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ORIGINAL PAPER

Numerical treatment of a well-posed Chebyshev Tau method for Bagley-Torvik equation with high-order of accuracy

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Abstract The main purpose of this study is to develop and analyze a new high-order operational Tau method based on the Chebyshev polynomials as basis functions for obtaining the numerical solution of Bagley-Torvik equation which has a important role in the fractional calculus. It is shown that some derivatives of the solutions of these equations have a singularity at origin. To overcome this drawback we first change the original equation into a new equation with a better regularity properties by applying a regularization process and thereby the operational Chebyshev Tau method can be applied conveniently. Our proposed method has two main advantages. First, the algebraic form of the Tau discretization of the problem has an upper triangular structure which can be solved by forward substitution method. Second, Tau approximation of the problem converges to the exact ones with a highly rate of convergence under a more general regularity assumptions on the input data in spite of the singularity behavior of the exact solution. Numerical results are presented which confirm the theoretical results obtained and efficiency of the proposed method.

Keywords Bagley-Torvik equation · Caputo derivative · Regularization · Tau method · Well-posedness

Mathematics Subject Classification (2010) 34A08 · 65L60

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1 Introduction

This paper deals with the numerical treatment of the following Bagley-Torvik equation

$$\begin{cases} Au''(t) + B\mathcal{D}^{\frac{3}{2}}u(t) + Cu(t) = f(t), & t \in \Lambda = [0, 1] \\ u(0) = u'(0) = 0, \\ A \neq 0, \quad B, C \in \mathbb{R}, \quad f(t) \in C(\Lambda). \end{cases} \tag{1}$$

In this study, $\mathcal{D}^q$ denotes the Caputo fractional derivative operator of order $q \in \mathbb{Q}^+$ given by (See [9, 17, 26])

$$\mathcal{D}^q u(t) = \mathcal{I}^{\lceil q \rceil - q}\left(u^{(\lceil q \rceil)}\right),$$

where the symbol $\lceil q \rceil$ is the smallest integer greater than or equal to q. $\mathcal{I}^\delta$ is the fractional integral operator from order $\delta \in \mathbb{Q}^+$ defined by

$$\mathcal{I}^\delta u(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} u(s) ds.$$

Here $\Gamma(\delta)$ is the well known Gamma function and $\mathbb{R}$, $\mathbb{Q}^+$ are (as usual) the set of real numbers and positive rational numbers respectively. $u(t)$ is an unknown function and $C(\Lambda)$ is the space of all continuous functions over Λ. It can be seen [9, 17, 26] that if $u(t)$ be a continuous function, we have

$$\mathcal{D}^q\left(\mathcal{I}^\delta u\right) = \mathcal{I}^{\delta-q}u, \quad 0 < q \leq \delta, \tag{2}$$

where the fractional integral $\mathcal{I}^{\delta-q}u$ exists.

Note that the homogeneous initial conditions in (1) are not restrictive because (1) with non-homogeneous initial conditions $u(0) = d_0$, $u'(0) = d_1$ can be converted to the following Bagley-Torvik equation with homogeneous initial conditions

$$\begin{cases} A\tilde{u}''(t) + B\mathcal{D}^{\frac{3}{2}}\tilde{u}(t) + C\tilde{u}(t) = \tilde{f}(t), \\ \tilde{u}(0) = \tilde{u}'(0) = 0, \\ A \neq 0, \quad B, C \in \mathbb{R}, \quad \tilde{f}(t) = f(t) - d_0 - d_1 t \in C(\Lambda), \end{cases}$$

by a simple transformation $\tilde{u}(t) = u(t) - d_0 - d_1 t$.

Such kind of equations arising in the motion of real physical systems, an immersed plate in a Newtonian fluid and in a micro-electro-mechanical system (MEMS) instrument that has been designed primarily to measure viscosity of fluids that are encountered during oil well exploration; see [1, 10, 13, 26]. This equation was originally proposed by the authors of [28] and it thoroughly investigated by Podlubny in his book [26]. The questions of existence and uniqueness of the solution to this problem have been discussed in [19].

Now, we try to describe the smoothness degree of the solutions of (1). To this end, we assume that $u''(t) = v(t)$. From (2) and homogeneous initial conditions in (1), we obtain

$$u(t) = \int_0^t (t-s)v(s)ds = \mathcal{I}^2 v, \qquad \mathcal{D}^{\frac{3}{2}} u = \mathcal{I}^{\frac{1}{2}} v, \tag{3}$$

Using the relations above we can rewrite (1) as the following Abel integral equation

$$Av(t) + \int_0^t (t-s)^{-\frac{1}{2}} K(t,s)v(s)ds = f(t), \tag{4}$$

where $K(t,s) = \frac{B}{\sqrt{\pi}} + C(t-s)^{\frac{3}{2}}$. In the following lemma we give the regularity result of (4).

Lemma 1.1 [4, 18] *Assume that* $A \neq 0$ *and constants* B, C *are chosen such that* $K(t,s) \in C^l(\Lambda \times \Lambda)$ *with* $K(t,t) \neq 0$ *and* $l \geq 1$. *If* $f(t) \in C^l(\Lambda)$ *then the regularity of the unique solution of* (4) *is described by*

$$v(t) \in C^l(0,1] \bigcap C(\Lambda), \quad \textit{with} \quad |v'(t)| \leq \frac{1}{\sqrt{t}}, \quad \textit{for} \ t \in (0,1],$$

and the solution $v(t)$ *can be written in the form*

$$v(t) = \sum_{(j,k)} \gamma_{j,k} t^{j+\frac{k}{2}} + V_l(t),$$

where the coefficients $\gamma_{j,k}$ *are some constants,* $V_l(t) \in C^l(\Lambda)$ *and* $(j,k) := \{(j,k) : j,k \in \mathbb{N}_0, j + \frac{k}{2} < l\}$. *Here* $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, *where* $\mathbb{N}$ *is(as usual) the set of natural numbers.*

From Lemma 1.1, we can conclude that the first derivative of $v(t)$ has a singularity near $t = 0^+$ and behaves like $\frac{1}{\sqrt{t}}$. Thus, from (3) the regularity of the solution $u(t)$ of (1) is given by

$$u(t) \in C^{l'}(0,1] \bigcap C^2(\Lambda) \quad \text{with} \quad |u'''(t)| \leq \frac{1}{\sqrt{t}} \quad \text{for} \quad t \in (0,1], \quad l' \geq 3. \tag{5}$$

Consequently, we have to develop a high-order method for obtaining the numerical solutions of (1). Many results on the numerical solutions of (1) were obtained by authors. In [6], authors studied the generalized Taylor collocation method to approximate the exact solutions of (1). In [7], authors presented the asymptotic stability conditions for the exact and discretized Bagley-Torvik equations in terms of its coefficients and a discretization stepsize. K. Diethelm and N. J. Ford [10] discussed some approaches for obtaining the numerical solutions of (1). Furthermore, In [27] authors applied a numerical method based on operational Haar wavelet to approximate (1). Other approaches for obtaining an approximate solution for (1) can be found in [19, 28, 29]. Recently, spectral methods have received considerable attention for solving fractional differential equations, specially Bagley-Torvik equation (1).

Bhrawy et al. [2, 3, 11] considered the shifted Jacobi and Laguerre spectral methods for solving multi-term fractional differential equations. Eslahchi et al. [12] applied collocation method for obtaining the numerical solution of the nonlinear fractional integro-differential equations. In [14, 16, 20, 21, 23, 24] authors proposed spectral Tau and collocation methods for obtaining the numerical solution of nonlinear fractional Riccati differential equations, fractional integro-differential equations and linear multi-term fractional differential equations where they also discussed the error analysis of the proposed methods. Moreover, Pedas and Tamme [25] introduced a new efficient spline collocation method for solving multi-term fractional differential equations. But many of the techniques mentioned above have at least two main difficulties. First, many of these methods or have not proper convergence analysis or if any, very restrictive assumptions including smoothness of the exact solution are considered. Second, their spectral discretizations lead to badly conditioned algebraic systems with full coefficient matrix which are very difficult to solve. Thus introducing and analyzing a well-posed spectral method with a high order of accuracy for obtaining the numerical solution of (1) is very important and novel in the area. In order to employ a well-posed and highly accurate numerical method for approximating the solution of the Bagley-Torvik equation (1) we propose a strategy mainly consisting in the following two steps:

- First, we follow a stable and highly accurate strategy for obtaining approximate solution $v_N(t)$ to the equivalent equation (4). To this end, Lemma 1.1 concludes the first derivative of the solution $v(t)$ has singularity at the origin and thereby its Tau discretization leads to very poor convergence results. Thus, in order to recover the high-order of convergence we use a regularization procedure that allows us to improve the smoothness of the given functions and then to approximate the solution with a satisfactory order of convergence using an operational Tau method. Other property of this methodology is that we can represent the Tau solution of the problem by solving a well-conditioned upper triangular linear algebraic system.
- Second, from (3), we can find the approximate solution of the Bagley-Torvik equation (1) by defining $u_N(t) = \mathcal{I}^2 v_N(t)$.

The reminder of this paper is organized as follows: In Section 2, we explain the numerical treatment of the problem. In Section 3, we analyze convergence behavior of the proposed method. In Section 4, we apply the proposed method developed in Section 2 to several numerical examples to confirm the theoretical predictions obtained in Section 3. In Section 5, we give our conclusions.

2 Numerical treatment of the problem

The main concern of this section is the numerical treatment of (4) by applying a well-posed operational Chebyshev Tau method and extend the approximation obtained to introduce the numerical solution of Bagley-Torvik equation (1). As we pointed out in the previous section because of singularity of the first derivative of the solution

$v(t)$ of (4) (from Lemma 1.1) its Tau discretization concludes very poor convergence results. In order to make it efficient for (4), the original equation will be changed into a new integral equation which possesses a better smoothness properties by applying a suitable variable transformation. To this end, we apply the variable transformation

$$t = x^2, \quad x = \sqrt{t}, \quad s = w^2, \quad w = \sqrt{s},$$

and change (4) as follows

$$A\bar{v}(x) + \int_0^x (x^2 - w^2)^{-\frac{1}{2}} \bar{K}(x, w)\bar{v}(w)dw = \bar{f}(x), \tag{6}$$

where

$$\bar{f}(x) = f(x^2), \quad \bar{K}(x, w) = 2wK(x^2, w^2),$$

and

$$\bar{v}(x) = v(x^2) = \sum_{(j,k)} \gamma_{jk} x^{2j+k} + V_l(x^2),$$

is the exact solution which has a more regularity properties compared with $v(t)$. Now, we present a well-posed operational Tau solution $\bar{v}_N(x)$ to the transformed equation (6) where Chebyshev polynomials have been employed as basis functions and define $v_N(t) = \bar{v}_N(x)$ as the approximate solution of (4).

Assume that $\bar{v}(x) = \sum_{i=0}^{\infty} a_i T_i(x) = a\underline{\mathcal{T}} = a\mathcal{T}\underline{X}$ is the Chebyshev series expansion of the exact solution of (6) where $a = [a_0, a_1, ..., a_N, ...]$ and $\underline{\mathcal{T}} := [T_0(x), T_1(x), ..., T_N(x), ...]^T$ is a shifted Chebyshev polynomial basis in Λ with degree $(T_i(x)) \leq i$ for $i = 0, 1, 2, \ldots$. Also, $\mathcal{T}$ is an infinitely non-singular lower triangular coefficient matrix and $\underline{X} = [1, x, x^2, ..., x^N, ...]^T$. Suppose that $\bar{v}_N(x)$ is a Chebyshev Tau approximation of degree N for $\bar{v}(x)$ as

$$\bar{v}_N(x) = \sum_{i=0}^{\infty} a_i T_i(x) = \underline{a}\, \underline{\mathcal{T}} = \underline{a}\mathcal{T}\underline{X}, \quad \underline{a} = [a_0, a_1, ..., a_N, 0, ...], \tag{7}$$

and $\bar{f}(x)$ is a given polynomial and consider

$$\bar{f}(x) = \sum_{i=0}^{\infty} \bar{f}_i x^i = \underline{f}\, \underline{X}, \quad \underline{f} = [\bar{f}_0, \bar{f}_1, ..., \bar{f}_N, 0, ...]. \tag{8}$$

If $\bar{f}(x)$ is not polynomial, then it can be approximated by polynomials to any degree of accuracy by Taylor series or any other suitable method.

We define

$$(\mathcal{L}\bar{v})(x) := A\bar{v}(x) + \int_0^x (x^2 - w^2)^{-\frac{1}{2}} \bar{K}(x, w)\bar{v}(w)dw, \tag{9}$$

and show how to replace the operator (9) by a matrix formulation of the operational Chebyshev Tau method in the following theorem.

Theorem 2.1 *Assume that the approximate solution $\bar{v}_N(x)$ is given by (7). Then we have*

$$(\mathcal{L}\bar{v}_N)(x) := \underline{a}\,\mathcal{B}\,\underline{\mathcal{T}},$$

where

$$\mathcal{B} = AI + \mathcal{T}\left(B\mathcal{B}^1 + C\mathcal{B}^2\right)\mathcal{T}^{-1},$$

I is the infinite identity matrix and

$$\mathcal{B}^1 := \begin{pmatrix} 0 & b_0^1 & 0 & \cdots & & & & \\ 0 & 0 & b_1^1 & 0 & \cdots & & & \\ 0 & 0 & 0 & b_2^1 & 0 & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \\ 0 & 0 & \cdots & & 0 & b_{N-2}^1 & \ddots & \\ 0 & 0 & \cdots & & & 0 & b_{N-1}^1 & \ddots \\ 0 & 0 & \cdots & & & & 0 & \ddots \\ \vdots & \vdots & & & & & & \ddots \end{pmatrix}, \quad b_i^1 = \frac{\Gamma\left(1+\frac{i}{2}\right)}{\Gamma\left(\frac{3+i}{2}\right)}, \quad i = 0, 1, 2, \ldots.$$

$$\mathcal{B}^2 := \begin{pmatrix} 0 & 0 & 0 & 0 & b_0^2 & 0 & \cdots & & & & \\ 0 & 0 & 0 & 0 & 0 & b_1^2 & 0 & \cdots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & b_2^2 & 0 & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & 0 & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & b_{N-5}^2 & \ddots & \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & b_{N-4}^2 & \ddots & \\ 0 & 0 & \cdots & \cdots & & 0 & 0 & 0 & 0 & \ddots & \\ 0 & 0 & \cdots & \cdots & & & 0 & 0 & 0 & \ddots & \\ 0 & 0 & \cdots & \cdots & & & & 0 & 0 & \ddots & \\ 0 & 0 & \cdots & \cdots & & & & & 0 & \ddots & \\ \vdots & \vdots & & & & & & & & \ddots & \end{pmatrix}, \quad b_i^2 = \frac{4}{8+6i+i^2}, \quad i = 0, 1, 2, \ldots .$$

Proof From relations (7) and (9) we can write

$$(\mathcal{L}\bar{v}_N) := \underline{a}\left(A\underline{\mathcal{T}} + \mathcal{T}\int_0^x (x^2 - w^2)^{-\frac{1}{2}}\bar{K}(x, w)\underline{X_w}\,dw\right), \tag{10}$$

where $\underline{X_w} := [1, w, w^2, ..., w^N, ...]$. Thus, the integral term of (10) can be rewritten as

$$\begin{aligned}\int_0^x (x^2-w^2)^{-\frac{1}{2}}\bar{K}(x,w)\underline{X_w}dw &= \frac{B}{\sqrt{\pi}}\int_0^x (x^2-w^2)^{-\frac{1}{2}}2w\underline{X_w}dw + C\int_0^x (x^2-w^2)2w\underline{X_w}dw \\ &= \frac{2B}{\sqrt{\pi}}\int_0^x (x^2-w^2)^{-\frac{1}{2}}\begin{bmatrix} w \\ w^2 \\ \vdots \\ w^{N+1} \\ \vdots \end{bmatrix}dw \\ &\quad +2C\int_0^x (x^2-w^2)\begin{bmatrix} w \\ w^2 \\ \vdots \\ w^{N+1} \\ \vdots \end{bmatrix}dw. \end{aligned} \tag{11}$$

By using the relations (See [22])

$$\frac{2}{\sqrt{\pi}}\int_0^x (x^2-w^2)^{-\frac{1}{2}}w^{1+i}dw = b_i^1 x^{1+i}, \quad i = 0, 1, ...,$$

$$2\int_0^x (x^2-w^2)w^{1+i}dw = b_i^2 x^{4+i}, \quad i = 0, 1, ...,$$

we can rewrite (11) as

$$\int_0^x (x^2-w^2)^{-\frac{1}{2}}\bar{K}(x,w)\underline{X_w}dw = \left(B\mathcal{B}^1 + C\mathcal{B}^2\right)\underline{X}. \tag{12}$$

Substituting (12) into (10) we obtain

$$\begin{aligned}(\mathcal{L}\bar{v}_N) &:= \underline{a}\left(A\underline{\mathcal{T}} + \mathcal{T}\left(B\mathcal{B}^1 + C\mathcal{B}^2\right)\underline{X}\right) \\ &= \underline{a}\left(AI + \mathcal{T}\left(B\mathcal{B}^1 + C\mathcal{B}^2\right)\mathcal{T}^{-1}\right)\underline{\mathcal{T}} \\ &= \underline{a}\,\mathcal{B}\,\underline{\mathcal{T}},\end{aligned}$$

and we can conclude the result. □

We are now ready to obtain the algebraic form of the Chebyshev Tau discretization of (6). According to the Theorem 2.1 and the relation (8) we obtain

$$\underline{a}\,\mathcal{B}\,\underline{\mathcal{T}} = \underline{f}\mathcal{T}^{-1}\underline{\mathcal{T}}.$$

Duo to the orthogonality of $\{T_j(x)\}_{j=0}^{\infty}$, after projecting the equation above on $\{T_j(x)\}_{j=0}^{N}$, we have

$$\underline{a}^N \mathcal{B}^N = \underline{f}^N \left(\mathcal{T}^N\right)^{-1}, \tag{13}$$

where $\underline{a}^N = [a_0, a_1, ..., a_N]$, $\underline{f}^N = [\bar{f}_0, \bar{f}_1, ..., \bar{f}_N]$ and $\mathcal{B}^N, \mathcal{T}^N$ are the principle submatrices of order $N+1$ from the matrices $\mathcal{B}$ and $\mathcal{T}$ respectively. Clearly, the solution of the linear system (13) gives us the unknowns $a_0, a_1, ..., a_N$.

Now, we give more details regarding the complexity analysis of the linear system (13). In other words, we will explain that how we can compute the unknowns $\underline{a}^N$ by a well-posed technique. Our approach is based upon the impressive paper [15].

Consider the linear system (13). Multiplying its both sides by $\mathcal{T}^N$ yields

$$\underline{a}^N \mathcal{B}^N \mathcal{T}^N = \underline{f}^N,$$

which can be rewritten as

$$\underline{a}^N \left(A\mathcal{T}^N + \mathcal{T}^N \left(B\mathcal{B}_N^1 + C\mathcal{B}_N^2\right)\right) = \underline{f}^N, \tag{14}$$

where $\mathcal{B}_N^1, \mathcal{B}_N^2$ are the principle submatrices of order $N+1$ from the matrices $\mathcal{B}^1, \mathcal{B}^2$ respectively. Defining

$$\underline{\underline{a}}^N = \underline{a}^N \mathcal{T}^N = [\bar{a}_0, \bar{a}_1, ..., \bar{a}_N], \tag{15}$$

in (14) we have

$$\underline{\underline{a}}^N \prod^N = \underline{f}^N, \tag{16}$$

where

$$\prod^N := AI^N + B\mathcal{B}_N^1 + C\mathcal{B}_N^2 := \begin{pmatrix}
A & Bb_0^1 & 0 & 0 & Cb_0^2 & 0 & \cdots & & & 0 \\
0 & A & Bb_1^1 & 0 & 0 & Cb_1^2 & 0 & \cdots & & 0 \\
0 & 0 & A & Bb_2^1 & 0 & 0 & Cb_2^2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & & A & Bb_{N-4}^1 & 0 & 0 & Cb_{N-4}^2 \\
0 & 0 & \cdots & \cdots & & & A & Bb_{N-3}^1 & 0 & 0 \\
0 & 0 & \cdots & \cdots & & & & A & Bb_{N-2}^1 & 0 \\
0 & 0 & \cdots & \cdots & & & & & A & Bb_{N-1}^1 \\
0 & 0 & \cdots & \cdots & & & & & & A
\end{pmatrix}, \tag{17}$$

is the non-singular $(N+1) \times (N+1)$ upper triangular matrix. I^N is the the principle submatrices of order $N+1$ from I.

The matrix $\prod^N$ has two important advantages. First, each columns of this matrix has at most three non-zero elements, i.e., $\prod^N$ has sparse structure. Second, since $\left\{b_i^1\right\}_{i=0}^{N-1}$ and $\left\{b_i^2\right\}_{i=0}^{N-1}$ are strictly descending sequences, condition number of this matrix remains bounded independently of approximation degree N for sufficiently large values of N. These properties deduce the well-posedness of the proposed strategy in obtaining the unknown vector $\underline{\underline{a}}^N$.

Now, we come to obtain the Chebyshev Tau representation $\bar{v}_N(x)$ in (7). To this end, from (15) it is sufficient that we find $\underline{\underline{a}}^N$ by solving (16) and set

$$\underline{a}^N = \underline{\underline{a}}^N \left(\mathcal{T}^N\right)^{-1}. \tag{18}$$

Due to the upper triangular structure of $\prod^N$, we can give components of the unknown vector $\underline{\underline{a}}^N$ by solving (16) using forward substitution method as

$$\begin{cases} \bar{a}_0 = \frac{1}{A}\bar{f}_0, \\ \bar{a}_i = \frac{1}{A}\left(\bar{f}_i - Bb^1_{i-1}\bar{a}_{i-1}\right), \quad i = 1, 2, 3, \\ \bar{a}_i = \frac{1}{A}\left(\bar{f}_i - Bb^1_{i-1}\bar{a}_{i-1} - Cb^2_{i-4}\bar{a}_{i-4}\right), \quad i = 4, 5, ..., N. \end{cases} \tag{19}$$

Finally the approximate solution $u_N(t)$ from the original equation (1) can be given as

$$u_N(t) = \mathcal{I}^2 v_N(t) = \mathcal{I}^2 \bar{v}_N(x).$$

3 Convergence analysis

In this section, we provide a suitable convergence analysis, which theoretically justifies the high-order rate of convergence of the proposed method when applied to the regularized equation (6).

We recall the following preliminaries which are needed in the sequel (See [5, 27])

- c_i will denote some generic positive constants that are independent on N.
- $\|.\|_\infty$ denotes the uniform norm and defines as $\|g\|_\infty = \max\limits_{x\in\Lambda} |g(x)|$.
- $L^2_{\alpha,\beta}(\Lambda)$ is the space of functions whose square is Lebesque integrable in Λ relative to the shifted Jacobi weight function $w^{\alpha,\beta}(x) = 2^{\alpha+\beta}x^\beta(1-x)^\alpha$ for the parameters $\alpha, \beta > -1$ with the norm $\|g\|^2_{\alpha,\beta} = (g, g)_{\alpha,\beta} := \int\limits_\Lambda g^2(x)w^{\alpha,\beta}(x)dx$. Here, $(., .)_{\alpha,\beta}$ is the weighted inner product formula.
- P_N be the space of all algebraic polynomials of degree up to N.
- $\mathcal{P}_N : L^2_{-\frac{1}{2},-\frac{1}{2}}(\Lambda) \to P_N$ is the Chebyshev orthogonal projection which is a mapping such that for any $g \in L^2_{-\frac{1}{2},-\frac{1}{2}}(\Lambda)$,

$$(g - \mathcal{P}_N g, \phi)_{-\frac{1}{2},-\frac{1}{2}} = 0, \quad \forall\phi \in P_N.$$

- For $r \geq 0$ and $0 \leq k \leq 1$ we denote by $C^{r,k}(\Lambda)$ the space of functions whose r-th derivatives are Holder continuous with exponent k, endowed with the usual norm

$$\|g\|_{r,k} := \max_{0\leq l\leq r}\max_{x\in\Lambda}|g^{(l)}(x)| + \max_{0\leq l\leq r}\sup_{x\neq y}\frac{|g^{(l)}(x) - g^{(l)}(y)|}{|x-y|^k}.$$

When $\gamma = 0$, $C^{k,0}(\Lambda)$ denotes the space of functions with k continuous derivatives on Λ, also denotes $C^k(\Lambda)$ and with norm $\|.\|_k$

In our analysis we shall apply the following Lemmas:

Lemma 3.1 [22, 27, 30] *Concerning the truncation error of Chebyshev series, the following estimates hold*

$$\|g - \mathcal{P}_N g\|_{-\frac{1}{2},-\frac{1}{2}} \leq c_1 N^{-s} \|g^{(s)}\|_{-\frac{1}{2}+s,-\frac{1}{2}+s}, \quad g^{(s)} \in L^2_{-\frac{1}{2}+s,-\frac{1}{2}+s}(\Lambda), \quad s \geq 0. \tag{20}$$

$$\|g - \mathcal{P}_N g\|_\infty \leq c_2 (1 + \log N) N^{-(k+\gamma)} \|g\|_{k,\gamma}, \quad g \in C^{k,\gamma}(\Lambda), \quad k \geq 0, \quad \gamma \in [0, 1]. \tag{21}$$

$$\|g - \mathcal{P}_N g\|_\infty \leq c_3 N^{\frac{3}{4}-s} \|g^{(s)}\|_{-\frac{1}{2},-\frac{1}{2}}, \quad g^{(s)} \in L^2_{-\frac{1}{2},-\frac{1}{2}}(\Lambda), \quad s \geq 1. \tag{22}$$

Lemma 3.2 [8, 27] *Consider the following linear weakly singular integral operator*

$$(\mathcal{M}g)(x) := \int_0^x (x-w)^{-\frac{1}{2}} M(x,w) g(w) dw,$$

where $M(x,w)$ is a given kernel function. If $0 < \gamma < \frac{1}{2}$, then for any continuous function $g(x)$, there exists a positive constant c_4 may depend on $\|M\|_{0,\gamma}$ and $\|M\|_\infty$ such that

$$\|\mathcal{M}g\|_{0,\gamma} \leq c_4 \|g\|_\infty.$$

Lemma 3.3 [8] *(Gronwall inequality) Assume that $g(x)$ is a non-negative, locally integrable function defined on Λ which satisfies*

$$g(x) \leq b(x) + d \int_0^x (x-w)^m w^n g(w) dw, \quad w \in \Lambda, m, n > -1,$$

where $b(x) \geq 0$ and $d \geq 0$. Then, there exists a constant c_5 such that

$$g(x) \leq b(x) + c_5 \int_0^x (x-w)^m w^n b(w) dw, \quad w \in \Lambda.$$

Now, we state and prove the main result of this section regarding the convergence of the proposed method for obtaining the numerical solution of the regularized equation (6).

Theorem 3.4 *Suppose that $\bar{v}_N(x)$ is the Chebyshev Tau solution of (6) given by (7). Then the following error estimates are hold*

$$\|\bar{e}_N\|_\infty \le c_{12} N^{\frac{3}{4}-s_1} \|\bar{v}^{(s_1)}\|_{-\frac{1}{2},-\frac{1}{2}}, \tag{23}$$

$$\|\bar{e}_N\|_{-\frac{1}{2},-\frac{1}{2}} \le c_{13} N^{-s_2} \|\bar{v}^{(s_2)}\|_{-\frac{1}{2}+s_2,-\frac{1}{2}+s_2} + c_{14}(1+\log N) N^{\frac{3}{4}-s_1-\gamma} \|\bar{v}^{(s_1)}\|_{-\frac{1}{2},-\frac{1}{2}}, \tag{24}$$

where $\gamma \in (0, \frac{1}{2})$ and $\bar{v}^{(s_1)} \in L^2_{-\frac{1}{2},-\frac{1}{2}}(\Lambda)$, $\bar{v}^{(s_2)} \in L^2_{-\frac{1}{2}+s_2,-\frac{1}{2}+s_2}(\Lambda)$ for $s_1 \ge 1$ and $s_2 \ge 0$. $\bar{e}_N(x) = \bar{v}(x) - \bar{v}_N(x)$ defines the error function.

Proof Considering (6), according to the proposed method we have

$$A\bar{v}_N(x) + \mathcal{P}_N \int_0^x (x^2 - w^2)^{-\frac{1}{2}} \bar{K}(x,w)\bar{v}_N(w)dw = \mathcal{P}_N \bar{f}. \tag{25}$$

Subtracting (6) from (25), we have

$$A\bar{e}_N(x) + \left(\int_0^x (x^2 - w^2)^{-\frac{1}{2}} \bar{K}(x,w)\bar{v}(w)dw - \mathcal{P}_N \int_0^x (x^2 - w^2)^{-\frac{1}{2}} \bar{K}(x,w)\bar{v}_N(w)dw \right) = e_{\mathcal{P}_N}(\bar{f}), \tag{26}$$

where $e_{\mathcal{P}_N}(\bar{f}) := \bar{f}(x) - \mathcal{P}_N \bar{f}$. Using some simple manipulations we can rewrite (26) as follows

$$\begin{aligned}
A\bar{e}_N(x) &= e_{\mathcal{P}_N}(\bar{f}) - (\mathcal{S}\bar{v} - \mathcal{P}_N \mathcal{S}\bar{v}_N) \\
&= e_{\mathcal{P}_N}(\bar{f}) - \left(e_{\mathcal{P}_N}(\mathcal{S}\bar{v}) + \mathcal{P}_N \mathcal{S}\bar{e}_N\right) \\
&= e_{\mathcal{P}_N}(\bar{f}) - \left(e_{\mathcal{P}_N}(\mathcal{S}\bar{v}) + \mathcal{S}\bar{e}_N - e_{\mathcal{P}_N}(\mathcal{S}\bar{e}_N)\right) \\
&= e_{\mathcal{P}_N}(\bar{f}) - e_{\mathcal{P}_N}(\bar{f} - A\bar{v}) - \left(\mathcal{S}\bar{e}_N - e_{\mathcal{P}_N}(\mathcal{S}\bar{e}_N)\right) \\
&= Ae_{\mathcal{P}_N}(\bar{v}) - \left(\mathcal{S}\bar{e}_N - e_{\mathcal{P}_N}(\mathcal{S}\bar{e}_N)\right),
\end{aligned}$$

which implies that

$$|\bar{e}_N(x)| \le c_6 \int_0^x (x-w)^{-\frac{1}{2}} |\bar{e}_N| dw + \left| e_{\mathcal{P}_N}(\bar{v}) + \frac{1}{A} e_{\mathcal{P}_N}(\mathcal{S}\bar{e}_N) \right|, \tag{27}$$

where $c_6 = \|\frac{\bar{K}(x,w)}{A\sqrt{x+w}}\|_\infty$ and

$$(\mathcal{S}\bar{e}_N)(x) := \int_0^x (x^2 - w^2)^{-\frac{1}{2}} \bar{K}(x,w)\bar{e}_N(w)dw.$$

Applying Gronwall inequality, i.e., Lemma 3.3 in (27) yields

$$\|\bar{e}_N\|_\infty \leq c_7 \left(\|e_{\mathcal{P}_N}(\bar{v})\|_\infty + \|e_{\mathcal{P}_N}(\mathcal{S}\bar{e}_N)\|_\infty\right). \tag{28}$$

Using (21) and Lemma 3.2 in (28) we can get

$$\begin{aligned}\|\bar{e}_N\|_\infty &\leq c_8 \left(\|e_{\mathcal{P}_N}(\bar{v})\|_\infty + (1+\log N)\, N^{-\gamma} \|\mathcal{S}\bar{e}_N\|_{0,\gamma}\right) \\ &\leq c_9 \left(\|e_{\mathcal{P}_N}(\bar{v})\|_\infty + (1+\log N)\, N^{-\gamma} \|\bar{e}_N\|_\infty\right).\end{aligned}$$

The first inequality (23) can be achieved by applying (22) in the equation above for sufficiently large values of N.

Now we prove the second estimate (24). To this end, using $\|.\|_{-\frac{1}{2},-\frac{1}{2}}$ norm instead of the uniform norm in (28) we have

$$\begin{aligned}\|\bar{e}_N\|_{-\frac{1}{2},-\frac{1}{2}} &\leq c_{10} \left(\|e_{\mathcal{P}_N}(\bar{v})\|_{-\frac{1}{2},-\frac{1}{2}} + \|e_{\mathcal{P}_N}(\mathcal{S}\bar{e}_N)\|_{-\frac{1}{2},-\frac{1}{2}}\right) \\ &\leq c_{10} \left(\|e_{\mathcal{P}_N}(\bar{v})\|_{-\frac{1}{2},-\frac{1}{2}} + \|e_{\mathcal{P}_N}(\mathcal{S}\bar{e}_N)\|_\infty\right).\end{aligned} \tag{29}$$

Applying (21) and Lemma 3.2 for $\|e_{\mathcal{P}_N}(\mathcal{S}\bar{e}_N)\|_\infty$ in (29) we obtain

$$\|\bar{e}_N\|_{-\frac{1}{2},-\frac{1}{2}} \leq c_{11} \left(\|e_{\mathcal{P}_N}(\bar{v})\|_{-\frac{1}{2},-\frac{1}{2}} + (1+\log N) N^{-\gamma} \|\bar{e}_N\|_\infty\right). \tag{30}$$

The second estimate (24) can be concluded by applying the estimates (20) and (23) in (30). □

4 Numerical results

In this section, we report the numerical results for some Bagley-Torvik equations (1) solved using the proposed numerical scheme. We illustrated three test problems to show the performance and efficiency of the operational Chebyshev Tau method which presented and analyzed in Sections 2 and 3 respectively. All of the calculations performed on a PC running *Mathematica* software. In the results obtained that follow we reported some crucial items regarding the condition number of the matrix $\prod^N$ $\left(\text{Condition number} := \left\|\prod^N\right\|_2 \left\|\left(\prod^N\right)^{-1}\right\|_2\right)$, order of convergence $\left(\text{Order} := \left|\frac{\log \frac{\varepsilon^{N_2}}{\varepsilon^{N_1}}}{\log \frac{N_2}{N_1}}\right|,\ \varepsilon^N = \|\bar{e}_N\|_{-\frac{1}{2},-\frac{1}{2}}\right)$ and L^2 and L_∞-norms of the errors obtained.

Example 4.1 Consider (1) with $f(t) := \frac{15}{4}\sqrt{t} + \frac{15}{8}\sqrt{\pi}t + t^{\frac{5}{2}}$ and $A = B = C = 1$. The exact solution of this problem is given by $u(t) = t^{\frac{5}{2}}$.

By applying the technique described in Section 2, with $N = 5$ we approximate the solution as follows

$$u_5(t) = \mathcal{I}^2\left(\bar{v}_5(x)\right), \quad x = \sqrt{t}, \tag{31}$$

where

$$\bar{v}_5(x) = \sum_{i=0}^{5} a_i T_i(x) = \underline{a}^5 \underline{\mathcal{T}}^5, \quad \underline{a}^5 := [a_0, a_1, ..., a_5],$$

is the Tau solution of the regularized equation (6) with $\bar{f}(x) = f(x^2)$. From (18) we have

$$\underline{a}^5 = \underline{\underline{a}}^5 \left(\mathcal{T}^5\right)^{-1},$$

where $\underline{\underline{a}}^5$ is the solution of the upper triangular system (16) with $\underline{f}^5 := [0, \frac{15}{4}, \frac{15}{8}\sqrt{\pi}, 0, 0, 1]$ and

$$\prod^5 = \begin{pmatrix} 1 & \frac{2}{\sqrt{\pi}} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{\sqrt{\pi}}{2} & 0 & 0 & \frac{4}{15} \\ 0 & 0 & 1 & \frac{4}{3\sqrt{\pi}} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{3\sqrt{\pi}}{8} & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{16}{15\sqrt{\pi}} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Applying (19) and then (18) we obtain

$$\underline{\underline{a}}^5 := \left[0, \frac{15}{4}, 0, 0, 0, 0\right], \quad \underline{a}^5 := \left[\frac{15}{8}, \frac{15}{8}, 0, 0, 0, 0\right].$$

Thus, the Tau representation $\bar{v}_5(x)$ can be presented by

$$\bar{v}_5(x) = a_0 T_0(x) + a_1 T_1(x) = \frac{15}{4} x.$$

From (31) the approximate solution of the Bagley-Torvik problem considered in this example is given by

$$u_5(t) = \mathcal{I}^2\left(\frac{15}{4} x\right) = \mathcal{I}^2\left(\frac{15}{4}\sqrt{t}\right) = t^{\frac{5}{2}},$$

which is the exact solution.

Table 1 Numerical treatment of Example 4.2

N	Numerical results		
	L_∞-error	L^2-error	Order
5	1.58×10^{-1}	4.01×10^{-2}	2.93
15	8.14×10^{-3}	1.6×10^{-3}	7.62
25	1.96×10^{-4}	3.27×10^{-5}	13.06
35	2.73×10^{-6}	4.03×10^{-7}	19.08
45	2.49×10^{-8}	3.33×10^{-9}	25.57
55	1.59×10^{-10}	1.97×10^{-11}	32.42
65	7.62×10^{-13}	8.74×10^{-14}	39.6
75	2.8×10^{-15}	3.02×10^{-16}	47.05

Table 2 Condition number of the matrix $\prod^N$ in the Example 4.2

N	Condition number
5	6.1994
15	6.85831
25	6.85832
35	6.85832
45	6.85832
55	6.85832
65	6.85832
75	6.85832
85	6.85832
95	6.85832

Example 4.2 Consider (1) with $A = B = 1$ and $C = -1$. We choose $f(t)$ so that the exact solution of the problem is $u(t) = \sin\left(t^2\sqrt{t}\right)$.

We solved the problem by applying the technique described in Section 2 and the results obtained are given in Tables 1 and 2. Thus, based on Tables 1 and 2 we can deduce that:

- The L_∞ and $L^2_{-\frac{1}{2},-\frac{1}{2}}$-norms errors decay with the approximation degree N which show that by employing the proposed method we obtained very accurate and reliable results.
- The condition number of the coefficient matrix $\prod^N$ for $N > 15$ remains bounded independently of approximation degree N which concludes the well-posedness of the proposed method.
- The order of convergence has increased with the approximation degree N which confirms the high-order of accuracy of the proposed method regardless of the singular behavior of the exact solution.

Table 3 Numerical treatment of Example 4.3

N	Numerical results		
	L_∞-error	L^2-error	Order
5	1.67×10^{-3}	5.29×10^{-4}	5.67
10	4.19×10^{-5}	1.04×10^{-5}	17.59
15	3.63×10^{-8}	8.28×10^{-9}	14.93
20	5.7×10^{-10}	1.13×10^{-10}	27.07
25	1.49×10^{-12}	2.69×10^{-13}	35.95
30	2.29×10^{-15}	3.83×10^{-16}	46.42

Table 4 Condition number of the matrix $\prod^N$ in the Example 4.3

N	Condition number
5	2.13825
15	2.19295
25	2.19295
35	2.19295
45	2.19295
55	2.19295
65	2.19295
75	2.19295
85	2.19295
95	2.19295

Example 4.3 [13] Consider (1) with $A = C = 1, \quad B = \beta\sqrt{\pi}, \quad f(t) = 0, \quad u(0) = 1, \quad u'(0) = 0$ which is developed for a Micro-Mechanical system (MEMS) instrument that has been designed primarily to measure the viscosity of fluids that are encountered during oil well exploration. Interpretation of the constant β is given is [13]. The exact solution of the problem may be expressed as

$$u(t) = 1 - \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(-\beta\sqrt{\pi})^j (j+k)! t^{2+2k+\frac{j}{2}}}{j!k!\left(2+2k+\frac{j}{2}\right)\Gamma\left(2+2k+\frac{j}{2}\right)}.$$

The asymptotic behavior of the exact solution is $u(t) = 1 - \frac{t^2}{2} + \frac{8t^{\frac{5}{2}}}{75} - O(t^3)$. We implement the proposed method to obtain an approximate solution for this problem with $\beta = \frac{1}{5}$ and report the results obtained in Tables 3 and 4. Indeed, from Tables 3 and 4 we can observe that:

- The proposed numerical scheme is stable and well-posed due to bounded condition numbers for the coefficient matrix $\prod^N$ independently of the approximation degree N.
- The numerical errors are reasonably accurate and as the degree of approximation gets larger the superiority of the new method becomes more evident.

5 Conclusion

This work has been concerned with the well-posed Chebyshev Tau method and its convergence analysis for Bagley Torvik type fractional differential equations. Due to the fact that some derivatives of the solutions of these equations usually have a weak singularity at origin, we proceed a suitable regularization process to improve the smoothness of the given functions and then to approximate the solution with a satisfactory order of convergence. The proposed method has two main advantages. First, the approximate solution obtains by solving a well-conditioned upper triangular linear algebraic system and then Tau representation of the problem possesses a high-order of convergence under a more general regularity assumptions on the input data.

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