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ORIGINAL PAPER

High strong order stochastic Runge-Kutta methods for Stratonovich stochastic differential equations with scalar noise

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Abstract This paper concerns the stochastic Runge-Kutta (SRK) methods with high strong order for solving the Stratonovich stochastic differential equations (SDEs) with scalar noise. Firstly, the new SRK methods with strong order 1.5 or 2.0 for the Stratonovich SDEs with scalar noise are constructed by applying colored rooted tree analysis and the theorem of order conditions for SRK methods proposed by Rößler (SIAM J. Numer. Anal. **48**(3), 922–952, 2010). Secondly, a specific SRK method with strong order 2.0 for the Stratonovich SDEs whose drift term vanishes is proposed. And another specific SRK method with strong order 1.5 for the Stratonovich SDEs whose drift and diffusion terms satisfy the commutativity condition is proposed. The two specific SRK methods need only to use one random variable and do not need to simulate the multiple Stratonovich stochastic integrals. Finally, the numerical results show that performance of our methods is better than those of well-known SRK methods with strong order 1.0 or 1.5.

Keywords Stratonovich stochastic differential equations · Stochastic Runge-Kutta methods · Strong convergence

Mathematics Subject Classification (2010) 60H35 · 65L06 · 65L20

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1 Introduction

In recent years, great progress has been made in the area of numerical methods for solving stochastic differential equations (SDEs). Runge-Kutta (RK) methods are a very important class of numerical methods for solving ordinary differential equations (ODEs). Therefore, recently there has been much interest in developing stochastic Runge-Kutta (SRK) methods for solving SDEs. For example, Some SRK methods converging in the strong sense or in the weak sense were proposed in [10, 13, 14]. And order conditions for these methods were obtained by comparing Taylor series of the exact and the numerical solutions. In analogy to the deterministic case, Burrage and Burrage [1–3] extended the classical rooted tree analysis in Butcher [5], and introduced colored trees (or stochastic trees) to calculate the order conditions of strong order SRK methods for the Stratonovich SDEs. Komori [11] applied the stochastic tree analysis to calculate the order conditions of weak order SRK methods for the Stratonovich SDEs. R$\ddot{o}$ßler [16, 17] applied the stochastic tree analysis to calculate the order conditions of weak order SRK methods for both the It$\hat{o}$ and the Stratonovich SDEs, and R$\ddot{o}$ßler [18] applied it to calculate the order conditions of strong order SRK methods for both the It$\hat{o}$ and the Stratonovich SDEs. Debrabant and Kværnø [8] introduced a unifying approach for the construction of stochastic B-series and gave order conditions of the weak and strong convergence for both the It$\hat{o}$ and the Stratonovich SDEs. Based on their work, the weak order SRK methods have been constructed, see, e.g., [9, 10, 13], and the strong order SRK methods have been constructed, see, e.g., [6, 7, 20]. However, up to now, it remains a challenging task to construct specific SRK methods with high strong order. This is due to that the order conditions of the high strong order SRK methods contain too many equations. The aim of the present paper is to make efforts in this direction and to construct new SRK methods with high strong order.

In this paper, new strong order SRK methods with several groups of independent internal stages are constructed. This technique can reduce the number of the equations in order conditions when we construct the SRK methods with high strong order. By applying the results of order conditions for the general class of strong order SRK methods in [18], some SRK methods with strong order 1.5 or 2.0 for solving the Stratonovich SDEs with scalar noise are constructed in Sections 3 and 4. In Section 5, some methods for approximating the multiple Stratonovich stochastic integrals are introduced because a multiple Stratonovich stochastic integral need to be approximated for the SRK methods with strong 2.0. In Section 6, two specific high strong order SRK methods applied to two specific types of the Stratonovich SDEs are proposed, and these methods do not need to simulate the multiple Stratonovich stochastic integrals. In Section 7, some numerical results are reported to illustrate the theoretical results.

We consider the Stratonovich autonomous SDE system with scalar noise

$$dy(t) = f(y(t))dt + g(y(t)) \circ dW(t), \quad y(t_0) = y_{t_0}, \quad t_0 \geq 0, \quad t \in [t_0, T], \quad y_{t_0} \in \mathbb{R}^d, \tag{1}$$

where $W(t)$ is a one-dimensional Wiener process, and the vector functions $f, g \in \mathbb{R}^d$ satisfy the uniform Lipschitz condition and guarantee the existence of a unique solution of the SDE (1). The SDE (1) can be written in integral form as

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y(s))ds + \int_{t_0}^{t} g(y(s)) \circ dW(s). \tag{2}$$

For the numerical methods for solving SDE (1), there are mainly two ways of measuring accuracy: strong convergence and weak convergence. In this paper, we consider the strong convergence.

Definition 1.1 ([4]) If y_N is the numerical approximation to $y(t_N)$ after N steps with constant stepsize $h = \frac{t_N - t_0}{N}$, then y_N is said to converge strongly to $y(t_N)$ with strong global order p if $\exists C > 0$ (independent of h) and δ_0 suth that

$$(E(||y(t_N) - y_N||^2))^{1/2} \leq Ch^p, \quad h \in (0, \delta_0). \tag{3}$$

Here p can be fractional.

2 A general class of SRK methods and its order conditions

2.1 A general class of SRK methods

Let $g^0(y) = f(y)$, $g^1(y) = g(y)$. By [18], a general s-stage SRK methods for the SDE (1) can be written as

$$\begin{aligned} y_{n+1} &= y_n + \sum_{i=1}^{s}\sum_{k=0}^{1}\sum_{v\in\mathscr{M}} z_i^{(k),(v)} g^k(H_i^v), \\ H_i^{(v)} &= y_n + \sum_{j=1}^{s}\sum_{l=0}^{1}\sum_{u\in\mathscr{M}} Z_{ij}^{(v),(l),(u)} g^l(H_i^u), \quad i = 1, 2, \ldots, s, v \in \mathscr{M}, \end{aligned} \tag{4}$$

where, $n = 0, 1, \ldots, N-1$, $y_0 = y(t_0)$, $\mathscr{M}$ is an arbitrary finite set of multi-indices, $0 \in \mathscr{M}$, and for $i, j = 1, 2, \ldots, s$,

$$z_i^{(k),(v)} = \sum_{\tau\in\mathscr{M}} {\gamma_i^{(\tau)}}^{(k),(v)} \theta_\tau^{(k)}(h), \; Z_{i,j}^{(v),(l),(u)} = \sum_{\tau\in\mathscr{M}} {C_{ij}^{(\tau)}}^{(v),(l),(u)} \theta_\tau^{(l)}(h),$$

here, ${\gamma_i^{(\tau)}}^{(k),(v)}, {C_{ij}^{(\tau)}}^{(v),(l),(u)} \in \mathbb{R}$, and $\theta_0^{(0)}(h) = h$, $\theta_\tau^{(k)}(h) \in L^2(\Omega)$, $(k = 0, 1)$ are some suitable random variables, and satisfy

$$E(\prod_{k=0}^{1} (\theta_{\tau_1}^{(k)}(h))^{p_1^k} \cdot \ldots \cdot (\theta_{\tau_q}^{(k)}(h))^{p_q^k}) = \mathcal{O}(h^{p_1^0 + \cdots + p_q^0 + (p_1^1 + \cdots + p_q^1)/2}), \tag{5}$$

where $q = |\mathscr{M}|$ is the element number of finite set $\mathscr{M}$, $p_i^0, p_i^1, i = 1, 2, \ldots, q$ are non-negative integers. Let $z^{(k),(v)} = (z_i^{(k),(v)})_{1\le i\le s} \in \mathbb{R}^s$, $Z^{(v),(l),(u)} = (Z_{i,j}^{(v),(l),(u)})_{1\le i,j\le s} \in \mathbb{R}^{s\times s}$. If $C_{ij}^{(\tau)^{(v),(l),(u)}} = 0$ $(j \ge i)$, then the method (4) is called explicit SRK method, otherwise it is called implicit SRK method.

2.2 Order conditions for the SRK method (4)

We need to apply the order-condition results of the SRK method (4) for solving the SDE (1) when we construct the new strong order SRK methods in the following sections. Therefore, we introduce the following theorem, and the details can be found in [8, 18].

Theorem 2.1 ([18]) *Let $f, g \in C^{2p+1}(\mathbb{R}^d, \mathbb{R}^d)$ and $p \in \frac{1}{2}N_0$ (N_0 denotes non-negative integer). Then the SRK method (4) has strong order p if (5) holds and the following conditions are fulfilled for arbitrary $\mathbf{t} \in \mathbb{TS}$, $\forall t, t+h \in [t_0, T]$:*

$$I_{\mathbf{t};t,t+h} = \Phi_S(\mathbf{t}; t, t+h) \quad P-a.s. \quad \rho(\mathbf{t}) \le p, \tag{6}$$

$$E(I_{\mathbf{t};t,t+h}) = E(\Phi_S(\mathbf{t}; t, t+h)) \qquad \rho(\mathbf{t}) = p + \frac{1}{2}. \tag{7}$$

Here, $\mathbb{TS}$ is the set of all stochastic trees, $\mathbf{t}$ denotes a stochastic tree in $\mathbb{TS}$, $\rho(\mathbf{t})$ denotes the order of $\mathbf{t}$, and $I_{\mathbf{t};t,t+h}$ denotes the corresponding multiple Stratonovich stochastic integral, $\Phi_S(\mathbf{t}; t, t+h)$ denotes the corresponding elementary weight. Let $\mathbf{t} \in \mathbb{TS}$, $\mathbf{t} = [\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_k]_l, l \in \{0, 1\}$, where, the case $l = 0$ denotes that $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_k$ are each joined by a single branch to deterministic node ($\bullet$), and the case $l = 1$ denotes that $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_k$ are each joined by a single branch to stochastic node ($\circ$). Then we can recursively define

$$\Psi^{(v)}(\mathbf{t}; t, t+h) = \sum_{u\in\mathscr{M}} Z^{(v),(l),(u)} \prod_{i=1}^{k} \Psi^{(u)}(\mathbf{t}_i; t, t+h) \;\; v \in \mathscr{M}, \tag{8}$$

$$\Phi_S(\mathbf{t}; t, t+h) = \sum_{v\in\mathscr{M}} z^{(l),(v)^T} \prod_{i=1}^{k} \Psi^{(v)}(\mathbf{t}_i; t, t+h), \tag{9}$$

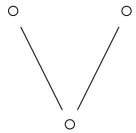
$$I_{\mathbf{t};t,t+h} = \int_t^{t+h} \prod_{i=1}^{k} I_{\mathbf{t}_i;t,s} \circ dW_s^l. \tag{10}$$

Here, let $\varnothing$ denote the empty tree, $\Psi^{(v)}(\varnothing; t, t+h) = e$, $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^s$, $\Phi_S(\varnothing; t, t+h) = 1$, $\circ\, dW_s^0 = ds$, $\circ\, dW_s^1 = \circ dW(s)$, $I_{\varnothing;t,t+h} = 1$, and $\tau_l = [\varnothing]_l$, $I_{\tau_l;t,t+h} = \int_t^{t+h} \circ dW_s^l$, $l \in \{0, 1\}$. By using the notation and formula in [11]

$$J_{j_1,j_2,\cdots,j_{\tilde{n}};t,t+h} = \int_t^{t+h} \int_t^{s_{\tilde{n}}} \cdots \int_t^{s_2} \circ dW_{s_1}^{j_1} \circ dW_{s_2}^{j_2} \cdots \circ dW_{s_{\tilde{n}}}^{j_{\tilde{n}}}, \tag{11}$$

$$\int_t^{t+h} X_s \circ dW_s^i \int_t^{t+h} Y_s \circ dW_s^j = \int_t^{t+h} X_s \left(\int_t^s Y_u \circ dW_u^j \right) \circ dW_s^i + \int_t^{t+h} \left(\int_t^s X_u \circ dW_u^i \right) Y_s \circ dW_s^j, \quad (12)$$

where, $i, j, j_1, j_2, \ldots, j_{\tilde{n}} \in \{0, 1\}$. Then we can calculate each $I_{\mathbf{t};t,t+h}$. In particular, for the stochastic trees

$\mathbf{t}_1 = \tau_0 : \bullet$ $\quad \mathbf{t}_2 = \tau_1 : \circ$ $\quad \mathbf{t}_3 = [\tau_1]_0 :$  $\quad \mathbf{t}_4 = [\tau_1, \tau_1]_1 :$ 

we have

$$\begin{aligned}
I_{\mathbf{t}_1;t,t+h} &= \int_t^{t+h} \circ dW_s^0 = J_{0;t,t+h}, \\
I_{\mathbf{t}_2;t,t+h} &= \int_t^{t+h} \circ dW_s^1 = J_{1;t,t+h}, \\
I_{\mathbf{t}_3;t,t+h} &= \int_t^{t+h} I_{\tau_1;t,s} \circ dW_s^0 = \int_t^{t+h} \int_t^s \circ dW_{s_1}^1 \circ dW_s^0 = J_{10;t,t+h}, \\
I_{\mathbf{t}_4;t,t+h} &= \int_t^{t+h} \left(I_{\tau_1;t,s} I_{\tau_1;t,s} \right) \circ dW_s^1 \\
&= \int_t^{t+h} \left(\int_t^s \circ dW_{s_1}^1 \int_t^s \circ dW_{s_1}^1 \right) \circ dW_s^1 \\
&= \int_t^{t+h} \left(\int_t^s \left(\int_t^{s_1} \circ dW_u^1 \right) \circ dW_{s_1}^1 + \int_t^s \left(\int_t^{s_1} \circ dW_u^1 \right) \circ dW_{s_1}^1 \right) \circ dW_s^1 \\
&= \int_t^{t+h} \int_t^s \int_t^{s_1} \circ dW_u^1 \circ dW_{s_1}^1 \circ dW_s^1 + \int_t^{t+h} \int_t^s \int_t^{s_1} \circ dW_u^1 \circ dW_{s_1}^1 \circ dW_s^1 \\
&= 2J_{111;t,t+h}.
\end{aligned}$$

3 SRK methods with strong order 1.5 for the SDE (1)

Burrage and Burrage [3] constructed a class of SRK methods with strong order 1.5 for the SDE (1). However, the order conditions of the strong order SRK methods in [3] are very complex. This leads to the fact that it is difficult to construct SRK methods with higher strong order. In this section, a new class of SRK methods with strong order 1.5 and with two groups of independent internal stages is proposed. The order conditions of the new methods are simpler than those of methods in [3], and this advantage is reflected more obviously when we construct SRK methods with strong order 2.0. For the SDE (1), we propose the SRK method

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{i_0=1}^{s_0}\alpha_{i_0}^{(0)}f(H_{i_0}^{(0)}) + J_1\sum_{i_0=1}^{s_0}\beta_{i_0}^{(0)}g(H_{i_0}^{(0)}) \\ &\quad + h\sum_{i_1=1}^{s_1}\alpha_{i_1}^{(1)}f(H_{i_1}^{(1)}) + \frac{J_{10}}{h}\sum_{i_1=1}^{s_1}\beta_{i_1}^{(1)}g(H_{i_1}^{(1)}), \\ H_{i_0}^{(0)} &= y_n + h\sum_{j=1}^{s_0}a_{i_0 j}^{(0)}f(H_j^{(0)}) + J_1\sum_{j=1}^{s_0}b_{i_0 j}^{(0)}g(H_j^{(0)}), \quad i_0 = 1,2,\ldots,s_0, \\ H_{i_1}^{(1)} &= y_n + h\sum_{j=1}^{s_1}a_{i_1 j}^{(1)}f(H_j^{(1)}) + \frac{J_{10}}{h}\sum_{j=1}^{s_1}b_{i_1 j}^{(1)}g(H_j^{(1)}), \ i_1 = 1,2,\ldots,s_1, \end{aligned} \tag{13}$$

where h denotes stepsize, $n = 0, 1, \ldots, N-1,\ y_0 = y(t_0)$, and

$$J_1 = J_{1;t_n,t_n+h} = \int_{t_n}^{t_n+h} \circ dW_s, \quad J_{10} = J_{10;t_n,t_n+h} = \int_{t_n}^{t_n+h}\int_{t_n}^{s} \circ dW_{s_1} ds.$$

In the rest of this paper, we use the abbreviation

$$J_{j_1,j_2,\ldots,j_{\tilde{n}}} = J_{j_1,j_2,\ldots,j_{\tilde{n}};t_n,t_n+h}, \quad j_1, j_2, \ldots, j_{\tilde{n}} \in \{0,1\}.$$

The SRK method (13) is a special case of the method (4) with $\mathscr{M} = \{0, 1\}$ and

$$\begin{aligned} z_{i_0}^{(0)(0)} &= h\alpha_{i_0}^{(0)}, \quad z_{i_0}^{(1)(0)} = J_1\beta_{i_0}^{(0)}, z_{i_1}^{(0)(1)} = h\alpha_{i_1}^{(1)}, \quad z_{i_1}^{(1)(1)} = \frac{J_{10}}{h}\beta_{i_1}^{(1)}, \\ Z_{i_0 j}^{(0)(0)(0)} &= ha_{i_0 j}^{(0)}, \ Z_{i_0 j}^{(0)(1)(0)} = J_1 b_{i_0 j}^{(0)}, \ Z_{i_0 j}^{(0)(0)(1)} = Z_{i_0 j}^{(0)(1)(1)} = 0, \\ Z_{i_1 j}^{(1)(0)(1)} &= ha_{i_1 j}^{(1)}, \ Z_{i_1 j}^{(1)(1)(1)} = \frac{J_{10}}{h}b_{i_1 j}^{(1)}, \ Z_{i_1 j}^{(1)(0)(0)} = Z_{i_1 j}^{(1)(1)(0)} = 0. \end{aligned} \tag{14}$$

Remark 3.1 In (13), s_0 and s_1 can be different numbers because the group $H_{i_0}^{(0)}$ and the group $H_{i_1}^{(1)}$ are independent.

Now, some formulas and expectation values of multiple Stratonovich stochastic integrals [11] are given in (15) and Table 1.

$$
\begin{aligned}
hJ_{11} &= J_{110} + J_{101} + J_{011}, & J_1 J_{10} &= J_{101} + 2J_{110}, \\
J_1 J_{01} &= J_{101} + 2J_{011}, & hJ_1 &= J_{10} + J_{01}, \\
J_{1\cdots 1} &= \frac{1}{k!} J_1^k \text{ (k is the length of the multi-index)}, & J_1 &= I_1, \\
J_{10} &= I_{10}, & J_{110} &= I_{110} + \frac{1}{2} I_{00}, \\
J_{101} &= I_{101}, & J_{011} &= I_{011} + \frac{1}{2} I_{00}, \\
I_{11} &= \frac{1}{2}(I_1^2 - h), & I_{111} &= \frac{1}{6}(I_1^3 - 3hI_1)
\end{aligned}
\tag{15}
$$

where $I_{j_1, j_2, \ldots, j_{\tilde{n}}}, \quad j_1, j_2, \ldots, j_{\tilde{n}} \in \{0, 1\}$ denotes the multiple It$\hat{o}$ stochastic integral that corresponds to $J_{j_1, j_2, \ldots, j_{\tilde{n}}}$.

Now, we can obtain the following results by applying Theorem 2.1.

Theorem 3.2 *Let $f, g \in C^4(\mathbb{R}^d, \mathbb{R}^d)$. Then the SRK method (13) converges strongly to solution of the SDE (1) with strong order 1.5 if the coefficients of the SRK method (13) satisfy the system of the following equtions*

$$
\begin{array}{ll}
1.\ \alpha^{(0)^T} e_0 + \alpha^{(1)^T} e_1 = 1, & 2.\ \beta^{(0)^T} e_0 = 1, \\
3.\ \beta^{(1)^T} e_1 = 0, & 4.\ \alpha^{(0)^T} A^{(0)} e_0 + \alpha^{(1)^T} A^{(1)} e_1 = \frac{1}{2}, \\
5.\ \alpha^{(0)^T} B^{(0)} e_0 = 0, & 6.\ \alpha^{(1)^T} B^{(1)} e_1 = 1, \\
7.\ \beta^{(0)^T} A^{(0)} e_0 = 1, & 8.\ \beta^{(1)^T} A^{(1)} e_1 = -1, \\
9.\ \beta^{(0)^T} B^{(0)} e_0 = \frac{1}{2}, & 10.\ \beta^{(1)^T} B^{(1)} e_1 = 0, \\
11.\ \beta^{(0)^T} (B^{(0)} e_0)^2 = \frac{1}{3}, & 12.\ \beta^{(1)^T} (B^{(1)} e_1)^2 = 0, \\
13.\ \beta^{(0)^T} (B^{(0)} (B^{(0)} e_0)) = \frac{1}{6}, & 14.\ \beta^{(1)^T} (B^{(1)} (B^{(1)} e_1)) = 0,
\end{array}
$$

$$
\begin{array}{l}
15.\ \alpha^{(0)^T} (B^{(0)} e_0)^2 + \frac{1}{3} \alpha^{(1)^T} (B^{(1)} e_1)^2 = \frac{1}{2}, \\
16.\ \alpha^{(0)^T} (B^{(0)} (B^{(0)} e_0)) + \frac{1}{3} \alpha^{(1)^T} (B^{(1)} (B^{(1)} e_1)) = \frac{1}{4}, \\
17.\ \beta^{(0)^T} (A^{(0)} (B^{(0)} e_0)) + \frac{1}{3} \beta^{(1)^T} (A^{(1)} (B^{(1)} e_1)) = 0, \\
18.\ \beta^{(0)^T} ((A^{(0)} e_0)(B^{(0)} e_0)) + \frac{1}{3} \beta^{(1)^T} ((A^{(1)} e_1)(B^{(1)} e_1)) = \frac{1}{4}, \\
19.\ \beta^{(0)^T} (B^{(0)} (A^{(0)} e_0)) + \frac{1}{3} \beta^{(1)^T} (B^{(1)} (A^{(1)} e_1)) = \frac{1}{4}, \\
20.\ 3\beta^{(0)^T} (B^{(0)} e_0)^3 + \frac{1}{3} \beta^{(1)^T} (B^{(1)} e_1)^3 = \frac{3}{4}, \\
21.\ 3\beta^{(0)^T} ((B^{(0)} (B^{(0)} e_0))(B^{(0)} e_0)) + \frac{1}{3} \beta^{(1)^T} ((B^{(1)} (B^{(1)} e_1))(B^{(1)} e_1)) = \frac{3}{8}, \\
22.\ 3\beta^{(0)^T} (B^{(0)} (B^{(0)} e_0)^2) + \frac{1}{3} \beta^{(1)^T} (B^{(1)} (B^{(1)} e_1)^2) = \frac{1}{4}, \\
23.\ 3\beta^{(0)^T} (B^{(0)} (B^{(0)} (B^{(0)} e_0))) + \frac{1}{3} \beta^{(1)^T} (B^{(1)} (B^{(1)} (B^{(1)} e_1))) = \frac{1}{8},
\end{array}
$$

where

$$A^{(0)}=(a^{(0)}_{i_0 j})_{s_0\times s_0},\quad A^{(1)}=(a^{(1)}_{i_1 j})_{s_1\times s_1},\quad B^{(0)}=(b^{(0)}_{i_0 j})_{s_0\times s_0},\quad B^{(1)}=(b^{(1)}_{i_1 j})_{s_1\times s_1}$$

$$\alpha^{(0)}=(\alpha^{(0)}_1,\alpha^{(0)}_2,\ldots,\alpha^{(0)}_{s_0})^T,\quad \alpha^{(1)}=(\alpha^{(1)}_1,\alpha^{(1)}_2,\ldots,\alpha^{(1)}_{s_1})^T$$

$$\beta^{(0)}=(\beta^{(0)}_1,\beta^{(0)}_2,\ldots,\beta^{(0)}_{s_0})^T,\quad \beta^{(1)}=(\beta^{(1)}_1,\beta^{(1)}_2,\ldots,\beta^{(1)}_{s_1})^T.$$

Proof First, for arbitrary $\mathbf{t}$ with $\rho(\mathbf{t})\le 1.5$, we prove that the following order condition

$$I_{\mathbf{t};t,t+h}=\Phi_S(\mathbf{t};t,t+h)\quad P-a.s.$$

holds. In fact, by (8)-(12), (14) and (15), we obtain that

$$\begin{aligned}
\text{for } \tau_0,\quad I_{\tau_0;t,t+h} &= \Phi_S(\tau_0;t,t+h) \;\Leftrightarrow\; h=h\alpha^{(0)^T}e_0+h\alpha^{(1)^T}e_1\\
&\Leftrightarrow \alpha^{(0)^T}e_0+\alpha^{(1)^T}e_1=1;\\
\text{for } \tau_1,\quad I_{\tau_1;t,t+h} &= \Phi_S(\tau_1;t,t+h) \;\Leftrightarrow\; J_1=J_1\beta^{(0)^T}e_0+\frac{J_{10}}{h}\beta^{(1)^T}e_1\\
&\Leftrightarrow \beta^{(0)^T}e_0=1,\ \beta^{(1)^T}e_1=0;\\
\text{for } [\tau_0]_0,\quad I_{[\tau_0]_0;t,t+h} &= \Phi_S([\tau_0]_0;t,t+h) \;\Leftrightarrow\; \frac{1}{2}h^2=h^2\alpha^{(0)^T}A^{(0)}e_0+h^2\alpha^{(1)^T}A^{(1)}e_1\\
&\Leftrightarrow \alpha^{(0)^T}A^{(0)}e_0+\alpha^{(1)^T}A^{(1)}e_1=\frac{1}{2};\\
\text{for } [\tau_1]_0,\quad I_{[\tau_1]_0;t,t+h} &= \Phi_S([\tau_1]_0;t,t+h) \;\Leftrightarrow\; J_{10}=hJ_1\alpha^{(0)^T}B^{(0)}e_0+h\frac{J_{10}}{h}\alpha^{(1)^T}B^{(1)}e_1\\
&\Leftrightarrow \alpha^{(0)^T}B^{(0)}e_0=0,\ \alpha^{(1)^T}B^{(1)}e_1=1;\\
\text{for } [\tau_0]_1,\quad I_{[\tau_0]_1;t,t+h} &= \Phi_S([\tau_0]_1;t,t+h) \;\Leftrightarrow\; J_{01}=hJ_1\beta^{(0)^T}A^{(0)}e_0+h\frac{J_{10}}{h}\beta^{(1)^T}A^{(1)}e_1\\
&\Leftrightarrow \beta^{(0)^T}A^{(0)}e_0=1,\ \beta^{(1)^T}A^{(1)}e_1=-1;\\
\text{for } [\tau_1]_1,\quad I_{[\tau_1]_1;t,t+h} &= \Phi_S([\tau_1]_1;t,t+h) \;\Leftrightarrow\; J_{11}=J_1^2\beta^{(0)^T}B^{(0)}e_0+(\frac{J_{10}}{h})^2\beta^{(1)^T}B^{(1)}e_1\\
&\Leftrightarrow \beta^{(0)^T}B^{(0)}e_0=\frac{1}{2},\ \beta^{(1)^T}B^{(1)}e_1=0;
\end{aligned}$$

$$\begin{aligned}
&\text{for } [\tau_1,\tau_1]_1,\quad I_{[\tau_1,\tau_1]_1;t,t+h}=\Phi_S([\tau_1,\tau_1]_1;t,t+h)\\
&\qquad\Leftrightarrow\; 2J_{111}=J_1^3\beta^{(0)^T}(B^{(0)}e_0)^2+(\frac{J_{10}}{h})^3\beta^{(1)^T}(B^{(1)}e_1)^2\\
&\qquad\Leftrightarrow\; \beta^{(0)^T}(B^{(0)}e_0)^2=\frac{1}{3},\quad \beta^{(1)^T}(B^{(1)}e_1)^2=0;\\
&\text{for } [[\tau_1]_1]_1,\quad I_{[[\tau_1]_1]_1;t,t+h}=\Phi_S([[\tau_1]_1]_1;t,t+h)\\
&\qquad\Leftrightarrow J_{111}=J_1^3\beta^{(0)^T}(B^{(0)}(B^{(0)}e_0))+(\frac{J_{10}}{h})^3\beta^{(1)^T}(B^{(1)}(B^{(1)}e_1))\\
&\qquad\Leftrightarrow\; \beta^{(0)^T}(B^{(0)}(B^{(0)}e_0))=\frac{1}{6},\quad \beta^{(1)^T}(B^{(1)}(B^{(1)}e_1))=0.
\end{aligned}$$

Next, for arbitrary $\mathbf{t}$ with $\rho(\mathbf{t})=2.0$, we prove that the following order condition

$$E(I_{\mathbf{t};t,t+h})=E(\Phi_S(\mathbf{t};t,t+h))$$

holds. In fact, by (8)-(12), (14) and (15) as well as Table 1, we obtain that

$$\begin{aligned}
\text{for } [\tau_1,\tau_1]_0,\quad & E[I_{[\tau_1,\tau_1]_0;t,t+h}] = E[\Phi_S([\tau_1,\tau_1]_0;t,t+h)] \\
& \Leftrightarrow \quad E[2J_{110}] = E[hJ_1^2\alpha^{(0)^T}(B^{(0)}e_0)^2 + h(\frac{J_{10}}{h})^2\alpha^{(1)^T}(B^{(1)}e_1)^2] \\
& \Leftrightarrow \quad \alpha^{(0)^T}(B^{(0)}e_0)^2 + \frac{1}{3}\alpha^{(1)^T}(B^{(1)}e_1)^2 = \frac{1}{2};
\end{aligned}$$

$$\begin{aligned}
\text{for } [[\tau_1]_1]_0,\quad & E[I_{[[\tau_1]_1]_0;t,t+h}] = E[\Phi_S([[\tau_1]_1]_0;t,t+h)] \\
& \Leftrightarrow \quad E[J_{110}] = E[hJ_1^2\alpha^{(0)^T}(B^{(0)}(B^{(0)}e_0)) + h(\frac{J_{10}}{h})^2\alpha^{(1)^T}(B^{(1)}(B^{(1)}e_1))] \\
& \Leftrightarrow \quad \alpha^{(0)^T}(B^{(0)}(B^{(0)}e_0)) + \frac{1}{3}\alpha^{(1)^T}(B^{(1)}(B^{(1)}e_1)) = \frac{1}{4}; \\
\text{for } [[\tau_1]_0]_1,\quad & E[I_{[[\tau_1]_0]_1;t,t+h}] = E[\Phi_S([[\tau_1]_0]_1;t,t+h)] \\
& \Leftrightarrow \quad E[J_{101}] = E[hJ_1^2\beta^{(0)^T}(A^{(0)}(B^{(0)}e_0)) + h(\frac{J_{10}}{h})^2\beta^{(1)^T}(A^{(1)}(B^{(1)}e_1))] \\
& \Leftrightarrow \quad \beta^{(0)^T}(A^{(0)}(B^{(0)}e_0)) + \frac{1}{3}\beta^{(1)^T}(A^{(1)}(B^{(1)}e_1)) = 0;
\end{aligned}$$

$$\begin{aligned}
\text{for } [\tau_0,\tau_1]_1,\quad & E[I_{[\tau_0,\tau_1]_1;t,t+h}] = E[\Phi_S([\tau_0,\tau_1]_1;t,t+h)] \\
& \Leftrightarrow \quad E[J_{101}+J_{011}] = E[hJ_1^2\beta^{(0)^T}((A^{(0)}e_0)(B^{(0)}e_0)) \\
& \qquad\qquad + h(\frac{J_{10}}{h})^2\beta^{(1)^T}((A^{(1)}e_1)(B^{(1)}e_1))] \\
& \Leftrightarrow \quad \beta^{(0)^T}((A^{(0)}e_0)(B^{(0)}e_0)) + \frac{1}{3}\beta^{(1)^T}((A^{(1)}e_1)(B^{(1)}e_1)) = \frac{1}{4};
\end{aligned}$$

$$\begin{aligned}
\text{for } [[\tau_0]_1]_1,\quad & E[I_{[[\tau_0]_1]_1;t,t+h}] = E[\Phi_S([[\tau_0]_1]_1;t,t+h)] \\
& \Leftrightarrow \quad E[J_{011}] = E[hJ_1^2\beta^{(0)^T}(B^{(0)}(A^{(0)}e_0))) + h(\frac{J_{10}}{h})^2\beta^{(1)^T}(B^{(1)}(A^{(1)}e_1))] \\
& \Leftrightarrow \quad \beta^{(0)^T}(B^{(0)}(A^{(0)}e_0)) + \frac{1}{3}\beta^{(1)^T}(B^{(1)}(A^{(1)}e_1)) = \frac{1}{4};
\end{aligned}$$

Table 1 Some expectation values of multiple Stratonovich stochastic integrals

expectation	values	expectation	values	expectation	values
J_1	0	J_1^4	$3h^2$	J_{101}^3	$\frac{1}{30}h^6$
J_{10}	0	J_{10}^4	$\frac{1}{3}h^6$	J_{001}	0
J_1^2	h	J_1^5	0	J_{010}	0
J_{10}^2	$\frac{1}{3}h^3$	$J_1^4J_{10}$	0	J_{100}	0
J_1^3	0	J_{110}	$\frac{1}{4}h^2$	J_{0111}	0
J_{10}^3	0	J_{101}	0	J_{1011}	0
$J_1^2J_{10}$	0	J_{011}	$\frac{1}{4}h^2$	J_{1101}	0
$J_1J_{10}^2$	0	J_{101}^2	$\frac{1}{12}h^4$	J_{1110}	0

$$
\begin{aligned}
&\text{for } [\tau_1,\tau_1,\tau_1]_1, \quad E[I_{[\tau_1,\tau_1,\tau_1]_1;t,t+h}] = E[\Phi_S([\tau_1,\tau_1,\tau_1]_1;t,t+h)]\\
&\qquad \Leftrightarrow \quad E[6J_{1111}] = E[J_1^4 \beta^{(0)^T}(B^{(0)}e_0)^3 + (\frac{J_{10}}{h})^4 \beta^{(1)^T}(B^{(1)}e_1)^3]\\
&\qquad \Leftrightarrow \quad 3\beta^{(0)^T}(B^{(0)}e_0)^3 + \frac{1}{3}\beta^{(1)^T}(B^{(1)}e_1)^3 = \frac{3}{4};\\
&\text{for } [[\tau_1]_1,\tau_1]_1, \quad E[I_{[[\tau_1]_1,\tau_1]_1;t,t+h}] = E[\Phi_S([[\tau_1]_1,\tau_1]_1;t,t+h)]\\
&\qquad \Leftrightarrow \quad E[3J_{1111}] = E[J_1^4 \beta^{(0)^T}((B^{(0)}(B^{(0)}e_0))(B^{(0)}e_0))\\
&\qquad\qquad + (\frac{J_{10}}{h})^4 \beta^{(1)^T}((B^{(1)}(B^{(1)}e_1))(B^{(1)}e_1))]\\
&\qquad \Leftrightarrow \quad 3\beta^{(0)^T}((B^{(0)}(B^{(0)}e_0))(B^{(0)}e_0))\\
&\qquad\qquad + \frac{1}{3}\beta^{(1)^T}((B^{(1)}(B^{(1)}e_1))(B^{(1)}e_1)) = \frac{3}{8};\\
&\text{for } [[\tau_1,\tau_1]_1]_1, \quad E[I_{[[\tau_1,\tau_1]_1]_1;t,t+h}] = E[\Phi_S([[\tau_1,\tau_1]_1]_1;t,t+h)]\\
&\qquad \Leftrightarrow \quad E[2J_{1111}] = E[J_1^4 \beta^{(0)^T}(B^{(0)}(B^{(0)}e_0)^2) + (\frac{J_{10}}{h})^4 \beta^{(1)^T}(B^{(1)}(B^{(1)}e_1)^2)]\\
&\qquad \Leftrightarrow \quad 3\beta^{(0)^T}(B^{(0)}(B^{(0)}e_0)^2) + \frac{1}{3}\beta^{(1)^T}(B^{(1)}(B^{(1)}e_1)^2) = \frac{1}{4};\\
&\text{for } [[[\tau_1]_1]_1]_1, \quad E[I_{[[[\tau_1]_1]_1]_1;t,t+h}] = E[\Phi_S([[[\tau_1]_1]_1]_1;t,t+h)]\\
&\qquad \Leftrightarrow \quad E[J_{1111}] = E[J_1^4 \beta^{(0)^T}(B^{(0)}(B^{(0)}(B^{(0)}e_0)))\\
&\qquad\qquad + (\frac{J_{10}}{h})^4 \beta^{(1)^T}(B^{(1)}(B^{(1)}(B^{(1)}e_1)))]\\
&\qquad \Leftrightarrow \quad 3\beta^{(0)^T}(B^{(0)}(B^{(0)}(B^{(0)}e_0))) + \frac{1}{3}\beta^{(1)^T}(B^{(1)}(B^{(1)}(B^{(1)}e_1))) = \frac{1}{8}.
\end{aligned}
$$

The proof is complete. □

Remark 3.3 Compared to the SRK methods with strong order 1.5 in [3], the number of the equations in Theorem 3.2 has been reduced by 5.

The SRK method (13) can be characterized by the extended Butcher tableau

$$
\begin{array}{c|c}
\alpha^{(0)^T} & \beta^{(0)^T} \\
\hline
A^{(0)} & B^{(0)} \\
\hline
A^{(1)} & B^{(1)} \\
\hline
\alpha^{(1)^T} & \beta^{(1)^T}
\end{array}
$$

By Theorem 3.2, a specific explicit method with strong order 1.5 is proposed, and it is denoted by SRKS1.5, *i.e*.

1	-1	1	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$
0				0			
$\frac{1}{2}$	0			$\frac{1}{2}$	0		
1	0	0		0	$\frac{1}{2}$	0	
3	0	0	0	0	0	1	0
0				0			
0	0			$\frac{3}{2}$	0		
1	0	0		$\frac{3}{2}$	0	0	
$-\frac{2}{3}$	$\frac{2}{3}$	0		0	1	-1	

4 SRK methods with strong order 2.0 for the SDE (1)

The multiple Stratonovich stochastic integrals of order 2.0 contain J_{1111},J_{110}, J_{101},J_{011}. As a result, compared to the SRK methods with strong order 1.5, the new random variable must be added when we construct the SRK methods with strong order 2.0. By (15), we obtain

$$J_{1111} = \frac{1}{24} J_1, \quad J_{110} = \frac{1}{2} J_1 J_{10} - \frac{1}{2} J_{101}, \quad J_{011} = \frac{1}{2} h J_1^2 - \frac{1}{2} J_1 J_{10} - \frac{1}{2} J_{101}.$$

Thus, it suffices to add J_{101}. Now three groups of independent internal stages are used for reducing the number of the equations in order conditions, and the SRK method

$$\begin{aligned}
y_{n+1} &= y_n + h \sum_{i_0=1}^{s_0} \alpha_{i_0}^{(0)} f(H_{i_0}^{(0)}) + J_1 \sum_{i_0=1}^{s_0} \beta_{i_0}^{(0)} g(H_{i_0}^{(0)}) \\
&\quad + h \sum_{i_1=1}^{s_1} \alpha_{i_1}^{(1)} f(H_{i_1}^{(1)}) + J_1 \sum_{i_1=1}^{s_1} \beta_{i_1}^{(1)} g(H_{i_1}^{(2)}) + \frac{J_{10}}{h} \sum_{i_1=1}^{s_1} \beta_{i_1}^{(2)} g(H_{i_1}^{(2)}) \\
&\quad + h \sum_{i_2=1}^{s_2} \alpha_{i_2}^{(2)} f(H_{i_2}^{(3)}) + \frac{J_{101}}{h^{3/2}} \sum_{i_2=1}^{s_2} \beta_{i_2}^{(3)} g(H_{i_2}^{(4)}), \\
H_{i_0}^{(0)} &= y_n + h \sum_{j=1}^{s_0} a_{i_0 j}^{(0)} f(H_j^{(0)}) + J_1 \sum_{j=1}^{s_0} b_{i_0 j}^{(0)} g(H_j^{(0)}), \qquad i_0 = 1, 2, \ldots, s_0, \qquad (16) \\
H_{i_1}^{(1)} &= y_n + J_1 \sum_{j=1}^{s_1} b_{i_1 j}^{(1)} g(H_j^{(2)}) + \frac{J_{10}}{h} \sum_{j=1}^{s_1} c_{i_1 j}^{(0)} g(H_j^{(2)}), \\
H_{i_1}^{(2)} &= y_n + h \sum_{j=1}^{s_1} a_{i_1 j}^{(1)} f(H_j^{(1)}) + J_1 \sum_{j=1}^{s_1} b_{i_1 j}^{(2)} g(H_j^{(2)}), \qquad i_1 = 1, 2, \ldots, s_1, \\
H_{i_2}^{(3)} &= y_n + \sqrt{h} \sum_{j=1}^{s_2} b_{i_2 j}^{(3)} g(H_j^{(4)}) + \frac{J_{101}}{h^{3/2}} \sum_{j=1}^{s_2} c_{i_2 j}^{(1)} g(H_j^{(4)}), \\
H_{i_2}^{(4)} &= y_n + h \sum_{j=1}^{s_2} a_{i_2 j}^{(4)} f(H_j^{(3)}) + \sqrt{h} \sum_{j=1}^{s_2} b_{i_2 j}^{(4)} g(H_j^{(4)}), \qquad i_2 = 1, 2, \ldots, s_2,
\end{aligned}$$

is proposed, where $n = 0, 1, \ldots, N-1$, $y_0 = y(t_0)$.

Remark 4.1 In (16), s_0, s_1 and s_2 can be different numbers because the group $H_{i_0}^{(0)}$, the group $H_{i_1}^{(1)}$, $H_{i_1}^{(2)}$ and the group $H_{i_2}^{(3)}$, $H_{i_2}^{(4)}$ are independent.

The SRK method (16) is a special case of the method (4) with $\mathcal{M} = \{0, 1, 2, 3, 4\}$ and

$$
\begin{aligned}
&z_{i_0}^{(0)(0)} = h\alpha_{i_0}^{(0)}, && z_{i_0}^{(1)(0)} = J_1\beta_{i_0}^{(0)},\\
&z_{i_1}^{(0)(1)} = h\alpha_{i_1}^{(1)}, && z_{i_1}^{(0)(2)} = 0,\\
&z_{i_1}^{(1)(1)} = 0, && z_{i_1}^{(1)(2)} = J_1\beta_{i_1}^{(1)} + \tfrac{J_{10}}{h}\beta_{i_1}^{(2)},\\
&z_{i_2}^{(0)(3)} = h\alpha_{i_2}^{(2)}, && z_{i_2}^{(0)(4)} = 0,\\
&z_{i_2}^{(1)(3)} = 0, && z_{i_2}^{(1)(4)} = \tfrac{J_{101}}{h^{3/2}}\beta_{i_2}^{(3)},\\
&Z_{i_0j}^{(0)(0)(0)} = ha_{i_0j}^{(0)}, && Z_{i_0j}^{(0)(0)(k)} = 0, \quad k = 1,2,3,4,\\
&Z_{i_0j}^{(0)(1)(0)} = J_1b_{i_0j}^{(0)}, && Z_{i_0j}^{(0)(1)(k)} = 0, \quad k = 1,2,3,4,\\
&Z_{i_1j}^{(1)(0)(k)} = 0, \quad k = 0,1,2,3,4, && \\
&Z_{i_1j}^{(1)(1)(2)} = J_1b_{i_1j}^{(1)} + \tfrac{J_{10}}{h}c_{i_1j}^{(0)}, && Z_{i_1j}^{(1)(1)(k)} = 0, \quad k = 0,1,3,4,\\
&Z_{i_1j}^{(2)(0)(1)} = ha_{i_1j}^{(1)}, && Z_{i_1j}^{(2)(0)(k)} = 0, \quad k = 0,2,3,4,\\
&Z_{i_1j}^{(2)(1)(2)} = J_1b_{i_1j}^{(2)}, && Z_{i_1j}^{(2)(1)(k)} = 0, \quad k = 0,1,3,4,\\
&Z_{i_2j}^{(3)(0)(k)} = 0, \quad k = 0,1,2,3,4, && \\
&Z_{i_2j}^{(3)(1)(4)} = \sqrt{h}b_{i_2j}^{(3)} + \tfrac{J_{101}}{h^{3/2}}c_{i_2j}^{(1)}, && Z_{i_2j}^{(3)(1)(k)} = 0, \quad k = 0,1,2,3,\\
&Z_{i_2j}^{(4)(0)(3)} = ha_{i_2j}^{(2)}, && Z_{i_2j}^{(4)(0)(k)} = 0, \quad k = 0,1,2,4,\\
&Z_{i_2j}^{(4)(1)(4)} = \sqrt{h}b_{i_2j}^{(4)}, && Z_{i_2j}^{(4)(1)(k)} = 0, \quad k = 0,1,2,3.
\end{aligned}
\tag{17}
$$

We can obtain the following results by applying Theorem 2.1.

Theorem 4.2 *Let $f, g \in C^5(\mathbb{R}^d, \mathbb{R}^d)$. Then the SRK method (16) converges strongly to the solution of the SDE (1) with strong order 2.0 if the coefficients of the SRK method (16) satisfy the system of the following equtions*

1. $\alpha^{(0)^T}e_0 + \alpha^{(1)^T}e_1 + \alpha^{(2)^T}e_2 = 1,$
2. $\beta^{(0)^T}e_0 + \beta^{(1)^T}e_1 = 1,$
3. $\beta^{(2)^T}e_1 = 0,$
4. $\beta^{(3)^T}e_2 = 0,$
5. $\alpha^{(0)^T}A^{(0)}e_0 = \frac{1}{2},$
6. $\alpha^{(0)^T}B^{(0)}e_0 + \alpha^{(1)^T}B^{(1)}e_1 = 0,$
7. $\alpha^{(1)^T}C^{(0)}e_1 = 1,$
8. $\alpha^{(2)^T}B^{(3)}e_2 = 0,$
9. $\alpha^{(2)^T}C^{(1)}e_2 = 0,$
10. $\beta^{(0)^T}A^{(0)}e_0 + \beta^{(1)^T}A^{(1)}e_1 = 1,$
11. $\beta^{(2)^T}A^{(1)}e_1 = -1,$
12. $\beta^{(3)^T}A^{(2)}e_2 = 0,$
13. $\beta^{(0)^T}B^{(0)}e_0 + \beta^{(1)^T}B^{(2)}e_1 = \frac{1}{2},$
14. $\beta^{(2)^T}B^{(2)}e_1 = 0,$
15. $\beta^{(3)^T}B^{(4)}e_2 = 0,$
16. $\beta^{(0)^T}(B^{(0)}e_0)^2 + \beta^{(1)^T}(B^{(2)}e_1)^2 = \frac{1}{3},$
17. $\beta^{(2)^T}(B^{(2)}e_1)^2 = 0,$
18. $\beta^{(3)^T}(B^{(4)}e_2)^2 = 0,$
19. $\beta^{(0)^T}(B^{(0)}(B^{(0)}e_0)) + \beta^{(1)^T}(B^{(2)}(B^{(2)}e_1)) = \frac{1}{6},$
20. $\beta^{(2)^T}(B^{(2)}(B^{(2)}e_1)) = 0,$
21. $\beta^{(3)^T}(B^{(4)}(B^{(4)}e_2)) = 0,$
22. $\alpha^{(0)^T}(B^{(0)}e_0)^2 + \alpha^{(1)^T}(B^{(1)}e_1)^2 = 0,$
23. $\alpha^{(1)^T}((B^{(1)}e_1)(C^{(0)}e_1)) = \frac{1}{2},$
24. $\alpha^{(1)^T}(C^{(0)}e_1)^2 = 0,$
25. $\alpha^{(2)^T}(B^{(3)}e_2)^2 = 0,$

26. $\alpha^{(2)^T}((B^{(3)}e_2)(C^{(1)}e_2)) = -\frac{1}{2}$,
27. $\alpha^{(2)^T}(C^{(1)}e_2)^2 = 0$,
28. $\alpha^{(0)^T}(B^{(0)}(B^{(0)}e_0)) + \alpha^{(1)^T}(B^{(1)}(B^{(2)}e_1)) = 0$,
29. $\alpha^{(1)^T}(C^{(0)}(B^{(2)}e_1)) = \frac{1}{2}$,
30. $\alpha^{(2)^T}(B^{(3)}(B^{(4)}e_2)) = 0$,
31. $\alpha^{(2)^T}(C^{(1)}(B^{(4)}e_2)) = -\frac{1}{2}$,
32. $\beta^{(0)^T}(A^{(0)}(B^{(0)}e_0)) + \beta^{(1)^T}(A^{(1)}(B^{(1)}e_1)) = 0$,
33. $\beta^{(1)^T}(A^{(1)}(C^{(0)}e_1)) + \beta^{(2)^T}(A^{(1)}(B^{(1)}e_1)) = 0$,
34. $\beta^{(2)^T}(A^{(1)}(C^{(0)}e_1)) = 0$,
35. $\beta^{(3)^T}(A^{(2)}(B^{(3)}e_2)) = 1$,
36. $\beta^{(3)^T}(A^{(2)}(C^{(1)}e_2)) = 0$,
37. $\beta^{(0)^T}((A^{(0)}e_0)(B^{(0)}e_0)) + \beta^{(1)^T}((A^{(1)}e_1)(B^{(2)}e_1)) = \frac{1}{2}$,
38. $\beta^{(2)^T}((A^{(1)}e_1)(B^{(2)}e_1)) = -\frac{1}{2}$,
39. $\beta^{(3)^T}((A^{(2)}e_2)(B^{(4)}e_2)) = \frac{1}{2}$,
40. $\beta^{(0)^T}(B^{(0)}(A^{(0)}e_0)) + \beta^{(1)^T}(B^{(2)}(A^{(1)}e_1)) = \frac{1}{2}$,
41. $\beta^{(2)^T}(B^{(2)}(A^{(1)}e_1)) = -\frac{1}{2}$,
42. $\beta^{(3)^T}(B^{(4)}(A^{(2)}e_2)) = -\frac{1}{2}$,

43. $\beta^{(0)^T}(B^{(0)}e_0)^3 + \beta^{(1)^T}(B^{(2)}e_1)^3 = \frac{1}{4}$,
44. $\beta^{(2)^T}(B^{(2)}e_1)^3 = 0$,
45. $\beta^{(3)^T}(B^{(4)}e_2)^3 = 0$,
46. $\beta^{(0)^T}((B^{(0)}(B^{(0)}e_0))(B^{(0)}e_0)) + \beta^{(1)^T}((B^{(2)}(B^{(2)}e_1))(B^{(2)}e_1)) = \frac{1}{8}$,
47. $\beta^{(2)^T}((B^{(2)}(B^{(2)}e_1))(B^{(2)}e_1)) = 0$,
48. $\beta^{(3)^T}((B^{(4)}(B^{(4)}e_2))(B^{(4)}e_2)) = 0$,
49. $\beta^{(0)^T}(B^{(0)}(B^{(0)}e_0)^2) + \beta^{(1)^T}(B^{(2)}(B^{(2)}e_1)^2) = \frac{1}{12}$,
50. $\beta^{(2)^T}(B^{(2)}(B^{(2)}e_1)^2) = 0$,
51. $\beta^{(3)^T}(B^{(4)}(B^{(4)}e_2)^2) = 0$,
52. $\beta^{(0)^T}(B^{(0)}(B^{(0)}(B^{(0)}e_0))) + \beta^{(1)^T}(B^{(2)}(B^{(2)}(B^{(2)}e_1))) = \frac{1}{24}$,
53. $\beta^{(2)^T}(B^{(2)}(B^{(2)}(B^{(2)}e_1))) = 0$,
54. $\beta^{(3)^T}(B^{(4)}(B^{(4)}(B^{(4)}e_2))) = 0$,
55. $\alpha^{(2)^T}(B^{(3)}(A^{(2)}e_2)) = 0$,
56. $\alpha^{(2)^T}(B^{(3)}e_2)^3 + \frac{1}{4}\alpha^{(2)^T}((B^{(3)}e_2)((C^{(1)}e_2)^2)) + \frac{1}{30}\alpha^{(2)^T}(C^{(1)}e_2)^3 = 0$,
57. $\alpha^{(2)^T}((B^{(3)}(B^{(4)}e_2))(B^{(3)}e_2)) + \frac{1}{12}\alpha^{(2)^T}((C^{(1)}(B^{(4)}e_2))(C^{(1)}e_2)) = 0$,
58. $\alpha^{(2)^T}(B^{(3)}(B^{(4)}e_2)^2) = 0$,
59. $\alpha^{(2)^T}(B^{(3)}(B^{(4)}(B^{(4)}e_2))) = 0$,
60. $\beta^{(3)^T}((A^{(2)}(C^{(1)}e_2))(B^{(4)}e_2)) = 0$,
61. $\frac{1}{6}\beta^{(3)^T}(A^{(2)}((B^{(3)}e_2)(C^{(1)}e_2))) + \frac{1}{30}\beta^{(3)^T}(A^{(2)}(C^{(1)}e_2)^2) = 0$,
62 $\beta^{(3)^T}(A^{(2)}(C^{(1)}(B^{(4)}e_2))) = 0$,
63 $\beta^{(3)^T}(B^{(4)}(A^{(2)}(C^{(1)}e_2))) = 0$.

The proof of Theorem 4.2 is analogous to the proof of Theorem 3.2, and the details can be found in Appendix A.

The SRK method (16) can be characterized by the extended Butcher tableau

$$
\begin{array}{c|c|c}
A^{(0)} & B^{(0)} & \\
\hline
 & B^{(1)} & C^{(0)} \\
\hline
A^{(1)} & B^{(2)} & \\
\hline
 & B^{(3)} & C^{(1)} \\
\hline
A^{(2)} & B^{(4)} & \\
\hline
{\alpha^{(0)}}^T & {\beta^{(0)}}^T & \\
\hline
{\alpha^{(1)}}^T & {\beta^{(1)}}^T & {\beta^{(2)}}^T \\
\hline
{\alpha^{(2)}}^T & {\beta^{(3)}}^T &
\end{array}
$$

By Theorem 4.2, a specific explicit SRK method with strong order 2.0 is proposed and it is denoted by SRKS2.0, which has coefficients

$$
{\alpha^{(0)}}^T = [\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}], \qquad {\beta^{(0)}}^T = [\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}],
$$

$$
A^{(0)} = \begin{bmatrix} 0 & & & \\ \frac{1}{2} & 0 & & \\ 0 & \frac{1}{2} & 0 & \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B^{(0)} = \begin{bmatrix} 0 & & & \\ \frac{1}{2} & 0 & & \\ 0 & \frac{1}{2} & 0 & \\ 0 & 0 & 1 & 0 \end{bmatrix},
$$

$$
{\alpha^{(1)}}^T = [-\frac{1}{12}, \frac{1}{12}, -1, 1, 0], \qquad {\beta^{(1)}}^T = [-\frac{5}{6}, \frac{1}{3}, \frac{1}{6}, -\frac{1}{3}, \frac{2}{3}],
$$

$$
{\beta^{(2)}}^T = [1, -1, 0, 1, -1],
$$

$$
B^{(1)} = \begin{bmatrix} 0 & & & & \\ -1 & 0 & & & \\ \frac{17}{24} & 0 & 0 & & \\ \frac{15}{24} & -\frac{1}{3} & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C^{(0)} = \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ -\frac{1}{2} & 0 & 0 & & \\ -\frac{1}{2} & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
A^{(1)} = \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0 & 1 & 0 & & \\ -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B^{(2)} = \begin{bmatrix} 0 & & & & \\ \frac{1}{2} & 0 & & & \\ 0 & 0 & 0 & & \\ \frac{1}{2} & 0 & 0 & 0 & \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix},
$$

$$
{\alpha^{(2)}}^T = [-\frac{4\sqrt{13}}{13}, \frac{4\sqrt{13}}{13}, \frac{4}{3}, -\frac{4}{3}], \qquad {\beta^{(3)}}^T = [1, -1, -1, 1],
$$

$$
B^{(3)} = \begin{bmatrix} 0 & & & \\ \frac{\sqrt{13}}{4} & 0 & & \\ \frac{3\sqrt{13}-10}{24} & \frac{1}{24} & 0 & \\ \frac{3\sqrt{13}+8}{24} & \frac{1}{24} & 0 & 0 \end{bmatrix}, \qquad C^{(1)} = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ -\frac{1}{4} & \frac{3}{4} & 0 & \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix},
$$

$$A^{(2)} = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ -1 & 0 & 0 & \\ -1-\frac{4}{\sqrt{13}} & \frac{4}{\sqrt{13}} & 0 & 0 \end{bmatrix}, \quad B^{(4)} = \begin{bmatrix} 0 & & & \\ -\frac{1}{2} & 0 & & \\ 0 & 0 & 0 & \\ -1 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

5 Approximation of multiple Stratonovich stochastic integrals

Three Stratonovich stochastic integral variables J_1, J_{10} and J_{101} are used in the SRK methods (13) and (16). We know that J_1 can be simulated by a $N(0, h)$ distributed random variable and J_{10} can be simulated by $J_{10} = \frac{1}{2}h(J_1 + \frac{1}{\sqrt{3}}\xi)$, where ξ is a $N(0, h)$ distributed random variable and is independent of J_1. However, multiple stochastic integrals J_{101} can not be simulated exactly. Therefore, in practical computation, we need to approximate J_{101}.

Lemma 5.1 ([14]) *Suppose that the one-step approximation*

$$\bar{X}_{t,x}(t+h) = x + A(t, x, h; w(\theta) - w(t), t \le \theta \le t+h) \tag{18}$$

generates a method with accuracy order p, and the function A in (18) contains the term of the form $Q(t, x) \cdot \xi(w(\theta) - w(t), t \le \theta \le t+h)$, where $\|Q(t, x)\| \le K(1 + \|x\|^2)^{\frac{1}{2}}$ ($0 < K$ is a constant), and ξ is a random variable depending on the Wiener processes on the interval $[t, t+h]$. Let $\xi = \eta + \zeta$, where η and ζ are random variables depending on the same Wiener processes on the same interval, and

$$|E\zeta| \le Kh^{p+1}, \quad (E\zeta^2)^{1/2} \le Kh^{p+\frac{1}{2}}. \tag{19}$$

Then the method based on the one-step approximation (18) and with $Q \cdot \xi$ replaced by $Q \cdot \eta$ has accuracy order p.

To the authors' knowledge, there are mainly two ways to approximate multiple Stratonovich stochastic integrals. One way is to use random Fourier series for Browian bridge processes based on the given Wiener processes [11]. Another way is to transform the integrals into a simple SDE, then the SDE is approximated by a simpler numerical method [14, 15]. In this paper, the second way is used.

Since

$$J_{101} = \int_{t_n}^{t_n+h} \int_{t_n}^{s} \int_{t_n}^{s_1} \circ dW_{s_2} ds_1 \circ dW_s, \tag{20}$$

where $W(s)$ is a standard Wiener process. From the definition of the standard Wiener process, without loss of generality, we only need to consider the case with $t_n = 0$. Since

$$\int_0^h \int_0^s \int_0^{s_1} \circ dW_{s_2} ds_1 \circ dW_s = W(h) \int_0^h W(s)ds - \int_0^h W^2(s)ds, \tag{21}$$

we only need to consider the following three random variables

$$W(h), \quad \int_0^h W(s)ds, \quad \int_0^h W^2(s)ds. \tag{22}$$

Let

$$V(s)=\frac{W(sh)}{\sqrt{h}},\quad 0\le s\le 1. \tag{23}$$

It is obvious that $V(s)$ is a standard Wiener process, and

$$W(h)=h^{\frac{1}{2}}V(1),\ \ \int_0^h W(s)ds=h^{\frac{3}{2}}\int_0^1 V(s)ds,\ \ \int_0^h W^2(s)ds=h^2\int_0^1 V^2(s)ds. \tag{24}$$

It is obvious that three random variables $V(1)$, $\int_0^1 V(s)ds$ and $\int_0^1 V^2(s)ds$ are the solution of the system of the equtions

$$\begin{array}{ll} dx=dV(s), & x(0)=0,\\ dy=xds, & y(0)=0,\\ dz=x^2ds, & z(0)=0 \end{array} \tag{25}$$

at the moment $s=1$.

Two approximation methods for (25) are given by Milstein [14]: the Euler method and the Taylor method with strong order 1.5.

Applying the Euler method with a constant stepsize h_1 to (25), we have

$$\begin{aligned} x_{k+1}&=x_k+\triangle V_k,\\ y_{k+1}&=y_k+x_kh_1,\\ z_{k+1}&=z_k+x_k^2h_1, \end{aligned} \tag{26}$$

where $0=s_0<s_1<\cdots<s_N=1, s_{k+1}-s_k=h_1=\frac{1}{N},\ \ \triangle V_k=V(s_{k+1})-V(s_k),\ \ k=0,1,\ldots,N-1$, and it is easy to obtain

$$\begin{array}{ll} |E(V(1)-x_N)|=0, & E(V(1)-x_N)^2=0,\\ \left|E(\int_0^1 V(s)ds-y_N)\right|=0, & E(\int_0^1 V(s)ds-y_N)^2=\frac{h_1^2}{3},\\ \left|E(\int_0^1 V^2(s)ds-z_N)\right|=\frac{h_1}{2}, & E(\int_0^1 V^2(s)ds-z_N)^2=\frac{11}{12}h_1^2-\frac{h_1^3}{3}. \end{array} \tag{27}$$

Applying the Taylor method with strong order 1.5 with a constant stepsize h_1 to (25), we have

$$\begin{aligned} x_{k+1}&=x_k+\triangle V_k,\\ y_{k+1}&=y_k+x_kh_1+\int_{s_k}^{s_{k+1}}(V(s)-V(s_k))ds,\\ z_{k+1}&=z_k+x_k^2h_1+2x_k\int_{s_k}^{s_{k+1}}(V(s)-V(s_k))ds+\frac{h_1^2}{2}, \end{aligned} \tag{28}$$

where

$$\int_{s_k}^{s_{k+1}}(V(s)-V(s_k))ds=\frac{1}{2}h_1(\triangle V_k+\frac{1}{\sqrt{3}}\eta_k). \tag{29}$$

Here $\eta_k,\ k=0,1,\ldots,N-1$ are some independent $N(0,h_1)$ distributed random variables and are independent of $\triangle V_k,\ k=0,1,\ldots,N-1$. It is not difficult to obtain

$$\begin{aligned}
&|E(V(1)-x_N)|=0, && E(V(1)-x_N)^2=0,\\
&\left|E(\int_0^1 V(s)ds-y_N)\right|=0, && E(\int_0^1 V(s)ds-y_N)^2=0,\\
&\left|E(\int_0^1 V^2(s)ds-z_N)\right|=0, && E(\int_0^1 V^2(s)ds-z_N)^2=\frac{h_1^3}{3}.
\end{aligned} \tag{30}$$

From (30), we only need to take $h_1=O(h^{\frac{1}{3}})$ so that $(E(\int_0^1 V^2(s)ds-z_N)^2)^{\frac{1}{2}}=O(h^{\frac{1}{2}})$. However, we need to take $h_1=O(h)$ so that $(E(\int_0^1 V(s)ds-y_N)^2)^{\frac{1}{2}}=O(h)$ for (27). This shows that the method (28) is more efficient than the method (26).

Theorem 5.2 *For the method SRKS2.0 with strong order 2.0, if the random variables J_1, J_{10}, J_{101} are replaced by $h^{\frac{1}{2}}x_N,\ h^{\frac{3}{2}}y_N$ and $h^{\frac{1}{2}}x_N h^{\frac{3}{2}}y_N-h^2 z_N$ respectively, where x_N, y_N, z_N can be found recurrently from (28) with the stepsize $h_1=O(h^{\frac{1}{3}})$, then the strong order of accuracy of the method SRKS2.0 remains 2.0.*

6 Two specific types of the SDE (1)

For some specific types of the SDE (1), the SRK methods with high strong order can be constructed even if we use only one random variable J_1. Thus we do not need to simulate the multiple Stratonovich stochastic integrals J_{10}, J_{101}, and the efficiency of the SRK methods can be greatly improved.

6.1 SDE (1) with $f(y(t))\equiv 0$

Since $f(y(t))\equiv 0$, the SDE (1) can be written as

$$dy(t)=g(y(t))\circ dW(t),\quad y(t_0)=y_{t_0},\quad 0\le t_0,\quad t\in[t_0,T],\quad y_{t_0}\in\mathbb{R}^d. \tag{31}$$

In this case, we need only to consider the stochastic trees with all nodes being stochastic node ($\circ$) (such as $\tau_1, [\tau_1]_1, [\tau_1,\tau_1]_1, [[\tau_1]_1]_1$). Thus we consider the SRK method

$$\begin{aligned}
y_{n+1}&=y_n+J_1\sum_{i=1}^{s}\beta_i g(H_i),\\
H_i&=y_n+J_1\sum_{j=1}^{s}b_{ij}g(H_j),\qquad i=1,2,\ldots,s,
\end{aligned} \tag{32}$$

where $n=0,1,\cdots,N-1,\ y_0=y(t_0)$. It is not difficult to obtain the following result.

Theorem 6.1 *Let $g \in C^5(\mathbb{R}^d, \mathbb{R}^d)$. Suppose that for the ODE corresponding to the SDE (31)*

$$dy(t) = g(y(t))dt, \quad y(t_0) = y_{t_0}, \tag{33}$$

the deterministic RK method corresponding to the SRK method (32)

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{i=1}^{s} \beta_i g(H_i), \\ H_i &= y_n + h\sum_{j=1}^{s} b_{ij} g(H_j), \qquad i = 1, 2, \ldots, s \end{aligned} \tag{34}$$

is of order p, then the strong order of the SRK method (32) for the SDE (31) is no less than p', here*

$$\begin{cases} p' = \frac{p}{2}, & p \text{ is even,} \\ p' = \frac{p-1}{2}, & p \text{ is odd.} \end{cases}$$

Proof Let $\mathscr{M} = \{0\}$, and

$$z_i^{(0)(0)} = 0, \; z_i^{(1)(0)} = J_1\beta_i,$$

$$Z_{ij}^{(0)(0)(0)} = 0, \; Z_{ij}^{(0)(1)(0)} = J_{(1)} b_{ij}.$$

Then the SRK method (32) is a special case of the method (4), and we can apply Theorem 2.1. Since

$$J_{1\cdots 1} = \frac{1}{k!} J_1^k \quad (k \text{ is the length of the multi-index}),$$

the Stratonovich stochastic integral has the same chain rule as the deterministic case. Thus it is not difficult to show that the order condition (6) is fulfilled for each stochastic tree $\mathbf{t}$ with $\rho(\mathbf{t}) \le \frac{p}{2}$ and all nodes being the stochastic node ($\circ$) if the order of the RK method (34) is p.

For each stochastic tree $\mathbf{t}$ with $\rho(\mathbf{t}) = \frac{p+1}{2}$ and all nodes being the stochastic node ($\circ$), since

$$E[J_1^{2k+1}] = 0, \quad E[J_1^{2k}] = \frac{2k!}{k!2^k} h^k,$$

and the corresponding multiple Stratonovich stochastic integral $I_{\mathbf{t};t_n,t_n+h}$ and the corresponding elementary weight $\Phi_S(\mathbf{t}; t_n, t_n+h)$ both contain the random factor J_1^{p+1}. It is obvious that the order condition (7) is fulfilled when p is even, and the strong order of the SRK method (32) is $\frac{p}{2}$. However, when p is odd, (7) is not fulfilled. Thus the strong order of the SRK method (32) is not $\frac{p}{2}$, and it is no less than $\frac{p-1}{2}$.

The proof is complete. □

The SRK method (32) can be characterized by the extended Butcher tableau

$$\begin{array}{c} B \\ \hline \beta^T \end{array}$$

By Theorem 6.1, a specific explicit method with strong order 2.0 for the SDE (31) is proposed, and it is denoted by SRKST1, *i.e.*

$$
\begin{array}{cccc}
0 & & & \\
\frac{1}{2} & 0 & & \\
0 & \frac{1}{2} & 0 & \\
0 & 0 & 1 & 0 \\
\hline
\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

6.2 SDE (1) satisfying the commutativity condition between the drift and diffusion terms

Consider the SDE (1) satisfying the commutativity condition

$$\sum_{k=1}^{d} \frac{\partial f_j}{\partial y_k}(y(t))g_k(y(t)) = \sum_{k=1}^{d} \frac{\partial g_j}{\partial y_k}(y(t)) f_k(y(t)), \quad j = 1, 2, \ldots, d. \tag{35}$$

In fact, the SDE (1) satisfying (35) exists, for example, the linear SDE

$$dy(t) = \sigma_1 y(t)dt + \sigma_2 y(t) \circ dW(t), \;\; y(t_0) = y_{t_0}, \;\; 0 \le t_0, \;\; t \in [t_0, T], \;\; y_{t_0} \in \mathbb{R}^d,$$

and the nonlinear SDE

$$\begin{aligned} dy(t) &= \sigma_1(1 + y(t)^2)dt + \sigma_2(1 + y(t)^2) \circ dW(t), \;\; y(t_0) \\ &= y_{t_0}, \;\; 0 \le t_0, \;\; t \in [t_0, T], \;\; y_{t_0} \in \mathbb{R}^d, \end{aligned}$$

where, $\sigma_1, \sigma_2 \in \mathbb{R}$.

We consider the SRK method

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{i=1}^{s} \alpha_i f(H_i) + J_1 \sum_{i=1}^{s} \beta_i g(H_i), \\ H_i &= y_n + h\sum_{j=1}^{s} a_{i,j} f(H_j) + J_1 \sum_{j=1}^{s} b_{i,j} g(H_j), \qquad i = 1, 2, \ldots, s, \end{aligned} \tag{36}$$

where $n = 0, 1, \ldots, N-1, \;\; y_0 = y(t_0)$.

The SRK method (36) is a special case of the method (4) with $\mathscr{M} = \{0\}$ and

$$z_i^{(0)(0)} = h\alpha_i, \;\; z_i^{(1)(0)} = J_1\beta_i, \;\; Z_{ij}^{(0)(0)(0)} = ha_{ij}, \;\; Z_{ij}^{(0)(1)(0)} = J_1 b_{ij}. \tag{37}$$

By [19], the strong order of the SRK method (36) for the SDE (1) can not exceed 1.5 when (35) is not true. However, it is not difficult to obtain the following result when (35) holds.

Theorem 6.2 *Let $f, g \in C^4(\mathbb{R}^d, \mathbb{R}^d)$ and (35) holds. Then the SRK method (36) converges strongly to solution of the SDE (1) with strong order 1.5 if the coefficients of the SRK method (36) satisfy the system of the following equtions*

$$
\begin{array}{ll}
1.\ \alpha^T e = 1, & 2.\ \beta^T e = 1,\\
3.\ \alpha^T Ae = \frac{1}{2}, & 4.\ \alpha^T Be + \beta^T Ae = 1,\\
5.\ \beta^T Be = \frac{1}{2}, & 6.\ \beta^T (Be)^2 = \frac{1}{3},\\
7.\ \beta^T (B(Be)) = \frac{1}{6}, & 8.\ \alpha^T (Be)^2 = \frac{1}{2},\\
9.\ \alpha^T (B(Be)) = \frac{1}{4}, & 10.\ \beta^T (A(Be)) = 0,\\
11.\ \beta^T ((Ae)(Be)) = \frac{1}{4}, & 12.\ \beta^T (B(Ae)) = \frac{1}{4},\\
13.\ \beta^T (Be)^3 = \frac{1}{4}, & 14.\ \beta^T ((B(Be))(Be)) = \frac{1}{8},\\
15.\ \beta^T (B(Be)^2) = \frac{1}{12}, & 16.\ \beta^T (B(B(Be))) = \frac{1}{24}.
\end{array}
$$

Proof Since (35) holds, it is not difficult to show that the two order conditions in Theorem 2.1 for the stochastic trees $[\tau_1]_0,\ [\tau_0]_1$

$$I_{[\tau_1]_0;t,t+h} = \Phi_S([\tau_1]_0; t, t+h), \quad I_{[\tau_0]_1;t,t+h} = \Phi_S([\tau_0]_1; t, t+h), \tag{38}$$

can be reduced to

$$I_{[\tau_1]_0;t,t+h} + I_{[\tau_0]_1;t,t+h} = \Phi_S([\tau_1]_0; t, t+h) + \Phi_S([\tau_0]_1; t, t+h). \tag{39}$$

More proof details can be found in Appendix B. □

Remark 6.3 If (35) holds, then the multiple random integration variable J_{10} in the SRK methods (13) and (16) is no longer the key factor to improve the strong order of the methods. In fact, $I_{[\tau_1]_0;t,t+h} = J_{10;t,t+h}$, $I_{[\tau_0]_1;t,t+h} = J_{01;t,t+h}$, $J_{10;t,t+h} + J_{01;t,t+h} = J_{1;t,t+h}$, thus the left side of the equality (39) becomes $J_{1;t,t+h}$, and the right side of the equality (39) need only to contain J_1 and does not need to contain J_{10}.

The SRK method (36) can be characterized by the extended Butcher tableau

$$\begin{array}{c|c} A & B \\ \hline \alpha^T & \beta^T \end{array}$$

By Theorem 6.2, a specific explicit method with strong order 1.5 is proposed and it is denoted by SRKST2, *i.e.*

$$
\begin{array}{ccccc|ccccc}
0 & & & & & 0 & & & & \\
1 & 0 & & & & \frac{1}{2} & 0 & & & \\
\frac{2}{3} & -\frac{1}{6} & 0 & & & 0 & \frac{1}{2} & 0 & & \\
-\frac{1}{3} & \frac{1}{3} & 0 & 0 & & 0 & 0 & 1 & 0 & \\
\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-\frac{1}{4} & 0 & 0 & \frac{1}{2} & \frac{3}{4} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0
\end{array}
$$

7 Numerical results

In this section, we compare the SRK methods SRKS1.5, SRKS2.0, SRKST1 and SRKST2 with the other three explicit methods which has been widely used. The three SRK methods are strong order 1.0 SRK method PL [3], strong order 1.5 SRK method G5 [3], and strong order 1.5 SRK method SRI1W1 [18]. Here, PL and G5 are two methods for solving the Stratonovich SDEs, and SRI1W1 is a method for solving the Itô SDEs.

We apply the SRK methods PL, G5, SRKS1.5, SRKS2.0, SRKST1 and SRKST2 to the Stratonovich SDEs and apply SRI1W1 to the corresponding Itô SDEs.

By (15), we know that the four Itô stochastic integral variables $I_1, I_{11}, I_{111}, I_{10}$ being used in SRI1W1 can be expressed by the two Stratonovich stochastic integral variables J_1, J_{10}. Thus, it suffices to simulate the three Stratonovich stochastic integrals J_1, J_{10}, J_{101} in practical computation.

We use Err to denote global errors, $i.e.$

$$Err = \frac{1}{m}\sum_{i=1}^{m} |y_{iN} - y_i(t_N)|, \tag{40}$$

here, $y_i(t_N)$ denotes the exact solution of a SDE at the endpoint $t = t_N$ in the ith trajectory, y_{iN} denotes the numerical approximation of $y_i(t_N)$, m denotes the number of the trajectories used in each numerical simulation. In this paper, $m = 5000$.

By (3), for the SRK method with strong order p, we have

$$Err \le Ch^p,$$

and

$$log_2(Err) \approx log_2(C) + p\, log_2(h). \tag{41}$$

Taking different values of the stepsize h, we can obtain a sequence of discrete points $(log_2(h), log_2(Err))$. By (41), it is not difficult to know that these discrete points $(log_2(h), log_2(Err))$ will approximately obey the linear distribution with slope p. Then, we can apply the least-square method to obtain the approximation value of p.

In order to investigate computational efficiency, the computational effort of each method must be considered.

In Table 2, h_1 is the stepsize of the method (28). N_f denotes the number of evaluations of the drift coefficient f in per step, N_g denotes the number of evaluations of the diffusion coefficient g in per step, and N_r denotes the number of necessary random variables which have to be simulated in per step. In this paper, we take the sum

Table 2 The computational complexity of each method

Methods	PL	G5	SRI1W	SRKST1	SRKST2	SRKS1.5	SRKS2.0
N_f	1	5	2	0	4	4	12
N_g	2	5	4	4	4	6	13
N_r	1	2	2	1	1	2	$\frac{2}{h_1}$

of the number of evaluations of the drift and diffusion coefficients as well as the number of necessary random variables which have to be simulated as the computational effort. Let $\hat{S}$ denotes the computational effort for each trajectory, then

$$\hat{S} = (N_f + N_g + N_r) \times N,$$

where N is the number of steps.

Example 1 Consider the Stratonovich SDE with $f(y(t)) \equiv 0$

$$dy(t) = \sigma_1(1 - (y(t))^2) \circ dW(t), \quad y(0) = 0, \quad t \in [0, T]. \tag{42}$$

The corresponding Itô SDE can be written as

$$dy(t) = -\sigma_1^2 y(t)(1 - (y(t))^2)dt + \sigma_1(1 - (y(t))^2)dW(t). \tag{43}$$

By [11], the exact solution can be written as

$$y(t) = \frac{(1 + y(0))exp(2\sigma_1 W(t)) + y(0) - 1}{(1 + y(0))exp(2\sigma_1 W(t)) + 1 - y(0)}. \tag{44}$$

Take $h = 2^{-7}, 2^{-8}, \ldots, 2^{-11}$. Because the stochastic integral J_{101} need to be used in the method SRKS2.0. We apply the method (28) to approximate the stochastic integral J_{101}. By Theorem 5.2, we take $h_1 = \frac{1}{G(h)}$, where $G(h) = \left\lceil h^{-\frac{1}{3}} \right\rceil$ ($\lceil \cdot \rceil$ is the ceiling function). The results with the corresponding step sizes are presented in Table 3 and Fig. 1. In Table 3 and the following tables, $a(b) = a \times 10^b$.

By Table 3, we know that the order of accuracy of the method SRKST1 reaches 2.0353, and this confirms the theoretical result. From the right-hand side of Fig. 1, we can show that the method SRKST1 is far more efficient than other methods. The accuracy of SRKS1.5 and SRKS2.0 is exactly the same as that of SRKST1, and this is due to the fact that the coefficients $\beta^{(0)^T}$, $B^{(0)}$ in SRKS1.5 and SRKS2.0 are the same as the coefficients β^T, B in SRKST1.

Example 2 Consider the Stratonovich SDE

$$dy(t) = -\sigma_1(1 - y(t)^2)dt + \sigma_2(1 - y(t)^2) \circ dW(t), \quad y(0) = 0, \quad t \in [0, T]. \tag{45}$$

Table 3 The global errors for (42) or (43) with $\sigma_1 = 2.0$, $T = 1$

h	PL	G5	SRI1W1	SRKST1	SRKS1.5	SRKS2.0
2^{-7}	0.0492	0.0085	0.0058	0.0014	0.0014	0.0014
2^{-8}	0.0248	0.0023	0.0019	3.3805(-4)	3.3805(-4)	3.3805(-4)
2^{-9}	0.0124	6.4648(-4)	6.9396(-4)	8.6269(-5)	8.6269(-5)	8.6269(-5)
2^{-10}	0.0057	1.8049(-4)	2.2825(-4)	1.9822(-5)	1.9822(-5)	1.9822(-5)
2^{-11}	0.0029	5.8792(-5)	8.2451(-5)	4.9965(-6)	4.9965(-6)	4.9965(-6)
p	1.0290	1.8023	1.5330	2.0353	2.0353	2.0353

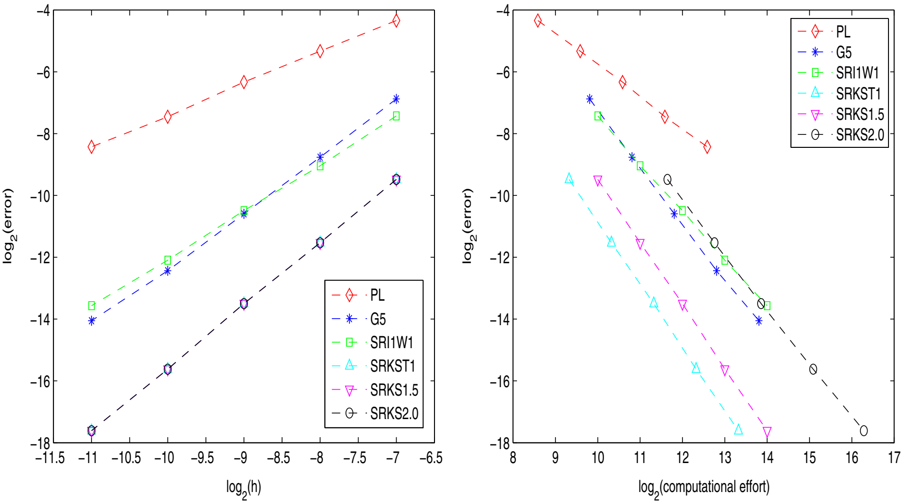


Fig. 1 Stepsize h vs. errors Err (left) and computational effort $\hat{S}$ vs. errors Err (right) for SDE (42) or (43)

It is obvious that (35) holds. The corresponding Itô SDE can be written as

$$dy(t) = -(\sigma_1 + (\sigma_2)^2 y(t))(1 - y(t)^2)dt + \sigma_2(1 - y(t)^2)dW(t). \tag{46}$$

By [11], the exact solution can be written as

$$y(t) = \frac{(1 + y(0))exp(-2\sigma_1 t + 2\sigma_2 W(t)) + y(0) - 1}{(1 + y(0))exp(-2\sigma_1 t + 2\sigma_2 W(t)) + 1 - y(0)}. \tag{47}$$

Take $h = 2^{-3}, 2^{-4}, \ldots, 2^{-11}$ and take $h_1 = \frac{1}{G(h)}$. The results with the corresponding step sizes are presented in Table 4 and Fig. 2.

Table 4 The global errors for (45) or (46) with $\sigma_1 = 2.0$, $\sigma_2 = 0.5$, $T = 1$

h	PL	G5	SRI1W1	SRKST2	SRKS1.5	SRKS2.0
2^{-3}	0.0226	0.0190	0.0059	0.0015	0.0067	0.0013
2^{-4}	0.0114	0.0034	0.0015	3.3681(-4)	0.0014	2.6633(-4)
2^{-5}	0.0057	7.8522(-4)	4.6826(-4)	8.7998(-5)	3.2800(-4)	6.0797(-5)
2^{-6}	0.0029	2.9136(-4)	1.5780(-4)	2.1627(-5)	8.2599(-5)	1.4629(-5)
2^{-7}	0.0014	1.4296(-4)	5.0028(-5)	5.1235(-6)	2.1103(-5)	3.5749(-6)
2^{-8}	7.0593(-4)	7.5256(-5)	1.8042(-5)	1.2968(-6)	5.6557(-6)	8.6513(-7)
2^{-9}	3.5585(-4)	3.9194(-5)	6.1007(-6)	3.2021(-7)	1.6142(-6)	2.1273(-7)
2^{-10}	1.7978(-4)	1.9790(-5)	2.1465(-6)	8.1065(-8)	5.0539(-7)	5.3278(-8)
2^{-11}	8.8998(-5)	1.0198(-5)	7.4956(-7)	2.0659(-8)	1.6593(-7)	1.3574(-8)
p	0.9993	1.2722	1.5961	2.0153	1.9119	2.0575

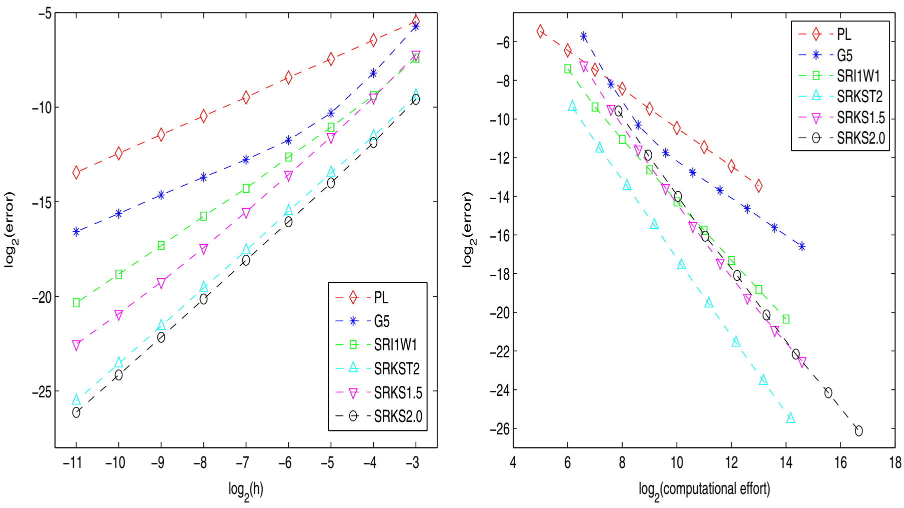


Fig. 2 Stepsize h vs. errors Err (**left**) and computational effort $\hat{S}$ vs. errors Err (**right**) for SDE (45) or (46)

For the Example 2, the order of accuracy of the method SRKST2 reaches 2.0153, which is higher than the theoretical order of the method SRKST2. However, the order of accuracy of the method G5 is lower than 1.5. In fact, Burrage and Burrage [3] take $E[J_{110}] = E[J_{011}] = 0$ when the method G5 is constructed. But by (15), we have $J_{110} = I_{110} + \frac{1}{2}I_{00}$, $J_{011} = I_{011} + \frac{1}{2}I_{00}$, then $E[J_{110}] = E[\frac{1}{2}I_{00}] = E[J_{011}] = \frac{1}{4}$. This means that the exact order of the method G5 is lower than 1.5. From the right-hand side of Fig. 2, we can show that the method SRKST2 is far more efficient than other methods.

Example 3 Consider two Stratonovich SDEs

$$dy(t) = (\sigma_1 y(t) - (y(t))^2)dt + \sigma_2 y(t) \circ dW(t),\ y(0) = 2,\ t \in [0, T], \tag{48}$$

$$dy(t) = \sigma_1 cos^2(y(t))tan(y(t))dt + \sqrt{2}cos^2(y(t)) \circ dW(t), \quad y(0) = 2, \quad t \in [0, T]. \tag{49}$$

The corresponding It $\hat{o}$ SDEs can be written as

$$dy(t) = ((\sigma_1 + \frac{1}{2}(\sigma_2)^2)y(t) - (y(t))^2)dt + \sigma_2 y(t)dW(t), \tag{50}$$

$$dy(t) = ((\frac{\sigma_1}{2} - \frac{1}{2})sin(2y(t)) - \frac{1}{4}sin(4y(t)))dt + \sqrt{2}cos^2(y(t))dW(t), \tag{51}$$

respectively. By [11], the exact solutions can be written as

Table 5 The global errors for (48) or (50) with $\sigma_1 = -3.0, \sigma_2 = 1, T = 1$

h	PL	G5	SRI1W1	SRKS1.5	SRKS2.0
2^{-3}	0.0499	0.0333	0.0137	0.0055	0.0027
2^{-4}	0.0257	0.0111	0.0038	0.0014	5.5453(-4)
2^{-5}	0.0132	0.0038	0.0012	3.3874(-4)	1.2791(-4)
2^{-6}	0.0067	0.0022	4.0053(-4)	8.6304(-5)	3.1132(-5)
2^{-7}	0.0033	0.0011	1.2368(-4)	2.0257(-5)	6.9325(-6)
2^{-8}	0.0017	5.8253(-4)	4.1943(-5)	5.3794(-6)	1.7157(-6)
2^{-9}	8.1537(-4)	2.9221(-4)	1.5691(-5)	1.2781(-6)	4.2454(-7)
2^{-10}	4.1203(-4)	1.5136(-4)	5.4569(-6)	3.5489(-7)	1.0237(-7)
2^{-11}	2.0218(-4)	7.3722(-5)	1.6944(-6)	9.2745(-8)	2.4610(-8)
p	0.9948	1.0531	1.6004	1.9894	2.0806

$$y(t) = \frac{y(0)exp(\sigma_1 t + \sigma_2 W(t))}{1 + y(0)\int_0^t exp(\sigma_1 s + \sigma_2 W(s))ds}, \tag{52}$$

$$y(t) = arctan(e^{\sigma_1 t} tan(y(0)) + \sqrt{2}e^{\sigma_1 t}\int_0^t e^{-\sigma_1 s} dW(s)), \tag{53}$$

respectively.

Because the exact solutions (52) and (53) contain the stochastic integrals, the numerical solutions of SRI1W1 with $h = 2^{-18}$ are used as the 'exact solutions'. And then we take $h = 2^{-3}, 2^{-4}, \ldots, 2^{-11}$ and take $h_1 = \frac{1}{G(h)}$. By comparing the numerical solutions of each SRK method with the 'exact solutions', we obtain the results with the corresponding step sizes. And they are presented in Tables 5, 6 and Figs. 3, 4.

Table 6 The global errors for (49) or (51) with $\sigma_1 = -3.0, T = 1$

h	PL	G5	SRI1W1	SRKS1.5	SRKS2.0
2^{-3}	0.0558	0.0563	0.0686	0.0405	0.0202
2^{-4}	0.0291	0.0190	0.0230	0.0109	0.0051
2^{-5}	0.0155	0.0083	0.0085	0.0029	0.0012
2^{-6}	0.0074	0.0037	0.0027	7.8181(-4)	3.2160(-4)
2^{-7}	0.0040	0.0019	0.0011	2.3100(-4)	8.3539(-5)
2^{-8}	0.0019	9.5050(-4)	3.5872(-4)	6.2501(-5)	2.0668(-5)
2^{-9}	9.7960(-4)	4.6569(-4)	1.3228(-4)	2.0916(-5)	5.3080(-6)
2^{-10}	4.7689(-4)	2.3801(-4)	4.5845(-5)	7.0530(-6)	1.3199(-6)
2^{-11}	2.5017(-4)	1.1584(-4)	1.6422(-5)	2.3854(-6)	3.2880(-7)
p	0.9821	1.0821	1.4992	1.7644	1.9829

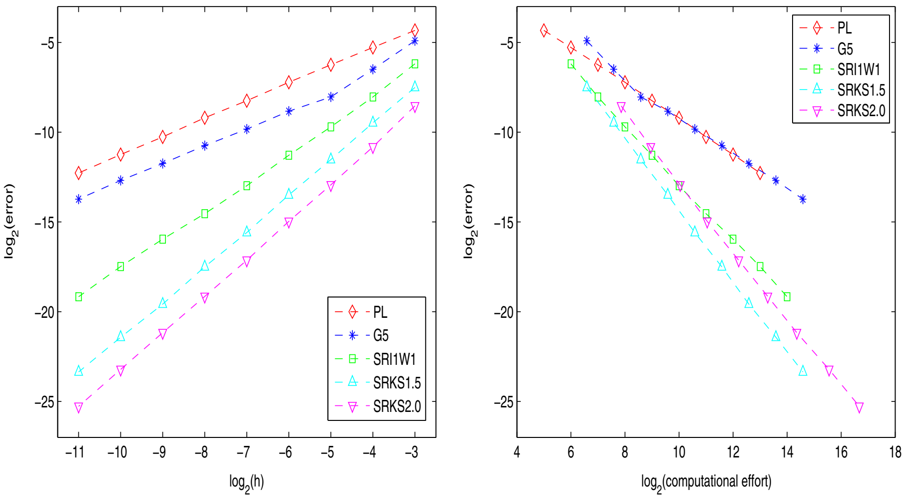


Fig. 3 Stepsize h vs. errors Err (*left*) and computational effort $\hat{S}$ vs. errors Err (*right*) for SDE (48) or (50)

In Tables 5 and 6, the order of accuracy of the method SRKS1.5 reaches 1.9894 and 1.7644 respectively, which are better than the theoretical result. And the order of accuracy of the method SRKS2.0 reaches 2.0806 and 1.9829 respectively, which are consistent with the theoretical result. From the right-hand sides of Figs. 3 and 4, we

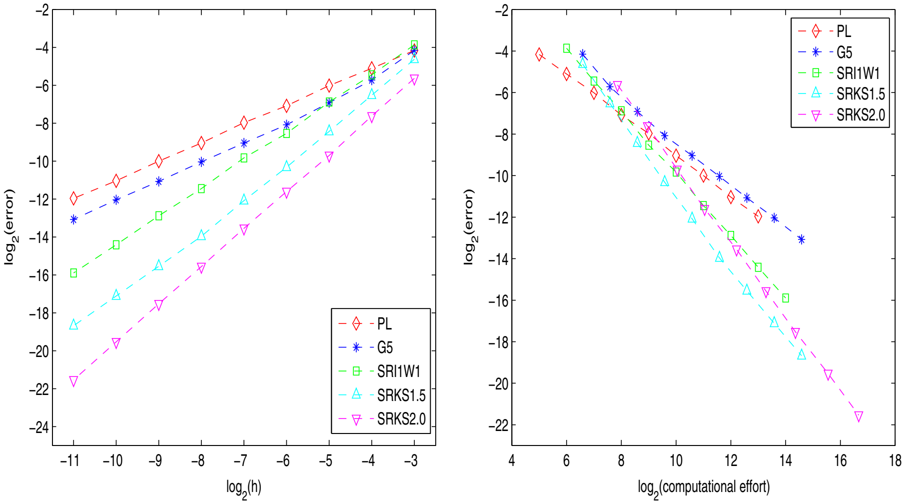


Fig. 4 Stepsize h vs. errors Err (*left*) and computational effort $\hat{S}$ vs. errors Err (*right*) for SDE (49) or (51)

can show that the method SRKS1.5 is more efficient than other methods for almost all step sizes. Although the order of accuracy of the method SRKS2.0 is higher than the method SRKS1.5, the method SRKS1.5 performs better than the method SRKS2.0. This is due to that the method SRKS2.0 needs to approximate the stochastic integral J_{101}. By Theorem 5.2, the computational effort to approximate the stochastic integral J_{101} is $O(h^{\frac{1}{3}})$ per step. Therefore, the effective order of the method SRKS2.0 is only 1.5 as $h \to 0$ after considering the overall computational work. Clearly, if a more efficient approximation method for the stochastic integral J_{101} would be available, then the effective order of the method SRKS2.0 may be improved up to 2.0.

8 Conclusions

In this paper, new high strong order SRK methods with several groups of independent internal stages for the Stratonovich SDEs with scalar noise are constructed. The advantage of independent internal stages is that they can reduced the complexity of order conditions of high strong order SRK methods. For two specific types of the Stratonovich SDEs, two explicit high strong order SRK methods SRKST1 and SRKST2 are constructed. These methods do not need to simulate the multiple Stratonovich stochastic integrals. From the right-hand sides of Figs. 1 and 2, it is not difficult to show that performance of the method SRKST1 is much better than other methods for the SDE (1) with $f(y(t)) \equiv 0$ and performance of the method SRKST2 is much better than other methods for the SDE (1) with (35). From the right-hand sides of Figs. 3 and 4, it is easy to show that the method SRKS1.5 performs best for the general SDE (1). Finally, it is worth mentioning that advantage of the method SRKS2.0 may be reflected more obviously if a more efficient approximation method for the multiple Stratonovich stochastic integrals would be available.

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Appendix A: Proof of Theorem 4.2

Proof First, we prove that for arbitrary $\mathbf{t}$ with $\rho(\mathbf{t}) \leq 2.0$,

$$I_{\mathbf{t};t,t+h} = \Phi_S(\mathbf{t}; t, t+h) \quad P - a.s.$$

holds. In fact, by (8)-(12), (15) and (17), we obtain that

$$\text{for } \tau_0,\quad I_{\tau_0;t,t+h} = \Phi_S(\tau_0;t,t+h) \quad\Leftrightarrow\quad h = h\alpha^{(0)^T}e_0 + h\alpha^{(1)^T}e_1 + h\alpha^{(2)^T}e_2$$
$$\Leftrightarrow\quad \alpha^{(0)^T}e_0 + \alpha^{(1)^T}e_1 + \alpha^{(2)^T}e_2 = 1;$$

$$\text{for } \tau_1,\quad I_{\tau_1;t,t+h} = \Phi_S(\tau_1;t,t+h)$$
$$\Leftrightarrow\quad J_1 = J_1(\beta^{(0)^T}e_0 + \beta^{(1)^T}e_1) + \frac{J_{10}}{h}\beta^{(2)^T}e_1 + \frac{J_{101}}{h^{3/2}}\beta^{(3)^T}e_2$$
$$\Leftrightarrow\quad \beta^{(0)^T}e_0 + \beta^{(1)^T}e_1 = 1,\quad \beta^{(2)^T}e_1 = 0,\quad \beta^{(3)^T}e_2 = 0;$$

$$\text{for } [\tau_0]_0,\quad I_{[\tau_0]_0;t,t+h} = \Phi_S([\tau_0]_0;t,t+h) \quad\Leftrightarrow\quad \frac{1}{2}h^2 = h^2\alpha^{(0)^T}A^{(0)}e_0$$
$$\Leftrightarrow\quad \alpha^{(0)^T}A^{(0)}e_0 = \frac{1}{2};$$

$$\text{for } [\tau_1]_0,\quad I_{[\tau_1]_0;t,t+h} = \Phi_S([\tau_1]_0;t,t+h)$$
$$\Leftrightarrow\quad J_{10} = hJ_1(\alpha^{(0)^T}B^{(0)}e_0 + \alpha^{(1)^T}B^{(1)}e_1) + h\frac{J_{10}}{h}\alpha^{(1)^T}C^{(0)}e_1 + h\sqrt{h}\alpha^{(2)^T}B^{(3)}e_2$$
$$+ h\frac{J_{101}}{h^{3/2}}\alpha^{(2)^T}C^{(1)}e_2$$
$$\Leftrightarrow\quad \alpha^{(0)^T}B^{(0)}e_0 + \alpha^{(1)^T}B^{(1)}e_1 = 0,\quad \alpha^{(1)^T}C^{(0)}e_1 = 1,\quad \alpha^{(2)^T}B^{(3)}e_2 = 0,$$
$$\alpha^{(2)^T}C^{(1)}e_2 = 0;$$

$$\text{for } [\tau_0]_1,\quad I_{[\tau_0]_1;t,t+h} = \Phi_S([\tau_0]_1;t,t+h)$$
$$\Leftrightarrow\quad J_{01} = hJ_1(\beta^{(0)^T}A^{(0)}e_0 + \beta^{(1)^T}A^{(1)}e_1) + h\frac{J_{10}}{h}\beta^{(2)^T}A^{(1)}e_1 + h\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}A^{(2)}e_2$$
$$\Leftrightarrow\quad \beta^{(0)^T}A^{(0)}e_0 + \beta^{(1)^T}A^{(1)}e_1 = 1,\quad \beta^{(2)^T}A^{(1)}e_1 = -1,\quad \beta^{(3)^T}A^{(2)}e_2 = 0;$$

$$\text{for } [\tau_1]_1,\quad I_{[\tau_1]_1;t,t+h} = \Phi_S([\tau_1]_1;t,t+h)$$
$$\Leftrightarrow\quad J_{11} = J_1^2(\beta^{(0)^T}B^{(0)}e_0 + \beta^{(1)^T}B^{(2)}e_1) + J_1\frac{J_{10}}{h}\beta^{(2)^T}B^{(2)}e_1 + \sqrt{h}\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}B^{(4)}e_2$$
$$\Leftrightarrow\quad \beta^{(0)^T}B^{(0)}e_0 + \beta^{(1)^T}B^{(2)}e_1 = \frac{1}{2},\quad \beta^{(2)^T}B^{(2)}e_1 = 0,\quad \beta^{(3)^T}B^{(4)}e_2 = 0;$$

$$\text{for } [\tau_1,\tau_1]_1,\quad I_{[\tau_1,\tau_1]_1;t,t+h} = \Phi_S([\tau_1,\tau_1]_1;t,t+h)$$
$$\Leftrightarrow\quad 2J_{111} = J_1^3(\beta^{(0)^T}(B^{(0)}e_0)^2 + \beta^{(1)^T}(B^{(2)}e_1)^2) + J_1^2\frac{J_{10}}{h}\beta^{(2)^T}(B^{(2)}e_1)^2$$
$$+ (\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(B^{(4)}e_2)^2$$
$$\Leftrightarrow\quad \beta^{(0)^T}(B^{(0)}e_0)^2 + \beta^{(1)^T}(B^{(2)}e_1)^2 = \frac{1}{3},\quad \beta^{(2)^T}(B^{(2)}e_1)^2 = 0,$$
$$\beta^{(3)^T}(B^{(4)}e_2)^2 = 0;$$

$$
\begin{aligned}
\text{for } [[\tau_1]_1]_1,\quad & I_{[[\tau_1]_1]_1;t,t+h} = \Phi_S([[\tau_1]_1]_1;t,t+h)\\
& \Leftrightarrow \quad J_{111} = J_1^3(\beta^{(0)^T}(B^{(0)}(B^{(0)}e_0)) + \beta^{(1)^T}(B^{(2)}(B^{(2)}e_1)))\\
&\qquad + J_1^2\frac{J_{10}}{h}\beta^{(2)^T}(B^{(2)}(B^{(2)}e_1)) + (\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(B^{(4)}(B^{(4)}e_2))\\
& \Leftrightarrow \quad \beta^{(0)^T}(B^{(0)}(B^{(0)}e_0)) + \beta^{(1)^T}(B^{(2)}(B^{(2)}e_1)) = \frac{1}{6},\\
& \quad \beta^{(2)^T}(B^{(2)}(B^{(2)}e_1)) = 0,\ \ \beta^{(3)^T}(B^{(4)}(B^{(4)}e_2)) = 0;\\
\text{for } [\tau_1,\tau_1]_0,\quad & I_{[\tau_1,\tau_1]_0;t,t+h} = \Phi_S([\tau_1,\tau_1]_0;t,t+h)\\
& \Leftrightarrow \quad 2J_{110} = hJ_1^2(\alpha^{(0)^T}(B^{(0)}e_0)^2 + \alpha^{(1)^T}(B^{(1)}e_1)^2)\\
&\qquad + 2hJ_1\frac{J_{10}}{h}\alpha^{(1)^T}((B^{(1)}e_1)(C^{(0)}e_1)) + h(\frac{J_{10}}{h})^2\alpha^{(1)^T}(C^{(0)}e_1)^2\\
&\qquad + h(\sqrt{h})^2\alpha^{(2)^T}(B^{(3)}e_2)^2 + 2h\sqrt{h}\frac{J_{101}}{h^{3/2}}\alpha^{(2)^T}((B^{(3)}e_2)(C^{(1)}e_2))\\
&\qquad + h(\frac{J_{101}}{h^{3/2}})^2\alpha^{(2)^T}(C^{(1)}e_2)^2\\
& \Leftrightarrow \quad \alpha^{(0)^T}(B^{(0)}e_0)^2 + \alpha^{(1)^T}(B^{(1)}e_1)^2 = 0,\\
& \quad \alpha^{(1)^T}((B^{(1)}e_1)(C^{(0)}e_1)) = \frac{1}{2},\ \ \alpha^{(1)^T}(C^{(0)}e_1)^2 = 0,\\
& \quad \alpha^{(2)^T}(B^{(3)}e_2)^2 = 0,\ \ \alpha^{(2)^T}((B^{(3)}e_2)(C^{(1)}e_2)) = -\frac{1}{2},\\
& \quad \alpha^{(2)^T}(C^{(1)}e_2)^2 = 0;\\
\text{for } [[\tau_1]_1]_0,\quad & I_{[[\tau_1]_1]_0;t,t+h} = \Phi_S([[\tau_1]_1]_0;t,t+h)\\
& \Leftrightarrow \quad J_{110} = hJ_1^2(\alpha^{(0)^T}(B^{(0)}(B^{(0)}e_0)) + \alpha^{(1)^T}(B^{(1)}(B^{(2)}e_1)))\\
&\qquad + hJ_1\frac{J_{10}}{h}\alpha^{(1)^T}(C^{(0)}(B^{(2)}e_1)) + h(\sqrt{h})^2\alpha^{(2)^T}(B^{(3)}(B^{(4)}e_2))\\
&\qquad + h\sqrt{h}\frac{J_{101}}{h^{3/2}}\alpha^{(2)^T}(C^{(1)}(B^{(4)}e_2))\\
& \Leftrightarrow \quad \alpha^{(0)^T}(B^{(0)}(B^{(0)}e_0)) + \alpha^{(1)^T}(B^{(1)}(B^{(2)}e_1)) = 0,\\
& \quad \alpha^{(1)^T}(C^{(0)}(B^{(2)}e_1)) = \frac{1}{2},\ \ \alpha^{(2)^T}(B^{(3)}(B^{(4)}e_2)) = 0,\\
& \quad \alpha^{(2)^T}(C^{(1)}(B^{(4)}e_2)) = -\frac{1}{2};\\
\text{for } [[\tau_1]_0]_1,\quad & I_{[[\tau_1]_0]_1;t,t+h} = \Phi_S([[\tau_1]_0]_1;t,t+h)\\
& \Leftrightarrow \quad J_{101} = hJ_1^2(\beta^{(0)^T}(A^{(0)}(B^{(0)}e_0)) + \beta^{(1)^T}(A^{(1)}(B^{(1)}e_1)))\\
&\qquad + hJ_1\frac{J_{10}}{h}(\beta^{(1)^T}(A^{(1)}(C^{(0)}e_1)) + \beta^{(2)^T}(A^{(1)}(B^{(1)}e_1)))\\
&\qquad + h(\frac{J_{10}}{h})^2\beta^{(2)^T}(A^{(1)}(C^{(0)}e_1)) + h\sqrt{h}\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(A^{(2)}(B^{(3)}e_2))\\
&\qquad + h(\frac{J_{101}}{h^{3/2}})^2\beta^{(3)^T}(A^{(2)}(C^{(1)}e_2))\\
& \Leftrightarrow \quad \beta^{(0)^T}(A^{(0)}(B^{(0)}e_0)) + \beta^{(1)^T}(A^{(1)}(B^{(1)}e_1)) = 0,\\
& \quad \beta^{(1)^T}(A^{(1)}(C^{(0)}e_1)) + \beta^{(2)^T}(A^{(1)}(B^{(1)}e_1)) = 0,\\
& \quad \beta^{(2)^T}(A^{(1)}(C^{(0)}e_1)) = 0,\ \ \beta^{(3)^T}(A^{(2)}(B^{(3)}e_2)) = 1,\\
& \quad \beta^{(3)^T}(A^{(2)}(C^{(1)}e_2)) = 0;
\end{aligned}
$$

$$\begin{aligned}
\text{for } [\tau_1,\tau_0]_1,\quad & I_{[\tau_1,\tau_0]_1;t,t+h} = \Phi_S([\tau_1,\tau_0]_1;t,t+h) \\
\Leftrightarrow\quad & J_{011}+J_{101} = hJ_1^2(\beta^{(0)^T}((A^{(0)}e_0)(B^{(0)}e_0)) + \beta^{(1)^T}((A^{(1)}e_1)(B^{(2)}e_1))) \\
& \qquad + hJ_1\frac{J_{10}}{h}\beta^{(2)^T}((A^{(1)}e_1)(B^{(2)}e_1)) \\
& \qquad + h\sqrt{h}\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}((A^{(2)}e_2)(B^{(4)}e_2)) \\
\Leftrightarrow\quad & \beta^{(0)^T}((A^{(0)}e_0)(B^{(0)}e_0)) + \beta^{(1)^T}((A^{(1)}e_1)(B^{(2)}e_1)) = \frac{1}{2}, \\
& \beta^{(2)^T}((A^{(1)}e_1)(B^{(2)}e_1)) = -\frac{1}{2},\ \beta^{(3)^T}((A^{(2)}e_2)(B^{(4)}e_2)) = \frac{1}{2};
\end{aligned}$$

$$\begin{aligned}
\text{for } [[\tau_0]_1]_1,\quad & I_{[[\tau_0]_1]_1;t,t+h} = \Phi_S([[\tau_0]_1]_1;t,t+h) \\
\Leftrightarrow\quad & J_{011} = hJ_1^2(\beta^{(0)^T}(B^{(0)}(A^{(0)}e_0)) + \beta^{(1)^T}(B^{(2)}(A^{(1)}e_1))) \\
& \qquad + hJ_1\frac{J_{10}}{h}\beta^{(2)^T}(B^{(2)}(A^{(1)}e_1)) + h\sqrt{h}\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(B^{(4)}(A^{(2)}e_2)) \\
\Leftrightarrow\quad & \beta^{(0)^T}(B^{(0)}(A^{(0)}e_0)) + \beta^{(1)^T}(B^{(2)}(A^{(1)}e_1)) = \frac{1}{2}, \\
& \beta^{(2)^T}(B^{(2)}(A^{(1)}e_1)) = -\frac{1}{2},\ \beta^{(3)^T}(B^{(4)}(A^{(2)}e_2)) = -\frac{1}{2};
\end{aligned}$$

$$\begin{aligned}
\text{for } [\tau_1,\tau_1,\tau_1]_1,\quad & I_{[\tau_1,\tau_1,\tau_1]_1;t,t+h} = \Phi_S([\tau_1,\tau_1,\tau_1]_1;t,t+h) \\
\Leftrightarrow\quad & 6J_{1111} = J_1^4(\beta^{(0)^T}(B^{(0)}e_0)^3 + \beta^{(1)^T}(B^{(2)}e_1)^3) + J_1^3\frac{J_{10}}{h}\beta^{(2)^T}(B^{(2)}e_1)^3 \\
& \qquad + (\sqrt{h})^3\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(B^{(4)}e_2)^3 \\
\Leftrightarrow\quad & \beta^{(0)^T}(B^{(0)}e_0)^3 + \beta^{(1)^T}(B^{(2)}e_1)^3 = \frac{1}{4},\ \ \beta^{(2)^T}(B^{(2)}e_1)^3 = 0, \\
& \beta^{(3)^T}(B^{(4)}e_2)^3 = 0;
\end{aligned}$$

$$\begin{aligned}
\text{for } [[\tau_1]_1,\tau_1]_1,\quad & I_{[[\tau_1]_1,\tau_1]_1;t,t+h} = \Phi_S([[\tau_1]_1,\tau_1]_1;t,t+h) \\
\Leftrightarrow\quad & 3J_{1111} = J_1^4(\beta^{(0)^T}((B^{(0)}(B^{(0)}e_0))(B^{(0)}e_0)) \\
& \qquad + \beta^{(1)^T}((B^{(2)}(B^{(2)}e_1))(B^{(2)}e_1))) \\
& \qquad + J_1^3\frac{J_{10}}{h}\beta^{(2)^T}((B^{(2)}(B^{(2)}e_1))(B^{(2)}e_1)) \\
& \qquad + (\sqrt{h})^3\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}((B^{(4)}(B^{(4)}e_2))(B^{(4)}e_2)) \\
\Leftrightarrow\quad & \beta^{(0)^T}((B^{(0)}(B^{(0)}e_0))(B^{(0)}e_0)) \\
& + \beta^{(1)^T}((B^{(2)}(B^{(2)}e_1))(B^{(2)}e_1)) = \frac{1}{8}, \\
& \beta^{(2)^T}((B^{(2)}(B^{(2)}e_1))(B^{(2)}e_1)) = 0, \\
& \beta^{(3)^T}((B^{(4)}(B^{(4)}e_2))(B^{(4)}e_2)) = 0;
\end{aligned}$$

$$
\begin{aligned}
\text{for } [[\tau_1,\tau_1]_1]_1,\quad & I_{[[\tau_1,\tau_1]_1]_1;t,t+h} = \Phi_S([[\tau_1,\tau_1]_1]_1;t,t+h)\\
\Leftrightarrow\quad & 2J_{1111} = J_1^4({\beta^{(0)}}^T(B^{(0)}(B^{(0)}e_0)^2) + {\beta^{(1)}}^T(B^{(2)}(B^{(2)}e_1)^2))\\
& \qquad + J_1^3\frac{J_{10}}{h}{\beta^{(2)}}^T(B^{(2)}(B^{(2)}e_1)^2) + (\sqrt{h})^3\frac{J_{101}}{h^{3/2}}{\beta^{(3)}}^T(B^{(4)}(B^{(4)}e_2)^2)\\
\Leftrightarrow\quad & {\beta^{(0)}}^T(B^{(0)}(B^{(0)}e_0)^2) + {\beta^{(1)}}^T(B^{(2)}(B^{(2)}e_1)^2) = \frac{1}{12},\\
& {\beta^{(2)}}^T(B^{(2)}(B^{(2)}e_1)^2) = 0, {\beta^{(3)}}^T(B^{(4)}(B^{(4)}e_2)^2) = 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [[[\tau_1]_1]_1]_1,\quad & I_{[[[\tau_1]_1]_1]_1;t,t+h} = \Phi_S([[[\tau_1]_1]_1]_1;t,t+h)\\
\Leftrightarrow\quad & J_{1111} = J_1^4({\beta^{(0)}}^T(B^{(0)}(B^{(0)}(B^{(0)}e_0))) + {\beta^{(1)}}^T(B^{(2)}(B^{(2)}(B^{(2)}e_1))))\\
& \qquad + J_1^3\frac{J_{10}}{h}{\beta^{(2)}}^T(B^{(2)}(B^{(2)}(B^{(2)}e_1)))\\
& \qquad + (\sqrt{h})^3\frac{J_{101}}{h^{3/2}}{\beta^{(3)}}^T(B^{(4)}(B^{(4)}(B^{(4)}e_2)))\\
\Leftrightarrow\quad & {\beta^{(0)}}^T(B^{(0)}(B^{(0)}(B^{(0)}e_0))) + {\beta^{(1)}}^T(B^{(2)}(B^{(2)}(B^{(2)}e_1))) = \frac{1}{24},\\
& {\beta^{(2)}}^T(B^{(2)}(B^{(2)}(B^{(2)}e_1))) = 0,\quad {\beta^{(3)}}^T(B^{(4)}(B^{(4)}(B^{(4)}e_2))) = 0.
\end{aligned}
$$

Next, we prove that for arbitrary $\mathbf{t}$ with $\rho(\mathbf{t}) = 2.5$,

$$E(I_{\mathbf{t};t,t+h}) = E(\Phi_S(\mathbf{t};t,t+h))$$

holds. In fact, by (8)-(12), (15) and (17) as well as Table 1, we obtain that

$$
\begin{aligned}
\text{for } [\tau_0,\tau_0]_1,\quad & E[I_{[\tau_0,\tau_0]_1;t,t+h}] = E[\Phi_S([\tau_0,\tau_0]_1;t,t+h)]\\
\Leftrightarrow\quad & E[2J_{001}] = E[h^2J_1({\beta^{(0)}}^T(A^{(0)}e_0)^2 + {\beta^{(1)}}^T(A^{(1)}e_1)^2)\\
& \qquad + h^2\frac{J_{10}}{h}{\beta^{(2)}}^T(A^{(1)}e_1)^2 + h^2\frac{J_{101}}{h^{3/2}}{\beta^{(3)}}^T(A^{(2)}e_2)^2]\\
\Leftrightarrow\quad & 0 = 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [\tau_0,\tau_1]_0,\quad & E[I_{[\tau_0,\tau_1]_0;t,t+h}] = E[\Phi_S([\tau_0,\tau_1]_0;t,t+h)]\\
\Leftrightarrow\quad & E[J_{010} + J_{100}] = E[h^2J_1{\alpha^{(0)}}^T((A^{(0)}e_0)(B^{(0)}e_0))]\\
\Leftrightarrow\quad & 0 = 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [[\tau_1]_0]_0,\quad & E[I_{[[\tau_1]_0]_0;t,t+h}] = E[\Phi_S([[\tau_1]_0]_0;t,t+h)]\\
\Leftrightarrow\quad & E[J_{100}] = E[h^2J_1{\alpha^{(0)}}^T(A^{(0)}(B^{(0)}e_0))]\\
\Leftrightarrow\quad & 0 = 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [[\tau_0]_1]_0, \quad & E[I_{[[\tau_0]_1]_0;t,t+h}] = E[\Phi_S([[\tau_0]_1]_0;t,t+h)] \\
& \Leftrightarrow \quad E[J_{010}] = E[h^2 J_1(\alpha^{(0)^T}(B^{(0)}(A^{(0)}e_0)) + \alpha^{(1)^T}(B^{(1)}(A^{(1)}e_1))) \\
& \qquad\qquad + h^2\frac{J_{10}}{h}\alpha^{(1)^T}(C^{(0)}(A^{(1)}e_1)) + h^2\sqrt{h}\alpha^{(2)^T}(B^{(3)}(A^{(2)}e_2)) \\
& \qquad\qquad + h^2\frac{J_{101}}{h^{3/2}}\alpha^{(2)^T}(C^{(1)}(A^{(2)}e_2))] \\
& \Leftrightarrow \quad \alpha^{(2)^T}(B^{(3)}(A^{(2)}e_2)) = 0; \\
\text{for } [[\tau_0]_0]_1, \quad & E[I_{[[\tau_0]_0]_1;t,t+h}] = (E[\Phi_S[[\tau_0]_0]_1;t,t+h]) \\
& \Leftrightarrow \quad E[J_{001}] = E[h^2 J_1 \beta^{(0)^T}(A^{(0)}(A^{(0)}e_0))] \\
& \Leftrightarrow \quad 0 = 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [\tau_1,\tau_1,\tau_1]_0, \quad & E[I_{[\tau_1,\tau_1,\tau_1]_0;t,t+h}] = E[\Phi_S([\tau_1,\tau_1,\tau_1]_0;t,t+h)] \\
& \Leftrightarrow E[6J_{1110}] = E[hJ_1^3(\alpha^{(0)^T}(B^{(0)}e_0)^3) + h\alpha^{(1)^T}(J_1(B^{(1)}e_1) + \frac{J_{10}}{h}(C^{(0)}e_1))^3 \\
& \qquad\qquad + h\alpha^{(2)^T}(\sqrt{h}(B^{(3)}e_2) + \frac{J_{101}}{h^{3/2}}(C^{(1)}e_2))^3] \\
& \Leftrightarrow \alpha^{(2)^T}(B^{(3)}e_2)^3 + \frac{1}{4}\alpha^{(2)^T}((B^{(3)}e_2)((C^{(1)}e_2)^2)) \\
& \quad + \frac{1}{30}\alpha^{(2)^T}(C^{(1)}e_2)^3 = 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [[\tau_1]_1,\tau_1]_0, \quad & E[I_{[[\tau_1]_1,\tau_1]_0;t,t+h}] = E[\Phi_S([[\tau_1]_1,\tau_1]_0;t,t+h)] \\
& \Leftrightarrow \quad E[3J_{1110}] = E[hJ_1^3(\alpha^{(0)^T}((B^{(0)}(B^{(0)}e_0))(B^{(0)}e_0)) \\
& \qquad\qquad + \alpha^{(1)^T}((B^{(1)}(B^{(2)}e_1))(B^{(1)}e_1))) \\
& \qquad\qquad + hJ_1^2\frac{J_{10}}{h}(\alpha^{(1)^T}((C^{(0)}(B^{(2)}e_1))(B^{(1)}e_1)) \\
& \qquad\qquad + \alpha^{(1)^T}((B^{(1)}(B^{(2)}e_1))(C^{(0)}e_1))) \\
& \qquad\qquad + hJ_1(\frac{J_{10}}{h})^2\alpha^{(1)^T}((C^{(0)}(B^{(2)}e_1))(C^{(0)}e_1)) \\
& \qquad\qquad + h(\sqrt{h})^3\alpha^{(2)^T}((B^{(3)}(B^{(4)}e_2))(B^{(3)}e_2)) \\
& \qquad\qquad + h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}(\alpha^{(2)^T}((C^{(1)}(B^{(4)}e_2))(B^{(3)}e_2)) \\
& \qquad\qquad + \alpha^{(2)^T}((B^{(3)}(B^{(4)}e_2))(C^{(1)}e_2))) \\
& \qquad\qquad + h\sqrt{h}(\frac{J_{101}}{h^{3/2}})^2\alpha^{(2)^T}((C^{(1)}(B^{(4)}e_2))(C^{(1)}e_2))] \\
& \Leftrightarrow \quad \alpha^{(2)^T}((B^{(3)}(B^{(4)}e_2))(B^{(3)}e_2)) \\
& \qquad + \frac{1}{12}\alpha^{(2)^T}((C^{(1)}(B^{(4)}e_2))(C^{(1)}e_2)) = 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [[\tau_1,\tau_1]_1]_0,\quad & E[I_{[[\tau_1,\tau_1]_1]_0;t,t+h}] = E[\Phi_S([[\tau_1,\tau_1]_1]_0;t,t+h)]\\
\Leftrightarrow\quad & E[2J_{1110}] = E[hJ_1^3(\alpha^{(0)^T}(B^{(0)}(B^{(0)}e_0)^2)+\alpha^{(1)^T}(B^{(1)}(B^{(2)}e_1)^2))\\
&\qquad + hJ_1^2\frac{J_{10}}{h}\alpha^{(1)^T}(C^{(0)}(B^{(2)}e_1)^2)\\
&\qquad + h(\sqrt{h})^3\alpha^{(2)^T}(B^{(3)}(B^{(4)}e_2)^2)\\
&\qquad + h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\alpha^{(2)^T}(C^{(1)}(B^{(4)}e_2)^2)]\\
\Leftrightarrow\quad & \alpha^{(2)^T}(B^{(3)}(B^{(4)}e_2)^2)=0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [[[\tau_1]_1]_1]_0,\quad & E[I_{[[[\tau_1]_1]_1]_0;t,t+h}] = E[\Phi_S([[[\tau_1]_1]_1]_0;t,t+h)]\\
\Leftrightarrow\quad & E[J_{1110}] = E[hJ_1^3(\alpha^{(0)^T}(B^{(0)}(B^{(0)}(B^{(0)}e_0)))\\
&\qquad + \alpha^{(1)^T}(B^{(1)}(B^{(2)}(B^{(2)}e_1))))\\
&\qquad + hJ_1^2\frac{J_{10}}{h}\alpha^{(1)^T}(C^{(0)}(B^{(2)}(B^{(2)}e_1)))\\
&\qquad + h(\sqrt{h})^3\alpha^{(2)^T}(B^{(3)}(B^{(4)}(B^{(4)}e_2)))\\
&\qquad + h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\alpha^{(2)^T}(C^{(1)}(B^{(4)}(B^{(4)}e_2)))]\\
\Leftrightarrow\quad & \alpha^{(2)^T}(B^{(3)}(B^{(4)}(B^{(4)}e_2)))=0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [\tau_0,\tau_1,\tau_1]_1,\quad & E[I_{[\tau_0,\tau_1,\tau_1]_1;t,t+h}] = E[\Phi_S([\tau_0,\tau_1,\tau_1]_1;t,t+h)]\\
\Leftrightarrow\quad & E[2J_{0111}+2J_{1101}+2J_{1011}] = E[hJ_1^3(\beta^{(0)^T}((A^{(0)}e_0)(B^{(0)}e_0)^2)\\
&\qquad + \beta^{(1)^T}((A^{(1)}e_1)(B^{(2)}e_1)^2))\\
&\qquad + hJ_1^2\frac{J_{10}}{h}\beta^{(2)^T}((A^{(1)}e_1)(B^{(2)}e_1)^2)\\
&\qquad + h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}((A^{(2)}e_2)(B^{(4)}e_2)^2)]\\
\Leftrightarrow\quad & 0=0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [[\tau_1]_0,\tau_1]_1,\quad & E[I_{[[\tau_1]_0,\tau_1]_1;t,t+h}] = E[\Phi_S([[\tau_1]_0,\tau_1]_1;t,t+h)]\\
\Leftrightarrow\quad & E[2J_{1101}+J_{1011}] = E[hJ_1^3(\beta^{(0)^T}((A^{(0)}(B^{(0)}e_0))(B^{(0)}e_0))\\
&\qquad + \beta^{(1)^T}((A^{(1)}(B^{(1)}e_1))(B^{(2)}e_1)))\\
&\qquad + hJ_1^2\frac{J_{10}}{h}(\beta^{(1)^T}((A^{(1)}(C^{(0)}e_1))(B^{(2)}e_1))\\
&\qquad + \beta^{(2)^T}((A^{(1)}(B^{(1)}e_1))(B^{(2)}e_1)))\\
&\qquad + hJ_1(\frac{J_{10}}{h})^2\beta^{(2)^T}((A^{(1)}(C^{(0)}e_1))(B^{(2)}e_1))\\
&\qquad + h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}((A^{(2)}(B^{(3)}e_2))(B^{(4)}e_2))\\
&\qquad + h\sqrt{h}(\frac{J_{101}}{h^{3/2}})^2\beta^{(3)^T}((A^{(2)}(C^{(1)}e_2))(B^{(4)}e_2))]\\
\Leftrightarrow\quad & \beta^{(3)^T}((A^{(2)}(C^{(1)}e_2))(B^{(4)}e_2))=0;
\end{aligned}
$$

$$\text{for } [[\tau_0]_1,\tau_1]_1, \quad E[I_{[[\tau_0]_1,\tau_1]_1;t,t+h}] = E[\Phi_S([[\tau_0]_1,\tau_1]_1;t,t+h)]$$

$$\begin{aligned}\Leftrightarrow \quad E[2J_{0111}+J_{1011}] = E[hJ_1^3(&\beta^{(0)^T}((B^{(0)}(A^{(0)}e_0))(B^{(0)}e_0))\\ &+\beta^{(1)^T}((B^{(2)}(A^{(1)}e_1))(B^{(2)}e_1)))\\ &+hJ_1^2\frac{J_{10}}{h}\beta^{(2)^T}((B^{(2)}(A^{(1)}e_1))(B^{(2)}e_1))\\ &+h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}((B^{(4)}(A^{(2)}e_2))(B^{(4)}e_2))]\end{aligned}$$

$$\Leftrightarrow \quad 0=0;$$

$$\text{for } [[\tau_1]_1,\tau_0]_1, \quad E[I_{[[\tau_1]_1,\tau_0]_1;t,t+h}] = E[\Phi_S([[\tau_1]_1,\tau_0]_1;t,t+h)]$$

$$\begin{aligned}\Leftrightarrow \quad E[J_{1101}+J_{0111}+J_{1011}] = E[hJ_1^3(&\beta^{(0)^T}((B^{(0)}(B^{(0)}e_0))(A^{(0)}e_0))\\ &+\beta^{(1)^T}((B^{(2)}(B^{(2)}e_1))(A^{(1)}e_1)))\\ &+hJ_1^2\frac{J_{10}}{h}\beta^{(2)^T}((B^{(2)}(B^{(2)}e_1))(A^{(1)}e_1))\\ &+h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}((B^{(4)}(B^{(4)}e_2))(A^{(2)}e_2))]\end{aligned}$$

$$\Leftrightarrow \quad 0=0;$$

$$\text{for } [[\tau_1,\tau_0]_1]_1, \quad E[I_{[[\tau_1,\tau_0]_1]_1;t,t+h}] = E[\Phi_S([[\tau_1,\tau_0]_1]_1;t,t+h)]$$

$$\begin{aligned}\Leftrightarrow \quad E[J_{1011}+J_{0111}] = E[hJ_1^3(&\beta^{(0)^T}(B^{(0)}((A^{(0)}e_0)(B^{(0)}e_0)))\\ &+\beta^{(1)^T}(B^{(2)}((A^{(1)}e_1)(B^{(2)}e_1))))\\ &+hJ_1^2\frac{J_{10}}{h}\beta^{(2)^T}(B^{(2)}((A^{(1)}e_1)(B^{(2)}e_1)))\\ &+h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(B^{(4)}((A^{(2)}e_2)(B^{(4)}e_2))))]\end{aligned}$$

$$\Leftrightarrow \quad 0=0;$$

$$\text{for } [[\tau_1,\tau_1]_0]_1, \quad E[I_{[[\tau_1,\tau_1]_0]_1;t,t+h}] = E[\Phi_S([[\tau_1,\tau_1]_0]_1;t,t+h)]$$

$$\begin{aligned}\Leftrightarrow \quad E[2J_{1101}] = E[hJ_1^3(&\beta^{(0)^T}(A^{(0)}(B^{(0)}e_0)^2)+\beta^{(1)^T}(A^{(1)}(B^{(1)}e_1)^2))\\ &+hJ_1^2\frac{J_{10}}{h}(2\beta^{(1)^T}(A^{(1)}((B^{(1)}e_1)(C^{(0)}e_1)))\\ &+\beta^{(2)^T}(A^{(1)}(B^{(1)}e_1)^2))\\ &+hJ_1(\frac{J_{10}}{h})^2(\beta^{(1)^T}(A^{(1)}(C^{(0)}e_1)^2)\\ &+2\beta^{(2)^T}(A^{(1)}((B^{(1)}e_1)(C^{(0)}e_1))))\\ &+h(\frac{J_{10}}{h})^3\beta^{(2)^T}(A^{(1)}(C^{(0)}e_1)^2)\\ &+h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(A^{(2)}(B^{(3)}e_2)^2)\\ &+2h\sqrt{h}(\frac{J_{101}}{h^{3/2}})^2\beta^{(3)^T}(A^{(2)}((B^{(3)}e_2)(C^{(1)}e_2)))\\ &+h(\frac{J_{101}}{h^{3/2}})^3\beta^{(3)^T}(A^{(2)}(C^{(1)}e_2)^2)]\end{aligned}$$

$$\Leftrightarrow \quad \frac{1}{6}\beta^{(3)^T}(A^{(2)}((B^{(3)}e_2)(C^{(1)}e_2)))+\frac{1}{30}\beta^{(3)^T}(A^{(2)}(C^{(1)}e_2)^2)=0;$$

$$
\begin{aligned}
\text{for } [[[\tau_1]_1]_0]_1, \quad & E[I_{[[[\tau_1]_1]_0]_1;t,t+h}] = E[\Phi_S([[[\tau_1]_1]_0]_1; t, t+h)] \\
\Leftrightarrow \quad & E[J_{1101}] = E[hJ_1^3(\beta^{(0)^T}(A^{(0)}(B^{(0)}(B^{(0)}e_0))) \\
& \qquad + \beta^{(1)^T}(A^{(1)}(B^{(1)}(B^{(2)}e_1)))) \\
& \qquad + hJ_1^2\frac{J_{10}}{h}(\beta^{(1)^T}(A^{(1)}(C^{(0)}(B^{(2)}e_1))) \\
& \qquad + \beta^{(2)^T}(A^{(1)}(B^{(1)}(B^{(2)}e_1)))) \\
& \qquad + hJ_1(\frac{J_{10}}{h})^2\beta^{(2)^T}(A^{(1)}(C^{(0)}(B^{(2)}e_1))) \\
& \qquad + h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(A^{(2)}(B^{(3)}(B^{(4)}e_2)))) \\
& \qquad + h\sqrt{h}(\frac{J_{101}}{h^{3/2}})^2\beta^{(3)^T}(A^{(2)}(C^{(1)}(B^{(4)}e_2))))] \\
\Leftrightarrow \quad & \beta^{(3)^T}(A^{(2)}(C^{(1)}(B^{(4)}e_2))) = 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [[[\tau_1]_0]_1]_1, \quad & E[I_{[[[\tau_1]_0]_1]_1;t,t+h}] = E[\Phi_S([[[\tau_1]_0]_1]_1; t, t+h)] \\
\Leftrightarrow \quad & E[J_{1011}] = E[hJ_1^3(\beta^{(0)^T}(B^{(0)}(A^{(0)}(B^{(0)}e_0))) \\
& \qquad + \beta^{(1)^T}(B^{(2)}(A^{(1)}(B^{(1)}e_1)))) \\
& \qquad + hJ_1^2\frac{J_{10}}{h}(\beta^{(1)^T}(B^{(2)}(A^{(1)}(C^{(0)}e_1))) \\
& \qquad + \beta^{(2)^T}(B^{(2)}(A^{(1)}(B^{(1)}e_1)))) \\
& \qquad + hJ_1(\frac{J_{10}}{h})^2\beta^{(2)^T}(B^{(2)}(A^{(1)}(C^{(0)}e_1))) \\
& \qquad + h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(B^{(4)}(A^{(2)}(B^{(3)}e_2)))) \\
& \qquad + h\sqrt{h}(\frac{J_{101}}{h^{3/2}})^2\beta^{(3)^T}(B^{(4)}(A^{(2)}(C^{(1)}e_2)))] \\
\Leftrightarrow \quad & \beta^{(3)^T}(B^{(4)}(A^{(2)}(C^{(1)}e_2))) = 0;
\end{aligned}
$$

$$
\begin{aligned}
\text{for } [[[\tau_0]_1]_1]_1, \quad & E[I_{[[[\tau_0]_1]_1]_1;t,t+h}] = E[\Phi_S([[[\tau_0]_1]_1]_1; t, t+h)] \\
\Leftrightarrow \quad & E[J_{0111}] = E[hJ_1^3(\beta^{(0)^T}(B^{(0)}(B^{(0)}(A^{(0)}e_0))) \\
& \qquad + \beta^{(1)^T}(B^{(2)}(B^{(2)}(A^{(1)}e_1)))) \\
& \qquad + hJ_1^2\frac{J_{10}}{h}\beta^{(2)^T}(B^{(2)}(B^{(2)}(A^{(1)}e_1))) \\
& \qquad + h(\sqrt{h})^2\frac{J_{101}}{h^{3/2}}\beta^{(3)^T}(B^{(4)}(B^{(4)}(A^{(2)}e_2))))] \\
\Leftrightarrow \quad & 0 = 0.
\end{aligned}
$$

For each stochastic tree $\mathbf{t}$ with $\rho(\mathbf{t}) = 2.5$ and all nodes being stochastic node ($\circ$) (such as $[\tau_1, \tau_1, \tau_1, \tau_1]_1$, $[[\tau_1, \tau_1, \tau_1]_1]_1$), since $E[J_{11111}] = 0$, $E[J_1^5] = E[J_1^4 \frac{J_{10}}{h}] = E[(\sqrt{h})^4 \frac{J_{101}}{h^{3/2}}] = 0$, it is not difficult to obtain that

$$E[(I_{\mathbf{t};t,t+h})] = E[(\Phi_S(\mathbf{t}; t, t+h))] \quad \Leftrightarrow \quad 0 = 0.$$

The proof is complete. □

Appendix B: Proof of Theorem 6.2

Proof First, we prove that for arbitrary $\mathbf{t}$ with $\rho(\mathbf{t}) \leq 1.5$

$$I_{\mathbf{t};t,t+h} = \Phi_S(\mathbf{t}; t, t+h) \quad P - a.s.$$

holds. In fact, by (8)-(12), (15) and (37), we obtain that

$$\begin{aligned}
&\text{for } \tau_0, \quad I_{\tau_0;t,t+h} = \Phi_S(\tau_0; t, t+h) \quad \Leftrightarrow \quad h = h\alpha^T e \quad \Leftrightarrow \quad \alpha^T e = 1;\\
&\text{for } \tau_1, \quad I_{\tau_1;t,t+h} = \Phi_S(\tau_1; t, t+h) \quad \Leftrightarrow \quad J_1 = J_1\beta^T e \quad \Leftrightarrow \quad \beta^T e = 1;\\
&\text{for } [\tau_0]_0, \quad I_{[\tau_0]_0;t,t+h} = \Phi_S([\tau_0]_0; t, t+h) \quad \Leftrightarrow \quad \frac{1}{2}h^2 = h^2\alpha^T Ae \quad \Leftrightarrow \quad \alpha^T Ae = \frac{1}{2};\\
&\text{for } [\tau_1]_0 and [\tau_0]_1, \quad I_{[\tau_1]_0;t,t+h} + I_{[\tau_0]_1;t,t+h} = \Phi_S(\tau_1]_0; t, t+h) + \Phi_S(\tau_0]_1; t, t+h)\\
&\qquad\qquad \Leftrightarrow \quad J_{10} + J_{01} = hJ_1\alpha^T Be + hJ_1\beta^T Ae \quad \Leftrightarrow \quad \alpha^T Be + \beta^T Ae = 1;\\
&\text{for } [\tau_1]_1, \quad I_{[\tau_1]_1;t,t+h} = \Phi_S([\tau_1]_1; t, t+h) \quad \Leftrightarrow \quad J_{11} = J_1^2\beta^T Be \quad \Leftrightarrow \quad \beta^T Be = \frac{1}{2};\\
&\text{for } [\tau_1, \tau_1]_1, \quad I_{[\tau_1,\tau_1]_1;t,t+h} = \Phi_S([\tau_1, \tau_1]_1; t, t+h) \quad \Leftrightarrow \quad 2J_{111} = J_1^3\beta^T (Be)^2\\
&\qquad\qquad \Leftrightarrow \quad \beta^T (Be)^2 = \frac{1}{3};\\
&\text{for } [[\tau_1]_1]_1, \quad I_{[[\tau_1]_1]_1;t,t+h} = \Phi_S([[\tau_1]_1]_1; t, t+h) \quad \Leftrightarrow J_{111} = J_1^3\beta^T (B(Be))\\
&\qquad\qquad \Leftrightarrow \quad \beta^T (B(Be)) = \frac{1}{6}.
\end{aligned}$$

Next, we prove that for arbitrary $\mathbf{t}$ with $\rho(\mathbf{t}) = 2.0$,

$$E(I_{\mathbf{t};t,t+h}) = E(\Phi_S(\mathbf{t}; t, t+h))$$

holds. In fact, by (8)-(12), (15) and (37) as well as Table 1, we obtain that

$$\text{for } [\tau_1,\tau_1]_0,\quad E[I_{[\tau_1,\tau_1]_0;t,t+h}] = E[\Phi_S([\tau_1,\tau_1]_0;t,t+h)] \;\Leftrightarrow\; E[2J_{110}] = E[hJ_1^2\alpha^T(Be)^2]$$
$$\Leftrightarrow\; \alpha^T(Be)^2 = \frac{1}{2};$$
$$\text{for } [[\tau_1]_1]_0,\quad E[I_{[[\tau_1]_1]_0;t,t+h}] = E[\Phi_S([[\tau_1]_1]_0;t,t+h)] \;\Leftrightarrow\; E[J_{110}] = E[hJ_1^2\alpha^T(B(Be))]$$
$$\Leftrightarrow\; \alpha^T(B(Be)) = \frac{1}{4};$$
$$\text{for } [[\tau_1]_0]_1,\quad E[I_{[[\tau_1]_0]_1;t,t+h}] = E[\Phi_S([[\tau_1]_0]_1;t,t+h)] \;\Leftrightarrow\; E[J_{101}] = E[hJ_1^2\beta^T(A(Be))]$$
$$\Leftrightarrow\; \beta^T(A(Be)) = 0;$$
$$\text{for } [\tau_0,\tau_1]_1,\quad E[I_{[\tau_0,\tau_1]_1;t,t+h}] = E[\Phi_S([\tau_0,\tau_1]_1;t,t+h)]$$
$$\Leftrightarrow\; E[J_{101}+J_{011}] = E[hJ_1^2\beta^T((Ae)(Be))] \;\Leftrightarrow\; \beta^T((Ae)(Be)) = \frac{1}{4};$$
$$\text{for } [[\tau_0]_1]_1,\quad E[I_{[[\tau_0]_1]_1;t,t+h}] = E[\Phi_S([[\tau_0]_1]_1;t,t+h)] \;\Leftrightarrow\; E[J_{011}]=E[hJ_1^2\beta^T(B(Ae)))]$$
$$\Leftrightarrow\; \beta^T(B(Ae))) = \frac{1}{4};$$
$$\text{for } [\tau_1,\tau_1,\tau_1]_1,\quad E[I_{[\tau_1,\tau_1,\tau_1]_1;t,t+h}] = E[\Phi_S([\tau_1,\tau_1,\tau_1]_1;t,t+h)]$$
$$\Leftrightarrow\; E[6J_{1111}] = E[J_1^4\beta^T(Be)^3] \;\Leftrightarrow\; \beta^T(Be)^3 = \frac{1}{4};$$
$$\text{for } [[\tau_1]_1,\tau_1]_1,\quad E[I_{[[\tau_1]_1,\tau_1]_1;t,t+h}] = E[\Phi_S([[\tau_1]_1,\tau_1]_1;t,t+h)]$$
$$\Leftrightarrow\; E[3J_{1111}] = E[J_1^4\beta^T((B(Be))(Be))]$$
$$\Leftrightarrow\; \beta^T((B(Be))(Be)) = \frac{1}{8};$$
$$\text{for } [[\tau_1,\tau_1]_1]_1,\quad E[I_{[[\tau_1,\tau_1]_1]_1;t,t+h}] = E[\Phi_S([[\tau_1,\tau_1]_1]_1;t,t+h)]$$
$$\Leftrightarrow\; E[2J_{1111}] = E[J_1^4\beta^T(B(Be)^2)] \;\Leftrightarrow\; \beta^T(B(Be)^2) = \frac{1}{12};$$
$$\text{for } [[[\tau_1]_1]_1]_1,\quad E[I_{[[[\tau_1]_1]_1]_1;t,t+h}] = E[\Phi_S([[[\tau_1]_1]_1]_1;t,t+h)]$$
$$\Leftrightarrow\; E[J_{1111}] = E[J_1^4\beta^T(B(B(Be)))] \;\Leftrightarrow\; \beta^T(B(B(Be))) = \frac{1}{24}.$$

The proof is complete. □

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