.
\end{aligned}
\tag{4.19}
$$

Using (4.6) (with $q(x) \equiv q$), (4.7), (4.18) and (4.19), we obtain

$$
\begin{aligned}
&\beta_2\tau\sum_{k=1}^{n} a_{n-k}\left\|\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}\right\|^2 + \mu d\left\|u^n\right\|_*^2 - \mu q\left\|\mathcal{H}_x u^n\right\|^2\\
&\quad\le \beta_2\tau\sum_{k=1}^{n-1} a_{n-k-1}\left\|\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}\right\|^2 + \mu d\left\|u^{n-1}\right\|_*^2 - \mu q\left\|\mathcal{H}_x u^{n-1}\right\|^2\\
&\qquad + \frac{2\tau\mu^2}{\beta_2 a_{n-1} + 4\mu\beta_1}\left\|\mathcal{H}_x g^{n-\frac{1}{2}}\right\|^2 + 2\tau\beta_2 a_{n-1}\|\mathcal{H}_x\psi^*\|^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\beta_2\tau\sum_{k=1}^{n} a_{n-k}\left\|\mathcal{H}_x\delta_t u^{k-\frac{1}{2}}\right\|^2 + \mu d\left\|u^n\right\|_*^2 - \mu q\left\|\mathcal{H}_x u^n\right\|^2\\
&\le \mu d\left\|u^0\right\|_*^2 - \mu q\left\|\mathcal{H}_x u^0\right\|^2 + \sum_{k=1}^{n}\frac{2\tau\mu^2}{\beta_2 a_{k-1} + 4\mu\beta_1}\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2 + 2\tau\beta_2 n^{2-\alpha}\|\mathcal{H}_x\psi^*\|^2.
\end{aligned}
\tag{4.20}
$$

Since

$$
\beta_2 a_{k-1} + 4\mu\beta_1 \ge \frac{\mu\left(\beta_2 + 4\beta_1 T^{\alpha-1}\Gamma(2-\alpha)\right)}{T^{\alpha-1}\Gamma(2-\alpha)} = \frac{\mu}{C}, \qquad \tau n^{2-\alpha} \le \frac{\mu T^{2-\alpha}}{\Gamma(3-\alpha)},
$$

we have from (4.20) that

$$
d\left\|u^n\right\|_*^2 - q\left\|\mathcal{H}_x u^n\right\|^2 \le d\left\|u^0\right\|_*^2 - q\left\|\mathcal{H}_x u^0\right\|^2 + 2\tau C\sum_{k=1}^{n}\left\|\mathcal{H}_x g^{k-\frac{1}{2}}\right\|^2 + \frac{2\beta_2 T^{2-\alpha}}{\Gamma(3-\alpha)}\|\mathcal{H}_x\psi^*\|^2.
$$

An application of Lemma 4.1 shows that the estimate (4.16) holds. □

Remark 4.1 The condition $q < \frac{16d}{3L^2}$ is automatically satisfied if $q \le 0$. The latter is certainly satisfied if the convection coefficient $p(x)$ in the original equation (1.1) is independent of x, i.e., $p(x) \equiv p$. This implies that for the fractional convection-diffusion-wave equation (1.1) with constant coefficients, the corresponding compact finite difference scheme (2.17) is unconditionally stable.

We now consider the convergence of the compact finite difference scheme (2.17). Let $e_i^n = U_i^n - u_i^n$. From (2.15) and (2.17), we get the following error equation:

$$\begin{cases} \beta_2 \mathcal{H}_x \left(\delta_t e_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \delta_t e_i^{k-\frac{1}{2}} \right) + \mu \beta_1 \mathcal{H}_x \delta_t e_i^{n-\frac{1}{2}} \\ \quad = \mu \left(d \delta_x^2 e_i^{n-\frac{1}{2}} + \mathcal{H}_x \left(q_i e_i^{n-\frac{1}{2}} \right) + (R_{xt})_i^{n-\frac{1}{2}} \right), \qquad 1 \le i \le M-1, \ \ 1 \le n \le N, \\ e_0^n = e_M^n = 0, \qquad 1 \le n \le N, \\ e_i^0 = 0, \qquad 0 \le i \le M. \end{cases} \tag{4.21}$$

Let C, C^* and C_1 be the constants in (4.5), (3.3) and (4.17). Using these constants, we define

$$C_2 = \left(\frac{8TLC^{*2}C}{d} \exp\left(\frac{3TC\|q\|_\infty^2 L^2}{2d} \right) \right)^{\frac{1}{2}}, \quad C_3 = \left(\frac{2TLCC^{*2}}{C_1} \right)^{\frac{1}{2}}. \tag{4.22}$$

Based on the error equation (4.21), we have the following convergence results.

Theorem 4.3 *Let U_i^n denote the value of the solution $u(x,t)$ of (2.1) at the mesh point (x_i, t_n) and let $u^n = (u_0^n, u_1^n, \ldots, u_M^n)$ be the solution of the compact finite difference scheme (2.17). Assume that the condition in Theorem 3.1 is satisfied. Then when $\tau \|q\|_\infty^2 \le \frac{4d}{3CL^2}$, we have*

$$\left\| U^n - u^n \right\|_* \le C_2 \left(\beta_2 \tau^{3-\alpha} + \beta_1 \tau^2 + h^4 \right), \qquad 1 \le n \le N. \tag{4.23}$$

Proof It follows from (4.21) and Theorem 4.1 that

$$\left\| e^n \right\|_*^2 \le \frac{8\tau C}{d} \exp\left(\frac{3TC\|q\|_\infty^2 L^2}{2d} \right) \sum_{k=1}^{n} \left\| (R_{xt})^{k-\frac{1}{2}} \right\|^2.$$

Applying Theorem 3.1, we get

$$\left\| e^n \right\|_*^2 \le C_2^2 \left(\beta_2 \tau^{3-\alpha} + \beta_1 \tau^2 + h^4 \right)^2.$$

The estimate (4.23) is proved. □

Theorem 4.4 *Let U_i^n denote the value of the solution $u(x,t)$ of (2.1) at the mesh point (x_i, t_n) and let $u^n = (u_0^n, u_1^n, \ldots, u_M^n)$ be the solution of the compact finite difference scheme (2.17). Assume that $q(x) \equiv q$ is independent of x and $q < \frac{16d}{3L^2}$. Also assume that the condition in Theorem 3.1 is satisfied. Then we have*

$$\left\| U^n - u^n \right\|_* \le C_3 \left(\beta_2 \tau^{3-\alpha} + \beta_1 \tau^2 + h^4 \right), \qquad 1 \le n \le N. \tag{4.24}$$

Proof The proof follows from (4.21) and Theorems 3.1 and 4.2. □

Combining Lemma 4.1 with Theorems 3.1 and 4.2, we get immediately the following two theorems concerning with the error estimates in the discrete H^1, L^2 and L^∞ norms.

Theorem 4.5 *Assume that the condition in Theorem* 4.3 *is satisfied. Then*

$$\begin{aligned}
\|U^n - u^n\|_1 &\le \frac{C_2\sqrt{3(8+L^2)}}{4}\left(\beta_2\tau^{3-\alpha} + \beta_1\tau^2 + h^4\right), \qquad 1 \le n \le N,\\
\|U^n - u^n\| &\le \frac{C_2 L\sqrt{3}}{4}\left(\beta_2\tau^{3-\alpha} + \beta_1\tau^2 + h^4\right), \qquad 1 \le n \le N,\\
\|U^n - u^n\|_\infty &\le \frac{C_2\sqrt{6L}}{4}\left(\beta_2\tau^{3-\alpha} + \beta_1\tau^2 + h^4\right), \qquad 1 \le n \le N.
\end{aligned} \tag{4.25}$$

Theorem 4.6 *Assume that the condition in Theorem* 4.4 *is satisfied. Then*

$$\begin{aligned}
\|U^n - u^n\|_1 &\le \frac{C_3\sqrt{3(8+L^2)}}{4}\left(\beta_2\tau^{3-\alpha} + \beta_1\tau^2 + h^4\right), \qquad 1 \le n \le N,\\
\|U^n - u^n\| &\le \frac{C_3 L\sqrt{3}}{4}\left(\beta_2\tau^{3-\alpha} + \beta_1\tau^2 + h^4\right), \qquad 1 \le n \le N,\\
\|U^n - u^n\|_\infty &\le \frac{C_3\sqrt{6L}}{4}\left(\beta_2\tau^{3-\alpha} + \beta_1\tau^2 + h^4\right), \qquad 1 \le n \le N.
\end{aligned} \tag{4.26}$$

Remark 4.2 In Theorem 4.5, the optimal error estimates (i.e., the error estimate with the same order as the truncation error) of the compact finite difference scheme (2.17) in the discrete L^2, H^1 and L^∞ norms are obtained under the mild condition $\tau\|q\|_\infty^2 \le \frac{4d}{3CL^2}$ for the general $q(x)$. Theorem 4.6 shows that this mild condition is no longer required to obtain the same optimal error estimates if $q(x) \equiv q$ is independent of x and $q < \frac{16d}{3L^2}$. In particular, this is the case for the fractional convection-diffusion-wave equation (1.1) with constant coefficients.

Remark 4.3 The constraint condition $q < \frac{16d}{3L^2}$ in Theorems 4.2 and 4.3 is easily verifiable for practical problems. If it does not hold we have the estimates (4.4) and (4.23) instead of the estimates (4.16) and (4.24), respectively, for the sufficiently small τ. When C_1 is very small, the estimates (4.16), (4.24) and (4.26) are poor. In this case, it is better to use the estimates (4.4), (4.23) and (4.25) for the sufficiently small τ. The restriction condition on τ in Theorems 4.1, 4.3 and 4.5 is only for the analysis of the stability and convergence of the compact finite difference scheme (2.17) with the general $q(x)$. One of the numerical experiments in the next section shows that it is only a sufficient condition. Improvement of this condition can be interesting both theoretically and computationally.

Remark 4.4 When d is a small positive parameter, say $0 < d \ll 1$, the problem (2.1) is singularly perturbed and characterized by its boundary layers where the solution varies rapidly (see [33] for classical singularly perturbed problems). The error estimates (4.23) and (4.25) are not applicable to such problems since C_2 becomes large. Due to the boundary layer behavior, it is difficult to solve efficiently singularly perturbed problems by most numerical methods using a uniform mesh (see [33]). A possible alternative approach is to use piecewise uniform meshes, i.e., Shishkin type meshes (see [33]). How to construct an efficient numerical method for solving fractional singularly perturbed problems will be our next consideration.

Remark 4.5 We now give a simple comment on the convergence order of the compact finite difference scheme (2.17). Since $3-\alpha < 2$, we see from (4.25) and (4.26) that the temporal convergence order of the compact finite difference scheme (2.17) is $3-\alpha$ if $\beta_2 \neq 0$; otherwise, it attains 2. This implies that the compact finite difference scheme (2.17) converges with the convergence order $\mathcal{O}(\tau^{3-\alpha}+h^4)$ for the fractional convection-diffusion-wave equation (1.1), and its convergence order increases to $\mathcal{O}(\tau^2+h^4)$ when the equation (1.1) is reduced to a classical convection-diffusion equation of integer order (i.e., $\beta_2 = 0$). It is very interesting to develop a compact finite difference scheme of order $\mathcal{O}(\tau^2+h^4)$ for the fractional convection-diffusion-wave equation (1.1) (i.e., the case for $\beta_2 \neq 0$) and establish the corresponding error estimates. This will be a subject of our future investigations.

Once we have the error estimate between the solution U_i^n of the transformed problem (2.1) and the solution u_i^n of the compact finite difference scheme (2.17), it is very straightforward to obtain the error estimate between the solutions of the original problem (1.1)–(1.3) and the compact finite difference scheme (2.17). Let $V_i^n = v(x_i, t_n)$ be the value of the solution $v(x,t)$ of the original problem (1.1)–(1.3) at the mesh point (x_i, t_n), and let $v_i^n = u_i^n/k_i$, where $k_i = k(x_i)$. Since $V_i^n = U_i^n/k_i$, we have from (4.25) or (4.26) that

$$\|V^n - v^n\|_\nu \leq C_4\left(\beta_2\tau^{3-\alpha} + \beta_1\tau^2 + h^4\right), \qquad 1 \leq n \leq N, \tag{4.27}$$

where the norm $\|\cdot\|_\nu$ stands for any of the norms $\|\cdot\|_1$, $\|\cdot\|$ and $\|\cdot\|_\infty$, and C_4 is a positive constant independent of the step sizes τ and h and the time level n. The estimate (4.27) will be used in our numerical experiments in the next section.

5 Applications and numerical results

In this section, we give some applications of the proposed compact finite difference method for three model problems. Our first example is a time fractional convection-diffusion-wave problem with damping, where the convection coefficient is spatially variable and the exact analytical solution $v(x,t)$ is explicitly known. This analytical solution is mainly used to compare with the computed solution $v_i^n = u_i^n/k_i$ to check the accuracy of the compact finite difference method, where $k_i = k(x_i)$ and u_i^n is the solution of the compact finite difference scheme (2.17). The second example is also a time fractional convection-diffusion-wave problem with damping, but the convection coefficient is a constant. We use this example to make some numerical comparisons of the compact finite difference scheme (2.17) with the finite difference scheme (28) given in [27] to demonstrate the high efficiency of the compact finite difference scheme (2.17). In the final example, we consider a time fractional convection-diffusion-wave problem with a large L compared with d. By means of this example, we show that the restriction condition on τ in Theorems 4.2, 4.3 and 4.5 is only a sufficient condition for the stability and convergence of the compact finite difference scheme (2.17) with the general $q(x)$.

To check the accuracy of the computed solution v_i^n, we compute L^2, H^1 and L^∞ norm errors:

$$\mathrm{e}_2(\tau, h) = \max_{0\le n\le N} \|V^n - v^n\|, \quad \mathrm{e}_\nu(\tau, h) = \max_{0\le n\le N} \|V^n - v^n\|_\nu \ (\nu = 1, \infty), \tag{5.1}$$

where $V_i^n = v(x_i, t_n)$. The temporal and spatial convergence orders are computed, respectively, by

$$\mathrm{order}_\nu^{\mathrm{t}}(\tau, h) = \log_2\left(\frac{\mathrm{e}_\nu(2\tau, h)}{\mathrm{e}_\nu(\tau, h)}\right), \qquad \mathrm{order}_\nu^{\mathrm{s}}(\tau, h) = \log_2\left(\frac{\mathrm{e}_\nu(\tau, 2h)}{\mathrm{e}_\nu(\tau, h)}\right), \quad \nu = 1, 2, \infty. \tag{5.2}$$

Table 1 The temporal errors and convergence orders of the compact finite difference scheme (2.17) for Example 5.1 ($h \approx \tau^{\frac{3-\alpha}{4}}$)

α	τ	$\mathrm{e}_1(\tau, h)$	$\mathrm{order}_1^{\mathrm{t}}(\tau, h)$	$\mathrm{e}_2(\tau, h)$	$\mathrm{order}_2^{\mathrm{t}}(\tau, h)$	$\mathrm{e}_\infty(\tau, h)$	$\mathrm{order}_\infty^{\mathrm{t}}(\tau, h)$
5/4	1/5	2.283320e-02		1.723803e-02		1.405934e-02	
	1/10	5.968381e-03	1.935722	4.916522e-03	1.809885	4.123869e-03	1.769458
	1/20	1.588305e-03	1.909851	1.407565e-03	1.804437	1.179934e-03	1.805292
	1/40	4.346612e-04	1.869525	4.042584e-04	1.799852	3.387652e-04	1.800349
	1/80	1.215439e-04	1.838414	1.167808e-04	1.791474	9.806325e-05	1.788501
	1/160	3.456831e-05	1.813956	3.376449e-05	1.790225	2.836018e-05	1.789846
	1/320	9.951144e-06	1.796516	9.827237e-06	1.780649	8.248315e-06	1.781695
3/2	1/5	5.094328e-02		3.597288e-02		2.794232e-02	
	1/10	1.581324e-02	1.687759	1.252104e-02	1.522556	1.036601e-02	1.430590
	1/20	5.108329e-03	1.630209	4.332446e-03	1.531100	3.520139e-03	1.558157
	1/40	1.672943e-03	1.610463	1.505348e-03	1.525085	1.246582e-03	1.497655
	1/80	5.603527e-04	1.577981	5.250619e-04	1.519538	4.349827e-04	1.518948
	1/160	1.912122e-04	1.551161	1.836975e-04	1.515156	1.519755e-04	1.517119
	1/320	6.605296e-05	1.533479	6.440742e-05	1.512033	5.322062e-05	1.513782
7/4	1/5	1.093827e-01		7.705109e-02		5.964775e-02	
	1/10	4.372347e-02	1.322904	3.287706e-02	1.228734	2.584664e-02	1.206491
	1/20	1.691744e-02	1.369897	1.390575e-02	1.241400	1.140806e-02	1.179923
	1/40	6.884574e-03	1.297072	5.833749e-03	1.253187	4.699823e-03	1.279375
	1/80	2.726847e-03	1.336133	2.452493e-03	1.250174	2.011458e-03	1.224365
	1/160	1.107637e-03	1.299749	1.029614e-03	1.252146	8.415658e-04	1.257094
	1/320	4.538547e-04	1.287183	4.324834e-04	1.251386	3.534918e-04	1.251399

Example 5.1 We first consider a time fractional convection-diffusion-wave problem with damping. This problem is governed by the equation (1.1) in the domain $(0, \pi) \times (0, 1]$ with $\beta_2 = \beta_1 = d = 1$ and

$$p(x) = -\sin x, \qquad f(x,t) = t^2\left(3 + t + \frac{6t^{1-\alpha}}{\Gamma(4-\alpha)}\right)\cos x + t^3 \sin^2 x. \quad (5.3)$$

The boundary and initial conditions are given by (1.2) and (1.3) with

$$\phi_0(t) = t^3, \qquad \phi_L(t) = -t^3, \qquad \varphi(x) = \psi(x) \equiv 0. \quad (5.4)$$

It is easy to check that $v(x,t) = t^3 \cos x$ is the solution of this problem and the function $q(x)$ in the problem (2.1) is given by $q(x) = -\frac{1}{4}\left(2\cos x + \sin^2 x\right)$.

We first test the temporal error and the temporal convergence order of the compact finite difference scheme (2.17) for different α. In this test, we let the spatial step $h \approx \tau^{\frac{3-\alpha}{4}}$ $(M = \lceil \pi\tau^{-\frac{3-\alpha}{4}} \rceil)$. Table 1 gives the errors $\mathrm{e}_\nu(\tau,h)(\nu = 1, 2, \infty)$ and the temporal convergence orders $\mathrm{order}_\nu^{\mathrm{t}}(\tau,h)(\nu = 1, 2, \infty)$ of the computed solution v_i^n for $\alpha = 5/4, 3/2, 7/4$ and different time step τ. As expected from our theoretical analysis, the computed solution v_i^n has the temporal accuracy of order $(3-\alpha)$.

We next compute the spatial error and the spatial convergence order. Table 2 presents the errors $\mathrm{e}_\nu(\tau,h)(\nu = 1, 2, \infty)$ and the spatial convergence orders $\mathrm{order}_\nu^{\mathrm{s}}(\tau,h)(\nu = 1, 2, \infty)$ of the computed solution v_i^n for $\alpha = 5/4, 3/2, 7/4$ and different spatial step h, where the time step $\tau \approx h^{\frac{4}{3-\alpha}}$ $(N = \lceil h^{-\frac{4}{3-\alpha}} \rceil)$. The data in this table demonstrate that the compact finite difference scheme (2.17) generates the fourth-order spatial accuracy. This coincides well with the analysis.

Table 2 The spatial errors and convergence orders of the compact finite difference scheme (2.17) for Example 5.1 $(\tau \approx h^{\frac{4}{3-\alpha}})$

α	h	$\mathrm{e}_1(\tau,h)$	$\mathrm{order}_1^{\mathrm{s}}(\tau,h)$	$\mathrm{e}_2(\tau,h)$	$\mathrm{order}_2^{\mathrm{s}}(\tau,h)$	$\mathrm{e}_\infty(\tau,h)$	$\mathrm{order}_\infty^{\mathrm{s}}(\tau,h)$
5/4	$\pi/4$	2.007541e-01		1.174402e-01		1.027557e-01	
	$\pi/8$	8.506234e-03	4.560765	6.749575e-03	4.120986	5.694532e-03	4.173497
	$\pi/16$	4.123673e-04	4.366519	3.835459e-04	4.137325	3.218611e-04	4.145067
	$\pi/32$	2.286029e-05	4.173014	2.242060e-05	4.096503	1.878749e-05	4.098594
	$\pi/64$	1.345212e-06	4.086939	1.338572e-06	4.066058	1.125026e-06	4.061741
3/2	$\pi/4$	2.640985e-01		1.537707e-01		1.324759e-01	
	$\pi/8$	1.218615e-02	4.437762	9.660234e-03	3.992579	8.049062e-03	4.040765
	$\pi/16$	6.008165e-04	4.342174	5.586492e-04	4.112043	4.615859e-04	4.124150
	$\pi/32$	3.478119e-05	4.110545	3.411072e-05	4.033645	2.818061e-05	4.033824
	$\pi/64$	2.121467e-06	4.035173	2.110982e-06	4.014239	1.748542e-06	4.010479
7/4	$\pi/4$	4.285346e-01		2.509337e-01		2.197285e-01	
	$\pi/8$	1.769530e-02	4.597974	1.399165e-02	4.164668	1.145233e-02	4.262010
	$\pi/16$	9.361196e-04	4.240529	8.701531e-04	4.007153	7.110124e-04	4.009622
	$\pi/32$	5.534022e-05	4.080293	5.426965e-05	4.003052	4.434236e-05	4.003117
	$\pi/64$	3.405653e-06	4.022325	3.388762e-06	4.001315	2.778334e-06	3.996394

Example 5.2 In order to make some numerical comparisons of the compact finite difference scheme (2.17) with the finite difference scheme (28) given in [27], we consider the problem given by Example 7 in [27]. This problem is governed by the equation (1.1) with the boundary and initial conditions (1.2) and (1.3) in the domain $(0, 1) \times (0, 1]$, where $\beta_2 = \beta_1 = d = 1$ and

$$p(x) = 1, \quad f(x,t) = 3t^2\left(1 + \frac{2t^{1-\alpha}}{\Gamma(4-\alpha)}\right)e^x, \quad \phi_0(t) = t^3,$$
$$\phi_L(t) = t^3 e, \quad \varphi(x) = \psi(x) \equiv 0. \tag{5.5}$$

Its exact analytical solution is known and is given by $v(x,t) = t^3 e^x$. For this example, the function $q(x)$ in the problem (2.1) is given by $q(x) = -\frac{1}{4}$ for all $x \in [0, 1]$.

We now use the compact finite difference scheme (2.17) in this paper and the finite difference scheme (28) in [27] to solve the above problem numerically. For comparison, Table 3 lists the error $e_\infty(\tau, h)$ and the temporal convergence order $\text{order}^t_\infty(\tau, h)$ of these two schemes for $\alpha = 5/4, 3/2, 7/4$ and different time step τ, while the error

Table 3 Comparisons of temporal accuracy between the scheme (2.17) and the scheme (28) in [27] for Example 5.2

		Scheme (2.17) ($h \approx \tau^{\frac{3-\alpha}{4}}$)		Scheme (28) in [27] ($h = \tau$)	
α	τ	$e_\infty(\tau, h)$	$\text{order}^t_\infty(\tau, h)$	$e_\infty(\tau, h)$	$\text{order}^t_\infty(\tau, h)$
5/4	1/5	1.516569e-02		2.020968e-01	
	1/10	4.293385e-03	1.820623	1.077003e-01	0.908025
	1/20	1.234304e-03	1.798418	5.546180e-02	0.957456
	1/40	3.543907e-04	1.800285	2.810787e-02	0.980520
	1/80	1.050106e-04	1.754806	1.414635e-02	0.990545
	1/160	3.038161e-05	1.789265	7.091633e-03	0.996240
	1/320	8.800219e-06	1.787587	3.549604e-03	0.998460
3/2	1/5	3.345199e-02		2.312428e-01	
	1/10	1.126876e-02	1.569763	1.221406e-01	0.920865
	1/20	3.916418e-03	1.524722	6.231016e-02	0.971004
	1/40	1.354654e-03	1.531610	3.132050e-02	0.992360
	1/80	4.716027e-04	1.522281	1.565125e-02	1.000829
	1/160	1.691014e-04	1.479683	7.801491e-03	1.004456
	1/320	5.919440e-05	1.514356	3.887714e-03	1.004828
7/4	1/5	7.705232e-02		2.818246e-01	
	1/10	3.182338e-02	1.275751	1.505801e-01	0.904267
	1/20	1.333995e-02	1.254333	7.698121e-02	0.967953
	1/40	5.702752e-03	1.226023	3.856938e-02	0.997050
	1/80	2.390355e-03	1.254433	1.910700e-02	1.013355
	1/160	1.022237e-03	1.225496	9.432427e-03	1.018400
	1/320	4.307661e-04	1.246753	4.652838e-03	1.019518

$e_\infty(\tau, h)$ and the spatial convergence order $\text{order}_\infty^{s}(\tau, h)$ for $\alpha = 5/4, 3/2, 7/4$ and different spatial step h are given in Table 4. It is seen from Table 3 that the compact finite difference scheme (2.17) given here is more accurate than the finite difference scheme (28) in [27] for the same time step τ. We also see from Table 4 that the compact finite difference scheme (2.17) in this paper possesses the fourth-order spatial accuracy, whereas the finite difference scheme (28) in [27] has only the first-order spatial accuracy.

Example 5.3 In this example, we consider the equation (1.1) in the domain $(0, 100) \times (0, 1]$ with $\beta_2 = \beta_1 = d = 1$ and

$$p(x) = \cos(\pi x), \quad f(x,t) = \left(\Gamma(2+\alpha)t + (1+\alpha)t^{\alpha} + \pi\,(\pi - \sin(\pi x))\,t^{1+\alpha}\right)\cos(\pi x). \tag{5.6}$$

The boundary and initial functions in (1.2) and (1.3) are chosen as

$$\phi_0(t) = \phi_L(t) = t^{1+\alpha}, \qquad \varphi(x) = \psi(x) \equiv 0. \tag{5.7}$$

This choice implies that $v(x,t) = t^{1+\alpha}\cos(\pi x)$ is the solution of this problem. Clearly, the function $q(x)$ in the problem (2.1) is given by $q(x) = -\frac{1}{4}\left(2\pi\sin(\pi x) + \cos^2(\pi x)\right)$ for this example.

We use the compact finite difference scheme (2.17) to solve the above problem numerically. In Table 5, we give the errors $e_\nu(\tau, h)(\nu = 1, 2, \infty)$ and the temporal convergence orders $\text{order}_\nu^{t}(\tau, h)(\nu = 1, 2, \infty)$ of the computed solution v_i^n for

Table 4 Comparisons of spatial accuracy between the scheme (2.17) and the scheme (28) in [27] for Example 5.2

		Scheme (2.17) ($\tau \approx h^{\frac{4}{3-\alpha}}$)		Scheme (28) in [27] ($\tau = h$)	
α	h	$e_\infty(\tau, h)$	$\text{order}_\infty^{s}(\tau, h)$	$e_\infty(\tau, h)$	$\text{order}_\infty^{s}(\tau, h)$
5/4	1/4	9.036314e-04		2.376788e-01	
	1/8	5.362622e-05	4.074724	1.318263e-01	0.850375
	1/16	3.212236e-06	4.061289	6.869237e-02	0.940416
	1/32	1.939886e-07	4.049534	3.503624e-02	0.971302
	1/64	1.184304e-08	4.033861	1.765607e-02	0.988684
3/2	1/4	1.338402e-03		2.728267e-01	
	1/8	8.224847e-05	4.024379	1.498762e-01	0.864213
	1/16	5.120146e-06	4.005732	7.746116e-02	0.952227
	1/32	3.183150e-07	4.007658	3.913657e-02	0.984956
	1/64	1.984025e-08	4.003953	1.957343e-02	0.999621
7/4	1/4	2.233499e-03		3.310531e-01	
	1/8	1.403319e-04	3.992391	1.841092e-01	0.846501
	1/16	8.872201e-06	3.983407	9.586803e-02	0.941440
	1/32	5.547141e-07	3.999476	4.824655e-02	0.990624
	1/64	4.210986e-08	3.719515	2.397248e-02	1.009047

Table 5 The temporal errors and convergence orders of the compact finite difference scheme (2.17) for Example 5.3 ($h \approx \tau^{\frac{3-\alpha}{4}}$)

α	τ	$e_1(\tau, h)$	$\text{order}_1^t(\tau, h)$	$e_2(\tau, h)$	$\text{order}_2^t(\tau, h)$	$e_\infty(\tau, h)$	$\text{order}_\infty^t(\tau, h)$
5/4	1/5	6.457124e-01		3.039362e-01		5.971827e-02	
	1/10	1.530110e-01	2.077257	7.698809e-02	1.981061	1.505691e-02	1.987747
	1/20	3.598995e-02	2.087969	2.094876e-02	1.877770	3.899111e-03	1.949208
	1/40	8.717690e-03	2.045576	5.917826e-03	1.823726	1.104963e-03	1.819148
	1/80	2.221518e-03	1.972400	1.718122e-03	1.784235	3.321866e-04	1.733932
	1/160	5.919316e-04	1.908043	5.044765e-04	1.767974	9.775818e-05	1.764704
	1/320	1.639968e-04	1.851763	1.491637e-04	1.757890	2.852672e-05	1.776903
3/2	1/5	9.913292e-01		4.719551e-01		9.160780e-02	
	1/10	2.934076e-01	1.756458	1.417491e-01	1.735310	2.751957e-02	1.735013
	1/20	8.536135e-02	1.781251	4.559106e-02	1.636516	8.567927e-03	1.683440
	1/40	2.505832e-02	1.768293	1.524824e-02	1.580108	3.049177e-03	1.490526
	1/80	7.487514e-03	1.742731	5.184603e-03	1.556337	1.027281e-03	1.569589
	1/160	2.334556e-03	1.681339	1.803769e-03	1.523219	3.607112e-04	1.509915
	1/320	7.491559e-04	1.639810	6.305137e-04	1.516415	1.203559e-04	1.583538
7/4	1/5	1.455163e+00		7.112095e-01		1.345126e-01	
	1/10	5.667429e-01	1.360415	2.743758e-01	1.374121	5.281255e-02	1.348788
	1/20	2.060476e-01	1.459717	1.054713e-01	1.379303	2.114363e-02	1.320658
	1/40	7.475571e-02	1.462722	4.206027e-02	1.326321	7.944462e-03	1.412201
	1/80	2.695190e-02	1.471797	1.693169e-02	1.312732	3.477183e-03	1.192030
	1/160	9.964135e-03	1.435571	6.952426e-03	1.284138	1.337687e-03	1.378178
	1/320	3.764678e-03	1.404218	2.880018e-03	1.271439	5.498090e-04	1.282738

$\alpha = 5/4, 3/2, 7/4$ and different time step τ, where the spatial step $h \approx \tau^{\frac{3-\alpha}{4}}$ ($M = \lceil 100\tau^{-\frac{3-\alpha}{4}} \rceil$). It is seen that the computed solution v_i^n has the temporal accuracy of order $(3-\alpha)$. The numerical results in Table 6 give the errors $e_\nu(\tau, h)(\nu = 1, 2, \infty)$ and the spatial convergence orders $\text{order}_\nu^s(\tau, h)(\nu = 1, 2, \infty)$ of the computed solution v_i^n for $\alpha = 5/4, 3/2, 7/4$ and different spatial step h, where the time step $\tau \approx h^{\frac{4}{3-\alpha}}$ ($N = \lceil h^{-\frac{4}{3-\alpha}} \rceil$). These results show that the compact finite difference scheme (2.17) generates the fourth-order spatial accuracy.

Since $L = 100$, $d = 1$ and $\|q\|_\infty = \frac{\pi}{2}$, the restriction condition on τ in Theorems 4.1, 4.3 and 4.5 for the present problem is reduced to

$$\tau \leq \frac{1}{1875\pi^2}\left(\frac{1}{\Gamma(2-\alpha)} + 4\right). \tag{5.8}$$

Clearly, this condition is not satisfied for all τ in Table 5 and some τ in Table 6. However, the corresponding numerical results in Tables 5 and 6 show that the compact finite difference scheme (2.17) is still stable and convergent. This implies that the restriction condition on τ in Theorems 4.1, 4.3 and 4.5 is only a sufficient condition

Table 6 The spatial errors and convergence orders of the compact finite difference scheme (2.17) for Example 5.3 ($\tau \approx h^{\frac{4}{3-\alpha}}$)

α	h	$e_1(\tau,h)$	$\text{order}_1^s(\tau,h)$	$e_2(\tau,h)$	$\text{order}_2^s(\tau,h)$	$e_\infty(\tau,h)$	$\text{order}_\infty^s(\tau,h)$
5/4	1/4	2.581770e-02		1.562794e-02		3.033780e-03	
	1/8	1.095566e-03	4.558612	8.960001e-04	4.124485	1.738498e-04	4.125203
	1/16	5.860527e-05	4.224502	5.518185e-05	4.021233	1.070731e-05	4.021174
	1/32	3.480066e-06	4.073844	3.425251e-06	4.009912	6.646437e-07	4.009871
	1/64	2.145757e-07	4.019556	2.137134e-07	4.002460	4.146840e-08	4.002497
3/2	1/4	2.588127e-02		1.576009e-02		3.147355e-03	
	1/8	1.077679e-03	4.585909	8.847259e-04	4.154901	1.769587e-04	4.152655
	1/16	5.776769e-05	4.221521	5.446596e-05	4.021804	1.089538e-05	4.021624
	1/32	3.441059e-06	4.069338	3.388135e-06	4.006791	6.778281e-07	4.006653
	1/64	2.123499e-07	4.018338	2.115175e-07	4.001643	4.231266e-08	4.001758
7/4	1/4	2.553825e-02		1.613469e-02		3.323958e-03	
	1/8	1.125783e-03	4.503659	9.448085e-04	4.094000	1.941036e-04	4.098004
	1/16	6.120311e-05	4.201180	5.815404e-05	4.022071	1.193691e-05	4.023325
	1/32	3.669129e-06	4.060096	3.620378e-06	4.005667	7.430968e-07	4.005736

for the stability and convergence of the compact finite difference scheme (2.17) with the general $q(x)$.

6 Concluding remarks

We have presented and analyzed a high-order compact finite difference method for a class of time fractional convection-diffusion-wave equations. The convection coefficients may be spatially variable, and the time fractional derivative is in the Caputo sense with the order α ($1 < \alpha < 2$). The class of the equations under consideration includes several important fractional differential equations such as time fractional convection-diffusion-wave/diffusion-wave equations with or without damping. It also includes classical convection-diffusion equations of integer order with spatially variable convection coefficients as its special case. We have proved that the proposed compact finite difference method is uniquely solvable, stable and convergent, and provided the optimal error estimates in the discrete H^1, L^2 and L^∞ norms. The error estimates show that the method has the fourth-order spatial accuracy and the $(3-\alpha)$-order temporal accuracy (or the second-order temporal accuracy for classical convection-diffusion equations of integer order). Numerical results confirm our theoretical analysis and show the efficiency of the proposed method.

In this paper, we use an indirect approach so that the scheme derived in this way has a very simple and practical form for the problems with spatially variable convection coefficients. The related theoretical analysis is also quite transparent. There are several ways in which our method can be extended. For example, the proposed method may be extended to the multi-dimensional problems under some suitable assumptions and to the equation (1.1) with a linear zero-order damping term $p_1(x)v(x,t)$ being added. However, since our method requires an exponential transformation to eliminate the convection term, the method for the present form may not be suitable for the convection-dominated problems, namely the problems of $|p(x)| \gg d$. Whether it can be extended to solve the convection-dominated problems efficiently will be an interesting subject.

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