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A modified positive-definite and skew-Hermitian splitting preconditioner for generalized saddle point problems from the Navier-Stokes equation

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Abstract In this paper, we extend the relaxed positive-definite and skew-Hermitian splitting preconditioner (RPSS) for generalized saddle-point problems in [J.-L. Zhang, C.-Q. Gu and K. Zhang, Appl. Math. Comput. 249(2014)468-479] by introducing an additional parameter. The spectral properties of the presented new preconditioned matrix for generalized saddle-point problem are investigated, meanwhile, the infinite termination merit of the iterative step is also discussed if the Krylov subspace method preconditioned by the modified positive-definite and skew-Hermitian splitting preconditioner (MPSS) is applied. Some numerical experiments illustrate that the efficiency of the proposed new preconditioner.

Keywords Saddle-point problems · MPSS preconditioner · Spectral properties · Krylov subspace · numerical test

Mathematics Subject Classifications (2010) 65F10 · 65F30 · 65F50

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1 Introduction

We are interested in finding the solution of large spare non-Hermitian saddle point problems with the following two-by-two block form:

$$\begin{bmatrix} A & B^* \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian matrix and its Hermitian part $H = \frac{1}{2}(A + A^*)$ is positive definite, $B \in \mathbb{C}^{m\times n}$ with $rank(B) = m$, that is, a matrix of full column rank, $C \in \mathbb{C}^{m\times m}$ is a Hermitian positive definite matrix, $x,\ f \in \mathbb{C}^n$ and $y,\ g \in \mathbb{C}^m$ are given vectors with $n \geq m$. The assumptions guarantee the existence and uniqueness of the solution of the system (1.1).

The above linear system (1.1) is the so called generalized saddle point problems which frequently arise in various scientific and engineering applications such as constrained optimization, fluid problems for incompressible fluids, incompressible elastic materials, mixed finite element of elliptic PDEs, constrained least squares problem, for general discussions, see [1–7]. For the classical example occurring in elasticity, see [8], for the occurrence of locking , see [9, 10]. In general, matrices A and B are large and sparse, the different iterative methods instead of direct methods are considered for solving the saddle problem (1.1). So, it is very vital to investigate various efficient iterative approaches [11–13]. When the matrix A is Hermitian positive definite and B is of full column rank, several efficient iterative methods have been presented in the recent years. For instance, Uzawa method [14, 15], the preconditioned Uzawa approach [16], SOR-like method [17], modified SSOR method[18], GSOR method [19], HSS method [3, 20], accelerated HSS method [21], and so forth. These approaches are stationary iterative methods, in general, which require much less computer memory than the Krylov subspace methods. But the preconditioned Krylov subspace methods are often very competitive than other methods, which leads to various efficient preconditioners proposed for solving the saddle point problems by many researches in recent works. As known, the favorable performance of convergence is related to a clustering of most of the eigenvalues around 1 and away from zero. So, a good preconditioner must be given as close as possible to the coefficient matrix. The preconditioners for saddle problems mainly can be considered in these cases, such as, block triangular preconditioners[2, 22], block diagonal preconditioners[23–25], fully factorized two-by-two block matrix precoditioners[26, 27], etc., see [28–40] for more detailed investigations.

The linear system (1.1) can be rewritten as the simple form

$$\mathcal{A}u = b, \tag{1.2}$$

where

$$\mathcal{A} = \begin{bmatrix} A & B^* \\ -B & C \end{bmatrix}, \quad u = \begin{bmatrix} x \\ y \end{bmatrix}, \quad b = \begin{bmatrix} f \\ -g \end{bmatrix}.$$

The above linear system (1.2) can be regarded as a classical saddle point problem when $C = 0$. Cao et al., in [41], proposed the following shift-splitting preconditioner for solving the classical saddle point problem:

$$\widehat{\mathcal{P}} = \frac{1}{2}(\alpha I + \mathcal{A}) = \frac{1}{2}\begin{bmatrix} \alpha I + A & B^* \\ -B & \alpha I \end{bmatrix}. \tag{1.3}$$

Moreover, Chen and Ma, in [42], presented a generalized shift-splitting (GSS) preconditioner

$$\mathcal{P}_{GSS} = \frac{1}{2}\begin{bmatrix} \alpha I + A & B^* \\ -B & \beta I \end{bmatrix} \tag{1.4}$$

with two parameters α and β. From the numerical examples in [42], we find the eigenvalues distribution of the preconditioned matrix $\mathcal{P}_{GSS}^{-1}\mathcal{A}$ gather more closely than those in [41], also the convergence performance is better.

To solve the saddle point problem (1.2) when $C = 0$, recently, Zhang et al. in [43], obtained a relaxed positive-definite and skew-Hermitian splitting (RPSS) preconditioner based on [5] proposed by Pan et al.. The RPSS preconditioner has the following form

$$\mathcal{P}_{RPSS} = \begin{bmatrix} A & \left(I + \frac{1}{\alpha}A\right)B^* \\ -B & \alpha I \end{bmatrix}. \tag{1.5}$$

The basic ideas are derived from the HSS method given by Bai et al. [3] and the ADI method introduced by Benner [44].

Inspired by [5, 43], in this paper, we construct an new preconditioner for solving the generalized saddle point problem (1.1) when A is non-Hermitian with a positive definite Hermitian part, which is refereed to the modified positive-definite and skew-Hermitian splitting (MPSS) preconditioner.

The remainder of the paper is organized as follows. In Section 2, we first consider a new splitting method and obtain a new preconditioner for the generalized saddle point problem (1.1). In Section 3, we analyze the spectral properties of the preconditioner and apply it to Krylov subspace methods in detailed. Some numerical experiments are given to illustrate that the new preconditioner is efficient in Section 4. At last, we end the paper with some conclusions in Section 5.

2 The MPSS preconditioner

As is known that the coefficient matrix $\mathcal{A}$ has the following splitting

$$\mathcal{A} = \mathcal{M} + \mathcal{N},$$

where

$$\mathcal{M} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} 0 & B^* \\ -B & 0 \end{bmatrix}, \tag{2.1}$$

meanwhile

$$\mathcal{A} = (\alpha I + \mathcal{M}) - (\alpha I - \mathcal{N}) = (\alpha I + \mathcal{N}) - (\alpha I - \mathcal{M}), \tag{2.2}$$

where I denotes the identity matrix with the appropriate dimension, and the shift parameter $\alpha > 0$. Analogously to the classical ADI method [44], we consider the following splitting

$$\mathcal{A} = (\alpha I + \mathcal{M}) - (\alpha I - \mathcal{N}) = (\beta I + \mathcal{N}) - (\beta I - \mathcal{M}),$$

where α and β are positive scalars. So the (1.2) can be written as

$$\begin{cases} (\alpha I + \mathcal{M})u^{k+\frac{1}{2}} = (\alpha I - \mathcal{N})u^k + b, \\ (\beta I + \mathcal{N})u^{k+1} = (\beta I - \mathcal{M})u^{k+\frac{1}{2}} + b, \end{cases}$$

where u^0 is the initial vector. By eliminating the intermediate vector $u^{k+\frac{1}{2}}$, one can obtain

$$u^{k+1} = \Gamma u^k + c, \tag{2.3}$$

where

$$\Gamma = (\beta I + \mathcal{N})^{-1}(\beta I - \mathcal{M})(\alpha I + \mathcal{M})^{-1}(\alpha I - \mathcal{N}), \tag{2.4}$$
$$c = (\alpha + \beta)\big(\beta I + \mathcal{N}\big)^{-1}(\alpha I + \mathcal{M})^{-1}b.$$

It is well known that the preconditioner $\mathcal{P}$ can be chosen by the splitting $\mathcal{A} = \mathcal{P} - \mathcal{Q}$ with a reversible matrix $\mathcal{P}$. From the (2.4), by the simple computation, we have

$$\mathcal{P} = \frac{1}{\alpha + \beta}(\alpha I + \mathcal{M})(\beta I + \mathcal{N}) \tag{2.5}$$

and

$$\mathcal{Q} = \frac{1}{\alpha + \beta}(\beta I - \mathcal{M})(\alpha I - \mathcal{N}).$$

Notice that the relation (2.1), we get

$$\mathcal{P} = \frac{1}{\alpha + \beta}\begin{bmatrix} \alpha I + A & 0 \\ 0 & \alpha I + C \end{bmatrix}\begin{bmatrix} \beta I & B^* \\ -B & \beta I \end{bmatrix} \tag{2.6}$$
$$= \frac{1}{\alpha + \beta}\begin{bmatrix} \alpha\beta I + \beta A & \alpha B^* + AB^* \\ -\alpha B - CB & \alpha\beta I + \beta C \end{bmatrix}. \tag{2.7}$$

Since the factor $\frac{1}{\alpha+\beta}$ has no effect on the preconditioned system

$$\mathcal{P}^{-1}\mathcal{A}u = c,$$

then, we can modify the parameter $\frac{1}{\alpha+\beta}$ to make the preconditioner could be as close as possible to the coefficient matrix $\mathcal{A}$. To this end, we firstly consider

$$\mathcal{P} = \frac{1}{\alpha}\begin{bmatrix} \alpha I + A & 0 \\ 0 & \alpha I + C \end{bmatrix}\begin{bmatrix} \beta I & B^* \\ -B & \beta I \end{bmatrix} = \begin{bmatrix} \beta I + \frac{\beta}{\alpha}A & B^* + \frac{1}{\alpha}AB^* \\ -B - \frac{1}{\alpha}CB & \beta I + \frac{\beta}{\alpha}C \end{bmatrix}, \tag{2.8}$$

moreover, the (1, 1)-block of (2.8) is relaxed as A, the (2, 1)-block of (2.8) is relaxed as $-B$ and the (2, 2)-block of (2.8) is relaxed as $\beta I + C$, it follows that

$$\mathcal{P}_{MPSS} = \begin{bmatrix} A & B^* + \frac{1}{\alpha}AB^* \\ -B & \beta I + C \end{bmatrix}, \tag{2.9}$$

then

$$\mathcal{Q} = \mathcal{P}_{MPSS} - \mathcal{A} = \begin{bmatrix} 0 & \frac{1}{\alpha}AB^* \\ 0 & \beta I \end{bmatrix}. \tag{2.10}$$

In fact, we find that the new preconditioner, the so-calling MPSS preconditioner, can be regarded as a modified version of the PRSS proposed by Zhang et al. in [43]. If set $\beta = \alpha$, then the MPSS is reduced to PRSS, so the MPSS is a generalized situation for PSSS in (1.5). When α approaches 0_+ (or $+\infty$), it will lead to the (2, 2)-block (or (1, 2)-block) of the precontitioner close to $\mathcal{A}$, but the other block will away from $\mathcal{A}$ as the relation of α and $\frac{1}{\alpha}$, so the choice of the optimal parameter α for (1.5) is very difficult just as the statement in [43]. Conversely, the MPSS precontitioner can overcome these defects due to the independence of the parameters α and β, this advantage will be shown in our numerical experiments part.

3 The spectral properties

In view of the eigenvalues distribution is closely related to convergence speed, we will analyze the eigenvalue problem associated with the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ (for convenience, denotes $\mathcal{P}$ instead of $\mathcal{P}_{MPSS}$ from now on).

Now consider the eigenvalue problem $\mathcal{P}^{-1}\mathcal{A}u = \lambda u$, i.e. $\mathcal{A}u = \lambda\mathcal{P}u$, then we get

$$\begin{bmatrix} A & B^* \\ -B & C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} A & B^* + \frac{1}{\alpha}AB^* \\ -B & \beta I + C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \tag{3.1}$$

In fact, $\mathcal{P}$ also can be factorized as

$$\mathcal{P} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & \left(A^{-1} + \frac{1}{\alpha}I\right)B^* \\ -B & \beta I + C \end{bmatrix} := \mathcal{P}_1\mathcal{P}_2, \tag{3.2}$$

where

$$\mathcal{P}_1 = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{P}_2 = \begin{bmatrix} I & \left(A^{-1} + \frac{1}{\alpha}I\right)B^* \\ -B & \beta I \end{bmatrix}. \tag{3.3}$$

Since

$$\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}\mathcal{P}_2\begin{bmatrix} I & -\left(A^{-1} + \frac{1}{\alpha}I\right)B^* \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & L \end{bmatrix}, \tag{3.4}$$

where

$$L := B\left(A^{-1} + \frac{1}{\alpha}I\right)B^* + \beta I + C, \tag{3.5}$$

we obtain

$$\mathcal{P}^{-1} = \mathcal{P}_2^{-1}\mathcal{P}_1^{-1} = \begin{bmatrix} I & -\left(A^{-1}+\frac{1}{\alpha}I\right)B^* \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & L^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}\begin{bmatrix} A^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{3.6}$$

$$= \begin{bmatrix} A^{-1}-\left(A^{-1}+\frac{1}{\alpha}I\right)B^*L^{-1}BA^{-1} & -\left(A^{-1}+\frac{1}{\alpha}I\right)B^*L^{-1} \\ L^{-1}BA^{-1} & L^{-1} \end{bmatrix}.$$

By utilizing (3.6), we show Algorithm 3.1 to compute the vector $z = (z_1^T, z_2^T)^T$, where $z_1 \in \mathbb{C}^n$, $z_2 \in \mathbb{C}^m$ as following.

Algorithm 3.1 *Computation of $z = \mathcal{P}^{-1}r$.*

1. *Solve $Aw_1 = r_1$ for w_1 and $Aq_i = B^*(:;i)$ for q_i, $i = 1, 2, \cdots, m$. $Q := (q_1, q_2, \cdots, q_m)$, where $B^*(:;i)$ denotes i_{th} column of matrix B^*.*
2. *Solve $\left(BQ + \frac{1}{\alpha}BB^* + \beta I + C\right)w_2 = Bw_1 + r_2$ for w_2, and $z_2 = w_2$.*
3. *Solve $Aw_3 = B^*w_2$ for w_3.*
4. *Compute $z_1 = w_1 - w_3 - \frac{1}{\alpha}B^*w_2$.*

Remark 3.1 The parameters α and β can be chosen with $\alpha \to \infty$ and $\beta \to 0$ from the above Algorithm 3.1. In this way, the preconditioner $\mathcal{P}$ will be sufficiently close to the coefficient matrix $\mathcal{A}$. Hence when the preconditioner matrix is applied to the Krylov subspace method, such as restarted GMRES method, we expect to obtain the rapid convergence of Algorithm 3.1.

Theorem 3.1 *The algebraic multiplicity of the unit eigenvalues of the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ has at least n. The remaining eigenvalues μ_i are determined by the matrix $L^{-1}BA^{-1}B^*$ and*

$$\mu_i = \frac{\alpha a + \alpha b}{\alpha a + d + \alpha\beta + \alpha b},$$

where

$$a := y^*B^*A^{-1}By, \quad b := y^*Cy, \quad d := y^*B^*By.$$

Furthermore, the eigenvalues μ_i are of the form

$$\mu_i = \frac{\sigma_i}{1+\sigma_i},$$

where the σ_i satisfy the generalized eigenvalue problem

$$BA^{-1}B^*z_i = \sigma_i\left(\beta I + \frac{1}{\alpha}BB^* + C\right)z_i.$$

Proof From the above (3.6), it has

$$\begin{aligned}
\mathcal{P}^{-1}\mathcal{A} &= \mathcal{P}^{-1}(\mathcal{P}-\mathcal{Q}) = I - \mathcal{P}^{-1}\mathcal{Q} \qquad (3.7)\\
&= I - \begin{bmatrix} A^{-1} - \left(A^{-1}+\frac{1}{\alpha}I\right)B^*L^{-1}BA^{-1} & -\left(A^{-1}+\frac{1}{\alpha}I\right)B^*L^{-1} \\ L^{-1}BA^{-1} & L^{-1}\end{bmatrix}\begin{bmatrix} 0 & \frac{1}{\alpha}AB^* \\ 0 & \beta I\end{bmatrix}\\
&= \begin{bmatrix} I & -\frac{1}{\alpha}B^* + \frac{1}{\alpha}\left(A^{-1}+\frac{1}{\alpha}I\right)B^*L^{-1}BB^* + \beta\left(A^{-1}+\frac{1}{\alpha}I\right)B^*L^{-1} \\ 0 & I - \frac{1}{\alpha}L^{-1}BB^* - \beta L^{-1}\end{bmatrix}\\
&:= \begin{bmatrix} I & K_1 \\ 0 & K_2\end{bmatrix},
\end{aligned}$$

it follows from (3.5) that

$$K_2 = L^{-1}BA^{-1}B^*. \qquad (3.8)$$

Multiplying both side of (3.1) from left with

$$\begin{bmatrix} I & 0 \\ BA^{-1} & I\end{bmatrix},$$

we have

$$\begin{bmatrix} A & B^* \\ 0 & BA^{-1}B^* + C\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = \lambda\begin{bmatrix} A & B^* + \frac{1}{\alpha}AB^* \\ 0 & BA^{-1}B^* + \frac{1}{\alpha}BB^* + \beta I + C\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix}, \qquad (3.9)$$

which implies that

$$\begin{cases} Ax + B^*y = \lambda Ax + \lambda\left(I + \frac{1}{\alpha}A\right)B^*y, \\ (BA^{-1}B^* + C)y = \lambda BA^{-1}B^*y + \frac{\lambda}{\alpha}BB^*y + \lambda\beta y + \lambda Cy. \end{cases} \qquad (3.10)$$

If $y = 0$, then from the first formula of (3.10) one get $Ax = \lambda Ax$, hence $\lambda = 1$. We now suppose that $y \neq 0$, without loss of generality, we suppose $\|y\|_2 = 1$. Multiplying both side of (3.10) from left with y^*, we obtain

$$\begin{aligned}
\lambda &= \frac{y^*BA^{-1}B^*y + y^*Cy}{y^*BA^{-1}B^*y + \frac{1}{\alpha}y^*BB^*y + \beta + y^*Cy}, \qquad (3.11)\\
&= \frac{\alpha a + \alpha b}{\alpha a + d + \alpha\beta + \alpha b}. \qquad (3.12)
\end{aligned}$$

Note that the Hermitian part of A is positive definite and C is a Hermitian positive definite matrix, it leads to

$$a > 0, \quad b > 0, \quad d > 0,$$

Table 1 Numerical results for MPSS

p	IT	CPU	RES
16	1(2)	$8.4921e-002$	$1.6983e-012$
24	1(2)	$5.2938e-001$	$2.1800e-012$
32	1(2)	$2.8168e+000$	$3.0896e-012$
40	1(2)	$8.8542e+001$	$4.1780e-012$

Table 2 Numerical results for Example 4.1 with ν=0.1

p		GMRES	RPSS	GSS	MPSS
16	IT	13(17)	1(3)	1(3)	1(2)
	CPU	$3.2643e-001$	$9.9150e-002$	$1.0114e-001$	$6.9734e-002$
	RES	$9.8892e-007$	$1.0854e-007$	$1.1711e-007$	$2.4161e-011$
24	IT	25(8)	1(3)	1(3)	1(2)
	CPU	$2.5810e+000$	$8.2307e-001$	$7.9690e-001$	$5.9164e-001$
	RES	$9.6872e-007$	$9.2053e-008$	$9.8568e-008$	$5.2263e-011$
32	IT	41(1)	1(3)	1(3)	1(2)
	CPU	$1.2089e+001$	$3.3295e+000$	$3.2791e+000$	$2.6445e+000$
	RES	$9.9907e-007$	$7.6883e-008$	$8.2414e-008$	$9.3254e-011$
40	IT	59(13)	1(3)	1(3)	1(2)
	CPU	$6.6671e+001$	$2.5422e+001$	$2.4465e+001$	$9.0135e+000$
	RES	$9.8916e-007$	$6.4781e-008$	$6.9934e-008$	$1.4495e-010$

so one can get in this case

$$0 < \lambda < 1.$$

To sum up

$$\lambda = 1 \;\; or \;\; \lambda = \frac{\alpha a + \alpha b}{\alpha a + d + \alpha\beta + \alpha b}.$$

Moreover, we let the non-unit eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ satisfy the eigenvalue problem

$$L^{-1}BA^{-1}B^{*}z_i = \mu_i z_i,$$

Table 3 Numerical results for Example 4.1 with ν=0.01

p		GMRES	RPSS	GSS	MPSS
16	IT	48(8)	1(3)	1(3)	1(2)
	CPU	$9.9440e-001$	$9.1679e-002$	$1.0144e-001$	$7.9235e-002$
	RES	$9.9697e-007$	$1.8310e-009$	$2.9710e-008$	$2.0346e-012$
24	IT	76(15)	1(3)	1(3)	1(2)
	CPU	$9.5826e+000$	$8.1288e-001$	$8.3005e-001$	$6.0878e-001$
	RES	$9.9702e-007$	$1.8373e-009$	$2.6079e-008$	$3.1103e-012$
32	IT	90(18)	1(3)	1(3)	1(2)
	CPU	$3.9226e+001$	$4.8638e+000$	$3.9461e+000$	$2.8644e+000$
	RES	$9.9903e-007$	$1.7335e-009$	$2.3877e-008$	$5.5362e-012$
40	IT	103(16)	1(3)	1(3)	1(2)
	CPU	$7.3612e+001$	$1.1832e+001$	$1.2637e+001$	$8.8247e+000$
	RES	$9.9352e-007$	$1.6013e-009$	$2.2405e-008$	$9.2630e-012$

i.e.

$$K_2 z_i = \mu_i z_i,$$

then

$$BA^{-1}B^* z_i = \mu_i L z_i = \mu_i \left(\beta I + \frac{1}{\alpha} BB^* + BA^{-1}B^* + C \right) z_i,$$

by transposition leads to

$$BA^{-1}B^* z_i = \frac{\mu_i}{1-\mu_i} \left(\beta I + \frac{1}{\alpha} BB^* + C \right) z_i,$$

we denote

$$\sigma_i := \frac{\mu_i}{1-\mu_i}, \ then \ \ \mu_i = \frac{\sigma_i}{1+\sigma_i}. \tag{3.13}$$

□

From the above relation of (2.9), (3.6) and (3.7), we know that iterative matrix

$$\Gamma = \mathcal{P}^{-1}\mathcal{Q} = \begin{bmatrix} 0 & -K_1 \\ 0 & I - K_2 \end{bmatrix},$$

evidently, the $\rho(\Gamma) < 1$ as (3.8) and (3.13), this means to the unconditional convergence of the iterative method.

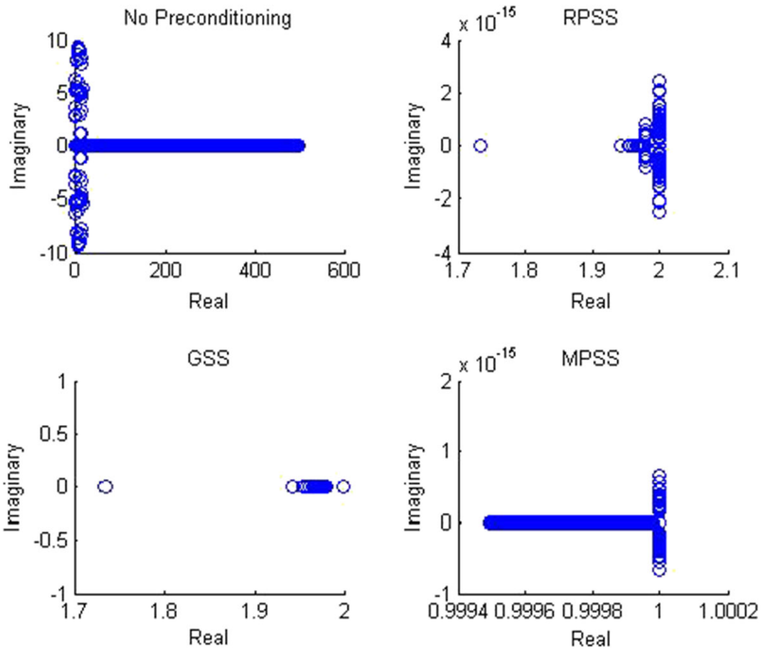

Fig. 1 The eigenvalues distribution of the preconditioned matrix for Example 4.1 with ν=0.1

Table 4 Numerical results for Example 4.2 with ν=1 (Q2–Q1)

Grids		GMRES	RPSS	GSS	MPSS
8×8	IT	24(1)	1(20)	2(5)	1(4)
	CPU	$7.5478e-002$	$2.1054e-002$	$2.5908e-002$	$3.9638e-003$
	RES	$9.9222e-007$	$8.0022e-007$	$7.3844e-007$	$1.2012e-008$
16×16	IT	111(17)	2(14)	3(3)	1(4)
	CPU	$1.6366e+000$	$5.9999e-001$	$7.2946e-001$	$7.0112e-002$
	RES	$9.9883e-007$	$5.9626e-007$	$7.2799e-007$	$3.9153e-007$

Theorem 3.2 *Let the MPSS preconditioner be defined in (2.9). Then the degree of minimal polynomial of preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ is at most $m+1$. Moreover, the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $m+1$.*

Proof From [11], the dimension of the Krylov subspace is closely related with the degree of the minimal polynomial. By the $\mathcal{P}^{-1}\mathcal{A}$ in (3.7) and the eigenvalue distribution described in Theorem 3.1, the characteristic polynomial of the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ is

$$(\mathcal{P}^{-1}\mathcal{A} - I)^n \Pi_{i=1}^m (\mathcal{P}^{-1}\mathcal{A} - \mu_i I).$$

As μ_i are the eigenvalues of the matrix K_2 $(i = 1, 2, \cdots, m)$, by the Hamilton-Cayley theorem, we obtain

$$\Pi_{i=1}^m (K_2 - \mu_i I) = 0.$$

So the polynomial

$$\begin{aligned}
&P_{m+1}(\mathcal{P}^{-1}\mathcal{A}) \\
&= (\mathcal{P}^{-1}\mathcal{A} - I)\Pi_{i=1}^m (\mathcal{P}^{-1}\mathcal{A} - \mu_i I) \\
&= \begin{bmatrix} 0 & K_1 \Pi_{i=1}^m (K_2 - \mu_i I) \\ 0 & (K_2 - I)\Pi_{i=1}^m (K_2 - \mu_i I) \end{bmatrix} = 0,
\end{aligned}$$

Table 5 Numerical results for Example 4.2 with ν=1 (Q2–P1)

Grids		GMRES	RPSS	GSS	MPSS
8×8	IT	10(1)	1(11)	1(13)	1(3)
	CPU	$3.0841e-002$	$1.0769e-002$	$1.2583e-002$	$3.4997e-003$
	RES	$9.8244e-007$	$8.7131e-007$	$2.5980e-007$	$1.4031e-009$
16×16	IT	23(17)	1(17)	2(2)	1(3)
	CPU	$4.7843e-001$	$4.0845e-001$	$5.4501e-001$	$7.3779e-002$
	RES	$9.8412e-007$	$7.8812e-007$	$6.1722e-007$	$2.1638e-007$

that is, the degree of the minimal polynomial of $\mathcal{P}^{-1}\mathcal{A}$ is at most $m+1$. Therefore, the dimension of the corresponding Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $m+1$ [11]. □

4 Numerical experiments

In this section, we report some numerical results to illustrate the effectiveness of the MPSS preconditioner for solving the generalized saddle point problem arising from a model Stokes equation in the sense of iteration step (denoted as 'IT'), elapsed CPU time in seconds (denoted as 'CPU'), and relative residual error (denoted as 'RES') defined by

$$\text{RES} := \frac{\sqrt{\|f - Ax^k - B^*y^k\|_2^2 + \|g - Bx^k\|_2^2}}{\sqrt{\|f\|_2^2 + \|g\|_2^2}}.$$

The MPSS preconditioner is applied to restarted GMRES method with restarting frequency 20, i.e., GMRES(20). The experiments have been carried out by MATLAB R2011b (7.13), Intel(R) Core(TM) i7-2670QM, CPU 2.20GHZ, RAM 8.GB PC Environment. Especially, when $C = 0$, i.e., the 2-by-2 block of the coefficient of (1.1) is zero, we compare the MPSS preconditioner with the RPSS in [43], GSS in[42] and no preconditioning situation.

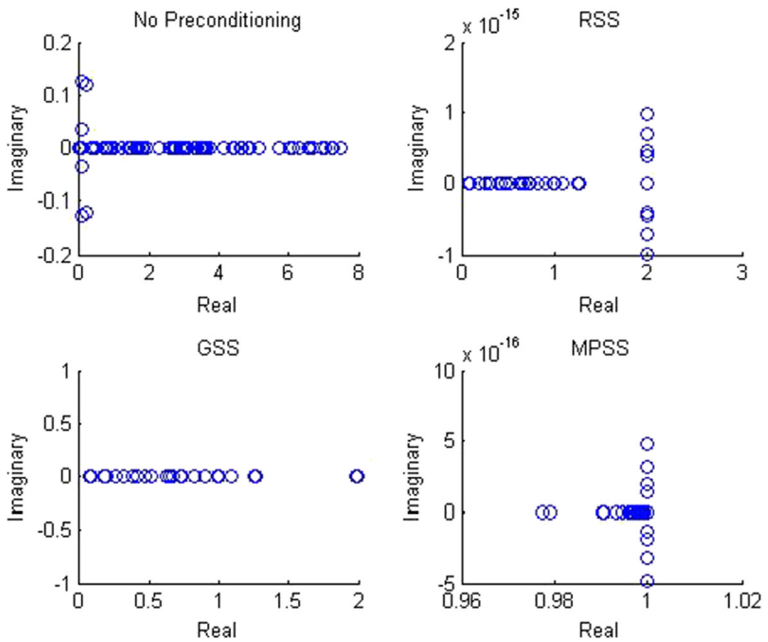

Fig. 2 The eigenvalues distribution of the preconditioned matrix for Example 4.2 with 8×8 grids (Q2–Q1)

Example 4.1 Nonsingular saddle point problem arising from a model Stokes equation [4], the coefficient matrix have the following form

$$A=\begin{bmatrix} I\otimes T+T\otimes I & 0 \\ 0 & I\otimes T+T\otimes I \end{bmatrix}\in R^{2p^2\times 2p^2},\quad B^T=\begin{bmatrix} I\otimes F \\ F\otimes I \end{bmatrix}\in R^{2p^2\times p^2},$$

where

$$T=\frac{\upsilon}{h^2}tridiag(-1,2,-1)\in R^{p\times p},\quad F=\frac{1}{h}tridiag(-1,-1,0)\in R^{p\times p},$$

with $\otimes$ being the Kronecker product and $h=\frac{1}{1+p}$ being the discretization mesh size.

In order to test the generalized saddle point problem (1.1), we choose the $C=A(1:p^2,1:p^2)$, the numerical results are listed in Table 1.

In fact, if we set $C=0$ in (1.1), we can compare MPSS preconditioner with RPSS , GSS and no preconditioning situation. In the following test problems, we only consider $C=0$. Now, we test two υ, i.e. υ=0.1, 0.01. For each υ, four different q are chosen, i.e., q=16, 24, 32, 40, the relative parameters $\alpha,\ \beta\in[0.0001,1]$. The numerical results are listed in Tables 2 and 3 with the IT, CPU, RES, where IT including inner iteration and outer iteration, particularly, the numbers in the brackets denote the inner iteration numbers. Figure 1 shows the eigenvalues of the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ of MPSS are clustered more closely than those of other three preconditioned matrices (no preconditioning situation can be regard as an identity

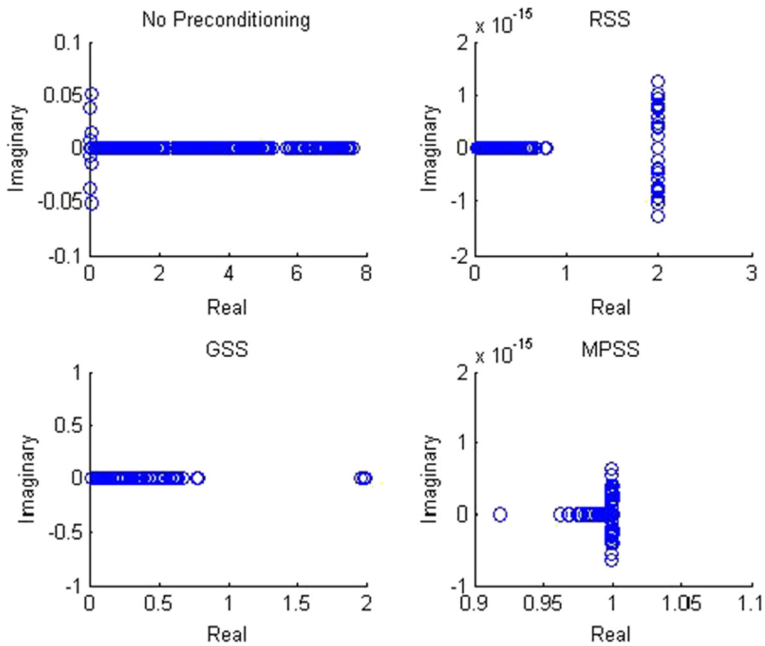

Fig. 3 The eigenvalues distribution of the preconditioned matrix for Example 4.2 with 16 × 16 grids (Q2–Q1)

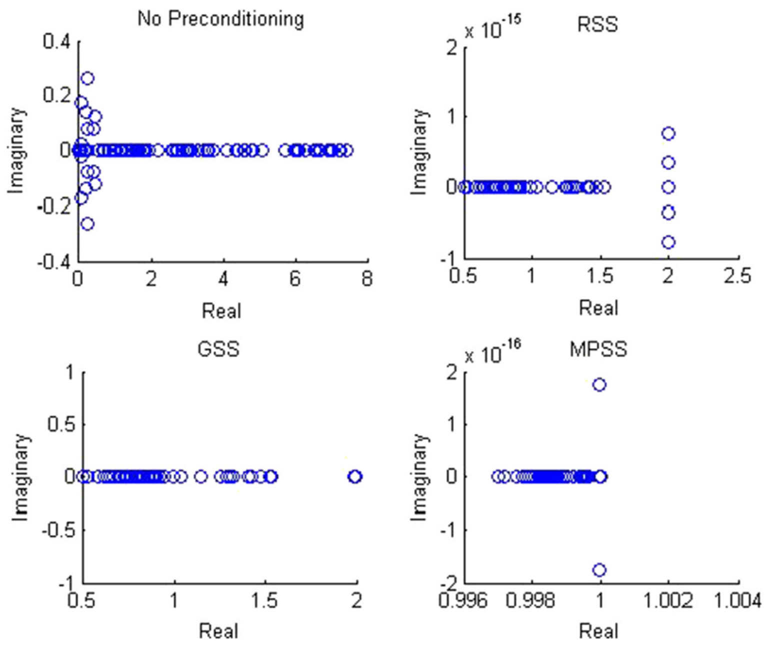

Fig. 4 The eigenvalues distribution of the preconditioned matrix for Example 4.2 with 8×8 grids (Q2–P1)

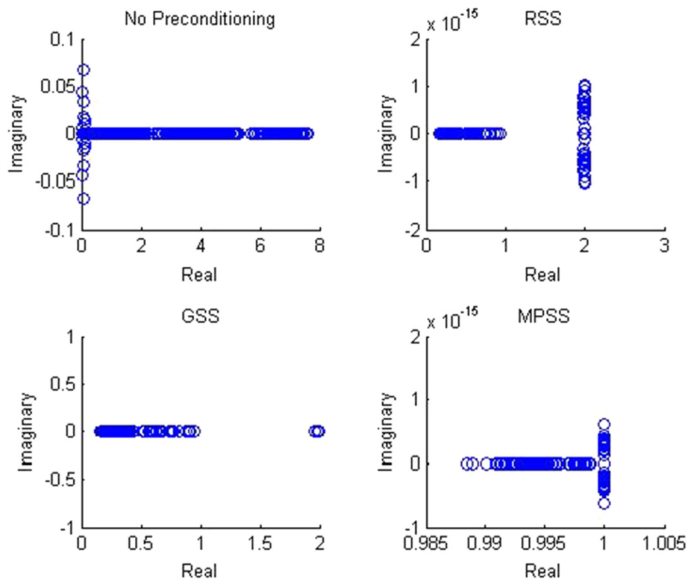

Fig. 5 The eigenvalues distribution of the preconditioned matrix for Example 4.2 with 16×16 grids (Q2–P1)

preconditioner). We can find that the eigenvalues of the MPSS are almost located in the interval (0.994, 1), so the convergence performance is very ideal.

Example 4.2 We test the problem which is obtained from the linearization of the steady-state Navier-Stokes equation with suitable boundary condition on $\partial\Omega$:

$$\begin{cases} -\nu\Delta u + w \cdot \nabla u + \nabla p = f, \\ \nabla \cdot u = 0, \quad \text{in } \Omega, \end{cases}$$

where $\nu > 0$, Δ, u, p, denote viscosity, Laplace operator, velocity and pressure of fluid, respectively.

The test grid generation in channel domain which is discretized with Q2-Q1 and Q2-P1 finite elements, respectively. the IFISS software package [45] is used in the example, the relative parameters $\alpha,\ \beta \in [0.0001, 1]$. From the Tables 4 and 5, and Figs. 2, 3, 4 and 5, we can furthermore show that the MPSS preconditioner is more favorable than the other three preconditioners.

5 Conclusion

In this paper, we have furthermore modified the PRSS preconditioner by introducing an additional parameter β and presented a modified positive-definite and skew-Hermitian splitting (MPSS) preconditioner for solving the generalized saddle point problems. It is a type of generalization of the RPSS preconditioner, if we chose $\beta = \alpha$, the MPSS preconditioner is reduced to the RPSS preconditioner. It is readily seen that the selection of the proper parameters can lead to the new preconditioner matrix closer to coefficient matrix $\mathcal{A}$ than the PRSS preconditioner, then the eigenvalues distribution of the preconditioned matrix must be gathered more closely in some interval . The spectral properties of the new preconditioner are analyzed in detail. We apply the new preconditioner to Krylov subspace method (here, the restarted GMRES(m) is employed). The unconditioned convergence property of the MPSS iterative has been derived. Numerical experiments of the model linearized Navier-Stokes equations are implemented to demonstrate the effectiveness of the proposed preconditioner.

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