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Parallel hybrid extragradient methods for pseudomonotone equilibrium problems and nonexpansive mappings

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Abstract In this paper we propose and analyze three parallel hybrid extragradient methods for finding a common element of the set of solutions of equilibrium problems involving pseudomonotone bifunctions and the set of fixed points of nonexpansive mappings in a real Hilbert space. Based on parallel computation we can reduce the overall computational effort under widely used conditions on the bifunctions and the nonexpansive mappings. A simple numerical example is given to illustrate the proposed parallel algorithms.

Keywords Equilibrium problem · Pseudomonotone bifunction · Lipschitz-type continuity · Nonexpansive mapping · Hybrid method · Parallel computation

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. The equilibrium problem for a bifunction $f : C \times C \to \Re \cup \{+\infty\}$, satisfying condition $f(x, x) = 0$ for every $x \in C$, is stated as follows:

$$\text{Find } x^* \in C \text{ such that } \quad f(x^*, y) \geq 0 \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(f)$. Problem (1) includes, as special cases, many mathematical models, such as, optimization problems, saddle point problems, Nash equilibrium point problems, fixed point problems, convex differentiable optimization problems, variational inequalities, complementarity problems, etc., see [5, 15]. In recent years, many methods have been proposed for solving equilibrium problems, for instance, see [8, 12, 20, 21, 23] and the references therein.

A mapping $T : C \to C$ is said to be nonexpansive if $||T(x) - T(y)|| \leq ||x - y||$ for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$.

Finding common elements of the solution set of an equilibrium problem and the fixed point set of a nonexpansive mapping is a task arising frequently in various areas of mathematical sciences, engineering, and economy. For example, we consider the following extension of a Nash-Cournot oligopolistic equilibrium model [9].

Assume that there are n companies that produce a commodity. Let x denote the vector whose entry x_j stands for the quantity of the commodity producing by company j. We suppose that the price $p_i(s)$ is a decreasing affine function of s with $s = \sum_{j=1}^{n} x_j$, i.e., $p_i(s) = \alpha_i - \beta_i s$, where $\alpha_i > 0$, $\beta_i > 0$. Then the profit made by company j is given by $f_j(x) = p_j(s)x_j - c_j(x_j)$, where $c_j(x_j)$ is the tax for generating x_j. Suppose that K_j is the strategy set of company j, Then the strategy set of the model is $K := K_1 \times \times ... \times K_n$. Actually, each company seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other companies is a parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept.

We recall that a point $x^* \in K = K_1 \times K_2 \times \cdots \times K_n$ is an equilibrium point of the model if

$$f_j(x^*) \geq f_j(x^*[x_j]) \; \forall x_j \in K_j, \; \forall j = 1, 2, \ldots, n,$$

where the vector $x^*[x_j]$ stands for the vector obtained from x^* by replacing x_j^* with x_j. By taking

$$f(x, y) := \psi(x, y) - \psi(x, x)$$

with

$$\psi(x, y) := -\sum_{j=1}^{n} f_j(x[y_j]), \tag{2}$$

the problem of finding a Nash equilibrium point of the model can be formulated as

$$x^* \in K : f(x^*, x) \geq 0 \; \forall x \in K. \tag{EP}$$

In practice each company has to pay a fee $g_j(x_j)$ depending on its production level x_j.

The problem now is to find an equilibrium point with minimum fee. We suppose that both tax and fee functions are convex for every j. The convexity assumption means that the tax and fee for producing a unit are increasing as the quantity of the production gets larger. The convex assumption on c_j implies that the bifunction f is monotone on K, while the convex assumption on g_j ensures that the solution-set of the convex problem

$$\min \left\{ g(x) = \sum_{j-1}^{n} g_j(x_j) : x \in K \right\}$$

coincides with fixed point-set of the nonexpansive proximal operator $P := (I + c\partial g)^{-1}$ with $c > 0$ [19].

Thus the problem of finding an equilibrium point with minimal cost is actually of the same kind as the problem studied in this paper.

Gradient based methods dealing with equilibrium problems as well as iteration methods for nonexpansive and pseudocontractive mappings have been studied by several authors (see, [6, 24–28] and the references therein).

For finding a common element of the set of solutions of monotone equilibrium problem (1) and the set of fixed points of a nonexpansive mapping T in Hilbert spaces, Tada and Takahashi [22] proposed the following hybrid method:

$$\begin{cases} x_0 \in C_0 = Q_0 = C, \\ z_n \in C \quad \text{such that} \quad f(z_n, y) + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \forall y \in C, \\ w_n = \alpha_n x_n + (1 - \alpha_n) T(z_n), \\ C_n = \{ v \in C : ||w_n - v|| \leq ||x_n - v|| \}, \\ Q_n = \{ v \in C : \langle x_0 - x_n, v - x_n \rangle \leq 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

According to the above algorithm, at each step for determining the intermediate approximation z_n we need to solve a strongly monotone regularized equilibrium problem

$$\text{Find } z_n \in C, \text{ such that } f(z_n, y) + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall y \in C. \tag{3}$$

If the bifunction f is only pseudomonotone, then subproblem (3) is not necessarily strongly monotone, even not pseudomonotone, hence the existing algorithms using the monotonicity of the subproblem, cannot be applied. To overcome this difficulty, Anh [1] proposed the following hybrid extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping T and the set of solutions of an equilibrium problem involving a pseudomonotone bifunction f.

$$\begin{cases} x_0 \in C, C_0 = Q_0 = C, \\ y_n = \arg\min \left\{ \lambda_n f(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C \right\}, \\ t_n = \arg\min \left\{ \lambda_n f(y_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C \right\}, \\ z_n = \alpha_n x_n + (1-\alpha_n)T(t_n), \\ C_n = \{v \in C : ||z_n - v|| \le ||x_n - v||\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, v - x_n \rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

Under certain assumptions, the strong convergence of the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ to $x^\dagger := P_{EP(f)\cap F(T)} x_0$ has been established.
Very recently, Anh and Chung [2] have proposed the following parallel hybrid method for finding a common fixed point of a finite family of relatively nonexpansive mappings $\{T_i\}_{i=1}^N$.

$$\begin{cases} x_0 \in C, C_0 = Q_0 = C, \\ y_n^i = J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) J T_i(x_n)\right), i = 1, \ldots, N, \\ i_n = \arg\max_{1\le i\le N} \left\{ \left\| y_n^i - x_n \right\| \right\}, \quad \bar{y}_n := y_n^{i_n}, \\ C_n = \{v \in C : \phi(v, \bar{y}_n) \le \phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle J x_0 - J x_n, x_n - v \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0, n \ge 0, \end{cases} \tag{4}$$

where J is the normalized duality mapping and $\phi(x, y)$ is the Lyapunov functional. This algorithm was extended, modified and generelized by Anh and Hieu [3] for a finite family of asymptotically quasi ϕ-nonexpansive mappings in Banach spaces.

According to algorithm (4), the intermediate approximations y_n^i can be found in parallel. Then the farthest element from x_n among all y_n^i, $i = 1, \ldots, N$, denoted by $\bar{y}_n$, is chosen. Using the element $\bar{y}_n$, the authors constructed two convex closed subsets C_n and Q_n containing the set of common fixed points F and seperating the initial approximation x_0 from F. The next approximation x_{n+1} is defined as the projection of x_0 onto the intersection $C_n \bigcap Q_n$.

The purpose of this paper is to propose three parallel hybrid extragradient algorithms for finding a common element of the set of solutions of a finite family of equilibrium problems for pseudomonotone bifunctions $\{f_i\}_{i=1}^N$ and the set of fixed points of a finite family of nonexpansive mappings $\left\{S_j\right\}_{j=1}^M$ in Hilbert spaces. We combine the extragradient method for dealing with pseudomonotone equilibrium problems (see, [1, 18]), and Mann's or Halpern's iterative algorithms for finding fixed points of nonexpansive mappings [11, 13], with parallel splitting-up techniques [2, 3], as well as hybrid methods (see, [1–3, 12, 17, 20, 21]) to obtain the strong convergence of iterative processes.

The paper is organized as follows: In Section 2, we recall some definitions and preliminary results. Section 3 deals with novel parallel hybrid algorithms and their convergence analysis. Finally, in Section 4, we illustrate the propesed parallel hybrid methods by considering a simple numerical experiment.

2 Preliminaries

In this section, we recall some definitions and results that will be used in the sequel. Let C be a nonempty closed convex subset of a Hilbert space H with an inner product $\langle ., . \rangle$ and the induced norm $||.||$. Let $T : C \to C$ be a nonexpansive mapping with the set of fixed points $F(T)$.

We begin with the following properties of nonexpansive mappings.

Lemma 1 [10] *Assume that $T : C \to C$ is a nonexpansive mapping. If T has a fixed point, then*

(i) *$F(T)$ is a closed convex subset of H.*
(ii) *$I - T$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$.*

Since C is a nonempty closed and convex subset of H, for every $x \in H$, there exists a unique element $P_C x$, defined by

$$P_C x = \arg\min \{\|y - x\| : y \in C\}.$$

The mapping $P_C : H \to C$ is called the metric (orthogonal) projection of H onto C. It is also known that P_C is firmly nonexpansive, or 1-inverse strongly monotone (1-ism), i.e.,

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2.$$

Besides, we have

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2. \tag{5}$$

Moreover, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \tag{6}$$

A function $f : C \times C \to \Re \cup \{+\infty\}$, where $C \subset H$ is a closed convex subset, such that $f(x, x) = 0$ for all $x \in C$ is called a bifunction. Throughout this paper we consider bifunctions with the following properties:

A1. f is pseudomonotone, i.e., for all $x, y \in C$,

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0;$$

A2. f is Lipschitz-type continuous, i.e., there exist two positive constants c_1, c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1||x - y||^2 - c_2||y - z||^2, \quad \forall x, y, z \in C;$$

A3. f is weakly continuous on $C \times C$;
A4. $f(x, .)$ is convex and subdifferentiable on C for every fixed $x \in C$.

A bifunction f is called monotone on C if for all $x, y \in C$, $\quad f(x, y) + f(y, x) \leq 0$. It is obvious that any monotone bifunction is a pseudomonotone one, but not vice versa. Recall that a mapping $A : C \to H$ is pseudomonotone if and only if the bifunction $f(x, y) = \langle A(x), y - x\rangle$ is pseudomonotone on C.

The following statements will be needed in the next section.

Lemma 2 [4] *If the bifunction f satisfies Assumptions $A1 - A4$, then the solution set $EP(f)$ is weakly closed and convex.*

Lemma 3 [7] *Let C be a convex subset of a real Hilbert space H and $g : C \to \Re$ be a convex and subdifferentiable function on C. Then, x^* is a solution to the following convex problem*

$$\min\{g(x) : x \in C\}$$

if and only if $\quad 0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(.)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.

Lemma 4 [17] *Let X be a uniformly convex Banach space, r be a positive number and $B_r(0) \subset X$ be a closed ball with center at origin and the radius r. Then, for any given subset $\{x_1, x_2, \ldots, x_N\} \subset B_r(0)$ and for any positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ with $\sum_{i=1}^N \lambda_i = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i < j$,*

$$\left\| \sum_{k=1}^{N} \lambda_k x_k \right\|^2 \leq \sum_{k=1}^{N} \lambda_k \|x_k\|^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

3 Main results

In this section, we propose three novel parallel hybrid extragradient algorithms for finding a common element of the set of solutions of equilibrium problems for pseudomonotone bifunctions $\{f_i\}_{i=1}^N$ and the set of fixed points of nonexpansive mappings $\left\{S_j\right\}_{j=1}^M$ in a real Hilbert space H.

In what follows, we assume that the solution set

$$F = \left(\cap_{i=1}^N EP(f_i)\right) \bigcap \left(\cap_{j=1}^M F(S_j)\right)$$

is nonempty and each bifunction f_i $(i = 1, \ldots, N)$ satisfies all the conditions $A1 - A4$.

Observe that we can choose the same Lipschitz coefficients $\{c_1, c_2\}$ for all bifunctions $f_i, i = 1, \ldots, N$. Indeed, condition $A2$ implies that $f_i(x, z) - f_i(x, y) - f_i(y, z) \leq c_{1,i}||x - y||^2 + c_{2,i}||y - z||^2 \leq c_1||x - y||^2 + c_2||y - z||^2$, where $c_1 = \max\left\{c_{1,i} : i = 1, \ldots, N\right\}$ and $c_2 = \max\left\{c_{2,i} : i = 1, \ldots, N\right\}$. Hence, $f_i(x, y) + f_i(y, z) \geq f_i(x, z) - c_1||x - y||^2 - c_2||y - z||^2$.

Further, since $F \neq \emptyset$, by Lemmas 1, 2, the sets $F(S_j)$ $j = 1, \ldots, M$ and $EP(f_i)$ $i = 1, \ldots, N$ are nonempty, closed and convex, hence the solution set F is a nonempty closed and convex subset of C. Thus, given any fixed element $x^0 \in C$ there exists a unique element $x^\dagger := P_F(x^0)$.

Algorithm 1 (Parallel Hybrid Mann-extragradient method)

Initialization. $x^0 \in C, 0 < \rho < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$, $n := 0$ and the sequence $\{\alpha_k\} \subset (0, 1)$ satisfies the condition $\limsup_{k\to\infty} \alpha_k < 1$.

Step 1. Solve N strongly convex programs in parallel

$$y_n^i = \operatorname{argmin}\{\rho f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\} \quad i = 1, \ldots, N.$$

Step 2. Solve N strongly convex programs in parallel

$$z_n^i = \operatorname{argmin}\{\rho f_i(y_n^i, y) + \frac{1}{2}||x_n - y||^2 : y \in C\} \quad i = 1, \ldots, N.$$

Step 3. Find among z_n^i, $\quad i = 1, \ldots, N$, the farthest element from x_n, i.e.,

$$i_n = \operatorname{argmax}\{||z_n^i - x_n|| : i = 1, \ldots, N\}, \bar{z}_n := z_n^{i_n}.$$

Step 4. Find intermediate approximations u_n^j in parallel

$$u_n^j = \alpha_n x_n + (1 - \alpha_n) S_j \bar{z}_n, \; j = 1, \ldots, M.$$

Step 5. Find among u_n^j, $\quad j = 1, \ldots, M$, the farthest element from x_n, i.e.,

$$j_n = \operatorname{argmax}\{||u_n^j - x_n|| : j = 1, \ldots, M\}, \bar{u}_n := u_n^{j_n}.$$

Step 6. Construct two closed convex subsets of C

$$C_n = \{v \in C : ||\bar{u}_n - v|| \leq ||x_n - v||\},$$
$$Q_n = \{v \in C : \langle x_0 - x_n, v - x_n \rangle \leq 0\}.$$

Step 7. The next approximation x_{n+1} is defined as the projection of x_0 onto $C_n \cap Q_n$, i.e.,

$$x_{n+1} = P_{C_n \cap Q_n}(x_0).$$

Step 8. If $x_{n+1} = x_n$ then stop. Otherwise, set $n := n + 1$ and go to **Step 1**.

For establishing the strong convergence of Algorithm 1, we need the following results.

Lemma 5 [1, 18] *Suppose that $x^* \in EP(f_i)$, and x_n, y_n^i, z_n^i, $i = 1, \ldots, N$, are defined as in Step 1 and Step 2 of Algorithm 1. Then*

$$||z_n^i - x^*||^2 \leq ||x_n - x^*||^2 - (1 - 2\rho c_1)||y_n^i - x_n||^2 - (1 - 2\rho c_2)||y_n^i - z_n^i||^2. \quad (7)$$

Lemma 6 *If Algorithm 1 reaches a step $n \geq 0$, then $F \subset C_n \cap Q_n$ and x_{n+1} is well-defined.*

Proof As mentioned above, the solution set F is closed and convex. Further, by definitions, C_n and Q_n are the intersections of halfspaces with the closed convex subset C, hence they are closed and convex.

Next, we verify that $F \subset C_n \bigcap Q_n$ for all $n \geq 0$. For every $x^* \in F$, by the convexity of $||.||^2$, the nonexpansiveness of S_j, and Lemma 5, we have

$$\begin{aligned} ||\bar{u}_n - x^*||^2 &= ||\alpha_n x_n + (1-\alpha_n) S_{j_n} \bar{z}_n - x^*||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||S_{j_n}\bar{z}_n - x^*||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||\bar{z}_n - x^*||^2 \\ &\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||x_n - x^*||^2 \\ &\leq ||x_n - x^*||^2. \end{aligned} \tag{8}$$

Therefore, $||\bar{u}_n - x^*|| \leq ||x_n - x^*||$ or $x^* \in C_n$. Hence $F \subset C_n$ for all $n \geq 0$. Now we show that $F \subset C_n \bigcap Q_n$ by induction. Indeed, we have $F \subset C_0$ as above. Besides, $F \subset C = Q_0$, hence $F \subset C_0 \bigcap Q_0$. Assume that $F \subset C_{n-1} \bigcap Q_{n-1}$ for some $n \geq 1$. From $x_n = P_{C_{n-1} \bigcap Q_{n-1}} x_0$ and (6), we get

$$\langle x_n - z, x_0 - x_n \rangle \geq 0, \forall z \in C_{n-1} \bigcap Q_{n-1}.$$

Since $F \subset C_{n-1} \bigcap Q_{n-1}$, $\langle x_n - z, x_0 - x_n \rangle \geq 0$ for all $z \in F$. This together with the definition of Q_n implies that $F \subset Q_n$. Hence $F \subset C_n \bigcap Q_n$ for all $n \geq 1$. Since F and $C_n \cap Q_n$ are nonempty closed convex subsets, $P_F x_0$ and $x_{n+1} := P_{C_n \cap Q_n}(x_0)$ are well-defined. □

Lemma 7 *If Algorithm 1 finishes at a finite iteration $n < \infty$, then x_n is a common element of two sets $\cap_{i=1}^N EP(f_i)$ and $\cap_{j=1}^M F(S_j)$, i.e., $x_n \in F$.*

Proof If $x_{n+1} = x_n$ then $x_n = x_{n+1} = P_{C_n \cap Q_n}(x_0) \in C_n$. By the definition of C_n, $||\bar{u}_n - x_n|| \leq ||x_n - x_n|| = 0$, hence $\bar{u}_n = x_n$. From the definition of j_n, we obtain

$$u_n^j = x_n, \forall j = 1, \ldots, M.$$

This together with the relations $u_n^j = \alpha_n x_n + (1-\alpha_n) S_j \bar{z}_n$ and $0 < \alpha_n < 1$ implies that $x_n = S_j \bar{z}_n$. Let $x^* \in F$. By Lemma 5 and the nonexpansiveness of S_j, we get

$$\begin{aligned} ||x_n - x^*||^2 &= ||S_j \bar{z}_n - x^*||^2 \\ &\leq ||\bar{z}_n - x^*||^2 \\ &\leq ||x_n - x^*||^2 - (1-2\rho c_1)||y_n^{i_n} - x_n||^2 - (1-2\rho c_2)||y_n^{i_n} - \bar{z}_n||^2. \end{aligned}$$

Therefore

$$(1-2\rho c_1)||y_n^{i_n} - x_n||^2 + (1-2\rho c_2)||y_n^{i_n} - \bar{z}_n||^2 \leq 0.$$

Since $0 < \rho < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, from the last inequality we obtain $x_n = y_n^{i_n} = \bar{z}_n$. Therefore $x_n = S_j \bar{z}_n = S_j x_n$ or $x_n \in F(S_j)$ for all $j = 1, \ldots, M$. Moreover, from the relation $x_n = \bar{z}_n$ and the definition of i_n, we also get $x_n = z_n^i$ for all $i = 1, \ldots, N$.

This together with the inequality (7) implies that $x_n = y_n^i$ for all $i = 1, \ldots, N$. Thus,

$$x_n = \text{argmin}\left\{\rho f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\right\}.$$

By [14, Proposition 2.1], from the last relation we conclude that $x_n \in EP(f_i)$ for all $i = 1, \ldots, N$, hence $x_n \in F$. Lemma 7 is proved. □

Lemma 8 *Let* $\{x_n\}, \{y_n^i\}, \{z_n^i\}, \left\{u_n^j\right\}$ *be (infinite) sequences generated by Algorithm 1. Then, there hold the relations*

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||x_n - u_n^j|| = \lim_{n\to\infty} ||x_n - z_n^i|| = \lim_{n\to\infty} ||x_n - y_n^i|| = 0,$$

and $\lim_{n\to\infty} ||x_n - S_j x_n|| = 0$.

Proof From the definition of Q_n and (6), we see that $x_n = P_{Q_n} x_0$. Therefore, for every $u \in F \subset Q_n$, we get

$$\|x_n - x_0\|^2 \le \|u - x_0\|^2 - \|u - x_n\|^2 \le \|u - x_0\|^2 . \tag{9}$$

This implies that the sequence $\{x_n\}$ is bounded. From (8), the sequence $\{\bar{u}_n\}$, and hence, the sequence $\left\{u_n^j\right\}$ are also bounded.

Observing that $x_{n+1} = P_{C_n \bigcap Q_n} x_0 \in Q_n$, $x_n = P_{Q_n} x_0$, from (5) we have

$$\|x_n - x_0\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x_0\|^2 . \tag{10}$$

Thus, the sequence $\{\|x_n - x_0\|\}$ is nondecreasing, hence there exists the limit of the sequence $\{\|x_n - x_0\|\}$. From (10) we obtain

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 .$$

Letting $n \to \infty$, we find

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{11}$$

Since $x_{n+1} \in C_n$, $||\bar{u}_n - x_{n+1}|| \le \|x_{n+1} - x_n\|$. Thus $||\bar{u}_n - x_n|| \le ||\bar{u}_n - x_{n+1}|| + ||x_{n+1} - x_n|| \le 2||x_{n+1} - x_n||$. The last inequality together with (11) implies that $||\bar{u}_n - x_n|| \to 0$ as $n \to \infty$. From the definition of j_n, we conclude that

$$\lim_{n\to\infty} \left\|u_n^j - x_n\right\| = 0 \tag{12}$$

for all $j = 1, \ldots, M$. Moreover, Lemma 5 shows that for any fixed $x^* \in F$, we have

$$\begin{aligned}
||u_n^j - x^*||^2 &= ||\alpha_n x_n + (1-\alpha_n) S_j \bar{z}_n - x^*||^2 \\
&\le \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||S_j \bar{z}_n - x^*||^2 \\
&\le \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||\bar{z}_n - x^*||^2 \\
&\le ||x_n - x^*||^2 \\
&-(1-\alpha_n)|| \left((1-2\rho c_1)||y_n^{i_n} - x_n||^2 + (1-2\rho c_2)||y_n^{i_n} - \bar{z}_n||^2\right).
\end{aligned}$$

Therefore

$$
\begin{aligned}
&(1-\alpha_n)(1-2\rho c_1)||y_n^{i_n}-x_n||^2+(1-2\rho c_2)||y_n^{i_n}-\bar{z}_n||^2\\
&\le ||x_n-x^*||^2-||u_n^j-x^*||^2\\
&=\left(||x_n-x^*||-||u_n^j-x^*||\right)\left(||x_n-x^*||+||u_n^j-x^*||\right)\\
&\le ||x_n-u_n^j||\left(||x_n-x^*||+||u_n^j-x^*||\right).
\end{aligned}
\tag{13}
$$

Using the last inequality together with (12) and taking into account the boundedness of two sequences $\left\{u_n^j\right\}$, $\{x_n\}$ as well as the condition $\limsup_{n\to\infty}\alpha_n<1$, we come to the relations

$$
\lim_{n\to\infty}\left\|y_n^{i_n}-x_n\right\|=\lim_{n\to\infty}\left\|y_n^{i_n}-\bar{z}_n\right\|=0 \tag{14}
$$

for all $i=1,\ldots,N$. From $||\bar{z}_n-x_n||\le||\bar{z}_n-y_n^{i_n}||+||y_n^{i_n}-x_n||$ and (14), we obtain $\lim_{n\to\infty}\|\bar{z}_n-x_n\|=0$. By the definition of i_n, we get

$$
\lim_{n\to\infty}\left\|z_n^i-x_n\right\|=0 \tag{15}
$$

for all $i=1,\ldots,N$. From Lemma 5 and (15), arguing similarly to (13) we obtain

$$
\lim_{n\to\infty}\left\|y_n^i-x_n\right\|=0 \tag{16}
$$

for all $i=1,\ldots,N$. On the other hand, since $u_n^j=\alpha_n x_n+(1-\alpha_n)S_j\bar{z}_n$, we have

$$
\begin{aligned}
||u_n^j-x_n|| &=(1-\alpha_n)||S_j\bar{z}_n-x_n||\\
&=(1-\alpha_n)||(S_jx_n-x_n)+(S_j\bar{z}_n-S_jx_n)||\\
&\ge(1-\alpha_n)\left(||S_jx_n-x_n||-||S_j\bar{z}_n-S_jx_n||\right)\\
&\ge(1-\alpha_n)\left(||S_jx_n-x_n||-||\bar{z}_n-x_n||\right).
\end{aligned}
$$

Therefore

$$
||S_jx_n-x_n||\le||\bar{z}_n-x_n||+\frac{1}{1-\alpha_n}||u_n^j-x_n||.
$$

The last inequality together with (12), (15) and the condition $\limsup_{n\to\infty}\alpha_n<1$ implies that

$$
\lim_{n\to\infty}\left\|S_jx_n-x_n\right\|=0, \tag{17}
$$

for all $j=1,\ldots,M$. The proof of Lemma 8 is complete. □

Lemma 9 *Let $\{x_n\}$ be the sequence generated by Algorithm 1. Suppose that $\bar{x}$ is a weak limit point of $\{x_n\}$. Then $\bar{x}\in F=\left(\bigcap_{i=1}^N EP(f_i)\right)\bigcap\left(\bigcap_{j=1}^M F(S_j)\right)$, i.e., $\bar{x}$*

is a common element of the set of solutions of equilibrium problems for bifunctions $\{f_i\}_{i=1}^N$ *and the set of fixed points of nonexpansive mappings* $\{S_j\}_{j=1}^M$.

Proof From Lemma 8 we see that $\{x_n\}$ is bounded. Then there exists a subsequence of $\{x_n\}$ converging weakly to $\bar{x}$. For the sake of simplicity, we denote the weakly convergent subsequence again by $\{x_n\}$, i.e., $x_n \rightharpoonup \bar{x}$. From (17) and the demiclosedness of $I - S_j$, we have $\bar{x} \in F(S_j)$. Hence, $\bar{x} \in \bigcap_{j=1}^M F(S_j)$. Noting that

$$y_n^i = \operatorname{argmin}\{\rho f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\},$$

by Lemma 3, we obtain

$$0 \in \partial_2 \left\{\rho f_i(x_n, y) + \frac{1}{2}||x_n - y||^2\right\} \left(y_n^i\right) + N_C\left(y_n^i\right).$$

Therefore, there exist $w \in \partial_2 f_i\left(x_n, y_n^i\right)$ and $\bar{w} \in N_C\left(y_n^i\right)$ such that

$$\rho w + x_n - y_n^i + \bar{w} = 0. \tag{18}$$

Since $\bar{w} \in N_C(y_n^i)$, $\langle \bar{w}, y - y_n^i \rangle \le 0$ for all $y \in C$. This together with (18) implies that

$$\rho\left\langle w, y - y_n^i\right\rangle \ge \left\langle y_n^i - x_n, y - y_n^i\right\rangle \tag{19}$$

for all $y \in C$. Since $w \in \partial_2 f_i\left(x_n, y_n^i\right)$,

$$f_i(x_n, y) - f_i(x_n, y_n^i) \ge \left\langle w, y - y_n^i\right\rangle, \forall y \in C. \tag{20}$$

From (19) and (20), we get

$$\rho\left(f_i(x_n, y) - f_i\left(x_n, y_n^i\right)\right) \ge \left\langle y_n^i - x_n, y - y_n^i\right\rangle, \forall y \in C. \tag{21}$$

Since $x_n \rightharpoonup \bar{x}$ and $||x_n - y_n^i|| \to 0$ as $n \to \infty$, we find $y_n^i \rightharpoonup \bar{x}$. Letting $n \to \infty$ in (21) and using assumption A3, we conclude that $f_i(\bar{x}, y) \ge 0$ for all $y \in C$ (i=1,…,N). Thus, $\bar{x} \in \bigcap_{i=1}^N EP(f_i)$, hence $\bar{x} \in F$. The proof of Lemma 9 is complete. □

Theorem 1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $\{f_i\}_{i=1}^N$ is a finite family of bifunctions satisfying conditions A1-A4 and $\{S_j\}_{j=1}^M$ is a finite family of nonexpansive mappings on C. Moreover, suppose that the solution set F is nonempty. Then, the (infinite) sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $x^\dagger = P_F x_0$.*

Proof It is directly followed from Lemma 6 that the sets F, C_n, Q_n are closed convex subsets of C and $F \subset C_n \bigcap Q_n$ for all $n \geq 0$. Moreover, from Lemma 8 we see that the sequence $\{x_n\}$ is bounded. Suppose that $\bar{x}$ is any weak limit point of $\{x_n\}$ and $x_{n_j} \rightharpoonup \bar{x}$. By Lemma 9, $\bar{x} \in F$. We now show that the sequence $\{x_n\}$ converges strongly to $x^\dagger := P_F x_0$. Indeed, from $x^\dagger \in F$ and (9), we obtain

$$||x_{n_j} - x_0|| \leq ||x^\dagger - x_0||.$$

The last inequality together with $x_{n_j} \rightharpoonup \bar{x}$ and the weak lower semicontinuity of the norm $||.||$ implies that

$$||\bar{x} - x_0|| \leq \lim\inf_{j\to\infty} ||x_{n_j} - x_0|| \leq \lim\sup_{j\to\infty} ||x_{n_j} - x_0|| \leq ||x^\dagger - x_0||.$$

By the definition of $x^\dagger$, $\bar{x} = x^\dagger$ and $\lim_{j\to\infty} ||x_{n_j} - x_0|| = ||x^\dagger - x_0||$. Since $x_{n_j} - x_0 \rightharpoonup \bar{x} - x_0 = x^\dagger - x_0$, the Kadec-Klee property of the Hilbert space H ensures that $x_{n_j} - x_0 \to x^\dagger - x_0$, hence $x_{n_j} \to x^\dagger$ as $j \to \infty$. Since $\bar{x} = x^\dagger$ is any weak limit point of $\{x_n\}$, the sequence $\{x_n\}$ converges strongly to $x^\dagger := P_F x_0$. The proof of Theorem 1 is complete. □

Corollary 1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $\{f_i\}_{i=1}^N$ is a finite family of bifunctions satisfying conditions $A1 - A4$, and the set $F = \bigcap_{i=1}^N EP(f_i)$ is nonempty. Let $\{x_n\}$ be the sequence generated in the following manner:*

$$\begin{cases} x_0 \in C_0 := C, Q_0 := C, \\ y_n^i = \text{argmin}\{\rho f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\} \quad i = 1, \dots, N, \\ z_n^i = \text{argmin}\{\rho f_i(y_n^i, y) + \frac{1}{2}||x_n - y||^2 : y \in C\} \quad i = 1, \dots, N, \\ i_n = \text{argmax}\{||z_n^i - x_n|| : i = 1, \dots, N\}, \bar{z}_n := z_n^{i_n}, \\ C_n = \{v \in C : ||\bar{z}_n - v|| \leq ||x_n - v||\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, v - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0, n \geq 0, \end{cases}$$

where $0 < \rho < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$. Then the sequence $\{x_n\}$ converges strongly to $x^\dagger = P_F x_0$.

Corollary 2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $\{A_i\}_{i=1}^N$ is a finite family of pseudomonotone and L-Lipschitz continuous mappings from C to H such that $F = \bigcap_{i=1}^N VI(A_i, C)$ is nonempty, where*

$VI(A_i, C) = \{x^* \in C : \langle A(x^*), y - x^*\rangle \geq 0,\ \forall y \in C\}$. Let $\{x_n\}$ be the sequence generated in the following manner:

$$\begin{cases} x_0 \in C_0 := C, Q_0 := C, \\ y_n^i = P_C\left(x_n - \rho A_i(x_n)\right) \quad i = 1, \ldots, N, \\ z_n^i = P_C\left(x_n - \rho A_i(y_n^i)\right) \quad i = 1, \ldots, N, \\ i_n = \operatorname{argmax}\{||z_n^i - x_n|| : i = 1, \ldots, N\}, \bar{z}_n := z_n^{i_n}, \\ C_n = \{v \in C : ||\bar{z}_n - v|| \leq ||x_n - v||\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, v - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0, n \geq 0, \end{cases}$$

where $0 < \rho < \frac{1}{L}$. Then the sequence $\{x_n\}$ converges strongly to $x^\dagger = P_F x_0$.

Proof Let $f_i(x, y) = \langle A_i(x), y - x\rangle$ for all $x, y \in C$ and $i = 1, \ldots, N$.
Since A_i is L-Lipschitz continuous, for all $x, y, z \in C$

$$\begin{aligned} f_i(x, y) + f_i(y, z) - f_i(x, z) &= \langle A_i(x), y - x\rangle + \langle A_i(y), z - y\rangle - \langle A_i(x), z - x\rangle \\ &= -\langle A_i(y) - A_i(x), y - z\rangle \\ &\geq -||A_i(y) - A_i(x)||||y - z|| \\ &\geq -L||y - x||||y - z|| \\ &\geq -\frac{L}{2}||y - x||^2 - \frac{L}{2}||y - z||^2. \end{aligned}$$

Therefore f_i is Lipschitz-type continuous with $c_1 = c_2 = \frac{L}{2}$. Moreover, the pseudomonotonicity of A_i ensures the pseudomonotonicity of f_i. Conditions A3, A4 are satisfied automatically. According to Algorithm 1, we have

$$\begin{aligned} y_n^i &= \operatorname{argmin}\{\rho\langle A_i(x_n), y - x_n\rangle + \frac{1}{2}||x_n - y||^2 : y \in C\}, \\ z_n^i &= \operatorname{argmin}\{\rho\left\langle A_i(y_n^i), y - y_n^i\right\rangle + \frac{1}{2}||x_n - y||^2 : y \in C\}. \end{aligned}$$

Or

$$\begin{aligned} y_n^i &= \operatorname{argmin}\left\{\frac{1}{2}||y - (x_n - \rho A_i(x_n))||^2 : y \in C\right\} = P_C(x_n - \rho A_i(x_n)), \\ z_n^i &= \operatorname{argmin}\left\{\frac{1}{2}||y - \left(x_n - \rho A_i\left(y_n^i\right)\right)||^2 : y \in C\right\} = P_C\left(x_n - \rho A_i\left(y_n^i\right)\right). \end{aligned}$$

Application of Theorem 1 with the above mentioned $f_i(x, y)$, $(i = 1, \ldots, N)$ and $S_j = I$, $(j = 1, \ldots, M)$ leads to the desired result. □

Remark 1 Putting $N = 1$ in Corollary 2, we obtain the corresponding result of Nadezhkina and Takahashi [16, Theorem 4.1].

Now, replacing Mann's iteration in Step 4 of Algorithm 1 by Halpern's one, we come to the following algorithm.

Algorithm 2 (Parallel hybrid Halpern-extragradient method)

Initialization. $x_0 \in C, 0 < \rho < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$, $n := 0$ and the sequence $\{\alpha_k\} \subset (0, 1)$ satisfies the condition $\lim_{k\to\infty} \alpha_k = 0$.

Step 1. Solve N strongly convex programs in parallel

$$y_n^i = \text{argmin}\{\rho f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\} \quad i = 1, \dots, N.$$

Step 2. Solve N strongly convex programs in parallel

$$z_n^i = \text{argmin}\{\rho f_i(y_n^i, y) + \frac{1}{2}||x_n - y||^2 : y \in C\} \quad i = 1, \dots, N.$$

Step 3. Find among $z_n^i, \quad i = 1, \dots, N$, the farthest element from x_n, i.e.,

$$i_n = \text{argmax}\{||z_n^i - x_n|| : i = 1, \dots, N\}, \bar{z}_n := z_n^{i_n}.$$

Step 4. Find intermediate approximations u_n^j in parallel

$$u_n^j = \alpha_n x_0 + (1 - \alpha_n) S_j \bar{z}_n, j = 1, \dots, M.$$

Step 5. Find among $u_n^j, \quad j = 1, \dots, M$, the farthest element from x_n, i.e.,

$$j_n = \text{argmax}\{||u_n^j - x_n|| : j = 1, \dots, M\}, \bar{u}_n := u_n^{j_n}.$$

Step 6. Construct two closed convex subsets of C

$$C_n = \{v \in C : ||\bar{u}_n - v||^2 \le \alpha_n||x_0 - v||^2 + (1 - \alpha_n)||x_n - v||^2\},$$
$$Q_n = \{v \in C : \langle x_0 - x_n, v - x_n\rangle \le 0\}.$$

Step 7. The next approximation x_{n+1} is defined as the projection of x_0 onto $C_n \cap Q_n$, i.e.,

$$x_{n+1} = P_{C_n \cap Q_n}(x_0).$$

Step 8. Put $n := n + 1$ and go to **Step 1**.

Remark 2 For Algorithm 2, the claim that x_n is a common solution of the equlibrium and fixed point problems, if $x_{n+1} = x_n$, in general is not true. So in practice, we need to use some "stopping rule" like if $n > n_{\max}$ for some chosen sufficiently large number $n_{\max}$, then stop.

Theorem 2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $\{f_i\}_{i=1}^N$ is a finite family of bifunctions satisfying conditions $A1 - A4$, and $\{S_j\}_{j=1}^M$ is a finite family of nonexpansive mappings on C. Moreover, suppose that the solution set F is nonempty. Then, the sequence $\{x_n\}$ generated by the Algorithm 2 converges strongly to $x^\dagger = P_F x_0$.*

Proof Arguing similarly as in the proof of Lemma 6 and Theorem 1, we conclude that F, C_n, Q_n are closed and convex. Besides, $F \subset C_n \cap Q_n$ for all $n \geq 0$. Moreover, the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{22}$$

Since $x_{n+1} \in C_{n+1}$,

$$||\bar{u}_n - x_{n+1}||^2 \leq \alpha_n ||x_0 - x_{n+1}||^2 + (1-\alpha_n)||x_n - x_{n+1}||^2.$$

Letting $n \to \infty$, from (22), $\lim_{n\to\infty} \alpha_n = 0$ and the boundedness of $\{x_n\}$, we obtain

$$\lim_{n\to\infty} ||\bar{u}_n - x_{n+1}|| = 0.$$

Proving similarly to (12) and (13), we get

$$\lim_{n\to\infty} ||u_n^j - x_n|| = 0, \quad j = 1, \ldots, M,$$

and

$$\begin{aligned}
&(1-\alpha_n)(1-2\rho c_1)||y_n^{i_n} - x_n||^2 + (1-2\rho c_2)||y_n^{i_n} - \bar{z}_n||^2 \\
&\leq \alpha_n (||x_0 - x^*||^2 - ||x_n - x^*||^2) \\
&+||x_n - u_n^j|| \left(||x_n - x^*|| + ||u_n^j - x^*||\right)
\end{aligned} \tag{23}$$

for each $x^* \in F$. Letting $n \to \infty$ in (23), one has

$$\lim_{n\to\infty} ||y_n^{i_n} - x_n|| = \lim_{n\to\infty} ||\bar{z}_n - x_n|| = 0, \quad j = 1, \ldots, N,$$

Repeating the proof of (15) and (16), we get

$$\lim_{n\to\infty} ||y_n^i - x_n|| = \lim_{n\to\infty} ||z_n^i - x_n|| = 0, \quad i = 1, \ldots, N.$$

Using $u_n^j = \alpha_n x_0 + (1-\alpha_n) S_j \bar{z}_n$, by a straightforward computation, we obtain

$$||S_j x_n - x_n|| \leq ||\bar{z}_n - x_n|| + \frac{1}{1-\alpha_n}||u_n^j - x_n|| + \frac{\alpha_n}{1-\alpha_n}||x_0 - x_n||,$$

which implies that $\lim_{n\to\infty} ||S_j x_n - x_n|| = 0$. The rest of the proof of Theorem 2 is similar to the arguments in the proofs of Lemma 9 and Theorem 1. □

Next replacing Steps 4 and 5 in Algorithm 1, consisting of a Mann's iteration and a parallel splitting-up step, by an iteration step involving a convex combination of the identity mapping I and the mappings S_j, $j = 1, \ldots, N$, we come to the following algorithm.

Algorithm 3 (Parallel hybrid iteration-extragradient method)

Initialization. $x^0 \in C, 0 < \rho < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$, $n := 0$ and the positive sequences $\left\{\alpha_{k,l}\right\}_{k=1}^{\infty}$ $(l = 0, \dots, M)$ satisfy the conditions: $0 \leq \alpha_{k,j} \leq 1$, $\sum_{j=0}^{M} \alpha_{k,j} = 1$, $\liminf_{k\to\infty} \alpha_{k,0}\alpha_{k,l} > 0$ for all $l = 1, \dots, M$.

Step 1. Solve N strongly convex programs in parallel

$$y_n^i = \operatorname{argmin}\{\rho f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\} \quad i = 1, \dots, N.$$

Step 2. Solve N strongly convex programs in parallel

$$z_n^i = \operatorname{argmin}\{\rho f_i(y_n^i, y) + \frac{1}{2}||x_n - y||^2 : y \in C\} \quad i = 1, \dots, N.$$

Step 3. Find among $z_n^i, \quad i = 1, \dots, N$, the farthest element from x_n, i.e.,

$$i_n = \operatorname{argmax}\{||z_n^i - x_n|| : i = 1, \dots, N\}, \bar{z}_n := z_n^{i_n}.$$

Step 4. Compute in parallel $u_n^j := S_j \bar{z}_n;\ j = 1, \dots, M$, and put

$$u_n = \alpha_{n,0} x_n + \sum_{j=1}^{M} \alpha_{n,j} u_n^j.$$

Step 5. Construct two closed convex subsets of C

$$C_n = \{v \in C : ||u_n - v|| \leq ||x_n - v||\},$$
$$Q_n = \{v \in C : \langle x_0 - x_n, v - x_n \rangle \leq 0\}.$$

Step 6. The next approximation x_{n+1} is determined as the projection of x_0 onto $C_n \cap Q_n$, i.e.,

$$x_{n+1} = P_{C_n \cap Q_n}(x_0).$$

Step 7. If $x_{n+1} = x_n$ then stop. Otherwise, set $n := n + 1$ and go to **Step 1**.

Remark 3 Arguing similarly as in the proof of Lemma 7, we can prove that if Algorithm 3 finishes at a finite iteration $n < \infty$, then $x_n \in F$, i.e., x_n is a common element of the set of solutions of equilibrium problems and the set of fixed points of nonexpansive mappings.

Theorem 3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $\{f_i\}_{i=1}^N$ is a finite family of bifunctions satisfying conditions $A1 - A4$, and $\left\{S_j\right\}_{j=1}^M$ is a finite family of nonexpansive mappings on C. Moreover, suppose that the solution set F is nonempty. Then, the (infinite) sequence $\{x_n\}$ generated by the Algorithm 3 converges strongly to $x^\dagger = P_F x_0$.*

Proof Arguing similarly as in the proof of Theorem 1, we can conclude that F, C_n, Q_n are closed convex subsets of C. Besides, $F \subset C_n \bigcap Q_n$ and

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||y_n^i - x_n|| = \lim_{n\to\infty} ||z_n^i - x_n|| = \lim_{n\to\infty} ||u_n - x_n|| = 0 \tag{24}$$

for all $i = 1, \ldots, N$. For every $x^* \in F$, by Lemmas 4 and 5, we have

$$\begin{aligned}
||u_n - x^*||^2 &= ||\alpha_{n,0}x_n + \sum_{j=1}^{M} \alpha_{n,j} S_j \bar{z}_n - x^*||^2 \\
&= ||\alpha_{n,0}(x_n - x^*) + \sum_{j=1}^{M} \alpha_{n,j} (S_j \bar{z}_n - x^*)||^2 \\
&\le \alpha_{n,0}||x_n - x^*||^2 + \sum_{j=1}^{M} \alpha_{n,j} ||S_j \bar{z}_n - x^*||^2 - \alpha_{n,0}\alpha_{n,l} g(||S_l \bar{z}_n - x_n||) \\
&\le \alpha_{n,0}||x_n - x^*||^2 + \sum_{j=1}^{M} \alpha_{n,j} ||\bar{z}_n - x^*||^2 - \alpha_{n,0}\alpha_{n,l} g(||S_l \bar{z}_n - x_n||) \\
&\le \alpha_{n,0}||x_n - x^*||^2 + \sum_{j=1}^{M} \alpha_{n,j} ||x_n - x^*||^2 - \alpha_{n,0}\alpha_{n,l} g(||S_l \bar{z}_n - x_n||) \\
&\le ||x_n - x^*||^2 - \alpha_{n,0}\alpha_{n,l} g(||S_l \bar{z}_n - x_n||).
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha_{n,0}\alpha_{n,l} g(||S_l \bar{z}_n - x_n||) &\le ||x_n - x^*||^2 - ||u_n - x^*||^2 \\
&\le \left(||x_n - x^*|| - ||u_n - x^*||\right)\left(||x_n - x^*|| + ||u_n - x^*||\right) \\
&\le ||x_n - u_n|| \left(||x_n - x^*|| + ||u_n - x^*||\right).
\end{aligned}$$

The last inequality together with (24), $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,l} > 0$ and the boundedness of $\{x_n\}, \{u_n\}$ implies that $\lim_{n\to\infty} g(||S_l \bar{z}_n - x_n||) = 0$. Hence

$$\lim_{n\to\infty} ||S_l \bar{z}_n - x_n|| = 0. \tag{25}$$

Moreover, from (24),(25) and $||S_l x_n - x_n|| \le ||S_l x_n - S_l \bar{z}_n|| + ||S_l \bar{z}_n - x_n|| \le ||x_n - \bar{z}_n|| + ||S_l \bar{z}_n - x_n||$ we obtain

$$\lim_{n\to\infty} ||S_l x_n - x_n|| = 0$$

for all $l = 1, \ldots, M$. The same argument as in the proofs of Lemma 9 and Theorem 1 shows that the sequence $\{x_n\}$ converges strongly to $x^\dagger := P_F x_0$. The proof of Theorem 3 is complete. □

Remark 4 Putting $M = N = 1$ in Theorems 1 and 3, we obtain the corresponding result announced in [1, Theorem 3.1].

4 Numerical experiment

Let $H = \Re^1$ be a Hilbert space with the standart inner product $\langle x, y\rangle := xy$ and the norm $||x|| := |x|$ for all $x, y \in H$. Consider the bifunctions defined on the set $C := [0, 1] \subset H$ by

$$f_i(x, y) := B_i(x)(y - x), i = 1, \ldots, N,$$

where $B_i(x) = 0$ if $0 \le x \le \xi_i$, and $B_i(x) = \exp(x - \xi_i) + \sin(x - \xi_i) - 1$ if $\xi_i \le x \le 1$. Here $0 < \xi_1 < \ldots < \xi_N < 1$. Obviously, conditions A3, A4 for the bifunctions f_i are satisfied. Further, since $B_i(x)$ is nondecreasing on [0, 1],

$$f_i(x, y) + f_i(y, x) = (x - y)(B_i(y) - B_i(x)) \le 0.$$

Thus, each bifunction f_i is monotone, and so is pseudomonotone. Moreover, $B_i(x)$ is 4-Lipschitz continuous. A straightforward calculation yields $f_i(x, y) + f_i(y, z) - f_i(x, z) = (y - z)(B_i(x) - B_i(y)) \ge -4|x - y||y - z| \ge -2(x - y)^2 - 2(y - z)^2$, which proves the Lipschitz-type continuity of f_i with $c_1 = c_2 = 2$. Finally,

$$f_i(x, y) = B_i(x)(y - x) \ge 0, \quad \forall y \in [0, 1]$$

if and only if $0 \le x \le \xi_i$, i.e., $EP(f_i) = [0, \xi_i]$. Therefore $\cap_{i=1}^N EP(f_i) = [0, \xi_1]$.

Define the mappings

$$S_j x := \frac{x^j \sin^{j-1}(x)}{2j - 1}, \quad j = 1, \ldots, M.$$

Clearly, $S_j : C \to C$ and

$$|S_j{}'(x)| = \frac{1}{2j - 1}|jx^{j-1}\sin^{j-1}(x) + (j - 1)x^j \sin^{j-2}(x)\cos(x)| \le 1.$$

Hence $S_j,\ j = 1, \ldots, M$ are nonexpansive mappings. Moreover, $F(S_1) = [0, 1]$ and $F(S_j) = \{0\},\ j = 2, \ldots, M$. Thus, the solution set

$$F = \left(\cap_{i=1}^N EP(f_i)\right) \bigcap \left(\cap_{j=1}^M F(S_j)\right) = \{0\}.$$

By Algorithm 1, we have

$$y_n^i = \arg\min\left\{\rho B_i(x_n)(y - x_n) + \frac{1}{2}(y - x_n)^2 : y \in [0; 1]\right\}. \tag{26}$$

A simple computation shows that (26) is equivalent to the following relation

$$y_n^i = x_n - \rho B_i(x_n), \quad i = 1, \ldots, N.$$

Similarly, we obtain

$$z_n^i = x_n - \rho B_i(y_n^i), \quad i = 1, \ldots, N. \tag{27}$$

From (27), we can find the itermediate approximation $\bar{z}_n$ which is the farthest from x_n among $z_n^i,\ i = 1, \ldots, N$. Therefore,

$$u_n^j = \alpha_n x_n + (1 - \alpha_n)\frac{\bar{z}_n^j \sin^{j-1}(\bar{z}_n)}{2j - 1},\ j = 1, \ldots, M. \tag{28}$$

From (28), we can find the intermediate approximation $\bar{u}_n$ which is farthest from x_n among u_n^j, $j = 1, \ldots, M$. By Lemma 7, if $x_n = \bar{u}_n$, $x_n = 0 \in F$. Otherwise, if $x_n > \bar{u}_n \geq 0$, by the proof of Theorem 1, $0 \in C_n$, i.e., $|\bar{u}_n| \leq |x_n|$, hence $0 \leq \bar{u}_n < x_n$. This together with the definitions of C_n and Q_n lead us to the following formulas:

$$C_n = \left[0, \frac{x_n + \bar{u}_n}{2}\right];$$
$$Q_n = [0, x_n].$$

Therefore

$$C_n \cap Q_n = \left[0, \min\left\{x_n, \frac{x_n + \bar{u}_n}{2}\right\}\right].$$

Since $\bar{u}_n \leq x_n$, we find $\frac{x_n+\bar{u}_n}{2} \leq x_n$. So

$$C_n \cap Q_n = \left[0, \frac{x_n + \bar{u}_n}{2}\right].$$

From the definition of x_{n+1} we obtain

$$x_{n+1} = \frac{x_n + \bar{u}_n}{2}.$$

Thus we come to the following algorithm:
Initialization. $x_0 := 1$; $n := 1$; $\rho := 1/5$; $\alpha_n := 1/n$; $\epsilon := 10^{-5}$; $\xi_i := i/(N+1)$, $i = 1, \ldots, N$; $N := 2 \times 10^6$; $M := 3 \times 10^6$.
Step 1. Find the intermediate approximations y_n^i in parallel ($i = 1, \ldots, N$).

$$y_n^i = \begin{cases} x_n & \text{if } \quad 0 \leq x_n \leq \xi_i, \\ x_n - \rho[\exp(x_n - \xi_i) + \sin(x_n - \xi_i) - 1] & \text{if } \quad \xi_i < x_n \leq 1. \end{cases}$$

Step 2. Find the intermediate approximations z_n^i in parallel ($i = 1, \ldots, N$).

$$z_n^i = \begin{cases} x_n & \text{if } \quad 0 \leq y_n^i \leq \xi_i, \\ x_n - \rho[\exp(y_n^i - \xi_i) + \sin(y_n^i - \xi_i) - 1] & \text{if } \quad \xi_i < y_n^i \leq 1. \end{cases}$$

Step 3. Find the element $\bar{z}_n$ which is farthest from x_n among z_n^i, $i = 1, \ldots, N$.

$$i_n = \arg\max\left\{|z_n^i - x_n| : i = 1, \ldots, N\right\}, \ \bar{z}_n = z_n^{i_n}.$$

Step 4. Find the intermediate approximations u_n^j in parallel

$$u_n^j = \alpha_n x_n + (1 - \alpha_n)\frac{\bar{z}_n^j \sin^{j-1}(\bar{z}_n)}{2j - 1}, \ j = 1, \ldots, M.$$

Step 5. Find the element $\bar{u}_n$ which is farthest from x_n among u_n^j, $j = 1, \ldots, M$.

$$j_n = \arg\max\left\{|u_n^j - x_n| : j = 1, \ldots, M\right\}, \ \bar{u}_n = z_n^{j_n}.$$

Step 6. If $|\bar{u}_n - x_n| \leq \epsilon$ then stop. Otherwise go to **Step 7.**
Step 7. $x_{n+1} = \frac{x_n+\bar{u}_n}{2}$.
Step 8. If $|x_{n+1} - x_n| \leq \epsilon$ then stop. Otherwise, set $n := n + 1$ and go to **Step 1.**

Table 1 Experiment with $\alpha_n = \frac{1}{n}$

TOL	PHMEM	
	T_p	T_s
10^{-5}	5.23	9.98
10^{-6}	5.86	11.25
10^{-8}	7.57	14.33

The numerical experiment is performed on a LINUX cluster 1350 with 8 computing nodes. Each node contains two Intel Xeon dual core 3.2 GHz, 2GBRam. All the programs are written in C.

For given tolerances we compare execution time of the parallel hybrid Mann-extragradient method (PHMEM) in parallel and sequential modes.

We use the following notations:

PHMEM	The parallel hybrid Mann-extragradient method
TOL	Tolerance $\|x_k - x^*\|$
T_p	Time for PHMEM's execution in parallel mode (2CPUs - in seconds)
T_s	Time for PHMEM's execution in sequential mode (in seconds)

According to the above experiment, in the most favourable cases the speed up and the efficiency of the parallel hybrid Mann-extragradient method are $S_p = T_s/T_p \approx 2$; $E_p = S_p/2 \approx 1$, respectively (Table 1).

5 Concluding remarks

In this paper we proposed three parallel hybrid extragradient methods for finding a common element of the set of solutions of equilibrium problems for pseudomonotone bifunctions $\{f_i\}_{i=1}^N$ and the set of fixed points of nonexpansive mappings $\left\{S_j\right\}_{j=1}^M$ in Hilbert spaces, namely:

– a parallel hybrid Mann-extragradient method;
– a parallel hybrid Halpern-extragradient method, and
– a parallel hybrid iteration-extragradient method.

The efficiency of the proposed parallel algorithms is verified by a simple numerical experiment on computing clusters.

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References

1. Anh, P.N.: A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems. Bull. Malays. Math. Sci. Soc. **36**(1), 107–116 (2013)
2. Anh, P.K., Chung, C.V.: Parallel hybrid methods for a finite family of relatively nonexpansive mappings. Numer. Funct. Anal. Optim. **35**(6), 649–664 (2014)
3. Anh, P.K., Hieu, D.V.: Parallel and sequential hybrid methods for a finite family of asymptotically quasi ϕ-nonexpansive mappings. J. Appl. Math. Comput. **48**(1), 241–263 (2015)
4. Bianchi, M., Schaible, S.: Generalized monotone bifunctions and equilibrium problems. J. Optim. Theory Appl. **90**, 31–43 (1996)
5. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Program. **63**, 123–145 (1994)
6. Chang, S.S., Kim, J.K., Wang, X.R.: Modified block iterative algorithm for solving convex feasibility problems in Banach spaces. J. Inequal. Appl. **2010**, 869–684 (2010). 10.1155/2010/869684
7. Daniele, P., Giannessi, F., Maugeri, A.: Equilibrium problems and variational models. Kluwer (2003)
8. Dinh, B.V., Hung, P.G., Muu, L.D.: Bilevel optimization as a regularization approach to pseudomonotone equilibrium problems. Numer. Funct. Anal. Optim. **35**(5), 539–563 (2014)
9. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer, Berlin (2002)
10. Goebel, K., Kirk, W.A.: Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Math, vol. 28. Cambridge University Press, Cambridge (1990)
11. Halpern, B.: Fixed points of nonexpanding maps. Bull. Amer. Math. Soc. **73**, 957–961 (1967)
12. Kang, J., Su, Y., Zhang, X.: Hybrid algorithm for fixed points of weak relatively nonexpansive mappings and applications. Nonlinear Anal. Hybrid Syst. **4**, 755–765 (2010)
13. Mann, W.R.: Mean value methods in iteration. Proc. Amer. Math. Soc. **4**, 506–510 (1953)
14. Mastroeni, G.: On auxiliary principle for equilibrium problems. Publ. Dipart. Math. Univ. Pisa **3**, 1244–1258 (2000)
15. Muu, L.D., Oettli, W.: Convergence of an adative penalty scheme for finding constrained equilibria. Nonlinear Anal. TMA **18**(12), 1159–1166 (1992)
16. Nadezhkina, N., Takahashi, W.: Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings. SIAM J. Optim. **16**, 1230–1241 (2006)
17. Plubtieng, S., Ungchittrakool, K.: Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2008**, 19 (2008). Art. ID 583082
18. Quoc, T.D., Muu, L.D., Hien, N.V.: Extragradient algorithms extended to equilibrium problems. Optimization **57**, 749–776 (2008)
19. Rockafellar R. T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **5**, 877–898 (1976)
20. Saewan, S., Kumam, P.: The hybrid block iterative algorithm for solving the system of equilibrium problems and variational inequality problems. Saewan and Kumam Springer Plus **2012** (2012). http://www.springerplus.com/content/1/1/8
21. Su, Y., Li, M., Zhang, H.: New monotone hybrid algorithm for hemi-relatively nonexpansive mappings and maximal monotone operators. Appl. Math. Comput. **217**(12), 5458–5465 (2011)
22. Tada, A., Takahashi, W.: Strong convergence theorem for an equilibrium problem and a nonexpansive mapping. In: Takahashi, W., Tanaka, T. (eds.) Nonlinear Analysis and Convex Analysis. Yokohama Publishers, Yokohama (2006)
23. Takahashi, S., Takahashi, W.: Viscosity approximation methods for equilibrium problems and fixed point in Hilbert space. J. Math. Anal. Appl. **331**(1), 506–515 (2007)
24. Tan, K.K., Xu, H.K.: Approximating fixed points of nonexpansive by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301–308 (1993)
25. Thakur, B.S., Postolache, M.: Existence and approximation of solutions for generalized extended nonlinear variational inequalities. J. Inequal. Appl. **2013** (2013). Art. No. 590
26. Yao, P.ostolache., M: Iterative methods for pseudomonotone variational inequalities and fixed point problems. J. Optim. Theory Appl. **155**(1), 273–287 (2012)
27. Yao, Y., Postolache, M., Liou, Y.C.: Variant extragradient-type method for monotone variational inequalities. Fixed Point Theory Appl. **2013** (2013). Art. No. 185
28. Yao, Y., Postolache, M., Liou, Y.C.: Coupling Ishikawa algorithms with hybrid techniques for pseudocontractive mappings. Fixed Point Theory Appl. **2013** (2013). Art. No. 211