Numer Algor (2016) 71:655–671
DOI 10.1007/s11075-015-0015-5

ORIGINAL PAPER

On SSOR iteration method for a class of block two-by-two linear systems

Zhao-Zheng Liang[1] · **Guo-Feng Zhang**[1]

Received: 11 December 2014 / Accepted: 4 June 2015 / Published online: 27 June 2015

Abstract In this paper, the optimal iteration parameters of the symmetric successive overrelaxation (SSOR) method for a class of block two-by-two linear systems are obtained, which result in optimal convergence factor. An accelerated variant of the SSOR (ASSOR) method is presented, which significantly improves the convergence rate of the SSOR method. Furthermore, a more practical way to choose iteration parameters for the ASSOR method has also been proposed. Numerical experiments demonstrate the efficiency of the SSOR and ASSOR methods for solving a class of block two-by-two linear systems with the optimal parameters.

Keywords Block two-by-two matrices · Complex symmetric linear systems · Symmetric SOR method · Convergence · Optimal parameter

Mathematics Subject Classification (2010) 65F10 · 65F50

This work was supported by the National Natural Science Foundation of China (No. 11271174, 11511130015).

✉ Guo-Feng Zhang
gf_zhang@lzu.edu.cn

Zhao-Zheng Liang
liangzz11@lzu.edu.cn

1 School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, People's Republic of China

1 Introduction

Consider the following block two-by-two linear system

$$Ax \equiv \begin{pmatrix} W & -T \\ T & W \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \equiv b, \tag{1}$$

where the matrices $W, T \in \mathbb{R}^{n\times n}$ are symmetric with at least one of them, e.g., W being positive definite. For the complex symmetric linear system

$$\mathscr{A}\tilde{x} = \tilde{b}, \quad \mathscr{A} \in \mathbb{C}^{n\times n} \quad \text{and} \quad \tilde{x}, \tilde{b} \in \mathbb{C}^n, \tag{2}$$

where $\mathscr{A}$ is a complex symmetric matrix of the form $\mathscr{A} = W + iT$. Let $\tilde{x} = u + iv$ and $\tilde{b} = p + iq$ with $u, v, p, q \in \mathbb{R}^n$. Then the complex linear system (2) can equivalently be written to the block two-by-two linear system (1) [1]. The linear system (1) can be regarded as a special case of the generalized saddle point problem [13]. Many scientific and engineering applications can lead to the linear system (1), for example, wave propagation [28], distributed control problems [25], structural dynamics [19], FFT-based solution of certain time-dependent PDEs [15], molecular scattering [20], lattice quantum chromo dynamics [20] and so on. For more examples about the practical backgrounds of this class of problems, we refer to [3, 12, 13] and the references therein.

A large variety of effective iteration methods have been proposed in the literature for solving the linear system (1), such as C-to-R iteration methods [1, 2, 5, 12, 17], preconditioned Krylov subspace methods [14, 16, 26], splitting iteration methods [11, 14, 22, 23, 27, 31], Hermitian and skew-Hermitian splitting (HSS) method and its variants [4, 6–9, 18, 21, 24, 30].

Applying the splitting $A = D - L - U$ with

$$D = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ -T & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix},$$

we have the following SSOR procedure for the block two-by-two linear system (1):

$$\begin{cases} (D-\omega L)\begin{pmatrix} u^{\left(k+\frac{1}{2}\right)} \\ v^{\left(k+\frac{1}{2}\right)} \end{pmatrix} = ((1-\omega)D + \omega U)\begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} + \omega \begin{pmatrix} p \\ q \end{pmatrix}, \\ (D-\omega U)\begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix} = ((1-\omega)D + \omega L)\begin{pmatrix} u^{\left(k+\frac{1}{2}\right)} \\ v^{\left(k+\frac{1}{2}\right)} \end{pmatrix} + \omega \begin{pmatrix} p \\ q \end{pmatrix}, \end{cases} \tag{3}$$

where ω is a positive parameter. Then it holds that

$$\begin{cases} \begin{pmatrix} u^{\left(k+\frac{1}{2}\right)} \\ v^{\left(k+\frac{1}{2}\right)} \end{pmatrix} = \mathscr{L}_\omega \begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} + \omega(D-\omega L)^{-1} b, \\ \begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix} = \mathscr{U}_\omega \begin{pmatrix} u^{\left(k+\frac{1}{2}\right)} \\ v^{\left(k+\frac{1}{2}\right)} \end{pmatrix} + \omega(D-\omega U)^{-1} b, \end{cases} \tag{4}$$

where

$$\begin{aligned}\mathscr{L}_\omega &= (D-\omega L)^{-1}((1-\omega)D+\omega U)\\ &= \begin{pmatrix} (1-\omega)I_n & \omega W^{-1}T \\ -\omega(1-\omega)W^{-1}T & (1-\omega)I_n-\omega^2 W^{-1}TW^{-1}T\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathscr{U}_\omega &= (D-\omega U)^{-1}((1-\omega)D+\omega L)\\ &= \begin{pmatrix} (1-\omega)I_n-\omega^2 W^{-1}TW^{-1}T & \omega(1-\omega)W^{-1}T \\ -\omega W^{-1}T & (1-\omega)I_n\end{pmatrix}.\end{aligned}$$

After eliminating $((u^{(k+\frac{1}{2})})^T, (v^{(k+\frac{1}{2})})^T)^T$, the SSOR iteration scheme (4) can be written as

$$\begin{pmatrix} u^{(k+1)} \\ v^{(k+1)} \end{pmatrix} = \mathscr{H}_\omega \begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} + \mathscr{M}_\omega^{-1} b, \tag{5}$$

where

$$\mathscr{H}_\omega = \mathscr{U}_\omega \mathscr{L}_\omega = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \tag{6}$$

is the iteration matrix of the SSOR method, with

$$\begin{cases} H_{11} = (1-\omega)^2 I_n - \omega^2(1-\omega)(2-\omega)W^{-1}TW^{-1}T, \\ H_{12} = \omega(1-\omega)(2-\omega)W^{-1}T - \omega^3(2-\omega)W^{-1}TW^{-1}TW^{-1}T, \\ H_{21} = -\omega(1-\omega)(2-\omega)W^{-1}T, \\ H_{22} = (1-\omega)^2 I_n - \omega^2(2-\omega)W^{-1}TW^{-1}T, \end{cases}$$

and

$$\mathscr{M}_\omega^{-1} = \omega(2-\omega)\begin{pmatrix} W^{-1}-\omega^2 W^{-1}TW^{-1}TW^{-1} & \omega W^{-1}TW^{-1} \\ -\omega W^{-1}TW^{-1} & W^{-1}\end{pmatrix}.$$

Based on the above discussions, the SSOR method for the block two-by-two linear system (1) can be described as follows.

The SSOR method Given any initial vectors $u^{(0)}, v^{(0)} \in \mathbb{R}^n$, and relaxation factor ω with $\omega > 0$. For $k = 0, 1, 2, \cdots$, until the iteration sequence $\{((u^{(k)})^T, (v^{(k)})^T)^T\}$ converges, compute

$$\begin{cases} v^{(k+1)} = (1-\omega)^2 v^{(k)} - \omega(2-\omega)W^{-1}((1-\omega)Tu^{(k)} + \omega TW^{-1}(Tv^{(k)}+p)-q), \\ u^{(k+1)} = (1-\omega)^2 u^{(k)} + \omega W^{-1}((1-\omega)Tv^{(k)} + Tv^{(k+1)} + (2-\omega)p). \end{cases} \tag{7}$$

Since the SSOR method is parameter dependent, the choice of optimal parameters which result in fast convergence rate is very important for the efficiency of this method. In this paper, the optimal parameters of the SSOR method for solving the block two-by-two linear system (1) are given. Furthermore, an accelerated variant of the SSOR (ASSOR) method is introduced. Numerical experiments reveal that the SSOR and ASSOR methods with the optimal parameters are of great feasibility and effectiveness.

The organization of this paper is as follows. In Section 2, we study the selection of optimal iteration parameters for the SSOR method. In Section 3, an accelerated variant of the SSOR method is given. Its convergence analysis and a practical way of choosing iteration parameters are discussed. In Section 4, numerical experiments are presented to examine the feasibility and effectiveness of the SSOR and ASSOR methods for solving the linear system (1). Finally, in Section 5, we give some concluding remarks.

2 The optimal parameters of the SSOR method

In this section, we discuss the convergence and the selection of optimal parameters for the SSOR method. We need the following lemmas.

Lemma 1 [29] *Both roots of the real quadratic equation* $\lambda^2 - \phi\lambda + \psi = 0$ *have modulus less than one if and only if* $|\psi| < 1$ *and* $|\phi| < 1 + \psi$.

Lemma 2 [27] *Let the matrices* W *and* $T \in \mathbb{R}^{n\times n}$ *be symmetric positive definite and symmetric, respectively. Then the eigenvalues of the matrix* $S := W^{-1}T$ *are all real.*

In the following, the spectral set and the spectral radius of a square matrix H are denoted by $\sigma(H)$ and $\rho(H)$, respectively. Based on Lemmas 1 and 2, we have the following convergence results about the SSOR method for the block two-by-two linear system (1).

Theorem 1 *Let the matrices* W *and* $T \in \mathbb{R}^{n\times n}$ *be symmetric positive definite and symmetric, respectively. Then the SSOR method* (7) *for the block two-by-two linear system* (1) *is convergent if one of the following conditions holds:*

i) *if* $\rho(S) \leqslant 1$, $0 < \omega < 2$ *should be satisfied;*
ii) *if* $\rho(S) > 1$, $0 < \omega < 1-\sqrt{\frac{\rho(S)-1}{\rho(S)+1}}$ *or* $1+\sqrt{\frac{\rho(S)-1}{\rho(S)+1}} < \omega < 2$ *should be satisfied.*

Proof From Lemma 2, we know that S has the spectral decomposition $S = V\Lambda V^{-1}$, where $V \in \mathbb{R}^{n\times n}$ is an invertible matrix and $\Lambda = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_n)$ with μ_i $(i = 1, 2, \cdots, n)$ being eigenvalues of S. Define

$$\mathscr{P} := \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}.$$

Then from (6), we get

$$\mathscr{P}\mathscr{H}_\omega\mathscr{P}^{-1} = \begin{pmatrix} (1-\omega)^2 I_n - \omega^2(1-\omega)(2-\omega)\Lambda^2 & \omega(1-\omega)(2-\omega)\Lambda - \omega^3(2-\omega)\Lambda^3 \\ -\omega(1-\omega)(2-\omega)\Lambda & (1-\omega)^2 I_n - \omega^2(2-\omega)\Lambda^2 \end{pmatrix},$$

which is similar to $\mathscr{H}_\omega$. So they have the same eigenvalues. It is easy to verify that the $2n$ eigenvalues of the above matrix satisfy the real quadratic equations

$$\lambda^2 - (2(1-\omega)^2 - \omega^2(2-\omega)^2\mu^2)\lambda + (1-\omega)^4 = 0, \quad (\mu = \mu_1, \mu_2, \cdots, \mu_n). \tag{8}$$

From Lemma 1, we know that the roots of the quadratic equations (8) hold $|\lambda| < 1$ if and only if

$$\begin{cases} |(1-\omega)^4| < 1, \\ |2(1-\omega)^2 - \omega^2(2-\omega)^2\mu^2| < 1 + (1-\omega)^4, \end{cases}$$

which are equivalent to

$$0 < \omega < 2 \quad \text{and} \quad (1 + (1-\omega)^2)^2 > \omega^2(2-\omega)^2\mu^2. \tag{9}$$

Since $\mu \leqslant \rho(S)$, the second inequality in (9) holds if

$$(1 + (1-\omega)^2)^2 > \omega^2(2-\omega)^2\rho^2(S).$$

By simple calculations, we have

$$(1 + \rho(S))\omega^2 - 2(1 + \rho(S))\omega + 2 > 0.$$

Solving the above inequality, we immediately obtain the convergence results. □

Remark 1 When $\rho(S) > 1$, we know that the convergence interval length of the SSOR method is $2(1 - \sqrt{\frac{\rho(S)-1}{\rho(S)+1}})$ from Theorem 1. Besides, we know that the GSOR method is convergent if $0 < \omega < \frac{2}{1+\rho(S)}$ from Theorem 1 in [27]. So the convergence interval length of the GSOR method is $\frac{2}{1+\rho(S)}$. Note that

$$2\left(1 - \sqrt{\frac{\rho(S)-1}{\rho(S)+1}}\right) - \frac{2}{1+\rho(S)} = \frac{2\left(\rho(S) - \sqrt{\rho(S)^2 - 1}\right)}{1+\rho(S)} > 0.$$

Then we know that the SSOR method has larger range of convergence than the GSOR method.

Next, by minimizing the spectral radius of the iteration matrix $\mathscr{H}_\omega$, which is defined in (6), we discuss the choice of the optimal relaxation parameters ω. The technique we used here is similar to Theorem 2 of [27].

Theorem 2 *Let the matrices W and $T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric, respectively. Assume that the conditions of Theorem 1 are satisfied. Then the optimal parameters of the SSOR method (7) are given by*

$$\omega_{opt} = 1 \pm \frac{\sqrt{\rho(S)^2 + 1} - 1}{\rho(S)}, \tag{10}$$

and the corresponding optimal convergence factor of the SSOR method is

$$\rho(\mathscr{H}_{\omega_{opt}}) = 1 - \frac{2}{\sqrt{\rho(S)^2 + 1} + 1}. \tag{11}$$

Proof From (8) we know that the eigenvalues λ of the iteration matrix $\mathscr{H}_\omega$ satisfy

$$\lambda - (1-\omega)^2 = \pm\omega(2-\omega)\mu\sqrt{-\lambda}. \tag{12}$$

It can be viewed as the intersection points of the straight line

$$f_\omega(\lambda) = -\frac{\lambda - (1-\omega)^2}{\omega(2-\omega)}$$

that passes through the point $(1, -1)$, and the parabolas

$$g(\lambda) = \pm\mu\sqrt{-\lambda}, \quad (\mu = \mu_1, \mu_2, \cdots, \mu_n).$$

The discriminant of (8) or (12) is denoted by

$$\Delta(\omega, \mu) := (\omega^2(2-\omega)^2\mu^2 + 4\omega(2-\omega) - 4)\omega^2(2-\omega)^2\mu^2.$$

When $\Delta(\omega, \mu) \geqslant 0$, the quadratic equation (8) has two real roots λ_1 and λ_2. For each μ, these roots are abscissas of the intersections of $f_\omega(\lambda)$ and $g(\lambda)$, as illustrated in Fig. 1.

The largest abscissa of the intersection point decreases when the slope of $f_\omega(\lambda)$ decreases until it becomes tangent to $g(\lambda)$. Under this condition, we have $\lambda_1 = \lambda_2$. Then $\Delta(\omega, \mu) = 0$, or equivalently,

$$\omega^2(2-\omega)^2\mu^2 + 4\omega(2-\omega) - 4 = 0 \quad \text{or} \quad \mu = 0. \tag{13}$$

If $\mu = 0$, $|\lambda_1| = |\lambda_2| = (1-\omega)^2$. So $\omega = 1$ is the best choice as $\lambda_1 = \lambda_2 = 0$.

If $\mu \neq 0$, (13) is equivalent to $\mu^2\tilde{\omega}^2 + 4\tilde{\omega} - 4 = 0$ with $\tilde{\omega} := \omega(2-\omega)$. From Theorem 1 we know that $\tilde{\omega} > 0$, so

$$\tilde{\omega} = \frac{2}{\sqrt{1+\mu^2}+1}.$$

Then by simple calculations, we have

$$\omega = 1 \pm \frac{\sqrt{\mu^2+1}-1}{\mu} \quad \text{and} \quad \lambda_1 = \lambda_2 = -(1-\omega)^2.$$

Note that $\pm\rho(S)\sqrt{-\lambda}$ is an envelope for all the curves $g(\lambda)$. So the minimum value of $\rho(\mathscr{H}_\omega)$ is attained at $(1-\omega)^2$ with

$$\omega = 1 \pm \frac{\sqrt{\rho(S)^2+1}-1}{\rho(S)} \quad \text{and} \quad \rho(\mathscr{H}_\omega) = 1 - \frac{2}{\sqrt{\rho(S)^2+1}+1}.$$

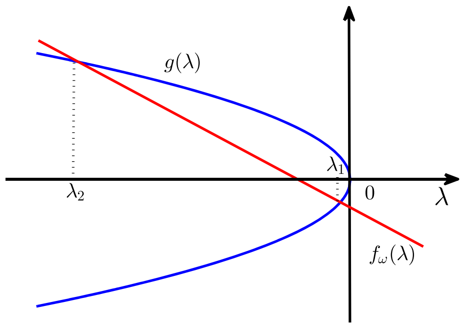

Fig. 1 Condition for minimization of $\rho(\mathscr{H}_\omega)$

When $\Delta(\omega, \mu) < 0$, the quadratic equation (8) has two conjugate complex roots λ_1 and λ_2. By some calculations, we have $|\lambda_1| = |\lambda_2| = (1-\omega)^2$ with

$$0 < \omega < 1 - \frac{\sqrt{\mu^2+1}-1}{\mu} \quad \text{or} \quad 1 + \frac{\sqrt{\mu^2+1}-1}{\mu} < \omega < 2.$$

Thus it holds

$$\rho(\mathscr{H}_\omega) = (1-\omega)^2 > 1 - \frac{2}{\sqrt{\rho(S)^2+1}+1}.$$

Combining with the above two situations, we obtain that (10) holds, and the optimal convergence factor is

$$\rho(\mathscr{H}_{\omega_{\rm opt}}) = (1-\omega_{\rm opt})^2 = 1 - \frac{2}{\sqrt{\rho(S)^2+1}+1}.$$

□

Remark 2 From Theorem 2, we see that the optimal convergence factor of the SSOR method here is the same as that of the GSOR method in [27]. However, the SSOR method has two choices for the optimal iteration parameter but the GSOR method has only a single choice. Thus, the SSOR method may be more practical to implement under certain situations.

Assume that the matrices W and T are symmetric positive definite and symmetric positive semi-definite, respectively. Then the eigenvalues of $S = W^{-1}T$ are all nonnegative. Denote the largest eigenvalue of S as $\mu_{\max}$. Based upon Theorem 2, we can obtain the following convergence conditions for the SSOR method, which can be used easily in practical applications.

Corollary 1 *Let the matrices W and $T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semi-definite, respectively. Then the SSOR method* (7) *for the block two-by-two linear system* (1) *is convergent if one of the following conditions holds:*

i) *if $\mu_{\max} \leqslant 1$, $0 < \omega < 2$ should be satisfied;*
ii) *if $\mu_{\max} > 1$, $0 < \omega < 1 - \sqrt{\frac{\mu_{\max}-1}{\mu_{\max}+1}}$ or $1 + \sqrt{\frac{\mu_{\max}-1}{\mu_{\max}+1}} < \omega < 2$ should be satisfied.*

Moreover, the optimal parameters of the SSOR method for the block two-by-two linear system (1) *are given by*

$$\omega_{opt} = 1 \pm \frac{\sqrt{\mu_{\max}^2+1}-1}{\mu_{\max}},$$

and the corresponding optimal convergence factor is

$$\rho(\mathscr{H}_{\omega_{opt}}) = 1 - \frac{2}{\sqrt{\mu_{\max}^2+1}+1}. \tag{14}$$

3 Accelerated variant of the SSOR method

From Theorem 2 and Corollary 1, we see that $\rho(\mathscr{H}_{\omega_{\rm opt}})$ defined as in (11) or (14), respectively, is increasing functions with respective to $\rho(S)$ or $\mu_{\max}$. To speed up the convergence rate of the SSOR method, $\rho(S)$ or $\mu_{\max}$ should be small enough. The accelerated variant of the SSOR method proposed in this part can help to realize this aim.

We consider the following preconditioned form of (1) [1, 23]

$$\begin{pmatrix} I & I \\ -I & I \end{pmatrix}\begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} I & I \\ -I & I \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix},$$

or equivalently,

$$\begin{pmatrix} \tilde{W} & -\tilde{T} \\ \tilde{T} & \tilde{W} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}, \tag{15}$$

where $\tilde{W} := W + T$, $\tilde{T} := T - W$ and $\tilde{p} := p + q$, $\tilde{q} := q - p$. Applying the SSOR method to the linear system (15), we have the ASSOR method for the linear system (1).

The ASSOR method Given any initial vectors $u^{(0)}, v^{(0)} \in \mathbb{R}^n$, and the relaxation factor ω with $\omega > 0$. For $k = 0, 1, 2, \cdots$, until the iteration sequence $\{((u^{(k)})^T, (v^{(k)})^T)^T\}$ converges, compute

$$\begin{cases} v^{(k+1)} = (1-\omega)^2 v^{(k)} - \omega(2-\omega)\tilde{W}^{-1}((1-\omega)\tilde{T}u^{(k)} + \omega\tilde{T}\tilde{W}^{-1}(Tv^{(k)} + \tilde{p}) - \tilde{q}), \\ u^{(k+1)} = (1-\omega)^2 u^{(k)} + \omega\tilde{W}^{-1}((1-\omega)\tilde{T}v^{(k)} + \tilde{T}v^{(k+1)} + (2-\omega)\tilde{p}). \end{cases} \tag{16}$$

Lemma 3 *Let the matrices W and T be symmetric positive definite and symmetric positive semi-definite, respectively. Denote $\tilde{S} := \tilde{W}^{-1}\tilde{T}$ with $\tilde{W} := W + T$ and $\tilde{T} := T - W$. Then*

$$\rho(\tilde{S}) = \max\left\{\left|\frac{1-\mu_{\min}}{1+\mu_{\min}}\right|, \left|\frac{1-\mu_{\max}}{1+\mu_{\max}}\right|\right\} < 1,$$

where $\mu_{\max}$ and $\mu_{\min}$ are the largest and smallest eigenvalue of $S = W^{-1}T$, respectively.

Proof Let (λ, x) be an eigenpair of $\tilde{S}$. Then $\tilde{S}x = \lambda x$, or equivalently,

$$(T - W)x = \lambda(W + T)x.$$

By simple calculations, we have $\lambda = \frac{\xi-1}{\xi+1}$ with $\xi := \frac{x^*Tx}{x^*Wx} \geqslant 0$. Since λ is an increasing function with respect to ξ, we can obtain the result of Lemma 3 obviously. □

Based on Theorems 1 and 2 and Lemma 3, we can obtain the following result.

Theorem 3 *Suppose the conditions of Lemma 3 are satisfied. Then the ASSOR method for the block two-by-two linear system (1) is convergent if $0 < \omega < 2$. The* *optimal parameters of the ASSOR method* (16) *for the block two-by-two linear system* (15) *are given by*

$$\omega_{opt} = 1 \pm \frac{\sqrt{\rho(\tilde{S})^2 + 1} - 1}{\rho(\tilde{S})}, \tag{17}$$

and the corresponding optimal convergence factor of the iteration matrix is

$$1 - \frac{2}{\sqrt{\rho(\tilde{S})^2 + 1} + 1}. \tag{18}$$

Remark 3 From Theorem 3, we see that the ASSOR method has a weaker condition to guarantee convergence comparing with the SSOR method. Besides, from Lemma 3, we know that $\rho(\tilde{S}) < 1$. Thus, when $\rho(S) > 1$, the convergence factor of the former is always smaller than that of the latter.

The SSOR and ASSOR methods are all parameter dependent, so the choice of iteration parameters is crucial for their implementations. The optimal parameters proposed in Theorem 2, Corollary 1 and Theorem 3 may be the best ones. However, all of them are related to the eigenvalues of S or $\tilde{S}$. When the problem size is large enough, it is not easy to calculate the eigenvalues of the two matrices. Now, we propose a more practical way for the choice of iteration parameters for the ASSOR method.

From Lemma 3, we know that $\rho(\tilde{S}) < 1$. Then the optimal parameter ω_{opt} in (17) satisfies

$$\omega_{\text{opt}} = 1 + \frac{\sqrt{\rho(S)^2 + 1} - 1}{\rho(S)} < \sqrt{2} \approx 1.4142$$

or

$$\omega_{\text{opt}} = 1 - \frac{\sqrt{\rho(S)^2 + 1} - 1}{\rho(S)} > 2 - \sqrt{2} \approx 0.5858.$$

The corresponding optimal convergence factor in (18) satisfies

$$\rho(\mathscr{H}_{\omega_{\text{opt}}}) = 1 - \frac{2}{\sqrt{\rho(\tilde{S})^2 + 1} + 1} < 1 - \frac{2}{\sqrt{2} + 1} \approx 0.1716.$$

Thus, we may simply choose $\omega = 1.41$ or $\omega = 0.59$ for the ASSOR method in the practical implements.

4 Numerical experiments

In this section, three examples from [7] are used to illustrate the numerical feasibility and effectiveness of the SSOR and ASSOR methods for solving the block two-by-two linear system (1). These methods are compared with the HSS [10], MHSS [7] and GSOR [27] methods. We denote the number of iteration steps as "IT", the elapsed

CPU time in seconds as “CPU”, and the relative residual norm as “RES”. Here, the “RES” is defined by

$$\text{RES} := \frac{\| b - Ax^{(k)} \|_2}{\| b \|_2},$$

where $x^{(k)} = ((u^{(k)})^T, (v^{(k)})^T)^T$ with $u^{(k)}, v^{(k)} \in \mathbb{R}^n$ being the current approximate solutions.

All numerical experiments are implemented in MATLAB (version 8.0.0.783 (R2012b)) with machine precision 10^{-16} on a personal computer with Genuine Inter (R) Dual-Core CPU T4300 2.09 GHz, and 1.93GB memory. We list the numerical results (IT, CPU times, and RES) for Examples 1–3. The initial guess for all examples are chosen to be zero vectors and the iteration is terminated if the current iterate $x^{(k)}$ satisfies RES $\leq 10^{-6}$.

Example 1 [1, 7] Consider the complex symmetric linear system of the form:

$$\left[\left(K + \frac{3-\sqrt{3}}{\tau} I_m\right) + i\left(K + \frac{3+\sqrt{3}}{\tau} I_m\right)\right]\tilde{x} = \tilde{b}, \tag{19}$$

where τ is the time step-size, and $K = I_m \otimes V_m + V_m \otimes I_m$ with $V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. K is the five-point centered difference approximation of the negative Laplacian operator $L = -\Delta$ with homogeneous Dirichlet boundary conditions on uniform mesh in the unit square $[0, 1] \times [0, 1]$. Here $\otimes$ is the Kronecker product symbol and $h = \frac{1}{m+1}$ is the discretization mesh-size.

This complex symmetric linear system arises in centered difference discretization of R_{22}-Padé approximations in the time integration of parabolic partial differential equations [1]. In this example, K is an $n \times n$ block diagonal matrix with $n = m^2$. In our tests, we take $\tau = h$. Furthermore, we normalize coefficient matrix and right-hand side of (19) by multiplying both by h^2. We take

$$W = K + \frac{3-\sqrt{3}}{\tau} I_m \quad \text{and} \quad T = K + \frac{3+\sqrt{3}}{\tau} I_m.$$

The right-hand vector $\tilde{b}$ is given with its jth entry

$$\tilde{b}_j = \frac{(1-i)j}{\tau(j+1)^2}, \quad j = 1, 2, \cdots, n.$$

Example 2 [7] Consider the complex symmetric linear system of the form:

$$[(-\nu^2 M + K) + i(\nu C_V + C_H)]\tilde{x} = \tilde{b}, \tag{20}$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and hysteretic damping matrices, respectively. ω is the driving circular frequency and K is defined the same as in Example 1.

This complex symmetric linear system arises in direct domain analysis of an n-degree-of-freedom (n-DOF) linear system [12]. In this example, K is an $n \times n$ block diagonal matrix with $n = m^2$. We take

$$W = -\nu^2 M + K \quad \text{and} \quad T = \nu C_V + C_H,$$

with $C_H = \mu K$ and μ being a damping coefficient, $M = I_m$, $C_V = 10 I_m$. In addition, we set $\nu = \pi$, $\mu = 0.02$ and the right-hand-side vector $\tilde{b}$ is chosen such that the exact solution of the linear system (20) is $(1+i)(1, 1, \cdots, 1)^T \in \mathbb{R}^n$. Similar to Example 1, the linear system is normalized by multiplying both sides with h^2.

Example 3 [7] Consider the complex symmetric linear system of the form

$$(W + iT)\tilde{x} = \tilde{b}$$

with $T = I_m \otimes V + V \otimes I_m$ and $W = 10(I_m \otimes V_c + V_c \otimes I_m) + 9(e_1 e_m^T + e_m e_1^T) \otimes I_m$,where $V = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}$, $V_c = V - e_1 e_m^T - e_m e_1^T \in \mathbb{R}^{m \times m}$ with e_1 and e_m being the first and the last unit vectors in $\mathbb{R}^m$, respectively. The right-hand-side vector $\tilde{b}$ is chosen such that the exact solution of this linear system is $(1+i)(1, 1, \cdots, 1)^T \in \mathbb{R}^n$.

This complex symmetric linear system is an artificially constructed one, which is challenging for iteration solvers. Here T and W correspond to the five-point centered difference approximation of the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions, respectively, on uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = \frac{1}{m+1}$.

In Table 1, the optimal iteration parameters of the tested methods for the above three examples are listed. The optimal parameters of the HSS and MHSS methods are those presented in [7], and those of the GSOR method are chosen based on Theorem 2 of [27]. As for the SSOR and ASSOR methods, the optimal parameters are chosen based on Theorems 2 and 3, which are

$$1 - \frac{\sqrt{\rho(S)^2 + 1} - 1}{\rho(S)} \quad \text{and} \quad 1 - \frac{\sqrt{\rho(\tilde{S})^2 + 1} - 1}{\rho(\tilde{S})},$$

respectively. From this table, we see that the optimal parameters decrease with the increasing of problems size, except for the ASSOR method. The optimal parameters for the ASSOR method keep almost unchanged with the increasing of problem size for each example. Besides, the optimal parameters of the GSOR and SSOR methods for Example 2 are also almost unchanged.

In Fig. 2, we depict the theoretical optimal parameters ω_{opt} of the GSOR, SSOR and ASSOR methods with respect to the changing of problem sizes. From Fig. 2, we see that, for each fixed m, the two optimal iteration parameters of the SSOR or ASSOR method are centered exactly on 1. The smaller optimal iteration parameters of the ASSOR method are always larger than those of the SSOR and GSOR methods for Examples 1 and 2. However, for Example 3, they are smaller than those of the SSOR and GSOR methods when m is small, and larger when m is large. Also,

Table 1 The optimal parameters for the tested methods

		Grid				
	Method	16 × 16	32 × 32	64 × 64	128 × 128	256 × 256
Example 1	HSS	0.81	0.55	0.37	0.28	0.20
	MHSS	1.06	0.75	0.54	0.40	0.30
	GSOR	0.55	0.49	0.45	0.43	0.41
	SSOR	0.33	0.29	0.26	0.24	0.24
	ASSOR	0.80	0.77	0.75	0.74	0.72
Example 2	HSS	0.42	0.23	0.12	0.07	0.04
	MHSS	0.21	0.08	0.04	0.02	0.01
	GSOR	0.45	0.45	0.45	0.44	0.44
	SSOR	0.26	0.26	0.26	0.26	0.26
	ASSOR	0.61	0.60	0.60	0.59	0.59
Example 3	HSS	4.41	2.71	1.61	0.93	0.53
	MHSS	1.61	1.01	0.53	0.26	0.13
	GSOR	0.90	0.76	0.56	0.35	0.19
	SSOR	0.69	0.52	0.34	0.19	0.10
	ASSOR	0.62	0.62	0.62	0.61	0.61

we note that when the problem size is large enough, ω_{opt} for all the three methods are almost unchanged. These phenomena are in accordance to the results in Table 1.

In order to see the role of the iteration parameter ω in the convergence behaviors of the SSOR and ASSOR methods, we illustrate the changing of the number of iteration steps with respect to ω for the three examples in Fig. 3. Here, the number of iteration steps are designated as 300 for the cases which are convergent with iteration steps higher than 300. From Fig. 3, we see that the graphs of the ASSOR method are rather flat near the minimum. What is more, the ASSOR method has larger convergence regions than the SSOR method. Thus, the ASSOR method is more insensitive with respective to ω comparing with the SSOR method.

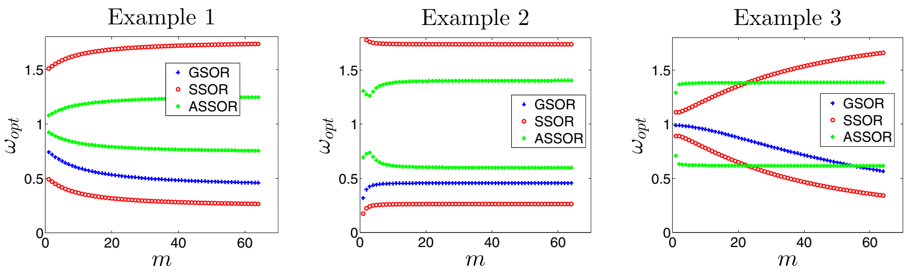

Fig. 2 The optimal ω_{opt} for GSOR, SSOR and ASSOR methods with varying m

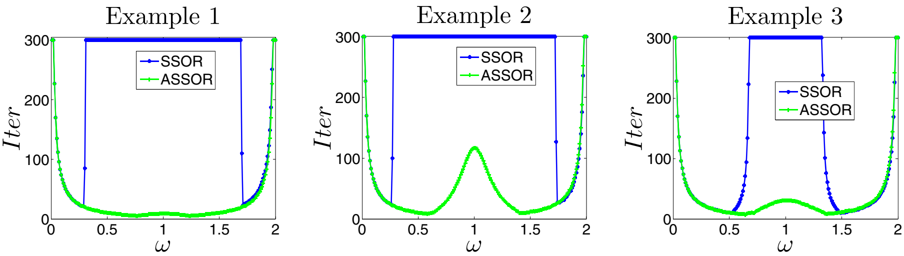

Fig. 3 Iteration steps for SSOR and ASSOR methods with varying ω when $m = 32$

Tables 2, 3 and 4 list the IT, CPU and RES for Examples 1–3 with the varying of problems size. The iteration parameters used in Tables 2–4 are chosen according to Table 1. Note that the HSS and MHSS methods are employed to solve the original complex symmetric linear system (2), while the GSOR and SSOR methods are employed to solve the equivalent block two-by-two linear system (1), and the ASSOR method is employed to solve the linear system (15).

From Tables 2–4, we see that for each example, the GSOR, SSOR and ASSOR methods with theoretical optimal parameters are superior to the HSS and MHSS methods. The GSOR and SSOR methods need less iteration steps and CPU times comparing with the HSS and MHSS methods. As for the GSOR and SSOR methods, their performances are much the same. The two methods need almost the same number of iteration steps, though the GSOR method needs less CPU times. However, the

Table 2 Numerical results for Example 1

Method		16 × 16	32 × 32	64 × 64	128 × 128	256 × 256
HSS	IT	44	65	97	136	191
	CPU	0.0287	0.1302	1.7926	17.8857	191.4391
	RES	9.16e-07	9.82e-07	9.84e-07	9.26e-07	9.72e-07
MHSS	IT	40	54	73	98	133
	CPU	0.0203	0.0901	0.8911	6.0287	46.1263
	RES	9.67e-07	9.61e-07	9.41e-07	9.35e-07	9.99e-07
GSOR	IT	19	22	24	26	27
	CPU	0.0098	0.0401	0.2504	1.4217	8.5623
	RES	9.02e-07	5.57e-07	8.01e-07	5.57e-07	8.88e-07
SSOR	IT	19	21	23	26	26
	CPU	0.0112	0.0501	0.3004	1.9129	12.5080
	RES	6.91e-07	5.63e-07	9.53e-07	6.33e-07	6.31e-07
ASSOR	IT	5	5	6	6	6
	CPU	0.0091	0.0124	0.0569	0.4917	2.1282
	RES	3.79e-07	6.59e-07	6.26e-08	9.74e-08	2.33e-07

Table 3 Numerical results for Example 2

Method		16 × 16	32 × 32	64 × 64	128 × 128	256 × 256
HSS	IT	86	153	287	540	1084
	CPU	0.0310	0.1302	1.7926	17.8857	865.4311
	RES	9.11e-07	9.85e-07	9.81e-07	9.99e-07	9.89e-07
MHSS	IT	34	38	50	81	139
	CPU	0.0223	0.0710	0.6209	4.9171	48.2594
	RES	9.67e-07	9.61e-07	9.41e-07	9.35e-07	9.99e-07
GSOR	IT	25	24	24	24	24
	CPU	0.0107	0.0398	0.2804	1.3419	7.7812
	RES	9.02e-07	5.57e-07	8.01e-07	5.57e-07	8.88e-07
SSOR	IT	19	21	23	26	26
	CPU	0.0114	0.0499	0.3749	1.8113	11.1560
	RES	9.64e-07	5.60e-07	9.73e-07	9.66e-07	9.65e-07
ASSOR	IT	8	9	9	8	8
	CPU	0.0093	0.0215	0.1031	0.7219	3.5156
	RES	4.93e-07	1.49e-07	5.80e-07	4.71e-07	3.42e-07

ASSOR method performs the best. It needs the least iteration steps and CPU times. Hence, the numerical results show that the SSOR and ASSOR methods with optimal parameters proposed in this paper can effectively solve the block two-by-two linear systems comparing with the HSS, MHSS and GSOR methods.

Table 4 Numerical results for Example 3

Method		16 × 16	32 × 32	64 × 64	128 × 128	256 × 256
HSS	IT	84	137	223	390	746
	CPU	0.0301	0.3605	4.1259	49.8517	674.0512
	RES	9.26e-07	9.28e-07	9.72e-07	9.86e-07	9.98e-07
MHSS	IT	53	76	130	246	468
	CPU	0.0221	0.2512	2.3277	24.0692	241.5209
	RES	9.67e-07	9.61e-07	9.41e-07	9.35e-07	9.99e-07
GSOR	IT	7	11	18	34	67
	CPU	0.0121	0.0301	0.2992	2.7439	31.8258
	RES	2.95e-07	2.46e-07	5.16e-07	7.32e-07	8.47e-07
SSOR	IT	6	10	17	33	66
	CPU	0.0131	0.0401	0.4206	4.0258	46.1263
	RES	9.68e-07	5.12e-07	8.00e-07	9.64e-07	9.38e-07
ASSOR	IT	8	8	8	8	8
	CPU	0.0104	0.0314	0.2250	1.0240	6.3181
	RES	4.30e-07	4.34e-07	4.31e-07	4.31e-07	4.33e-07

Table 5 Numerical results of the ASSOR method with $\omega = 0.59$

		16×16	32×32	64×64	128×128	256×256
Example 1	IT	8	8	8	8	8
	CPU	0.0119	0.0146	0.0571	0.5804	2.2953
	RES	6.37e-07	6.40e-07	6.38e-07	6.37e-07	6.37e-07
Example 2	IT	8	9	8	8	8
	CPU	0.0102	0.0286	0.1213	0.7813	3.7109
	RES	6.74e-07	1.60e-07	7.72e-07	4.71e-07	3.42e-07
Example 3	IT	8	8	8	8	8
	CPU	0.0132	0.0382	0.2551	1.1423	6.9354
	RES	6.10e-07	5.45e-07	4.98e-07	4.77e-07	4.64e-07

In practical applications, it is a tough thing to calculate the optimal ω_{opt} when the problem size is too large. In Table 5 we list the IT, CPU and RES for Examples 1–3 with the varying of problems size. The iteration parameters for the ASSOR method are 0.59 based on the discussions at the end of Section 3. The numerical results of Table 5 is to show that $\omega = 0.59$ can be considered as a reasonable approximation of the optimal ω_{opt}.

For comparison, we observe from the numerical results in Table 5 that the performances of the ASSOR methods with $\omega = 0.59$ are almost the same as those with the optimal parameters, see numerical results of the ASSOR methods in Tables 2–4. From the numerical results of Table 5 we see that $\omega = 0.59$ can be considered as a reasonable approximation of the optimal ω_{opt}. Thus, we may simply choose $\omega = 0.59$ in practical computation as a substitution for the ASSOR method.

5 Conclusions

In this paper, we shed some light on the choice of optimal iteration parameters of the SSOR method for solving a class of block two-by-two linear systems. The optimal iteration parameters of this method are obtained under suitable convergence analysis. Besides, an accelerated variant of the SSOR method, namely the ASSOR method, is established, which is more effective than the SSOR method. Numerical experiments show that the SSOR and ASSOR methods with optimal parameters proposed in this paper can solve the block two-by-two linear systems more effectively comparing with the HSS, MHSS and GSOR methods. Hence, this work gives a better parameter choice for the SSOR method to solve a class of block two-by-two linear systems.

Acknowledgments We would like to express our sincere thanks to the two unknown reviewers for their careful reading of the manuscript. Their useful comments and valuable suggestions greatly improve the quality of the paper.

References

1. Axelsson, O., Kucherov, A.: Real valued iterative methods for solving complex symmetric linear systems. Numer. Linear Algebra Appl. **7**, 197–218 (2000)
2. Axelsson, O., Neytcheva, M., Ahmad, B.: A comparison of iterative methods to solve complex valued linear algebraic systems. Numer. Algorithm **66**(4), 811–841 (2014)
3. Bai, Z.-Z.: Structured preconditioners for nonsingular matrices of block two-by-two structures. Math. Comput. **75**(254), 791–815 (2006)
4. Bai, Z.-Z.: Rotated block triangular preconditioning based on PMHSS. Sci. China Math. **56**, 2523–2538 (2013)
5. Bai, Z.-Z.: On preconditioned iteration methods for complex linear systems. J. Eng. Math. doi:10.1007/s10665-013-9670-5 (2015)
6. Bai, Z.-Z., Benzi, M., Chen, F.: On preconditioned MHSS iteration methods for complex symmetric liinear systems. Numer. Algorithm **56**, 297–317 (2011)
7. Bai, Z.-Z., Benzi, M., Chen, F.: Modified HSS iteration methods for a class of complex symmetric liinear systems. Computing **87**, 93–111 (2014)
8. Bai, Z.-Z., Benzi, M., Chen, F., Wang, Z.-Q.: Preconditioned MHSS iteration method for a class of block two-by-two linear systems with applications to distributed control problems. IMA J. Numer. Anal. **33**, 343–369 (2013)
9. Bai, Z.-Z., Chen, F., Wang, Z.-Q.: Additive block diagonal preconditioning for block two-by-two linear systems of skew-Hamiltonian coefficient matrices. Numer. Algorithm **62**, 655–675 (2013)
10. Bai, Z.-Z., Golub, G.H., Ng, M.K.: Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. SIAM J. Matrix Anal. Appl. **24**(3), 603–626 (2003)
11. Bai, Z.-Z., Parlett, B.N., Wang, Z.-Q.: On generalized successive overrelaxation methods for augmented linear systems. Numer. Math. **102**(1), 1–38 (2005)
12. Benzi, M., Bertaccini, D.: Block preconditioning of real-valued iterative algorithms for complex linear systems. IMA J. Numer. Anal. **28**, 598–618 (2008)
13. Benzi, M., Golub, G.H., Liesen, J.: Numerical solution of saddle point problems. Acta Numer. **14**, 1–137 (2005)
14. Bertaccini., D.: Efficient solvers for sequences of complex symmetric linear systems. Electr. Trans. Numer. Anal. **18**, 49–64 (2004)
15. Betts, J.T.: Practical methods foroptimal controlusing nonlinear programming. SIAM, Philadelphia (2001)
16. Bunse, G.M., Stöver, R.: On a conjugate gradient-type method for solving complex symmetric linear systems. Linear Algebra Appl. **287**(1), 105–123 (1999)
17. Day, D., Heroux, M.A.: Solving complex-valued linear systems via equivalent real formulations. SIAM J. Sci. Comput. **23**(2), 480–498 (2001)
18. Dehghan, M., Dehghani, M.M., Hajarian, M.: A generalized preconditioned MHSS method for a class of complex symmetric linear systems. J. Math. Model. Anal. **18**(4), 561–576 (2013)
19. Feriani, A., Perotti, F., Simoncini, V.: Iterative system solvers for the frequency analysis of linear mechanical systems. Methods Appl. Mech. Eng. **190**, 1719–1739 (2000)
20. Frommer, A., Lippert, T., Medeke, B., Schilling, K.: Numerical challenges in lattice quantum chromodynamics. Lecture notes in computational science and engineering, vol. 15. Springer, Heidelberg (2000)
21. Guo, X.-X., Wang, S.: Modified HSS iteration method for a class of non-symmetric positive definite linear systems. Appl. Math. Comput. **218**, 10122–10128 (2012)
22. Golub, G.H., Wu, X., Yuan, J.-Y.: SOR-like methods for augmented systems. BIT Numer. Math. **41**(1), 71–85 (2001)
23. Hezari, D., Edalatpour, V., Salkuyeh, D.K.: Preconditioned GSOR iterative method for a class of complex symmetric system of linear equations. Numer. Linear Algebera Appl. doi:10.1002/nla.1987 (2015)
24. Lang, C., Ren, Z.-R.: Inexact rotated block triangular preconditioners for a class of block two-by-two matrices. J. Eng. Math. doi:10.1007//s13160-014-0140-x (2013)
25. Lass, O., Vallejos, M., Borzì, A., Douglas, C.C.: Implementation and analysis of multigrid schemes with finite elements for elliptic optimal control problems. Computing **84**, 27–48 (2009)
26. Saad, Y.: Iterative Methods for Sparse Linear Systems. SIAM, Philadephia (2003)

27. Salkuyeh, D.K., Hezari, D., Edalatpour, V.: Generalized successive overrelaxation iterative method for a class of complex symmetric linear system of equations. Int. J. Comput. Math. **92**(4), 802–815 (2015)
28. Sommerfeld, A.: Partial Differential Equations. Academic Press, New York (1949)
29. Young, D.M.: Iterative Solution for Large Linear Systems. Academic Press, New York (1971)
30. Wu, S.-L.: Several variants of the Hermitian and skew-Hermitian splitting method for a class of complex symmetric linear systems. Numer. Linear Algebera Appl. **22**(2), 338–356 (2015)
31. Zhang, G.-F., Zheng, Z.: A parameterized splitting iteration methods for complex symmetric linear systems. Jpn. J. Indust. Appl. Math. **31**, 265–278 (2014)