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An iterative algorithm for solving split feasibility problems and fixed point problems in Banach spaces

Y. Shehu[1] · O. S. Iyiola[2] · C. D. Enyi[3]

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Abstract The purpose of this paper is to study split feasibility problems and fixed point problems concerning left Bregman strongly relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces. We suggest an iterative scheme for the problem and prove strong convergence theorem of the sequences generated by our scheme under some appropriate conditions in real p-uniformly convex and uniformly smooth Banach spaces. Finally, we give numerical examples of our result to study its efficiency and implementation. Our result complements many recent and important results in this direction.

Keywords Strong convergence · Split feasibility problem · Uniformly convex · Uniformly smooth · Fixed point problem · Left Bregman strongly nonexpansive mappings

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✉ Y. Shehu
yekini.shehu@unn.edu.ng

O. S. Iyiola
osiyiola@uwm.edu

C. D. Enyi
cenyi@kfupm.edu.sa

1 Department of Mathematics, University of Nigeria, Nsukka, Nigeria

2 Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI, USA

3 Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, KFUPM, Dhahran, Dammam, Saudi Arabia

1 Introduction

Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. The *split feasibility problem* (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q. \tag{1.1}$$

We assume that SFP (1.1) has a nonempty solution set $\Omega := \{y \in C : Ay \in Q\} = C \cap A^{-1}(Q)$. Then, we have that Ω is a closed and convex subset of E_1.

The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [5] for modelling inverse problems which arise from phase retrievals, medical image reconstruction and recently in modelling of intensity modulated radiation therapy [3]. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, for example, [4, 12, 15, 17, 30–33] and references therein).

In solving SFP (1.1) in p-uniformly convex real Banach spaces which are also uniformly smooth, Schöpfer et al. [22] proposed the following algorithm: For $x_1 \in E_1$ and $n \geq 1$, set

$$x_{n+1} = \Pi_C J_{E_1}^*[J_{E_1}(x_n) - t_n A^* J_{E_2}(Ax_n - P_Q(Ax_n))], \tag{1.2}$$

where Π_C denotes the Bregman projection and J the duality mapping. Clearly the above algorithm covers the Byrne's CQ algorithm [3]

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), n \geq 1,$$

which is found to be a gradient-projection method (GPM) in convex minimization as a special case. They established the *weak convergence* of algorithm (1.2) under the condition that E_1 is p-uniformly convex, uniformly smooth and the duality mapping of E_1 is sequentially weak-to-weak-continuous.

We remark here that the condition that the duality mapping of E_1 is sequentially weak-to-weak-continuous assumed in [22] excludes some important Banach spaces, such as the classical $L_p(2 < p < \infty)$ spaces.

Recently, Wang [28] modified the above algorithm (1.2) and proved strong convergence for the following multiple-sets split feasibility problem (MSSFP) (please, see [15]): find $x \in E_1$ satisfying

$$x \in \bigcap_{i=1}^{r} C_i, Ax \in \bigcap_{j=1+r}^{r+s} Q_j, \tag{1.3}$$

where r, s are two given integers, $C_i, i = 1, \ldots, r$ is a closed convex subset in E_1, and $Q_j, j = r+1, \ldots, r+s$, is a closed convex subset in E_2. He defined for each $n \in \mathbb{N}$,

$$T_n(x) = \begin{cases} \Pi_{C_{i(n)}}(x), & 1 \leq i(n) \leq r, \\ J_{E_1}^*[J_{E_1}(x) - t_n A^* J_{E_2}(Ax - P_{Q_{j(n)}}(Ax))], & r+1 \leq i(n) \leq r+s, \end{cases}$$

where $i : \mathbb{N} \to I$ is the cyclic control mapping

$$i(n) = n \mod (r + s) + 1,$$

and t_n satisfies

$$0 < t \leq t_n \leq \left(\frac{q}{C_q ||A||^q} \right)^{\frac{1}{q-1}}, \tag{1.4}$$

with C_q a constant defined as in Lemma 2.1 and proposed the following algorithm: For any initial guess $x_1 = \bar{x}$, define $\{x_n\}$ recursively by

$$\begin{cases} y_n = T_n x_n \\ D_n = \{w \in E_1 : \Delta_p(y_n, w) \leq \Delta_p(x_n, w)\} \\ E_n = \{w \in E_1 : \langle x_n - w, J_p(\bar{x}) - J_p(x_n) \geq 0\} \\ x_{n+1} = \Pi_{D_n \cap E_n}(\bar{x}). \end{cases} \tag{1.5}$$

Using the idea in the work of Nakajo and Takahashi [16], he proved the following strong convergence theorem in p-uniformly convex Banach spaces which is also uniformly smooth.

Theorem 1.1 *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP (1.3) has a nonempty solution set Ω. Let the sequence $\{x_n\}_{n=1}^\infty$ be generated by (1.5). Then $\{x_n\}_{n=1}^\infty$ converges strongly to the Bregman projection of $\bar{x}$ onto the solution set Ω.*

The main advantage of result of Wang [28] is that the weak-to-weak continuity of the duality mapping, assumed in [22] is dispensed with and strong convergence result was achieved. On the other hand, to implement the algorithm (1.5) of Wang [28], one has to calculate, at each iteration, the Bregman projection onto the intersection of two half spaces D_n and E_n.

The class of left Bregman firmly nonexpansive mappings associated with the Bregman distance induced by a convex function was introduced and studied by Martin-Marques et al. [14]. Examples of left Bregman firmly nonexpansive mappings are given in [14]. If C is a nonempty and closed subset of int(dom f), where f is a Legendre and Fréchet differentiable function, and $T : C \to$ int (dom f) is a left Bregman strongly nonexpansive mapping, it is proved that $F(T)$ is closed (see [14]). In addition, they have shown that this class of mappings is closed under composition and convex combination and proved weak convergence of the Picard iterative method to a fixed point of a mapping under suitable conditions (see [13]). However, Picard iteration process has only *weak convergence*.

The classes of firmly nonexpansive operators and strongly nonexpansive operators (see, for example, [2, 10]) are of utmost importance in fixed point, monotone mapping, and convex optimization theories in view of Minty's Theorem regarding the correspondence between firmly nonexpansive operators and maximal monotone mappings. In this connection, see Section 7 of the paper by S. Reich [20]. Furthermore, the class of strongly nonexpansive operators, which contains the class of firmly nonexpansive operators, presents the advantage of its being closed under compositions, whereas this property fails for firmly nonexpansive operators (see, for example, [18]). A related class of operators comprises the quasi-nonexpansive operators. These operators still enjoy relevant fixed point properties although nonexpansivity is only required for each fixed point. A basic example of a firmly nonexpansive operator is the nearest point projection onto a closed and convex subset of a Hilbert space. For details on examples and applications of firmly nonexpansive operators and strongly nonexpansive operators, please see [14] and the references contained therein.

Our aim in this paper is to construct an iterative scheme for solving problem (1.1) which is also a fixed point of a left Bregman strongly nonexpansive mapping T. Thus, let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and T be a left Bregman strongly nonexpansive mapping of C into C. We construct an iterative scheme for solving the following problem: find

$$x \in C \cap F(T) \text{ such that } Ax \in Q. \tag{1.6}$$

We assume in this paper that the problem (1.6) has solutions. Furthermore, our problem (1.6) extends some recent problems studied by many authors in the literature.

Suppose that $T = I$, the identity map, then $F(T) = C$ and in this case, our problem (1.6) reduces to SFP (1.1). If $C = E_1$, then problem (1.6) reduces to: find $x \in F(T)$ such that $Ax \in Q$. If furthermore, $F(S) \subseteq Q$, for some nonlinear operator S, then our problem (1.6) reduces to split common fixed point problems (SCFPP). Finally, let $A = I$, $C = E_1 = E_2 = Q$, then our problem (1.6) reduces fixed point problem for T.

In this paper, we shall prove strong convergence of the sequence generated by our scheme for solving problem (1.6) in p-uniformly convex real Banach spaces which are also uniformly smooth. Also, we give numerical result to demonstrate the performance and convergence of our iterative scheme. Our result complements the result of Shehu et al. [25] and many other recent results in the literature.

2 Preliminaries

Let E_1 and E_2 be real Banach spaces and let $A : E_1 \to E_2$ be a bounded linear operator. The *dual (adjoint)* operator of A, denoted by A^*, is a bounded linear operator defined by $A^* : E_2^* \to E_1^*$

$$\langle A^*\bar{y}, x\rangle := \langle \bar{y}, Ax\rangle, \quad \forall x \in E_1, \bar{y} \in E_2^*$$

and the equalities $||A^*|| = ||A||$ and $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ are valid, where $\mathcal{R}(A)^\perp := \{x^* \in E_2^* : \langle x^*, u\rangle = 0, \quad \forall u \in \mathcal{R}(A)\}$. For more details on bounded linear operators and their duals, please see [9, 26, 27].

Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let E be a real Banach space. The *modulus of convexity* $\delta_E : [0, 2] \to [0, 1]$ is defined as

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{||x+y||}{2} : ||x|| = 1 = ||y||, ||x-y|| \geq \epsilon\right\}.$$

E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$; *p-uniformly convex* if there is a $c_p > 0$ so that $\delta_E(\epsilon) \geq c_p \epsilon^p$ for any $\epsilon \in (0, 2]$. The *modulus of smoothness* $\rho_E(\tau) : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_E(\tau) = \left\{\frac{||x+\tau y|| + ||x - \tau y||}{2} - 1 : ||x|| = ||y|| = 1\right\}.$$

E is called *uniformly smooth* if $\lim_{n\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$; *$q$-uniformly smooth* if there is a $C_q > 0$ so that $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$. The L_p space is 2-uniformly convex for $1 < p \leq 2$ and p-uniformly convex for $p \geq 2$. It is known that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth (see [11]).

The q-uniformly smooth spaces have the following estimate [29].

Lemma 2.1 *(Xu, [29]) Let $x, y \in E$. If E is q-uniformly smooth, then there is a $C_q > 0$ so that*

$$||x - y||^q \leq ||x||^q - q\langle y, J_E^q(x)\rangle + C_q||y||^q.$$

Here and hereafter, we assume that E is a p-uniformly convex and uniformly smooth, which implies that its dual space, E^*, is q-uniformly smooth and uniformly convex. In this situation, it is known that the duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the duality mapping of E^* (see [1, 8, 19]). Here the *duality mapping* $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = ||x||^p, ||\bar{x}|| = ||x||^{p-1}\}.$$

The duality mapping J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_E^p x_n, y\rangle \to \langle J_E^p x, y\rangle$$

holds true for any $y \in E$. It is worth noting that the $\ell_p (p > 1)$ space has such a property, but the $J_E^p (p > 2)$ space does not share this property.

Given a Gâteaux differentiable convex function $f : E \to \mathbb{R}$, the *Bregman distance* with respect to f is defined as:

$$\Delta_f(x, y) = f(y) - f(x) - \langle f'(x), y - x\rangle, \quad x, y \in E$$

It is worth noting that the duality mapping J_p is in fact the derivative of the function $f_p(x) = (\frac{1}{p})||x||^p$. Then the Bregman distance with respect to f_p is given by

$$\begin{aligned}\Delta_p(x, y) &= \frac{1}{q}||x||^p - \langle J_E^p x, y\rangle + \frac{1}{p}||y||^p \\ &= \frac{1}{p}(||y||^p - ||x||^p) + \left\langle J_E^p x, x - y\right\rangle \\ &= \frac{1}{q}(||x||^p - ||y||^p) - \left\langle J_E^p x - J_E^p y, x\right\rangle.\end{aligned}$$

Given $x, y, z \in E$, one can easily get

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \left\langle z - y, J_E^p x - J_E^p z\right\rangle, \tag{2.1}$$

$$\Delta_p(x, y) + \Delta_p(y, x) = \left\langle x - y, J_E^p x - J_E^p y\right\rangle. \tag{2.2}$$

Generally speaking, the Bregman distance is not a metric due to the absence of symmetry, but it has some distance-like properties. For the p-uniformly convex space, the metric and Bregman distance has the following relation (see [22]):

$$\tau||x - y||^p \leq \Delta_p(x, y) \leq \left\langle x - y, J_E^p x - J_E^p y\right\rangle, \tag{2.3}$$

where $\tau > 0$ is some fixed number.

It is easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a p-uniformly convex and uniformly smooth E, then $x_n - y_n \to 0, \quad n \to \infty$ implies that $\Delta_p(x_n, y_n) \to 0, \quad n \to \infty$.

Let C be a nonempty, closed and convex subset of E. The metric projection

$$P_C x = \text{argmin}_{y\in C}||x - y||, \quad x \in E,$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\left\langle J_E^p(x - P_C x), z - P_C x\right\rangle \leq 0, \quad \forall z \in C. \tag{2.4}$$

Likewise, one can define the Bregman projection:

$$\Pi_C x = \text{argmin}_{y\in C}\Delta_p(x, y), \quad x \in E,$$

as the unique minimizer of the Bregman distance (see [21]). The Bregman projection can also be characterized by a variational inequality:

$$\left\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\right\rangle \leq 0, \quad \forall z \in C, \tag{2.5}$$

from which one has

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \quad \forall z \in C. \tag{2.6}$$

In Hilbert spaces, the metric projection and the Bregman projection with respect to f_2 are coincident, but in general they are different. More importantly, the metric projection can not share the decent property (2.6) as the Bregman projection in Banach spaces.

Following [1, 6], we make use of the function $V_p : E^* \times E \to [0, +\infty)$ associated with f_p, which is defined by

$$V_p(\bar{x}, x) = \frac{1}{q}||\bar{x}||^q - \langle \bar{x}, x\rangle + \frac{1}{p}||x||^p, \forall x \in E, \bar{x} \in E^*.$$

(Recall that E is a p-uniformly convex and uniformly smooth, which implies that its dual space, E^*, is q-uniformly smooth and uniformly convex). Then V_p is nonnegative and

$$V_p(\bar{x}, x) = \Delta_p(J_{E^*}^q(\bar{x}), x) \tag{2.7}$$

for all $x \in E$ and $\bar{x} \in E^*$. Moreover, using the subdifferential inequality for $f(x) = \frac{1}{q}||x||^q, \quad x \in E^*$, we have

$$\left\langle J_E^q(x), y\right\rangle \leq \frac{1}{q}||x + y||^q - \frac{1}{q}||x||^q, \quad \forall x, y \in E^*. \tag{2.8}$$

Using (2.8), we have for all $\bar{x}, \bar{y} \in E^*$ and $x \in E$ that

$$\begin{aligned} V_p(\bar{x} + \bar{y}, x) &= \frac{1}{q}||\bar{x} + \bar{y}||^q - \langle \bar{x} + \bar{y}, x\rangle + \frac{1}{p}||x||^p \\ &\geq \frac{1}{q}||\bar{x}||^q + \left\langle \bar{y}, J_{E^*}^q(\bar{x})\right\rangle - \langle \bar{x} + \bar{y}, x\rangle + \frac{1}{p}||x||^p \\ &= \frac{1}{q}||\bar{x}||^q - \langle \bar{x}, x\rangle + \frac{1}{p}||x||^p + \langle \bar{y}, J_{E^*}^q(\bar{x})\rangle \\ &\quad -\langle \bar{y}, x\rangle \\ &= \frac{1}{q}||\bar{x}||^q - \langle \bar{x}, x\rangle + \frac{1}{p}||x||^p + \left\langle \bar{y}, J_{E^*}^q(\bar{x}) - x\right\rangle \\ &= V_p(\bar{x}, x) + \left\langle \bar{y}, J_{E^*}^q(\bar{x}) - x\right\rangle. \end{aligned}$$

In other words,

$$V_p(\bar{x}, x) + \left\langle \bar{y}, J_{E^*}^q(\bar{x}) - x\right\rangle \leq V_p(\bar{x} + \bar{y}, x) \tag{2.9}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see, for example, [23, 24]). In addition, V_p is convex in the first variable since $\forall z \in E$, $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$,

$$\begin{aligned}
\Delta_p\left(J_{E^*}^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right), z\right) &= V_p\left(\sum_{i=1}^N t_i J_E^p(x_i), z\right)\\
&= \frac{1}{q}\Big|\Big|\sum_{i=1}^N t_i J_E^p(x_i)\Big|\Big|^q - \left\langle \sum_{i=1}^N t_i J_E^p(x_i), z\right\rangle + \frac{1}{p}||z||^p\\
&\le \frac{1}{q}\sum_{i=1}^N t_i ||J_E^p(x_i)||^q - \frac{1}{q}\sum_{i=1}^N t_i \langle J_E^p(x_i), z\rangle + \frac{1}{p}||z||^p\\
&= \frac{1}{q}\sum_{i=1}^N t_i ||x_i||^{(p-1)q} - \frac{1}{q}\sum_{i=1}^N t_i \langle J_E^p(x_i), z\rangle + \frac{1}{p}||z||^p\\
&= \frac{1}{q}\sum_{i=1}^N t_i ||x_i||^{p} - \frac{1}{q}\sum_{i=1}^N t_i \langle J_E^p(x_i), z\rangle + \frac{1}{p}||z||^p\\
&= \sum_{i=1}^N t_i \Delta_p(x_i, z).
\end{aligned}$$

Thus, for all $z \in E$,

$$\Delta_p\left(J_{E^*}^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right), z\right) \le \sum_{i=1}^N t_i \Delta_p(x_i, z), \tag{2.10}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Let C be a convex subset of int domf_p, where $f_p(x) = \left(\frac{1}{p}\right)||x||^p$, $2 \le p < \infty$ and let T be a self-mapping of C. A point $p \in C$ is said to be an *asymptotic fixed point* (please, see [7, 18] of T if C contains a sequence $\{x_n\}_{n=1}^\infty$ which converges weakly to p and $\lim_{n\to\infty}||x_n - Tx_n|| = 0$ (see [7, 18]). The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Definition 2.2 Recalling that the Bregman distance is not symmetric, we define the following operators.

Definition 2.3 A nonlinear mapping T with a nonempty asymptotic fixed point set is said to be: (i) *left Bregman strongly nonexpansive* (L-BSNE) (see [13, 14]) with respect to a nonempty $\widehat{F}(T)$ if

$$\Delta_p(Tx, \bar{x}) \le \Delta_p(x, \bar{x}), \quad \forall x \in C, \quad \bar{x} \in \widehat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $\bar{x} \in \widehat{F}(T)$ and

$$\lim_{n\to\infty}(\Delta_p(x_n, \bar{x}) - \Delta_p(Tx_n, \bar{x})) = 0,$$

it follows that

$$\lim_{n\to\infty} \Delta_p(x_n, Tx_n) = 0.$$

According to Martin-Marquez et al. [13, 14], a left Bregman strongly nonexpansive mapping T with respect to a nonempty $\widehat{F}(T)$ is called *strictly left Bregman strongly nonexpansive mapping*. (ii) An operator $T : C \to E$ is said to be: *left Bregman firmly nonexpansive* (L-BFNE) if

$$\left\langle J_p^E(Tx) - J_p^E(Ty), Tx - Ty\right\rangle \leq \left\langle J_p^E(Tx) - J_p^E(Ty), x - y\right\rangle$$

for any $x, y \in C$, or equivalently,

$$\Delta_p(Tx, Ty) + \Delta_p(Ty, Tx) + \Delta_p(x, Tx) + \Delta_p(y, Ty) \leq \Delta_p(x, Ty) + \Delta_p(y, Tx).$$

Remark 2.4 It should be pointed out at this point that using our definition of $\Delta_f(x, y)$ given above, we see that our definitions of left Bregman strongly nonexpansive mapping and left Bregman firmly nonexpansive mapping in Definition 2.3 coincide with the definitions of left Bregman strongly nonexpansive mapping and left Bregman firmly nonexpansive mapping given in [13, 14]. Kindly observe the order of x, y in our definitions here and in the results of [13, 14].

The class of left Bregman strongly nonexpansive mappings is of particular significance in fixed point, iteration and convex optimization theories mainly because it is closed under composition. For more information and examples of L-BSNE and L-BFNE operators, please see [13, 14]. From [13, 14], we know that every left Bregman firmly nonexpansive mapping is left Bregman strongly nonexpansive if $F(T) = \widehat{F}(T)$.

We next state the following lemmas which will be used in the sequel.

Lemma 2.5 *(Xu [29]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 1,$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 1)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

We shall adopt the following notations in this paper:

- $x_n \to x$ means that $x_n \to x$ strongly;
- $x_n \rightharpoonup x$ means that $x_n \to x$ weakly;
- $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak w-limit set of the sequence $\{x_n\}_{n=1}^{\infty}$.

In this paper, we assume that E_1 and E_2 are p-uniformly convex real Banach spaces which are also uniformly smooth, E_1^* is q-uniformly smooth real Banach space which is also uniformly convex where $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We further denote by $J_{E_1}^p$ and $J_{E_2}^p$ the duality mappings of E_1 and E_2 respectively and $J_{E_1^*}^q$ the duality mapping of $E_1{}^*$.

3 Main results

Theorem 3.1 *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP (1.1) has a nonempty solution set Ω. Let T be a left Bregman strongly nonexpansive mapping of C into C such that $F(T) = \widehat{F}(T)$ and $F(T) \cap \Omega \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. For a fixed $u \in E_1$, let sequence $\{x_n\}_{n=1}^{\infty}$ be iteratively generated by $u_1 \in E_1$,*

$$\begin{cases} x_n = \Pi_C J_{E_1^*}^q [J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_C J_{E_1^*}^q (\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(Tx_n)), \quad n \geq 1. \end{cases} \tag{3.1}$$

Suppose the following conditions are satisfied:
(a) $\lim_{n\to\infty} \alpha_n = 0$*;*
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*
(c) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q ||A||^q}\right)^{\frac{1}{q-1}}$.
Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element $x^* \in F(T) \cap \Omega$, where $x^* = \Pi_{F(T)\cap\Omega} u$.

Proof Let $x^* \in \Omega$. Suppose $w_n := Au_n - P_Q(Au_n), \quad \forall n \geq 1$. Suppose $v_n := J_{E_1^*}^q [J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(Au_n - P_Q(Au_n))], \quad \forall n \geq 1$. Then, we have $x_n = \Pi_C v_n, \quad \forall n \geq 1$. Also, it follows from (2.4) that

$$\begin{aligned} \langle J_{E_2}^p(w_n), Au_n - Ax^* \rangle &= ||Au_n - P_Q(Au_n)||^p + \langle J_{E_2}^p(w_n), P_Q(Au_n) - Ax^* \rangle \\ &\geq ||Au_n - P_Q(Au_n)||^p = ||w_n||^p, \end{aligned} \tag{3.2}$$

which, with Lemma 2.1, yields

$$\begin{aligned} \Delta_p(x_n, x^*) &\leq \Delta_p(v_n, x^*) = \Delta_p\left(J_{E_1^*}^q\left[J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(w_n)\right], x^*\right) \\ &= \frac{1}{q}||J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(w_n)||^q - \langle J_{E_1}^p(u_n), x^* \rangle \\ &\quad + t_n \left\langle J_{E_2}^p(w_n), Ax^* \right\rangle + \frac{1}{p}||x^*||^p \\ &\leq \frac{1}{q}||J_{E_1}^p(u_n)||^q - t_n \langle Au_n, J_{E_2}^p(w_n) \rangle + \frac{C_q (t_n ||A||)^q}{q} ||J_{E_2}^p(w_n)||^q \\ &\quad - \left\langle J_{E_1}^p(u_n), x^* \right\rangle + t_n \left\langle J_{E_2}^p(w_n), Ax^* \right\rangle + \frac{1}{p}||x^*||^p \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q}||u_n||^p - \langle J^p_{E_1}(u_n), x^*\rangle + \frac{1}{p}||x^*||^p + t_n\left\langle J^p_{E_2}(w_n), Ax^* - Au_n\right\rangle \\
&\quad + \frac{C_q(t_n||A||)^q}{q}||J^p_{E_2}(w_n)||^q \\
&= \Delta_p(u_n, x^*) + t_n\left\langle J^p_{E_2}(w_n), Ax^* - Au_n\right\rangle + \frac{C_q(t_n||A||)^q}{q}||J^p_{E_2}(w_n)||^q \\
&\le \Delta_p(u_n, x^*) - \left(t_n - \frac{C_q(t_n||A||)^q}{q}\right)||w_n||^p. \qquad (3.3)
\end{aligned}
$$

Using the condition (c), we have

$$\Delta_p(x_n, x^*) \le \Delta_p(u_n, x^*), \quad \forall n \ge 1.$$

Now, using (3.1), we have

$$
\begin{aligned}
\Delta_p(x_{n+1}, x^*) \le \Delta_p(u_{n+1}, x^*) &\le \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(Tx_n, x^*) \\
&\le \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(x_n, x^*) \qquad (3.4) \\
&\le \max\{\Delta_p(u, x^*), \Delta_p(x_n, x^*)\} \\
&\;\;\vdots \\
&\le \max\{\Delta_p(u, x^*), \Delta_p(x_1, x^*)\}.
\end{aligned}
$$

Hence, $\{x_n\}_{n=1}^{\infty}$ is bounded.

Let $y_n := J^q_{E_1^*}\left(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n)\right), \;\; n \ge 1$. From condition (i), we obtain

$$
\begin{aligned}
\Delta_p(y_n, Tx_n) &\le \alpha_n\Delta_p(u, Tx_n) + (1-\alpha_n)\Delta_p(Tx_n, Tx_n) \\
&= \alpha_n\Delta_p(u, Tx_n) \to 0, \quad n \to \infty.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\Delta_p(x_{n+1}, x^*) &\le \Delta_p(v_{n+1}, x^*) \le \Delta_p(u_{n+1}, x^*) \\
&= V_p\left(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n), x^*\right) \\
&\le V_p\left(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n) - \alpha_n\left(J^p_{E_1}(u) - J^p_{E_1}(x^*)\right), x^*\right) \\
&\quad + \alpha_n\left\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), y_n - x^*\right\rangle \\
&= V_p\left(\alpha_n J^p_{E_1}(x^*) + (1-\alpha_n)J^p_{E_1}(Tx_n), x^*\right) \\
&\quad + \alpha_n\left\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), y_n - x^*\right\rangle \\
&\le \alpha_n V_p\left(J^p_{E_1}(x^*), x^*\right) + (1-\alpha_n)V_p\left(J^p_{E_1}(Tx_n), x^*\right) \\
&\quad + \alpha_n\left\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), y_n - x^*\right\rangle \\
&= (1-\alpha_n)\Delta_p(Tx_n, x^*) + \alpha_n\left\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), y_n - x^*\right\rangle \\
&\le (1-\alpha_n)\Delta_p(x_n, x^*) + \alpha_n\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), y_n - x^*\rangle\rangle. \qquad (3.5)
\end{aligned}
$$

The rest of the proof will be divided into two parts.

Case 1 Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Delta_p(x_n, x^*)\}_{n=n_0}^{\infty}$ is non-increasing. Then $\{\Delta_p(x_n, x^*)\}_{n=1}^{\infty}$ converges and $\Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*) \to 0, \quad n \to \infty$. Observe that

$$\Delta_p(x_{n+1}, x^*) \leq \Delta_p(u_{n+1}, x^*) \leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(Tx_n, x^*).$$

It then follows that

$$\begin{aligned}\Delta_p(x_n, x^*) - \Delta_p(Tx_n, x^*) &= \Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*) \\ &\quad + \Delta_p(x_{n+1}, x^*) - \Delta_p(Tx_n, x^*) \\ &\leq \Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*) \\ &\quad + \alpha_n(\Delta_p(u, x^*) - \Delta_p(Tx_n, x^*)) \to 0, \quad n \to \infty. \quad (3.6)\end{aligned}$$

It then follows that

$$\lim_{n\to\infty} \Delta_p(x_n, Tx_n) = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to z. Since $F(T) = \widehat{F}(T)$, we have $z \in F(T)$.

Next, we show that $z \in \Omega$. Now, from (3.3), we obtain

$$\left(t_n - \frac{C_q(t_n||A||)^q}{q}\right)||Au_n - P_Q(Au_n)||^p \leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*). \quad (3.7)$$

Also, from (3.4), we have

$$\Delta_p(u_{n+1}, x^*) \leq \alpha_n \Delta_p(u, x^*) + \Delta_p(x_n, x^*). \quad (3.8)$$

Putting (3.7) into (3.8), we have

$$\begin{aligned}\left(t_n - \frac{C_q(t_n||A||)^q}{q}\right)||Au_n - P_Q(Au_n)||^p &\leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*) \\ &\leq \alpha_{n-1}\Delta_p(u, x^*) + \Delta_p(x_{n-1}, x^*) \\ &\quad - \Delta_p(x_n, x^*). \quad (3.9)\end{aligned}$$

By condition (c) and (3.9), we have

$$\begin{aligned} 0 &< t\left(1 - \frac{C_q k^{q-1}||A||^q}{q}\right)||Au_n - P_Q(Au_n)||^p \\ &\leq \left(t_n - \frac{C_q(t_n||A||)^q}{q}\right)||Au_n - P_Q(Au_n)||^p \\ &\leq \alpha_{n-1}\Delta_p(u, x^*) + \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*) \to 0, n \to \infty.\end{aligned}$$

Hence, we obtain

$$\lim_{n\to\infty} ||Au_n - P_Q(Au_n)|| = 0. \quad (3.10)$$

Since $v_n := J_{E_1^*}^q \left[J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(Au_n - P_Q(Au_n)) \right], \quad \forall n \geq 1$, then we have

$$0 \leq ||J_{E_1}^p(v_n) - J_{E_1}^p(u_n)|| \leq t_n||A^*||||J_{E_2}^p(Au_n - P_Q(Au_n))|| \leq \left(\frac{q}{C_q||A||^q} \right)^{\frac{1}{q-1}} ||A^*||||Au_n - P_Q(Au_n)|| \to 0, n \to \infty. \quad (3.11)$$

Therefore, we obtain

$$\lim_{n\to\infty} ||J_{E_1}^p(v_n) - J_{E_1}^p(u_n)|| = 0.$$

Since $J_{E_1^*}^q$ is also norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} ||v_n - u_n|| = 0.$$

Furthermore,

$$||J_{E_1^*}^q \left[J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(Au_n - P_Q(Au_n)) \right] - u_n|| = ||v_n - u_n|| \to 0, n \to \infty.$$

Since $J_{E_1}^p$ is norm-to-norm uniformly continuous on bounded sets, then

$$\begin{aligned} t||A^* J_{E_2}^p(Au_n - P_Q(Au_n))|| &\leq t_n||A^* J_{E_2}^p(Au_n - P_Q(Au_n))|| \\ &= ||J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(Au_n - P_Q(Au_n)) \\ &\quad - J_{E_1}^p(u_n)|| \to 0, n \to \infty. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} ||A^* J_{E_2}^p(Au_n - P_Q(Au_n))|| = 0. \quad (3.12)$$

Furthermore, we have from (2.6) and (3.4) that

$$\begin{aligned} \Delta_p(v_n, x_n) &= \Delta_p(v_n, \Pi_C v_n) \leq \Delta_p(v_n, x^*) - \Delta_p(x_n, x^*) \\ &\leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*) \\ &\leq \alpha_{n-1} M^* + \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*) \to 0, n \to \infty, \quad (3.13) \end{aligned}$$

for some $M^* > 0$. By (2.3), we have that

$$\lim_{n\to\infty} ||v_n - x_n|| = 0.$$

Hence,

$$||x_n - u_n|| \leq ||v_n - u_n|| + ||v_n - x_n|| \to 0, n \to \infty.$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in \omega_w(x_n)$. Now, since $x_{n_j} \rightharpoonup z$ and $\lim_{n\to\infty} ||x_n - u_n|| = 0$, we obtain that $u_{n_j} \rightharpoonup z$. From (2.2), (2.5) and (2.3), we have that

$$\begin{aligned} \Delta_p(z, \Pi_C z) &\leq \left\langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), z - \Pi_C z \right\rangle \\ &= \left\langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), z - u_{n_j} \right\rangle + \left\langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), u_{n_j} - \Pi_C u_{n_j} \right\rangle \\ &\quad + \left\langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), \Pi_C u_{n_j} - \Pi_C z \right\rangle \\ &\leq \left\langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), z - u_{n_j} \right\rangle + \left\langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), u_{n_j} - \Pi_C u_{n_j} \right\rangle. \end{aligned}$$

As $j \to \infty$, we obtain that $\Delta_p(z, \Pi_C z) = 0$. Thus, $z \in C$. Let us now fix $x \in C$. Then, $Ax \in Q$ and

$$\begin{aligned} ||(I - P_Q)Au_{n_j}||^p &= \langle J_{E_2}^p(Ax_n - P_Q(Au_{n_j})), Ax_n - P_Q(Au_{n_j})\rangle \\ &= \langle J_{E_2}^p(Ax_n - P_Q(Au_{n_j})), Au_{n_j} - Ax\rangle \\ &\quad + \langle J_{E_2}^p(Ax_n - P_Q(Au_{n_j})), Ax - P_Q(Au_{n_j})\rangle \\ &\leq \langle J_{E_2}^p(Ax_n - P_Q(Au_{n_j})), Au_{n_j} - Ax\rangle \\ &\leq M||A^*(I - P_Q)Au_{n_j}||^{p-1} \to 0, n \to \infty, \end{aligned}$$

where $M > 0$ is sufficiently large number. It then follows from (2.4) that

$$\begin{aligned} ||(I - P_Q)Az||^p &= \langle J_{E_2}^p(Az - P_Q(Az)), Az - P_Q(Az)\rangle \\ &= \langle J_{E_2}^p(Az - P_Q(Az)), Az - Au_{n_j}\rangle \\ &\quad + \langle J_{E_2}^p(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle \\ &\quad + \langle J_{E_2}^p(Az - P_Q(Az)), P_Q(Au_{n_j}) - P_Q(Az)\rangle \\ &\leq \langle J_{E_2}^p(Az - P_Q(Az)), Az - Au_{n_j}\rangle \\ &\quad + \langle J_{E_2}^p(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle. \end{aligned}$$

Also, since $Au_{n_j} \rightharpoonup Az$, we have that

$$\lim_{n\to\infty} ||(I - P_Q)Az|| = 0.$$

Thus, $Az \in Q$. This implies that $z \in \Omega$ and hence $z \in F(T) \cap \Omega$.

Furthermore, we have that

$$\Delta_p(x_n, y_n) \leq \alpha_n \Delta_p(x_n, u) + (1 - \alpha_n)\Delta_p(x_n, Tx_n) \to 0, \quad n \to \infty. \tag{3.14}$$

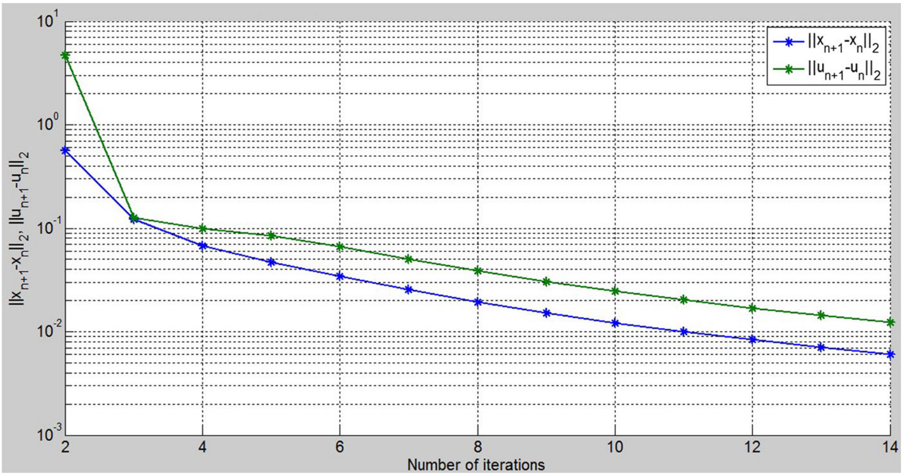

Fig. 1 Example 4.1 case I: ($u = (1, 1, 1)$, $u_1 = (3, 0, 4)$ and $t_n = 0.0137$)

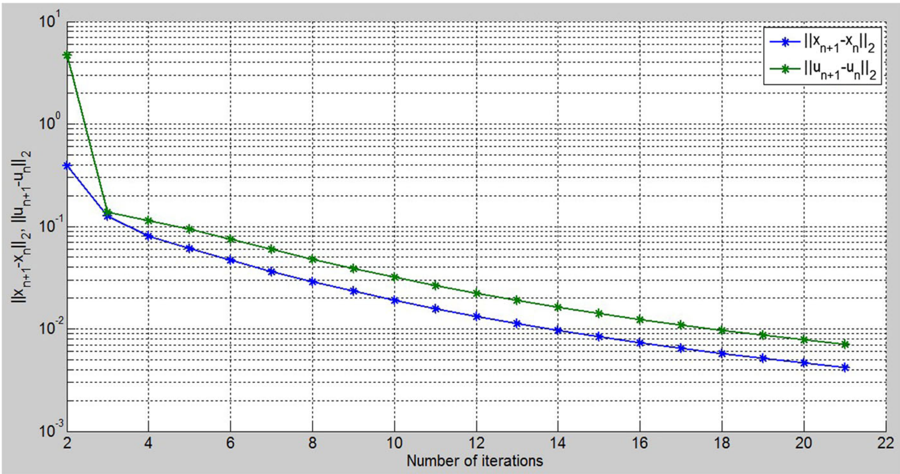

Fig. 2 Example 4.1 case I: ($u = (1, 1, 1)$, $u_1 = (3, 0, 4)$ and $t_n = 0.01$)

By (2.3), it follows that $||x_n - y_n|| \to 0, \ \ n \to \infty$.

Let $p := \Pi_{F(T)\cap\Omega}u$. We next show that $\limsup_{n\to\infty}\left\langle J^p_{E_1}(u) - J^p_{E_1}(p), y_n - p\right\rangle \leq 0$. To show the inequality $\limsup_{n\to\infty}\left\langle J^p_{E_1}(u) - J^p_{E_1}(p), y_n - p\right\rangle \leq 0$, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle J^p_{E_1}(u) - J^p_{E_1}(p), x_n - p\right\rangle = \lim_{j\to\infty}\left\langle J^p_{E_1}(u) - J^p_{E_1}(p), x_{n_j} - p\right\rangle.$$

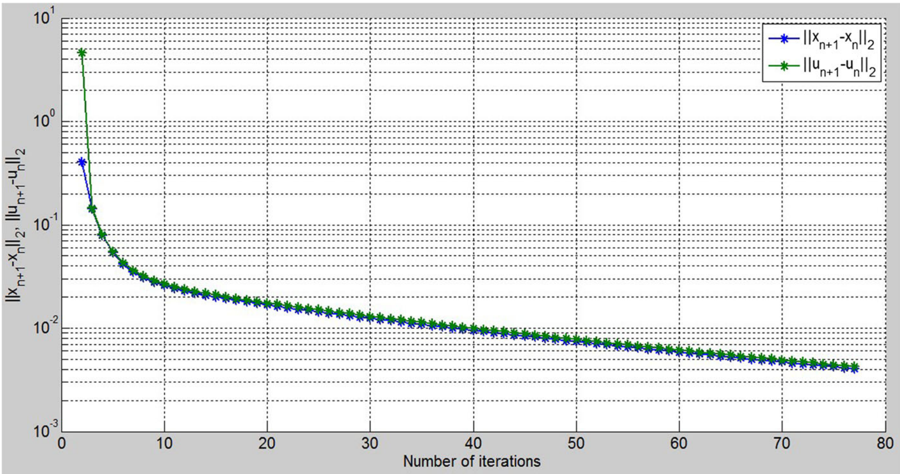

Fig. 3 Example 4.1 case I: ($u = (1, 1, 1)$, $u_1 = (3, 0, 4)$ and $t_n = 0.001$)

Table 1 Example 4.1 case I: ($t_n = 0.0137$ and $t_n = 0.01$)

t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$	t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$
0.0137	8.9205e-04	2	0.5621	4.7161	0.01	0.0013	2	0.3916	4.6617
		3	0.1214	0.1260			3	0.1259	0.137
		4	0.0677	0.0990			4	0.0811	0.1146
		5	0.0473	0.0855			5	0.0607	0.0946
		6	0.0342	0.0662			6	0.0467	0.0757
		7	0.0254	0.0505			7	0.0365	0.0602
		8	0.0194	0.0389			8	0.0289	0.0481
		9	0.0152	0.0307			9	0.0232	0.0389
		10	0.0122	0.0247			10	0.019	0.0319
		11	0.01	0.0202			11	0.0157	0.0265
		12	0.0083	0.0169			12	0.0132	0.0223
		13	0.0071	0.0143			13	0.0112	0.0189
		14	0.0061	0.0123			14	0.0097	0.0163
							15	0.0084	0.0141
							16	0.0073	0.0124
							17	0.0065	0.0109
							18	0.0058	0.0097
							19	0.0052	0.0087
							20	0.0046	0.0078
							21	0.0042	0.0071

By $||x_n - y_n|| \to 0, \quad n \to \infty$ and (2.5), we obtain

$$\limsup_{n\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(p), y_n - p\rangle \le \limsup_{n\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(p), x_n - p\rangle \le 0. \quad (3.15)$$

Now, using (3.15), (3.5) and Lemma 2.5, we obtain $\Delta_p(x_n, p) \to 0, \quad n \to \infty$. Hence, $x_n \to p, \quad n \to \infty$.

Case 2 Assume that $\{\Delta_p(x_n, x^*)\}_{n=1}^{\infty}$ is not monotonically decreasing sequence. Set $\Gamma_n = \Delta_p(x_n, x^*), \quad \forall n \ge 1$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}.$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \le \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \forall n \ge n_0.$$

Furthermore, we obtain

$$\begin{aligned}\Delta_p(x_{\tau(n)}, x^*) - \Delta_p(Tx_{\tau(n)}, x^*) &= D_f(x^*, x_{\tau(n)}) - D_f(x^*, x_{\tau(n)+1}) \\ &\quad + D_f(x^*, x_{\tau(n)+1}) - \Delta_p(Tx_{\tau(n)}, x^*) \\ &\le \Delta_p(x_{\tau(n)}, x^*) - \Delta_p(x_{\tau(n)+1}, x^*) \\ &\quad + \alpha_n(\Delta_p(u, x^*) - \Delta_p(x_{\tau(n)}, x^*)) \to 0, \quad n \to \infty.\end{aligned}$$

It then follows that

$$\lim_{n\to\infty} \Delta_p(x_{\tau(n)}, Tx_{\tau(n)}) = 0.$$

After a similar conclusion from (3.10), it is easy to see that

$$||Au_{\tau(n)} - P_Q u_{\tau(n)}|| \to 0, n \to \infty.$$

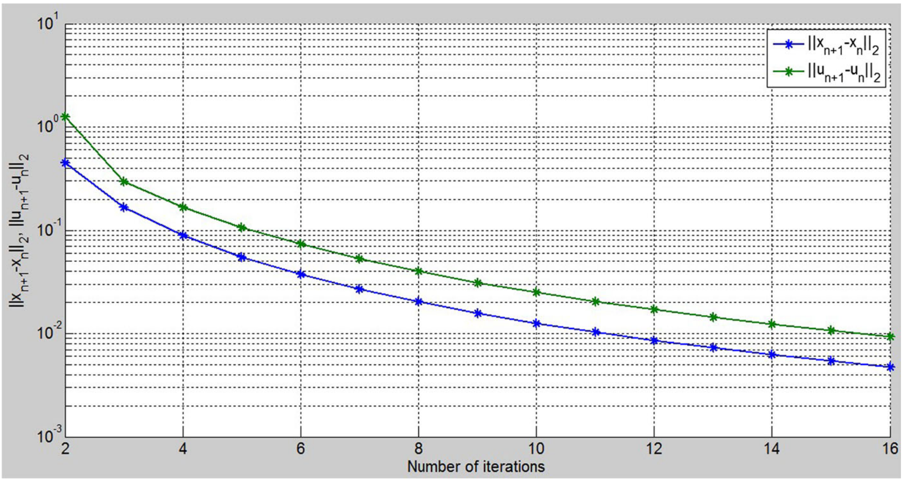

Fig. 4 Example 4.1 case II: ($u = (1, 1, 1)$, $u_1 = (1, 2, 1)$ and $t_n = 0.0137$)

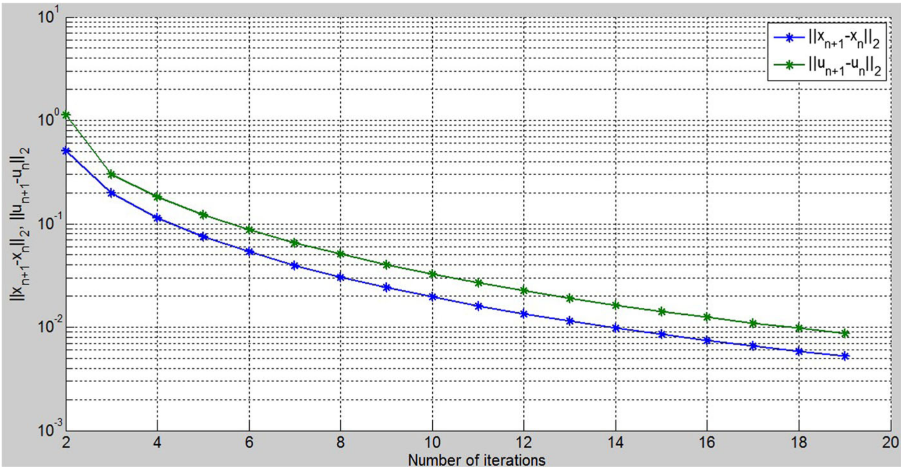

Fig. 5 Example 4.1 case II: ($u = (1, 1, 1)$, $u_1 = (1, 2, 1)$ and $t_n = 0.01$)

By the similar argument as above in Case 1, we conclude immediately that

$$\lim_{n\to\infty} ||A^* J_{E_2}^p (Au_{\tau(n)} - P_Q(Au_{\tau(n)}))|| = 0.$$

and

$$\limsup_{n\to\infty} \langle J_{E_1}^p(u) - J_{E_1}^p(x^*), y_{\tau(n)} - x^* \rangle \leq 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$ which converges weakly to $z \in C$ and $Az \in Q$. From (3.5) we have that

$$\Delta_p(x_{\tau(n)+1}, x^*) \leq (1 - \alpha_{\tau(n)})\Delta_p(x_{\tau(n)}, x^*) + \alpha_n \langle J_{E_1}^p(u) - J_{E_1}^p(x^*), y_{\tau(n)} - x^* \rangle$$

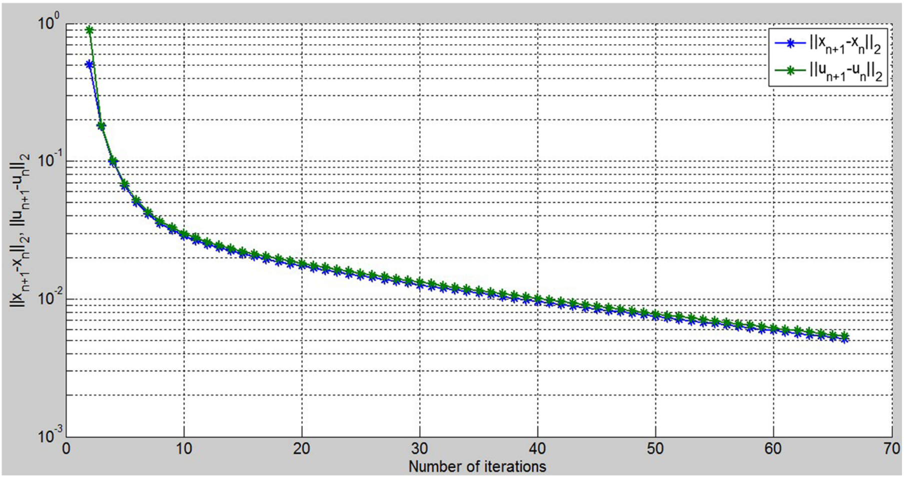

Fig. 6 Example 4.1 case II: ($u = (1, 1, 1)$, $u_1 = (1, 2, 1)$ and $t_n = 0.001$)

Table 2 Example 4.1 case II: ($t_n = 0.0137$ and $t_n = 0.01$)

t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$	t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$
0.0137	8.6527e-04	2	0.4518	1.2532	0.01	0.0010	2	0.5073	1.1291
		3	0.1667	0.2965			3	0.1998	0.3020
		4	0.0888	0.1675			4	0.1143	0.1809
		5	0.0551	0.1067			5	0.0754	0.1222
		6	0.0374	0.0732			6	0.0535	0.0879
		7	0.0269	0.0529			7	0.0398	0.0659
		8	0.0202	0.0399			8	0.0306	0.0509
		9	0.0158	0.0311			9	0.0242	0.0403
		10	0.0126	0.0249			10	0.0196	0.0326
		11	0.0103	0.0204			11	0.0161	0.0269
		12	0.0086	0.0170			12	0.0135	0.0225
		13	0.0073	0.0144			13	0.0114	0.0191
		14	0.0062	0.0123			14	0.0098	0.0164
		15	0.0054	0.0107			15	0.0085	0.0142
		16	0.0047	0.0094			16	0.0074	0.0125
							17	0.0066	0.0110
							18	0.0058	0.0098
							19	0.0052	0.0087

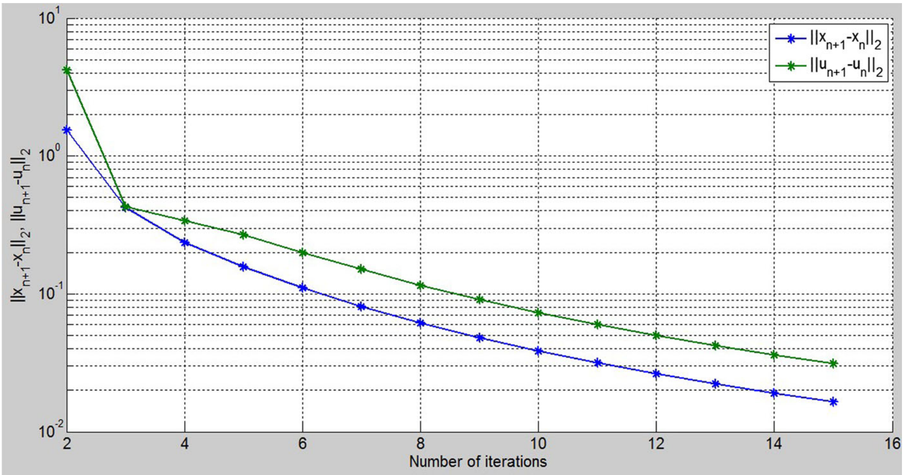

Fig. 7 Example 4.1 case III: ($u = (1, 5, 1)$, $u_1 = (2, 1, 4)$ and $t_n = 0.0137$)

which implies by Lemma 2.5

$$\lim_{n\to\infty} \Delta_p(x_{\tau(n)}, x^*) = 0 \tag{3.16}$$

and $\lim_{n\to\infty} \Delta_p(x_{\tau(n)+1}, x^*) = 0$. Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to x^*. This completes the proof. □

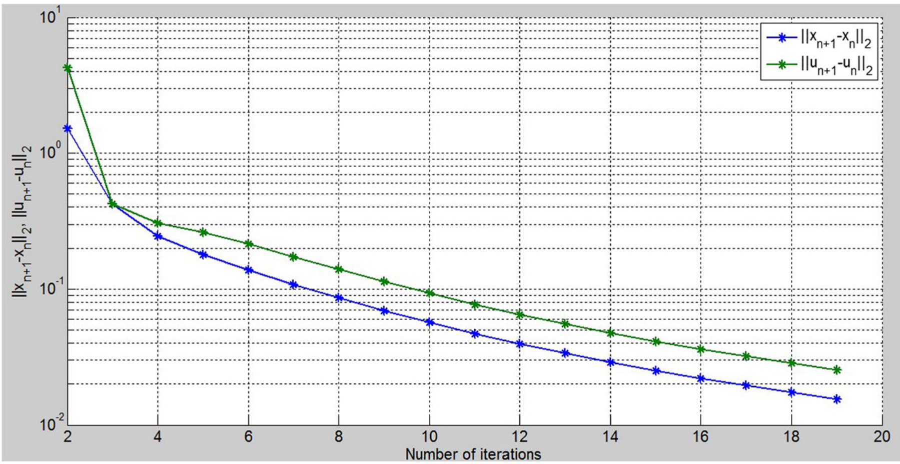

Fig. 8 Example 4.1 case III: ($u = (1, 5, 1)$, $u_1 = (2, 1, 4)$ and $t_n = 0.01$)

Corollary 3.2 *Let E_1 and E_2 be two L_p spaces with $2 \leq p < \infty$. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Let T be a left Bregman strongly nonexpansive mapping of C into C such that $F(T) = \widehat{F}(T)$ and $F(T) \cap \Omega \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. For a fixed $u \in E_1$, let sequence $\{x_n\}_{n=1}^{\infty}$, be iteratively generated by $u_1 \in E$,*

$$\begin{cases} x_n = \Pi_C J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_C J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n) J^p_{E_1}(Tx_n)), \quad n \geq 1. \end{cases}$$

Suppose the following conditions are satisfied:
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*
(c) $0 < t \leq t_n \leq k < \left(\frac{q}{c_q ||A||^q}\right)^{\frac{1}{q-1}}$.
Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element $x^* \in F(T) \cap \Omega$, where $x^* = \Pi_{F(T)\cap\Omega} u$.

Next, using the idea in [13], we consider the mapping $T : C \to C$ defined by $T = T_m T_{m-1} ... T_1$, where $T_i (i = 1, 2, \ldots, m)$ are left Bregman strongly nonexpansive mappings on E. We know from Proposition 3.4 (page 602) of [13] that

$$\left(\cap_{i=1}^m F(T_i)\right) = F(T).$$

Using Theorem 3.1, we have the following corollary.

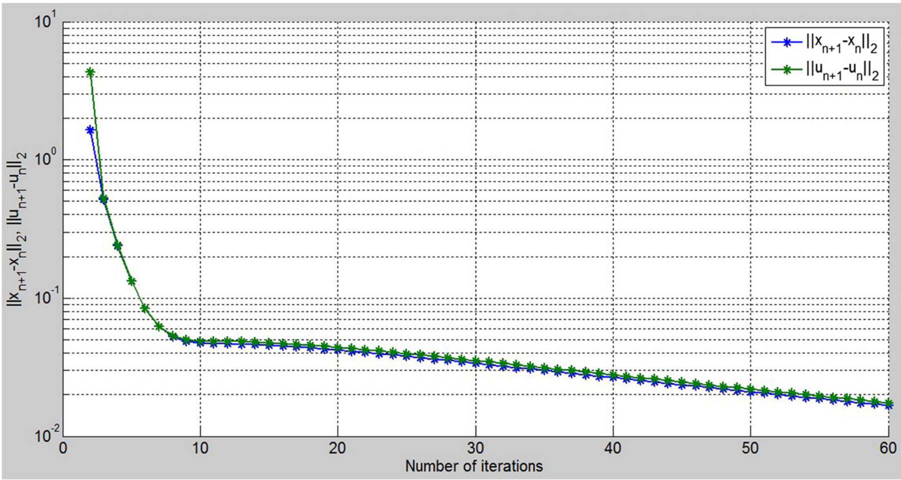

Fig. 9 Example 4.1 case III: $(u = (1, 5, 1)$, $u_1 = (2, 1, 4)$ and $t_n = 0.001)$

Table 3 Example 4.1 case III: ($t_n = 0.0137$ and $t_n = 0.01$)

t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$	t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$
0.0137	8.8625e-04	2	1.542	4.2209	0.01	0.0010	2	1.5204	4.2411
		3	0.4239	0.4278			3	0.4233	0.4234
		4	0.2365	0.3417			4	0.2448	0.3059
		5	0.1572	0.2676			5	0.1797	0.2611
		6	0.1108	0.2003			6	0.1384	0.2140
		7	0.0813	0.1506			7	0.1085	0.1725
		8	0.0617	0.1156			8	0.0862	0.1391
		9	0.0483	0.0908			9	0.0695	0.1131
		10	0.0387	0.0730			10	0.0568	0.0929
		11	0.0317	0.0598			11	0.0471	0.0773
		12	0.0264	0.0499			12	0.0396	0.0650
		13	0.0224	0.0422			13	0.0337	0.0553
		14	0.0192	0.0362			14	0.0290	0.0476
		15	0.0166	0.0314			15	0.0251	0.0413
							16	0.0220	0.0362
							17	0.0194	0.0320
							18	0.0173	0.0284
							19	0.0155	0.0255

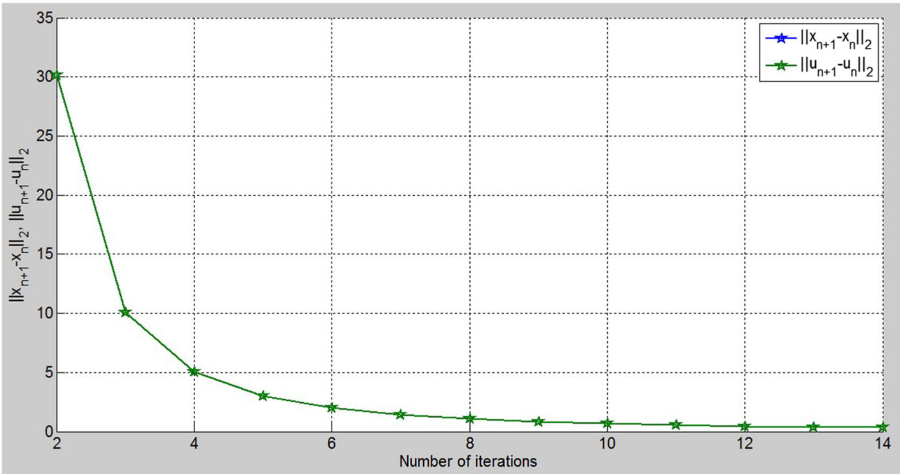

Fig. 10 Example 4.2 case I: ($u = \frac{5}{2}t^2 - 2t, u_1 = 3\sin(t)$ and $t_n = 1.0 \times 10^9$)

Corollary 3.3 *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Let $T_i (i = 1, 2, \dots, m)$ be a sequence of left Bregman strongly nonexpansive mapping of C into C such that $F(T_i) = \widehat{F}(T_i)$ and $(\cap_{i=1}^m F(T_i)) \cap \Omega \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in (0, 1). For a fixed $u \in E_1$, let sequence $\{x_n\}_{n=1}^\infty$ be iteratively generated by $u_1 \in E$,*

$$\begin{cases} x_n = \Pi_C J_{E_1^*}^q [J_{E_1}^p (u_n) - t_n A^* J_{E_2}^p (Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_C J_{E_1^*}^q (\alpha_n J_{E_1}^p (u) + (1 - \alpha_n) J_{E_1}^p (T_m T_{m-1} ... T_1 x_n)), \quad n \geq 1. \end{cases}$$

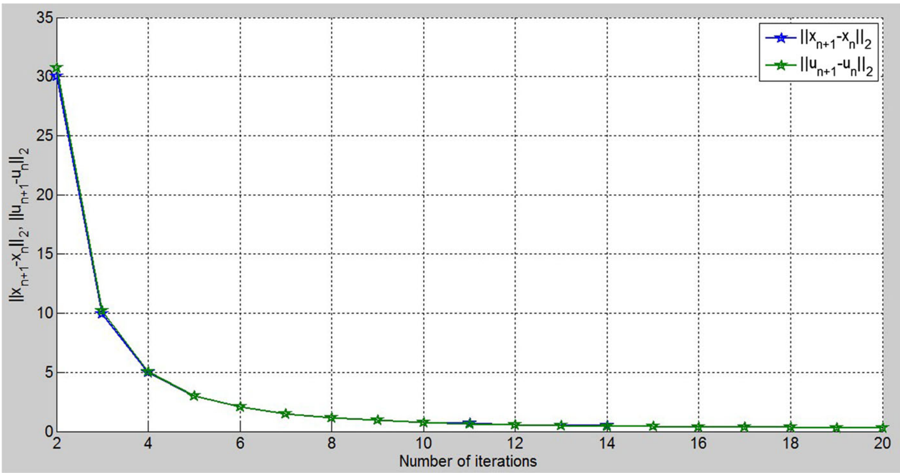

Fig. 11 Example 4.2 case I: ($u = \frac{5}{2}t^2 - 2t, u_1 = 3\sin(t)$ and $t_n = 0.1$)

Suppose the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and*

(c) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to an element* $x^* \in (\cap_{i=1}^{m} F(T_i)) \cap \Omega$, *where* $x^* = \Pi_{(\cap_{i=1}^{m} F(T_i))\cap\Omega} u$.

4 Numerical example

In this section, we present some preliminary numerical results. All codes were written in Matlab 2012b and run on Hp i-5 Dual-Core laptop.

Example 4.1 We give a numerical example in $(\mathbb{R}^3, ||.||_2)$ of the problem considered in Theorem 3.1 in the previous section. Now take

$$C := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x\rangle \geq b\},$$

where $a = (2, -1, 5)$ and $b = 1$, then

$$\Pi_C(x) = P_C(x) = \frac{b - \langle a, x\rangle}{||a||_2^2} a + x.$$

Let

$$Q := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x\rangle = b\},$$

where $a = (3, 5, 7)$ and $b = 2$ then

$$P_Q(x) = \max\left\{0, \frac{b - \langle a, x\rangle}{||a||_2^2}\right\} a + x.$$

Furthermore, let $T = P_C$ (which is an example of a left Bregman strongly nonexpansive mapping, please see [13, 14], $\alpha_n = \frac{1}{n+1}$ and $A = \begin{pmatrix} 5 & -5 & -7 \\ -4 & 2 & -4 \\ -7 & -4 & 5 \end{pmatrix}$, then our iterative scheme (3.1) becomes

$$\begin{cases} x_n = P_C[u_n - t_n A^T(Au_n - P_Q(Au_n))] \\ u_{n+1} = P_C\left(\frac{u}{n+1} + \left(1 - \frac{1}{n+1}\right)(P_C x_n)\right), \quad n \geq 1. \end{cases}$$

We make different choices of u_1, u and t_n. The stopping criterion for all testing methods was taken as:

$$\frac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-2}.$$

We note here that in each case, we omit tables for very small values of t_n.

Case I: Take $u = (1, 1, 1)$ and $u_1 = (3, 0, 4)$ and then consider $t_n = 0.0137$, $t_n = 0.01$ and $t_n = 0.001$. The graphs using our algorithm (3.1) with these t_ns are

Table 4 Example 4.2 case I: ($t_n = 0.0137$ and $t_n = 0.01$)

t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$	t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$
10^{-9}	0.0095	2	30.1656	30.1656	0.1	0.0093	2	30.0439	30.7715
		3	10.0552	10.0552			3	9.9520	10.1759
		4	5.0276	5.0276			4	4.9907	5.0409
		5	3.0166	3.0166			5	3.0294	3.0363
		6	2.0110	2.0110			6	2.0572	2.0534
		7	1.4365	1.4365			7	1.5055	1.4990
		8	1.0773	1.0773			8	1.1628	1.1558
		9	0.8379	0.8379			9	0.9357	0.9289
		10	0.6703	0.6703			10	0.7777	0.7713
		11	0.5485	0.5485			11	0.6635	0.6576
		12	0.4571	0.4571			12	0.5785	0.5730
		13	0.3867	0.3867			13	0.5136	0.5085
		14	0.3315	0.3315			14	0.4631	0.4582
							15	0.4231	0.4185
							16	0.3909	0.3866
							17	0.3649	0.3607
							18	0.3435	0.3394
							19	0.3258	0.3218
							20	0.3111	0.3072

given respectively in Figs. 1, 2 and 3 while Table 1 shows the numerical values for two t_ns only.

Case II: Take $u = (1, 1, 1)$ and $u_1 = (1, 2, 1)$ and then consider $t_n = 0.0137$, $t_n = 0.01$ and $t_n = 0.001$. The graphs using our algorithm (3.1) with these t_ns are given respectively in Figs. 4, 5 and 6 while Table 2 shows the numerical values for two t_ns only.

Case III: Take $u = (1, 5, 1)$ and $u_1 = (2, 1, 4)$ and then consider $t_n = 0.0137$, $t_n = 0.01$ and $t_n = 0.001$. The graphs using our algorithm (3.1) with these t_ns are given respectively in Figs. 7, 8 and 9 while Table 3 shows the numerical values for two t_ns only.

Remark 4.1 We make the following comments from Example 4.1.
1. By the choice of our stopping criterion, we get less number of iterations required for the convergence. For example, we observe that if the stopping criterion is taken as:

$$\frac{\max\{||x_n - P_C x_n||, ||Ax_n - P_Q(Ax_n)||\}}{\max\{||x_1 - P_C x_1||, ||Ax_1 - P_Q(Ax_1)||\}} < 10^{-4},$$

we get very large iterations in thousands in many cases. If the stopping criterion is taken as

$$\frac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-4},$$

we get about 194 iterations in some cases. Furthermore, using the choice of our stopping criterion $\frac{||x_{n+1}-x_n||}{||x_2-x_1||} < 10^{-2}$, if t_n is chosen very small and close to zero, we require many iteration steps but when t_n is chosen such that it is a bit away from zero but close to $\frac{2}{||A||^2}$, we require less iterations for convergence.

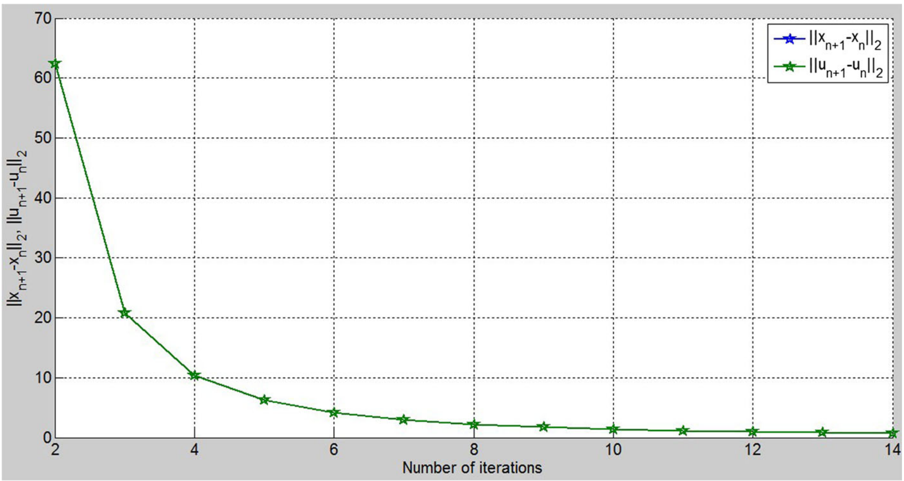

Fig. 12 Example 4.2 case II: ($u = \frac{5}{2}t^2 - 2t$, $u_1 = \exp(2t)$ and $t_n = 1.0 \times 10^9$)

2. We observe from the numerical analysis of our result of the tables and graphs that we realise fast convergence when t_n is taken close to $\frac{2}{||A||^2}$ and the more the iteration steps are, the more slowly the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge to the solution of our problem. Furthermore, we see that the sequence $\{x_n\}_{n=1}^{\infty}$ converges faster to the solution than $\{u_n\}_{n=1}^{\infty}$.

3. We also notice that the choice of u_1, either close to u or not, does not have significant effect on the convergent rate of both sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$.

Example 4.2 Here, we take $E_1 = L_2([0, 1]) = E_2$ with the inner product given as

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

Now, let

$$C := \{x \in L_2([0, 1]) : \langle x, a\rangle = b\},$$

where $a = 2t^2, \quad b = 0$. Then

$$P_C(x) = \max\left\{0, \frac{b - \langle a, x\rangle}{||a||_2^2}\right\} a + x.$$

Also, let

$$Q := \{x \in L_2([0, 1]) : \langle x, c\rangle \geq d\},$$

where $c = \frac{t}{3}, \quad d = -1$. Then

$$\Pi_Q(x) = P_Q(x) = \frac{d - \langle c, x\rangle}{||c||_2^2} c + x.$$

Let us assume that

$$A : L_2([0, 1]) \to L_2([0, 1]), \qquad (Ax)(t) = \frac{x(t)}{2}.$$

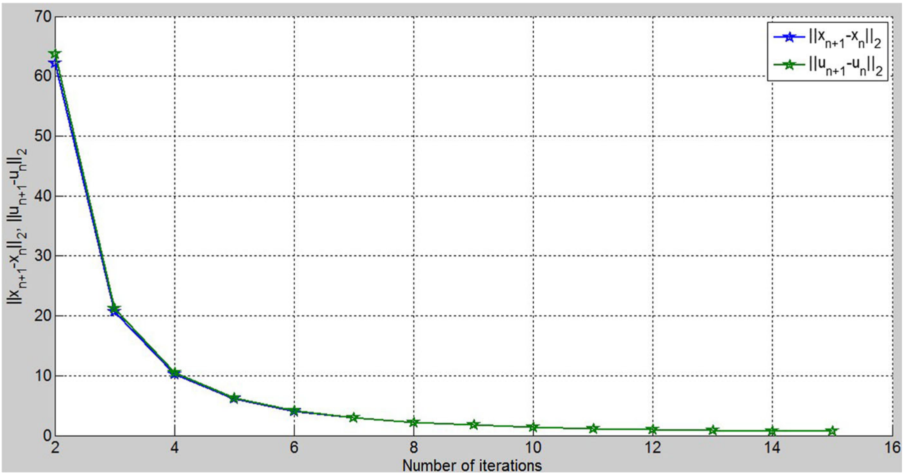

Fig. 13 Example 4.2 case II: ($u = \frac{5}{2}t^2 - 2t$, $u_1 = \exp(2t)$ and $t_n = 0.1$)

Table 5 Example 4.2 case II: ($t_n = 0.0137$ and $t_n = 0.01$)

t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$	t_n	Time	Iter.	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|u_{n+1} - u_n\|\|_2$
10^{-9}	0.0095	2	62.3415	62.3415	0.1	0.0099	2	62.1964	63.7764
		3	20.7805	20.7805			3	20.6378	21.1620
		4	10.3903	10.3903			4	10.25	10.5103
		5	6.2342	6.2342			5	6.1246	6.2510
		6	4.1561	4.1561			6	4.0887	4.1462
		7	2.9686	2.9686			7	2.9363	2.9664
		8	2.2265	2.2265			8	2.2222	2.2386
		9	1.7317	1.7317			9	1.7499	1.7589
		10	1.3854	1.3854			10	1.4217	1.4265
		11	1.1335	1.1335			11	1.1847	1.1869
		12	0.9446	0.9446			12	1.0083	1.0088
		13	0.7993	0.7993			13	0.8735	0.8730
		14	0.6851	0.6851			14	0.7684	0.7672
							15	0.6850	0.6833

Then A is a bounded linear operator and $A^* = A$. Suppose that we take operator T in Theorem 3.1 as $T := P_C$, the metric projection on C. Then the problem considered in Theorem 3.1 reduces to:

$$\text{find } x \in F(T) \cap C(= C) \text{ such that } Ax \in Q. \tag{4.1}$$

We observe that if Ω denotes the set of solutions of (4.1), then $\Omega \neq \emptyset$, since $x^* = 0 \in \Omega$. Furthermore, our iterative scheme (3.1) becomes

$$\begin{cases} x_n = P_C[u_n - t_n A^*(Au_n - P_Q(Au_n))] \\ u_{n+1} = P_C\left(\frac{u}{n+1} + \left(1 - \frac{1}{n+1}\right)(P_C x_n)\right), \quad n \geq 1. \end{cases}$$

We make different choices of u_1 and t_n with a choice of $u = \frac{5}{2}t^2 - 2t$ and the same stopping criterion as used in the Example 4.1.

Case I: Take $u_1 = 3sin(t)$ and then consider both $t_n = 1.0 \times 10^{-9}$ and $t_n = 0.1$. The graphs for both t_ns are presented respectively in Figs. 10 and 11 while Table 4 shows the numerical values for both cases with the same choice of u_1.

Case II: Take $u_1 = \exp(2t)$ and then consider $t_n = 1.0 \times 10^{-9}$ and $t_n = 0.1$. The graphs for both t_ns are presented respectively in Figs. 12 and 13 while Table 5 shows the numerical values for both cases with the same choice of u_1.

Remark 4.2 We make the following comments from Example 4.2. We observe that different choices of t_n and u_1 have no effect in terms of cpu time for the convergence of our algorithm but when t_n is taken close to zero, we have small reduction in the number of iterations in some cases with relatively the same cpu time.

References

1. Alber, Y.I.: Metric and generalized projection operator in Banach spaces: properties and applications, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type vol 178 of Lecture Notes in Pure and Applied Mathematics, pp, vol. 15-50. USA, Dekker, New York, NY (1996)
2. Bauschke, H.H., Combettes, P.L.: Convex analysis and monotone operator theory in hilbert spaces. Springer, New York (2011)
3. Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Problems **18**(2), 441–453 (2002)
4. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Problems **20**(1), 103–120 (2004)
5. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in a product space. Numerical Algorithms **8**(2-4), 221–239 (1994)
6. Censor, Y., Lent, A.: An iterative row-action method for interval convex programming. J. Optim. Theory Appl. **34**, 321?353 (1981)
7. Censor, Y., Reich, S.: Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization. Optimization **37**, 323–339 (1996)
8. Cioranescu, I.: Geometry of banach spaces, duality mappings and nonlinear problems. Kluwer Academic Dordrecht (1990)
9. Dunford, N., Schwartz, J.T.: Linear operators I. Wiley?Interscience, New York (1958)
10. Goebel, K., Kirk, W.A.: Topics in metric fixed point theory. In: Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, (1990)
11. Lindenstrauss, J., Tzafriri, L.: Classical banach spaces II. Springer, Berlin (1979)

12. Maingé, P.E.: The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces. Comput. Math. Appl. **59**(1), 74–79 (2010)
13. Martín-Márquez, V., Reich, S., Sabach, S.: Bregman strongly nonexpansive operators in reflexive Banach spaces. J. Math. Anal. Appl. **400**, 597–614 (2013)
14. Martín-Márquez, V., Reich, S., Sabach, S.: Right Bregman nonexpansive operators in banach spaces. Nonlinear Anal. **75**, 5448–5465 (2012)
15. Masad, E., Reich, S.: A note on the multiple-set split convex feasibility problem in Hilbert space. J. Nonlinear Convex Anal. **8**, 367–371 (2007)
16. Nakajo, K., Takahashi, W.: Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. J. Math. Anal. Appl. **279**, 372–379 (2003)
17. Qu, B., Xiu, N.: A note on the CQ algorithm for the split feasibility problem. Inverse Problems **21**(5), 1655–1665 (2005)
18. Reich, S.: A weak convergence theorem for the alternating method with Bregman distances. In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, pp. 313-318 (1996)
19. Reich, S.: Book Review: Geometry of Banach spaces, duality mappings and nonlinear problems. Bull. Amer. Math. Soc. **26**, 367–370 (1992)
20. Reich, S.: Extension problems for accretive sets in Banach spaces. J. Functional Anal. **26**, 378–395 (1977)
21. Schöpfer, F.: Iterative regularization method for the solution of the split feasibility problem in Banach spaces. PhD thesis, Saarbrücken (2007)
22. Schöpfer, F., Schuster, T., Louis, A.K.: An iterative regularization method for the solution of the split feasibility problem in Banach spaces. Inverse Problems **24**, 055008 (2008)
23. Shehu, Y.: Strong convergence theorem for Multiple Sets Split Feasibility Problems in Banach Spaces, Under review: Numerical Functional Analysis and Optimization
24. Shehu, Y.: A cyclic iterative method for solving Multiple Sets Split Feasibility Problems in Banach Spaces, Under review: Quaestiones Mathematicae
25. Shehu, Y., Ogbuisi, F.U., Iyiola, O.S.: Convergence Analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces, In press: Optimization. doi:10.1080/02331934.2015.1039533
26. Takahashi, W.: Nonlinear Functional Analysis-Fixed Point Theory and Applications, Yokohama Publishers Inc., Yokohama. (in Japanese) (2000)
27. Takahashi, W.: Nonlinear functional analysis. Yokohama Publishers, Yokohama (2000)
28. Wang, F.: A new algorithm for solving the multiple-sets split feasibility problem in Banach spaces. Numer. Funct. Anal. Optim. **35**, 99–110 (2014)
29. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**(2), 1127–1138 (1991)
30. Xu, H.-K.: A variable Krasnoselskii-Mann algorithm and the multiple-set split feasibility problem. Inverse Problems **22**(6), 2021–2034 (2006)
31. Yang, Q.: The relaxed CQ algorithm solving the split feasibility problem. Inverse Problems **20**(4), 1261–1266 (2004)
32. Yang, Q., Zhao, J.: Generalized KM theorems and their applications. Inverse Problems **22**(3), 833–844 (2006)
33. Yao, Y., Jigang, W., Liou, Y.-C.: Regularized methods for the split feasibility problem. Abstr. Appl Anal. **2012**(ID), 140679 (2012). 13